

Renormalisation group

Link: <https://www.youtube.com/watch?v=4a4S-283c1o>

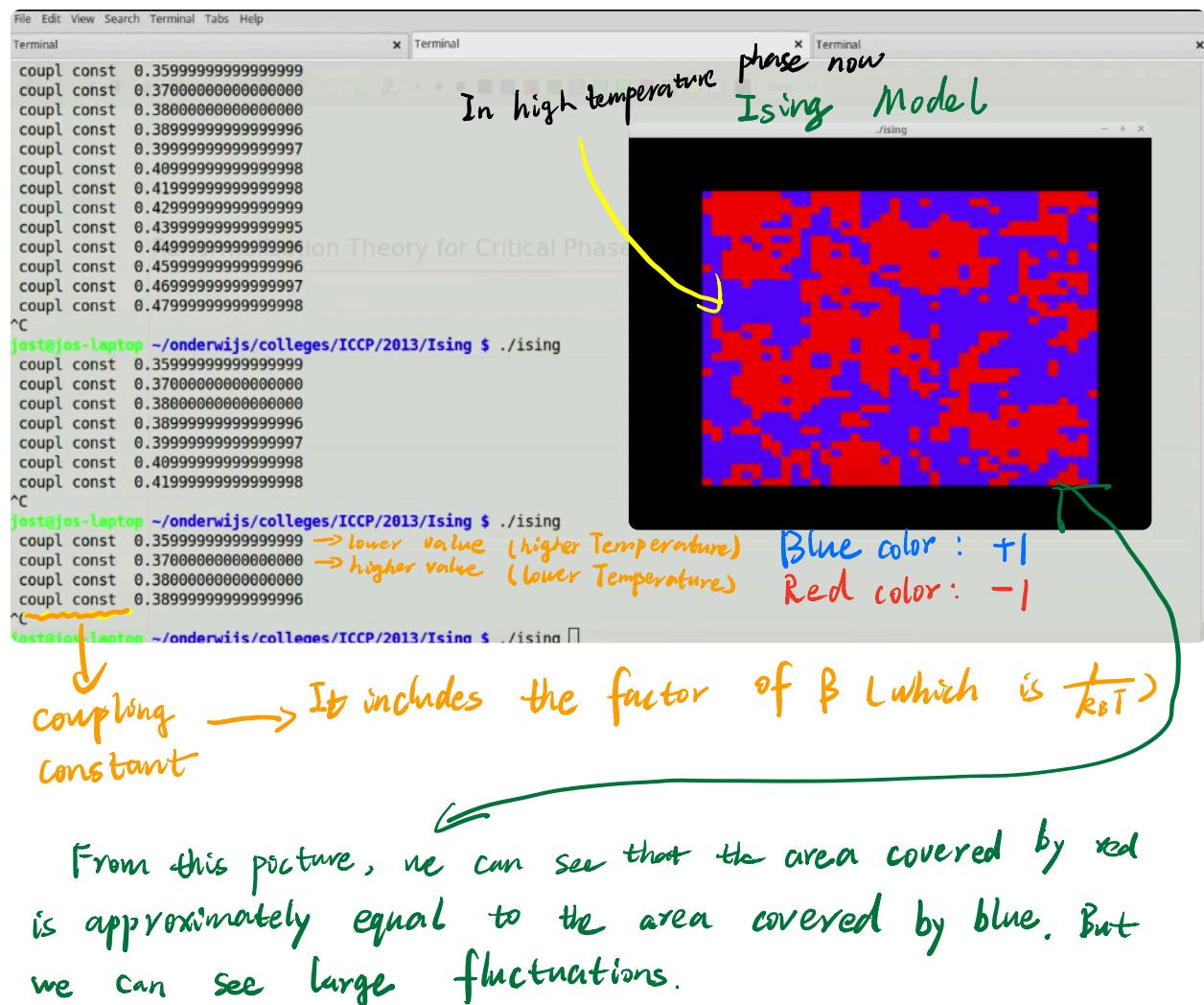
Critical phase transition accepts a very weird behavior which is characterized by all kinds of divergences and those divergences are quantified by critical exponents.

Main feature of renormalization theory which explains how these different critical exponents are related.

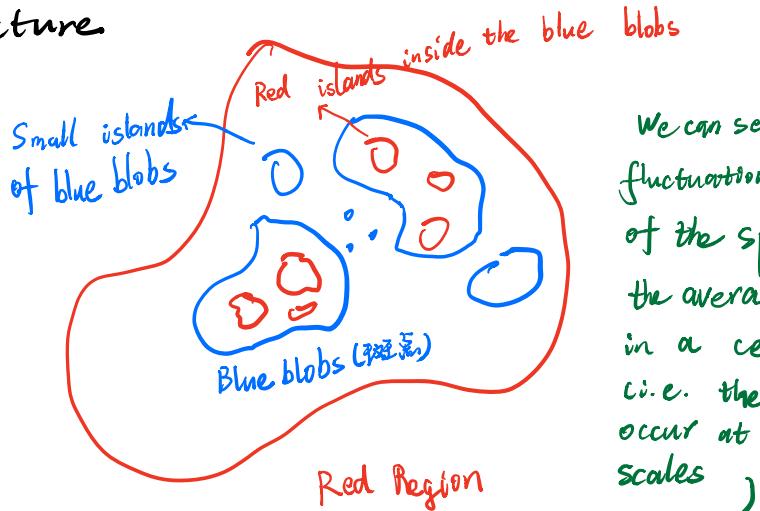
Renormalization Theory for Critical Phase Transitions

Using Ising model

(In Ising Model, there spins and these spins can only take on 2 different values, +1 or -1. And there is a Nearest Neighbor interaction which **tries to keep the spins aligned**. So if there is a +1 at a lattice site, then the neighboring sites also want to be +1. But the temperature will compete!)



Droplet picture



We can see the fluctuations / deviations of the spins from the average value in a certain region (i.e. the islands) occur at all length scales

In this case, we can say that the system seems to have some scale invariance and this scale invariance forms the basis of the idea of renormalization theory.

Critical Phenomena & Critical Exponents



They quantize the non-analytic behavior that we see in different quantities in the thermodynamic limit when we approach a critical temperature (T_c)

magnetization: m , which vanishes when we approach T_c from below T_c .

$$m \sim (T - T_c)^\beta \quad T < T_c$$

$$C_h \sim |T - T_c|^{-\alpha} \quad (0 < \alpha < 1)$$

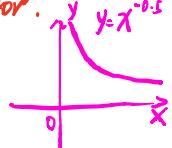
Correlation length $\xi \sim |T - T_c|^{-\nu}$

magnetic susceptibility $\chi \sim |T - T_c|^{-\gamma}$

$m \sim h^{\frac{1}{\delta}}$

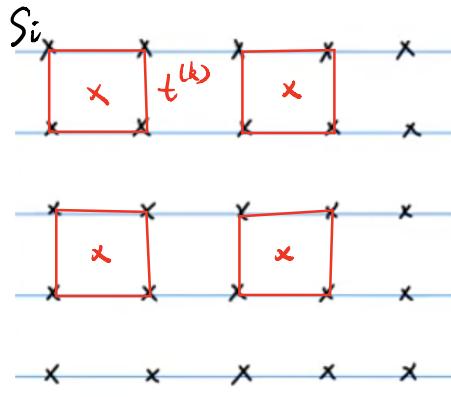
where h is the field of the magnetization with the field itself at a critical point.

C_h : specific heat capacity at constant net magnetic field.
 $(C_h \rightarrow \infty)$ when $T \rightarrow T_c$, C_h diverges.
 ξ also diverges with ν .
 entry of the correlation function $\Rightarrow g(r) \sim \frac{1}{r^{d-2+\eta}}$



There are 6 critical exponents above, with some weird features.

1. They are all 2 dimensions, very simple rational numbers.
2. They seem to be related. If we go from one model to another, we find different critical exponents but there are several relations between them that always seem to hold. And it's precisely that point with **these scaling relations** between the critical exponents that is addressed using renormalization theory.



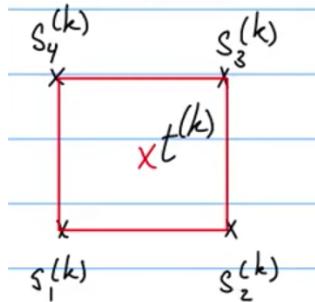
In that renormalization theory, we try to exploit the scale invariance of the system and we do that by starting with the Ising lattice formulated on the square lattice.

Here we have the Ising spins on each site and what we want to do is that we want to map the system onto a coarse-grained system and we do that as the following way:

We identify brackets (or the red squares in the left figure) in the lattice and they are separated by a lattice constant. Remember we have spins at each of the lattice site (black crosses). But we want to replace those by new spins that live at the center of each bracket. So we are going to introduce new spins that live in the center of the brackets (red squares) --- red crosses. And we can see that the red crosses are 2 lattice constants apart.

We call the original spins on the fine-grained lattice site letter S_i . And the spins on the red crosses are called $t^{(k)}$.

k identifies the site and $t(k)$ is the spin on that site.



Here we have an enlarged view of the bracket which is surrounded by 4 fine-grained spins. We want our new $t(k)$ to assume the same values as the original spin so that is:

$$t^{(k)} = \pm 1$$

(just like $S_i^{(k)}$)

So how do we choose the value of $t(k)$?

If $S_1^{(k)} + S_2^{(k)} + S_3^{(k)} + S_4^{(k)} > 0$ then $t^{(k)} = 1$

If $\dots - - - - - < 0$ then $t^{(k)} = -1$

If $\dots - - - - - = 0$ then $t^{(k)} = \pm 1$

Equal Probabilities

$$W(t^{(k)}, S_1^{(k)}, S_2^{(k)}, S_3^{(k)}, S_4^{(k)}) = 1 \text{ for } \sum_i S_i^{(k)} > 0, t^{(k)} = +1$$

given

$$1 \quad " \quad \sum_i S_i^{(k)} < 0, t^{(k)} = -1$$

$$0.5 \quad " \quad \sum_i S_i^{(k)} = 0, t^{(k)} = +1$$

$$0.5 \quad " \quad \sum_i S_i^{(k)} = 0, t^{(k)} = -1$$

We can formulate the recipe in terms of a probability density $W(t(k), S1(k), S2(k), S3(k), S4(k))$ and W is a function of $t(k)$ and 4 surrounding spins. We suppose that the 4 surrounding spins have given values and if those 4 given values add up to a positive number, then the probability to have $t(k)$ is 1. etc...

$$\Rightarrow \sum_{t^{(k)}=\pm 1} W(t^{(k)}, S_1^{(k)}, S_2^{(k)}, S_3^{(k)}, S_4^{(k)}) = 1 \text{ for any set of } S_i^{(k)}$$

We can see that for any given set of $S_i(k)$, if we sum over the 2 possibilities for $t(k)$, then it always ends up to 1.

i.e. If $\sum_i S_i^{(k)} > 0$, then $W(t^{(k)} = +1) = 1$

$$W(t^{(k)} = -1) = 0$$

Thus $W(t^{(k)} = +1) + W(t^{(k)} = -1) = W(t^{(k)} = \pm 1) = 1 + 0 = 1$

Now we defined a "new" Hamiltonian, which governs the behavior of the $t(k)$ (It is not defined in terms of the s spins but in terms of the new spins $t(k)$.) This new Hamiltonian describes the system in terms of the coarse-grained spins

The new Hamiltonian: (\mathcal{H}')

probability density W

$$e^{-\beta \mathcal{H}'(t^{(k)})} = \sum_{\{S_i^{(k)} = \pm 1\}} e^{-\beta \mathcal{H}(S_i)} W(t^{(k)}; S_1^{(k)} \dots S_4^{(k)})$$

We take first the Boltzmann factor of the original model so we have a configuration of fine spins S_i . (The superscript k is left out because all the spins in the system). We calculate the appropriate Boltzmann weight and we sum all of the possible configurations of spins in the system.

Because we have summed over all the configurations of fine-grained spins over the lattice, the final result will ONLY depend on this $t(k)$ ($t(k)$ is the only dependence in the final expression!)

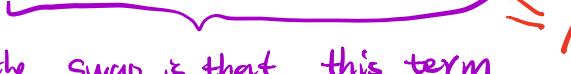
Thus, this new Hamiltonian \mathcal{H}' is defined in terms of $t^{(k)}$, NOT $S_i(k)$

Partition function for the new Hamiltonian:

$$Z' = \sum_{\{t^{(k)}\}} e^{-\beta H(t^{(k)})} = \sum_{\{t^{(k)}=\pm 1\}} \sum_{\{s_i^{(k)}=\pm 1\}} e^{-\beta H(\{s_i\})} W(t^{(k)}, s_1^{(k)} \dots s_4^{(k)}).$$

 Swap the 2 of them.

$$= \sum_{\{s_i^{(k)}=\pm 1\}} \sum_{\{t^{(k)}=\pm 1\}} W(t^{(k)}, s_1^{(k)} \dots s_4^{(k)}) e^{-\beta H(\{s_i\})} = Z$$

Reason of the swap is that this term
Only depends on $t^{(k)}$. 

(from previous analysis)

$$\left(\Rightarrow \sum_{t^{(k)}=\pm 1} W(t^{(k)}, s_1^{(k)}, s_2^{(k)}, s_3^{(k)}, s_4^{(k)}) = 1 \text{ for any set of } s_i^{(k)} \right)$$

$$= \sum_{\{s_i^{(k)}=\pm 1\}} e^{-\beta H(\{s_i\})} = Z \Rightarrow Z(\{t^k\}) = Z(\{s_i^{(k)}\})$$

They are the same.

Thus, we have succeeded in reformulating the problem in terms of $t(k)$ such that the partition function and therefore the thermodynamic behavior is the same for both models.

Conclusion: So, transforming from "fine" to "coarse" spins leaves the partition function invariant!

So we carry out the procedure that we have outlined above. So for any fine-grained configuration $S_i(k)$, we construct the coarse-grained configuration using the W function that we have defined above. Then we will find a Hamiltonian which describes the interaction between the $t(k)$ case

Supposing we carry out the coarsening procedure for the Ising model with nearest neighbor couplings and the next-nearest neighbor couplings.

$$-\beta H\{s_i\} = J \sum_{\langle ij \rangle} s_i s_j + K \sum_{\langle\langle ij \rangle\rangle} s_i s_j$$

2 dominant terms

$$-\beta H\{t^{(k)}\} = J \sum_{\langle kk' \rangle} t^{(k)} t^{(k')} + K \sum_{\langle\langle kk' \rangle\rangle} t^{(k)} t^{(k')} + \text{several other couplings.}$$

↑ ↑

Nearest Neighbour Coupling between the coarse-grained spins.

Next Nearest Neighbour Coupling between the coarse-grained spins.

Several other terms.
i.e. terms coupling 3 or 4 spins

We just truncate the Hamiltonian beyond the terms that we have already present in the original Hamiltonian. Whether that's a justified procedure is not known yet, but we'll come back to this point later.

↑
We assumed that we can neglect those terms.

Note: the J & J' , K & K' in these 2 Hamiltonians are NOT necessarily same.

For example, if we are in high temperature phase and there is already a weak coupling between s_i & s_j . If we take the bracket's form (coarse-grained) which are even farther apart, it's likely that this J' is going to be smaller than J . Same situation for K .

$$\begin{array}{ccc} J & \xrightarrow{\text{Map into}} & J' \\ & \longrightarrow & \\ K & \longrightarrow & K' \end{array}$$

Given the fact that these mappings exist. It means that when we start from a point J, K in the (J, K) space, the renormalization transformation will map us onto new values J' & K' .

We see that J, K map into J' and K' . This mapping induces a flow in J, K space.

So from each point we move to another one and then we move again to another one if you would carry out another renormalization transformation. So we move from point to point.

We take $B = 0$. Later we consider weak fields.
 (B: the magnetic field. $B=0$, otherwise we wouldn't have a critical system)

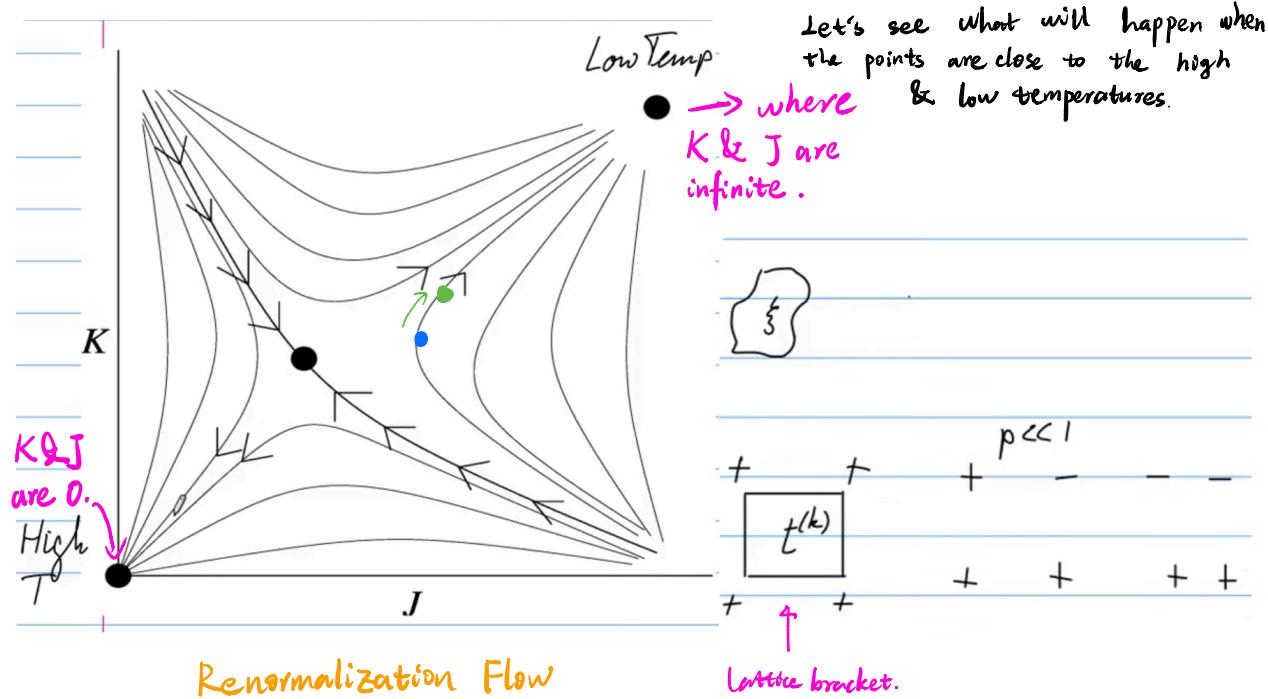
New lattice constant = $L \times$ old lattice constant

\downarrow

$L=2$ in our example

L : Rescaling length
 (or called as scaling factor)

The criticality ceases to exist when B is nonzero.



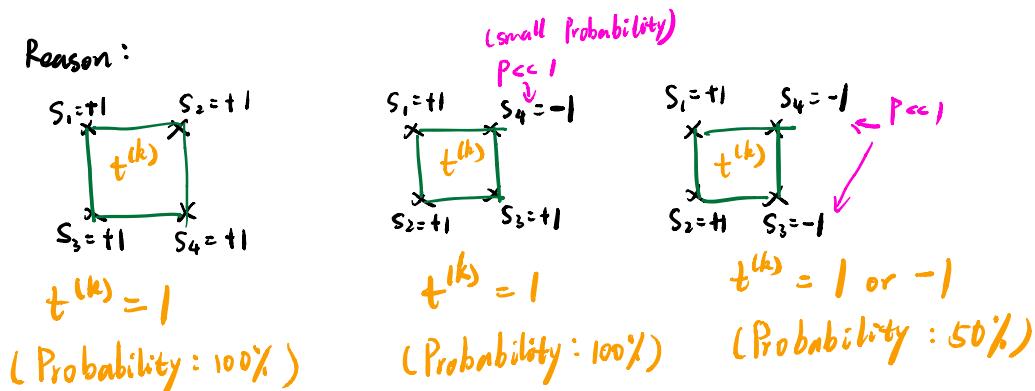
Let's say going from one point (•) to another (•) that is what happens when we go from fine-grained spins to coarse-grained spins. Now the points where the coupling constants are very large.

Let's start with a low temperature point. At low temperatures, the vast majority of the spins is the same, say it is +1 in our case. And there are small islands of -1. So whenever we go from the S_i to the $t(k)$, if we consider a lattice bracket (shown in the right above figure), it is almost impossible that there are 2 S_i which are -1 because the probability to have an S_i which deviates from the majority value of +1 only lattice is very small.

So there is a very very big probability, very close to 1, that is, $t(k)$ in the brackets would also be +1. Because everywhere in the lattice there is a vast majority of +1 spins, all the $t(k)$ will become to 1. \implies So they will become even STRONGER +1 than the $S_i(k)$ already were.

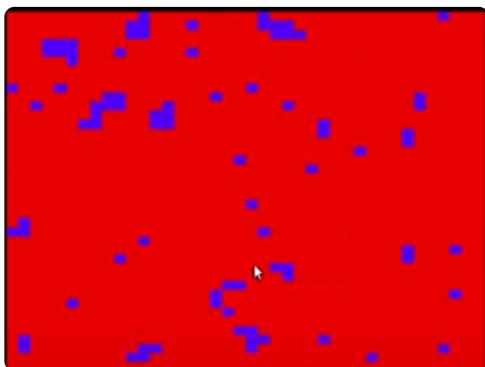
That means we are going from a low temperature phase with the majority of +1 to a even lower temperature phase because $t(k)$ case has a higher probability to be +1. 

Reason:



These explains the Point (•)  to Point (•) upward direction. (shown above)

Renormalization will move the system towards a lower temperature.



So considering the initial example at very low temperatures, if we zoom out the picture (Coarser grains), we would rarely see the blue dots. (Details are no longer seen)

At high temperatures, we have islands of a small size which can be +1 or -1. Actually it is kind of patchwork (拼湊物) that our entire system is composed of clusters that are either blue or red clusters. Now we are going to coarse grain and because their lattice constant is getting larger, the effective size of the domains of one spin measured in units of that lattice constant is getting smaller.

Suppose we have a red block in the system:

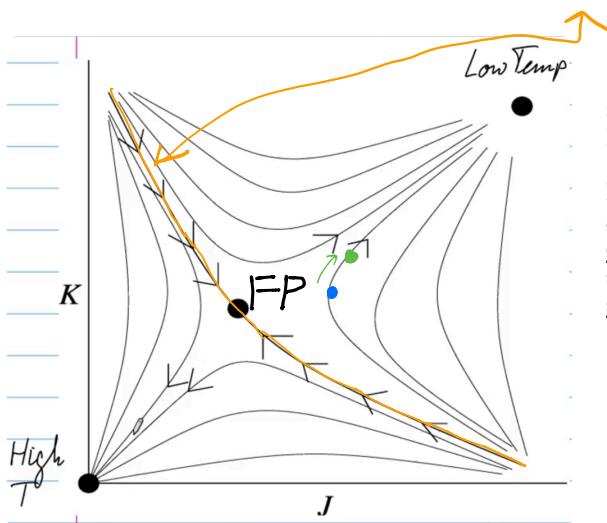
Red block in the system (with a certain size ξ , measured in lattice constants of the fine-grain system). Then I'm going to make the lattice constant twice as big and I'm measuring the size here again in terms of the new lattice constant. (in new units of distance). It means that effectively the blob (ξ_{new}) is going to be smaller. And the smaller the blobs, the higher the temperature. That explains why when we are at higher temperature, the renormalization transformation brings us to a even higher temperature.



As a short summary, in the low-temperature phase, we have a majority of, say, red spins, with small islands of blue spins. If we are going to look at the average spins on larger areas, it's almost certainly going to be red, and the blue part is going to be reduced. When the lattice is becoming more and more homogeneous, it will be filled with one color, which means we are going to lower temperature.

If we are at high temperatures, we have domains of certain size which can be blue or red. If we apply the renormalization transformation, then measured in the new lattice constant (which is twice as big), we see that the domains shrink. And shrinking domains means that we are going to higher temperature because the situation is going to be more and more chaotic.

Now given this trend of low temperature flow towards one point (Right-top corner) and high temperature flow toward the opposite point (Left-bottom corner), as shown in the picture above, the only possibility for the remainder is that there is some like which separates the high temperature region from the low temperature region. And by continuity, the flow at that line (from left-top corner to the right-bottom corner) should be like the line shown in the picture.



So that completes the renormalization flow in a qualitative sense, and what is very important here is that usually on this line, it is possible that we find a so-called **fixed point (FP)**.

If we are at the fixed point, then the system does NOT change anymore upon renormalization. So the effective coupling constants for the coarse-grained spins turn out to be the same as the effective coupling spins for the fine-grained spins.

$$T_{\text{low}} \xrightarrow{l} T'_{\text{lower}} < T_{\text{low}}$$

$$T_{\text{high}} \xrightarrow{l} T'_{\text{higher}} > T_{\text{high}}$$

l : rescaling factor

Correlation length $\xi(J, K)$ = average size of region with single spin (+1 or -1)

The argument for the high temperature involves a correlation length, which is the average size of the region of +1 or -1 spin. So it's a region of one color (red or blue) in the simulation.

We measure $\xi(J, K)$ in units of the lattice constant.

If the lattice constant becomes larger (coarse-grained), then the correlation length $\xi(J, K)$ measured in units of the lattice constant will be smaller.

Scaling by a factor l therefore implies

$$\xi(J', K') = \xi(J, K)/l$$

The free energy F is extensive : $F \sim N$ ($\#$ of sites)

Free energy per site : $f = \frac{F}{N}$

↑ Intensive Quantity

The free energy quantity is an extensive quality so it scales with the number of sites in the lattice.

Previously we have seen that after a renormalization transition is performed, the partition function does not change.

We have seen : $Z_{N/l^d}(J', K') = Z_N(J, K)$

$\rightarrow d$: dimension

Therefore : $F(J', K') = F(J, K)$

The old one before
renormalization
↓

$$\text{so : } f(J', K') = \frac{F(J', K')}{N/l^d} = l^d \frac{F(J, K)}{N} = l^d f(J, K).$$

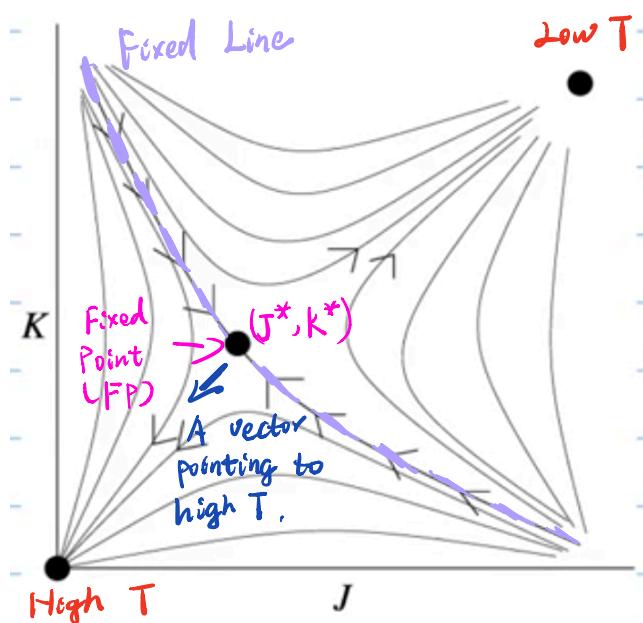
The new free energy
per lattice site in terms of the coupling constants J' & K' .

What we can infer from this is that: The free energy is directly defined in terms of the partition function. Since the partition function are the same, then the free energy are the same. So if we consider the free energy per lattice site, we should divide the free energy by number of the sites, in this case the number of $t(k)$ spins --- $N/[l^d(t)]$.

So the scaling factor for the free energy per lattice site is l^d .

As a short summary:

$$\begin{aligned}\mathcal{E}(J', K') &= \mathcal{E}(J, K) / l \\ f(J', K') &= l^d f(J, K)\end{aligned}$$



Let's denote the Fixed Point as (J^*, K^*) .

To analyze the behavior close to the F.P., we define:

$$J^* + \Delta J, K^* + \Delta K$$

Renormalization Transformation (RT)

$$\underline{\underline{J^* + \Delta J'}}, \underline{\underline{K^* + \Delta K'}}$$

J^* & K^* would NOT change according to the definition of Fixed Point.
 ΔJ & ΔK will become into $\Delta J'$ & $\Delta K'$

We can say that

$$\begin{pmatrix} \Delta J' \\ \Delta k' \end{pmatrix} = (A) \begin{pmatrix} \Delta J \\ \Delta k \end{pmatrix}$$

Now to first order Taylor expansion, we can expect that the relation between $\Delta J'$ & $\Delta k'$ is linear.

(A) is a (2×2) matrix which relates the deviations after the renormalization to the deviations from the fixed point before.

If we diagonalize A , we'll find useful information about the mapping. We find 2 eigenvalues: λ, μ .

And if we inspect the flow diagram (above), we can identify those 2 eigenvectors corresponding to those because we expect that a vector that is pointing to (high T) (see above diagram) will become larger but not change its value too much. So there should be a vector in the vicinity of this one which is the eigenvector and you see that it will grow because we move away from the fixed point. So there will be one eigenvalue, and that's $\lambda < \mu > 1$

It's the same if we look at minus of that eigenvector, of course it will have the same eigenvector

and it will move the opposite direction (towards to the low T point). So $\mu > 1$.

On the other hand, there is a direction along the fixed line (in this color, from Left-top to Right-bottom), and we see the points on the fixed line (These points move back towards the fixed points!)

This means that $\Delta J'$ & $\Delta k'$ are smaller than ΔJ & Δk .

They correspond to an eigenvalue $\lambda < 1$.

Thus, from the flow diagram: $\lambda < 1$, $\mu > 1$.

\uparrow \uparrow
 Corresponds to the corresponds to the
 fixed line. direction (perhaps)
perpendicular to
the fixed line.

λ & μ depend on l

Applying rescaling twice: eigenvalues λ^2, μ^2
 rescaling length l^2

Now let's see what happens if we would apply 2 renormalization transformations in succession. We have the eigenvalues λ^2 & μ . and obviously they depend on the rescaling length l .

Now what we could do is to apply 2 renormalization transformations so that means that we rescale twice, and that corresponds to rescale in length not of l but of l^2 . So the rescaling length in that case is l^2 . If we look what happens to the eigenvalues, if we apply the renormalization transformation twice, we apply in fact the matrix a twice --- A^2 .

So we also get λ^2 & μ^2

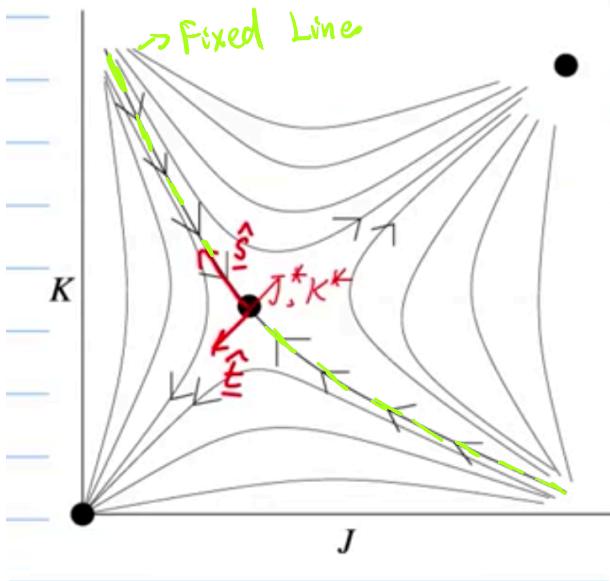
Thus,

$$\left. \begin{array}{l} \lambda(l^2) = \lambda^2(l) \\ \mu(l^2) = \mu^2(l) \end{array} \right\} \Rightarrow \begin{array}{l} \lambda \& \mu \text{ should have the form} \\ \lambda = l^y \\ \mu = l^z \end{array}$$

↓
since $y < 1$,
 $z > 0$

y, z : scaling dimensions.

They are the exponents of the expressions for λ & μ in terms of l .



Now going back to the flow diagram.

There are 2 unit vectors which are eigenvectors of the transformation A .

There is an eigenvector \hat{t} which points along this direction (Left-bottom to Right-Top), which corresponds to the eigenvalue bigger than 1. (because it's moving away from the fixed point) (μ)

And we have the eigenvector \hat{s} . which moves along the fixed line. Once we move away from the fixed point, we tend to go back. So that corresponds to λ . ($\lambda < 1$)