Additional Reading Notes for SDE + Brown Motion

04/19/2020

1) Stochastic differential equation “随机微分方程”

1-1) “随机微分方程” (https://zh.wikipedia.org/wiki/随机微分方程)

“随机微分方程是微分方程的扩展。一般微分方程的对象为可导函数，并以其建立等式。然而，随机过程函数本身的导数不可定义，所以一般解微分方程的概念不适用于随机微分方程。随机微分方程多用于对一些多样化现象进行建模，比如不停变动的股票价格，部分物理现象如热扰动等。”

Einstein & Marian Smoluchowski ---> Langevin ---> Kiyosi Itô & Ruslan Stratonovich

Background [edit]

Early work on SDEs was done to describe Brownian motion in Einstein's famous paper, and at the same time by Smoluchowski. However, one of the earlier works related to Brownian motion is credited to Bachelier (1900) in his thesis 'Theory of Speculation'. This work was followed upon by Langevin. Later Itô and Stratonovich put SDEs on more solid mathematical footing.

(Fun facts: Paul Langevin is the PhD student of Pierre Curie. Langevin is the advisor for Irène Joliot-Curie (Daughter of Pierre Curie & Madame Curie) and Louis de Broglie (who came out of the famous “*Matter and wave–particle duality*” theory))

一般而言，随机微分方程的解是一随机过程函数，但解方程需要先定义随机过程函数的微分。最常见的定义为根据伊藤清所创，假设B为布朗运动，则对于某函数H，作以下定积分之定义：

$$\int_0^t H\,dB = \lim_{n\to\infty} \sum_{t_{i-1},t_i\in\pi_n} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}).$$

此称为伊藤积分。伊藤式的随机微分方程常用于在金融数学中。

1-2) Stochastic differential equation (https://en.wikipedia.org/wiki/Stochastic_differential_equation)

“A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. SDEs are used to model various phenomena such as unstable stock prices or physical systems subject to thermal fluctuations. Typically, SDEs contain a variable which represents random white noise calculated as the derivative of Brownian motion or the Wiener process.

Use in physics (See also: Langevin equation)

In physics, SDEs have widest applicability ranging from molecular dynamics to neurodynamics and to the dynamics of astrophysical objects. More specifically, **SDEs describe all dynamical systems, in which quantum effects are either unimportant or can be taken into account as perturbations. SDEs can be viewed as a generalization of the dynamical systems theory to models with noise**. This is an important generalization because real systems cannot be completely isolated from their environments and for this reason always experience external stochastic influence.

There are standard techniques for transforming higher-order equations into several coupled first-order equations by introducing new unknowns. Therefore, the following is the most general class of SDEs:

$$\frac{dx(t)}{dt} = F(x(t)) + \sum_{\alpha=1}^{n} g_\alpha(x(t))\xi^\alpha(t),$$

where $x \in X$ is the position in the system in its phase (or state) space, X, assumed to be a differentiable manifold, the $F \in TX$ is a flow vector field representing deterministic law of evolution, and $g_\alpha \in TX$ is a set of vector fields that define the coupling of the system to Gaussian white noise, ξ^α. If X is a linear space and g are constants, the system is said to be subject to additive noise, otherwise it is said to be subject to multiplicative noise. This term is somewhat misleading as it has come to mean the general case even though it appears to imply the limited case in which $g(x) \propto x$.

For a fixed configuration of noise, SDE has a unique solution differentiable with respect to the initial condition. Nontriviality of stochastic case shows up when one tries to average various objects of interest over noise configurations. In this sense, an SDE is not a uniquely defined entity when noise is multiplicative and when the SDE is understood as a continuous time limit of a stochastic difference equation. In this case, SDE must be complemented (补充) by what is known as "interpretations of SDE" such as Itô or a Stratonovich interpretations of SDEs. Nevertheless, when SDE is viewed as a continuous-time stochastic flow of diffeomorphisms, it is a uniquely defined mathematical object that corresponds to Stratonovich approach to a continuous time limit of a stochastic difference equation.

In physics, the main method of solution is to find the **probability distribution function** as a function of time using the equivalent Fokker–Planck equation (FPE). The Fokker–Planck equation is a deterministic partial differential equation. **It tells how the probability distribution function evolves in time** similarly to how the Schrödinger equation gives the time evolution of the quantum wave function or the diffusion equation gives the time evolution of chemical concentration. Alternatively, **numerical solutions can be obtained by Monte Carlo simulation.** Other techniques include the path integration that draws on the analogy between statistical physics and quantum mechanics (for example, the Fokker-Planck equation can be transformed into the Schrödinger equation by rescaling a few variables) or by writing down ordinary differential equations for the statistical moments of the probability distribution function.

(Notes: **Monte Carlo Methods**: Monte Carlo methods is mainly used for finding the solution to differential equations (or integrations?) in the field of simulation)

Use in probability and mathematical finance

The notation used in probability theory (and in many applications of probability theory, for instance mathematical finance) is slightly different. This notation makes the exotic nature of the random function of time η_m in the physics formulation more explicit. It is also the notation used in publications on numerical methods for solving stochastic differential equations. In strict mathematical terms, η_m cannot be chosen as an ordinary function, but only as a generalized function. The mathematical formulation treats this complication with less ambiguity than the physics formulation.

A typical equation is of the form

$$\mathrm{d}X_t = \mu(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}B_t,$$

where B denotes a Wiener process (Standard Brownian motion). This equation should be interpreted as an informal way of expressing the corresponding integral equation

$$X_{t+s} - X_t = \int_t^{t+s} \mu(X_u, u)\mathrm{d}u + \int_t^{t+s} \sigma(X_u, u)\,\mathrm{d}B_u.$$

The equation above characterizes the behavior of the continuous time stochastic process X_t as the sum of an ordinary Lebesgue integral and an Itô integral. A heuristic (but very helpful) interpretation of the stochastic differential equation is that in a small time interval of length δ the stochastic process X_t changes its value by an amount that is normally distributed with expectation $\mu(X_t, t)\,\delta$ and variance $\sigma(X_t, t)^2\,\delta$ and is independent of the past behavior of the process. This is so because the increments of a Wiener process are independent and normally distributed. The function μ is referred to as the drift coefficient, while σ is called the diffusion coefficient. The stochastic process X_t is called a diffusion process, and satisfies the Markov property.

The formal interpretation of an SDE is given in terms of what constitutes a solution to the SDE. There are two main definitions of a solution to an SDE, a strong solution and a weak solution. Both require the existence of a process X_t that solves the integral equation version of the SDE. The difference between the two lies in the underlying probability space $(\Omega,\ \mathcal{F},\ P)$. A weak solution consists of a probability space and a process that satisfies the integral equation, while a strong solution is a process that satisfies the equation and is defined on a given probability space.

An important example is the equation for geometric Brownian motion

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t.$$

which is the equation for the dynamics of the price of a stock in the Black–Scholes options pricing model of financial mathematics.

There are also more general stochastic differential equations where the coefficients μ and σ depend not only on the present value of the process X_t, but also on previous values of the process and possibly on present or previous values of other processes too. In that case the solution process, X, is not a Markov process, and it is called an Itô process and not a diffusion process. When the coefficients depends only on present and past values of X, the defining equation is called a stochastic delay differential equation.

Additionally

Stochastic calculus [edit]

Brownian motion or the Wiener process was discovered to be exceptionally complex mathematically. The Wiener process is almost surely nowhere differentiable; thus, it requires its own rules of calculus. There are two dominating versions of stochastic calculus, the Itô stochastic calculus and the Stratonovich stochastic calculus. Each of the two has advantages and disadvantages, and newcomers are often confused whether the one is more appropriate than the other in a given situation. Guidelines exist (e.g. Øksendal, 2003) and conveniently, one can readily convert an Itô SDE to an equivalent Stratonovich SDE and back again. Still, one must be careful which calculus to use when the SDE is initially written down.

Numerical solutions [edit]

Numerical solution of stochastic differential equations and especially stochastic partial differential equations is a young field relatively speaking. Almost all algorithms that are used for the solution of ordinary differential equations will work very poorly for SDEs, having very poor numerical convergence. A textbook describing many different algorithms is Kloeden & Platen (1995).

Methods include the Euler–Maruyama method, Milstein method and Runge–Kutta method (SDE).

2) YouTube video (Outline of Stochastic Calculus)
https://www.youtube.com/watch?v=rvYfNz2H3Uk
1. Brownian motion is a fractal (分形的).
A curve or geometric figure, each part of which has the same statistical character as the whole. Fractals are useful in modeling structures (such as eroded coastlines or snowflakes) in which similar patterns recur at progressively smaller scales, and in describing partly random or chaotic phenomena such as crystal growth, fluid turbulence, and galaxy formation.

For the ordinary function, if we zoom in its curve, it will look closer to a straight line (tangent line) ---> Differentiable!

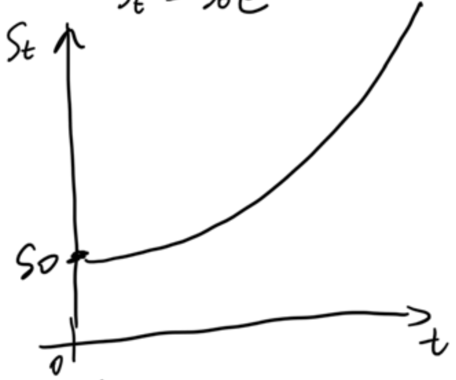
Thus, for the Brownian motion curve, we cannot really differentiate it. Its curve will never become straight no matter how we zoom it in.

Stochastic Calculus

$$\parallel$$

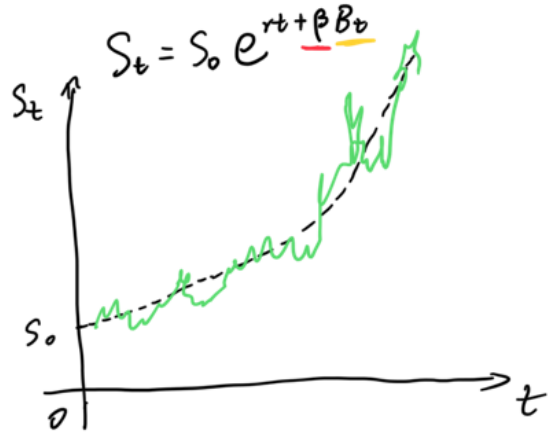
Ordinary Calculus + Randomness

$S_t = S_0 e^{rt}$



$$\frac{ds}{dt} = r \underbrace{S_0 e^{rt}} = r S_t$$

$S_t = S_0 e^{rt + \beta B_t}$



β : is a constant

B_t : Brownian Motion

It's hard to differentiate this Brownian Curve

Stochastic Calculus Comes in

To Accomodate the Randomness

Randomness does NOT mean there is no order at all! In fact, randomness could have some distribution.

In this case, for the right curve,

if we are going to calculate $\frac{ds}{dt}$ using the previous ordinary calculus method, what do we get? $\longrightarrow$ $\frac{ds}{dt} = \left[r + \beta \frac{dB_t}{dt} \right] S_0 e^{rt + \beta B_t}$

$$= \left[r + \beta \frac{dB_t}{dt} \right] S_t$$

$$= r S_t + \beta \underbrace{\frac{dB_t}{dt}}_{\text{Problem comes! Does NOT work here!}} S_t$$

Thus, we need to introduce new method: Differential Form.

(1) Rewrite $\underbrace{\frac{dS_t}{dt} = r S_t}_{\text{for left figure}}$ into $dS_t = r S_t \, dt$.

(2) $\int dS_t = \int r S_t \, dt$.

$\longrightarrow$ for the left figure (ordinary case. No Brownian motion.)

For the right figure (with Brownian motion)

(1) Rewrite $\frac{dS_t}{dt} = r S_t + \beta S_t \frac{d\beta_t}{dt}$ into

$$dS_t = r S_t \, dt + \beta S_t \, d\beta_t$$ $\longrightarrow$ * No issues now!

Notes:
For ordinary function (no Brownian motion)

Differentiation $\xleftrightarrow{\text{Inverse to each other}}$ Integration

(2) Here, for our new case (with Brownian Motion)

We are gonna start with Integration first

SDE
$\downarrow$
Differentiation $\longleftarrow$ Integration