So instead of using K & J, we can also express each point here in terms of these new eigenvectors that we use as basis vectors $\hat{t}$ & $\hat{s}$.

We can write any vectors in the J, K plane as (in particular the vectors close to the fixed point):

$$\underline{v} = t\,\hat{t} + s\,\hat{\underline{s}} \quad , \quad \text{so } (J, K) \Longleftrightarrow (t, s)$$

because the 2 eigenvectors form the basis.

where $\hat{s}$, $\hat{t}$ are the eigenvectors corresponding to λ & μ:

$$(A)\,\hat{\underline{s}} = \lambda\,\hat{s}$$

$$(A)\,\hat{t} = \mu\,\hat{t}$$

some general vector

$$\text{So: } (A)(s\hat{\underline{s}} + t\hat{\underline{t}}) = s\lambda\hat{\underline{s}} + t\mu\hat{\underline{t}} = s\,l^{y}\hat{\underline{s}} + t\,l^{z}\hat{\underline{t}}$$

$$(A^n)(s\hat{\underline{s}} + t\hat{\underline{t}}) = s\lambda^n\hat{\underline{s}} + t\mu^n\hat{\underline{t}} = s\,l^{yn}\hat{\underline{s}} + t\,l^{zn}\hat{\underline{t}}$$

$$\text{So} \quad s^{(n)} = s\,l^{yn} \qquad y<0$$

$$t^{(n)} = t\,l^{zn} \qquad z>0$$

So we can conclude that the coefficient of $\hat{s}$ after n transformations is given by the original $s \times l^{yn}$. Same for $\hat{t}$.

Recall that we have $\xi(J', K') = \xi(J, K)/l$

$\longrightarrow \quad \xi(s', t') = \xi(s, t)/l$

(which is just a reparameterization)

Now let's see what happens if we perform the renormalization transformation n times.

After n rescalings:

$$\xi(l^{ny} s, l^{nz} t) = \xi(s, t)/l^n$$

Now we are going to apply a special trick

Choose t & n such that $\underline{l^{nz} t = 1}$, $\Rightarrow l^n = \underline{t^{-1/z}}$

(We are starting close to the fixed point, which means that the s & t are very small. But after many many rescalings, with this eigenvector t which has a eigenvalue which is biger than 1, we can move arbitrarily far away.

So we take n large such that $\underline{l^{nz} t = 1}$)

$$\xi(t^{-y/z} s, 1) = \xi(s, t)\, t^{1/z}$$

$$\uparrow$$

$l^{ny} s = (l^n)^y s = (t^{-1/z})^y s = t^{-y/z} s$

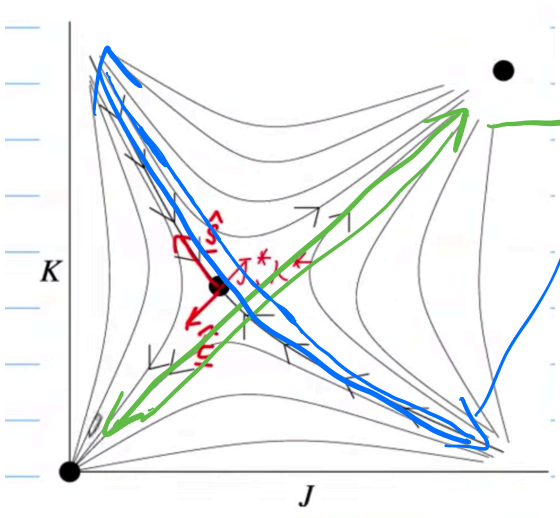
Note: $y < 0$, $\therefore \frac{-y}{z} > 0$ $(z > 0)$

t is small, s is small as well. Thus approximately $t^{-y/z} s \approx 0$.

Thus, $\Rightarrow \quad \xi(s,t) = t^{-1/z}\,\underbrace{\xi(0,1)}$

(If we move closer and closer to the fixed point, we find this dependence more reliable.)

This is just a number, which is the given correlation length at the point (0, 1)



Now going back to the diagram.

This direction (A).

Direction (B)

It's clear that the direction (A) changes the value of K & J. Now K & J are fixed coupling constants, but they include the effect of β, which is $\frac{1}{k_B T}$. And that means that K & J include the temperature. So moving in this direction (A), means that we are changing the temperature.

temperature direction.

For direction (B), it's the direction which corresponds to changing the relative importance of the nearest neighbour and the next nearest neighbour coupling.

$\underline{v} = t\hat{t} + s\hat{s}$, so $(J, K) \Leftrightarrow (t, s)$ (from above)

So the parameter t, (along direction (A)), we can safely identify that

$$t \underset{\substack{\uparrow \\ \text{proportional}}}{\sim} |T - T_c|.$$

Therefore, $\xi \sim \frac{1}{|T-T_c|^{1/z}}$

(since $\xi(s,t) = t^{-1/z}\,\xi(0,1)$)

And by definition, $\xi \sim |T-T_c|^{-\nu}$)

critical exponents

describe the divergence of the correlation length.

$\Downarrow$

$$\nu = \frac{1}{z}$$

So what we have see is that if we perform a renormalization transformation, we linearize it around a fixed point and recalculate the eigenvalues of the linearized transformation. Then these eigenvalues via the critical dimensions in this case tell us something about a critical experience which which is *v*.

Now recall that this is just a global discussion about critical phenomena and we do not want to be specific with respect to the renormalization transformation or the model so that means that the *Z* is a completely unknown quantity for us so we have not obtained any information about *v*. Nevertheless, if we combine this result with other results that we are about to derive now, it turns out that we find very useful relations.

Now consider the free energy per site $f(s,t)$

$$f(s',t') = l^d f(s,t)$$

fine-grained Model

Coarse-grained model

Extend the model with a (weak) magnetic field h.

Then the Hamiltonian becomes:

Extra term

$$-\beta H(\{s_i\}) = J\sum_{\langle ij\rangle} s_i s_j + K \sum_{\langle\langle ij\rangle\rangle} s_i s_j + h\sum_i s_i$$

Magnetic field

Then we move from 2 variables: J & K to 3 variables : J, K & h.

Matrix (A) now becomes 3×3 matrix

$\longrightarrow$ 3 eigenvalues λ, μ, κ

means that we switch on the magnetic field

$\lambda = l^y$, $\mu = l^z$, $\kappa = l^{\nu}$, $\nu > 0$, $y < 0$, $z > 0$

(It's ν, NOT υ.)

if $\kappa < 1$, it means that we switch on a magnetic field that renormalization brings us back to the zero field case. It means that the term associated with the magnetic field would shrink under renormalization but because we know the critical behavior is destroyed when we switch on a magnetic field, we anticipate that κ is > 1.

Previously we have seen that λ & μ can be written as $\lambda = l^y$ & $\mu = l^z$. So for κ, we choose $\kappa = l^{\nu}$.

(It's ν, NOT υ)

3rd critical dimension

And because κ is expected to be larger than 1, thus, $\nu > 0$.

The scaling argument for the free energy is still valid.

$$f(s', t', h') = l^d f(s, t, h)$$

$$\longrightarrow f(l^{ny} s, l^{nz} t, l^{n\nu} h) = l^{nd} f(s, t, h)$$

(Same as before)

Choose t (small) & n (large) such that

Close to the Fixed Point

$\ell^{nz} t = 1$. Then $\ell^{ny} s \to 0$ (s small, $y<0$, n large).

1st, if $h=0$ $\Longrightarrow f(s,t,0) = \ell^{-nd} f(0,1,0) = t^{d/z} f(0,1,0)$

It's just a given number

(also, $\ell^{-n} = t^{1/z}$)

t is the deviation from the critical temperature.

Having the free energy, we can then obtain the specific heat, so that's our aim.

Recall $t \sim |T-T_c|$

Entropy : $S = \left.\frac{\partial F}{\partial T}\right)_{N,h}$ and $C_h = T\frac{\partial S}{\partial T} = T\frac{\partial^2 F}{\partial T^2}$

$\frac{\partial}{\partial t} \longleftrightarrow \frac{\partial}{\partial T}$ hence $C_h \sim t^{d/z-2}$

(since T_c is just a constant temperature.)

Also, we know that $C_v = t^{-\alpha}$ (by definition)

$\Rightarrow \nu = \frac{1}{z} \Rightarrow \boxed{\alpha = 2 - d\nu}$

The relation between the critical exponents

Scaling Relation

(First example of the scaling relations).

(The scaling relation only involves the dimension & the critical exponents themselves. It does NOT involve the scaling

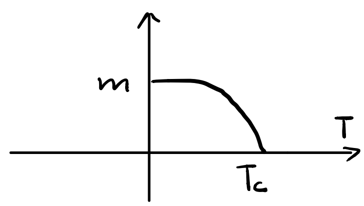
(It's ν, NOT υ)

dimensions (Those are the numbers y, z, ν, which are the numbers that occur in the eigenvalues of the renormalization transformation close to the fixed point.)

The scaling relations are just relations between the critical exponents & they are independent of the specific eigenvalues that we find in diagonalizing our matrix (A).

Magnetization

$$m \sim t^{\beta}$$



$t \sim (T_c - T)$

(β: critical exponent, which describes how the magnetization vanishes when we approached a critical temperature from below)

$$m = \left.\frac{\partial f}{\partial h}\right|_{h=0}$$

We have the following scaling relation for the free energy per site.

$$f(s,t,h) = l^{-nd} f(l^{ny} s, \underline{l^{nz} t}, l^{n\nu} h)$$

$$= t^{d/z} f(0, 1, t^{-\nu/z} h)$$

(again we choose t small & n large such that $l^{nz} t = 1$.)

$$\Downarrow$$

$l^{-n} = t^{\frac{1}{z}}$ (see above)

$$\longrightarrow \frac{\partial f}{\partial h} = t^{d/z - \nu/z} \frac{\partial f}{\partial h}(0, 1, t^{-\nu/z} h)$$

(chain rule)

dimensions

This relation again involves a scaling dimension which is related to the eigenvalue of the renormalization transformation close to the critical point.

Recall $\upsilon = 1/z \longrightarrow \beta = (d - \nu)\upsilon$

critical exponents

We could combine this relation to another one in the following page to get some useful information.

In principle, we do NOT know that value ν.

Magnetic Susceptibility : χ

as a function of temperature. By definition

$$\chi = \frac{\partial m}{\partial h}\Big|_{h=0} \sim t^{d/z - 2v/z}\, \underbrace{\frac{\partial^2 f}{\partial h^2}(0,1,0)}_{\text{It's just a number}} \sim t^{-\gamma}$$

$$\Rightarrow -\gamma = (d-2v)\,\nu$$

critical exponents

$$\left.\begin{aligned} \beta &= (d-v)\,\nu \\ -\gamma &= (d-2v)\,\nu \end{aligned}\right\} \Rightarrow \gamma + 2\beta = d\nu \quad ①$$

(γ, β, ν are critical exponents)

Remember the previous scaling relation : $\alpha = 2 - d\nu$ ②

The 1st scaling relation we discovered.

We get $\alpha + 2\beta + \gamma = 2$

The 2nd scaling relation we discovered.

The next critical exponent δ:

$$m \sim h^{1/\delta} \quad (T = T_c)$$

if we are exactly at the critical point.

Thus, $(t = 0)$

Choose h (small) and n (large) such that

We used a number of iterations.

n: The number of the renormalization transformations.

$$l^{nv} h = 1 \longrightarrow l^n = h^{-1/v}$$

It's the last argument of the free energy per site.

函数的自变量

$$f(l^{ny}s,\ l^{nz}t,\ \underbrace{l^{nv}h}_{=1})$$

So we use the usual scaling relation for the free energy per site.

After n coarsening, by this relation

$$f(s,0,h) = l^{-nd} f(0,0,1) = h^{d/\nu} f(0,0,1)$$

The original f on the fine grid

$l^n = h^{-1/\nu}$

$$m = \frac{\partial f}{\partial h} \sim h^{d/\nu - 1} f(0,0,1)$$

$$\Downarrow$$

$$\frac{1}{\delta} = \frac{d}{\nu} - 1$$

since $\begin{pmatrix} \beta = (d-\nu)\nu \\ -\gamma = (d-2\nu)\nu \end{pmatrix} \Rightarrow \boxed{\beta(\delta - 1) = \gamma}$

3rd scaling relation.

Next we are going to consider the critical exponent of the correlation function.

It gives us the dependence on the length of the correlations in the system and it's defined as $g(r)$:

Correlation function exponent:

$$g(r) \sim \frac{1}{r^{d-2+\eta}}$$

dimension

A new critical exponent for the Ising Model

We find a relation for η along different lines!

$$\text{magnetization} \quad \langle m_j \rangle = \frac{\sum_{\{S_i\}} e^{-\beta H_0 + h\sum_i S_i} S_j}{\sum_{\{S_i\}} e^{-\beta H_0 + h\sum_i S_i}} = \langle m \rangle$$

The starting point is the formula for the magnetization and we now use the Ising Model as an illustration. It's governed by some Hamiltonian which, apart from the field, is just H_0 (the Ising Hamiltonian without a magnetic field). And we extract the external magnetic field term and put it explicitly in the latter part of the expression.

Then we want to find the average value of the spin at some site J which, due to the translation invariance of the model, is just a magnetization. We take the spin at site J, multiply it by the Boltzmann factor and divide it by the partition function for the proper normalization.

$$\left.\frac{\partial m}{\partial h}\right|_{h=0} = \frac{\sum_{\{S_i\}} e^{-\beta H_0} \left(\sum_k S_k\right) S_j}{\sum_{\{S_i\}} e^{-\beta H_0}} - \frac{\sum_{\{S_i\}} e^{-\beta H_0} \left(\sum_k S_k\right) \sum_{\{S_i\}} e^{-\beta H_0} S_j}{\left(\sum_{\{S_i\}} e^{-\beta H_0}\right)^2}$$

So now let's consider these susceptibility which is the derivative of m with respect to h. The idea it that we can relate the susceptibility to the correlation function.

$$\left.\frac{\partial m}{\partial h}\right|_{h=0} = \sum_k \langle S_k S_j \rangle - \left(\sum_k \langle S_k \rangle\right) \langle S_j \rangle$$

$$= \sum_k \left(\langle S_k S_j \rangle - \langle S_k \rangle \langle S_j \rangle\right) = \sum_k g(k,j)$$

$g(k,j)$: The correlation function between site k & site j.

$g(r) \sim \frac{1}{r^{d-2+\eta}}$ holds only for $r < \xi$

Because ξ diverges at the critical point ($\xi \to \infty$), this expression is always correct at the critical point

$$\sim \int_{r<\xi} g(r)\, d^d r$$

(from above, we have $g(r) \sim \frac{1}{r^{d-2+\eta}}$)

True at the critical point

If we are just below the critical point,

If we are in the continuum limit then the correlations are long-ranged, we do not have to carry out that sum in a discreet way. We can just replace it by an integral and we get all the contributions of g(r) of this correlation function over the entire space. Here we have restricted that to values of r that are smaller than the correlation length ξ and that requires some explanation.

After $r > \xi$, there is an exponential cutoff for this expression $g(r) \sim \frac{1}{r^{d-2+\eta}}$.

Next.

$$\int_{r<\xi} g(r)\, d^d r \sim \int_0^\xi g(r)\, r^{d-1} dr = \int_0^\xi \frac{r^{d-1}\, dr}{r^{d-2+\eta}}$$

we turn that into a radial integral

$$= \xi^{2-\eta} = t^{-\nu(2-\eta)} \quad ①$$

$(\xi = t^{-\nu})$

We also have $\left.\frac{\partial m}{\partial h}\right|_{h=0} = \chi \sim t^{-\gamma}$ ②

(by definition)

Combining these 2 expressions ① & ②,

we get

$$\boxed{\gamma = \nu(2-\eta)}$$

Our last critical exponent relation

(Scaling relation)

Recap:

Scaling Relations:

1. $\alpha + 2\beta + \gamma = 2$
2. $d\nu = 2 - \alpha$
3. $\beta(\delta - 1) = \gamma$
4. $\gamma = \nu(2 - \eta)$

Ising 2D

(1) $\alpha = 0$
(2) $\beta = \frac{1}{8}$
(3) $\delta = 15$
(4) $\gamma = \frac{7}{4}$
(5) $\nu = 1$
(6) $\eta = \frac{1}{4}$

Thus, there are only 2 Independent critical exponents.

We have studied through the renormalization transformation, and everything derived from the properties of that transformation close to the fixed point. And close to the fixed point, we have a series of eigenvalues, i.e. λ & μ (seen above). One is > 1 and the other is < 1. And associated with those were scaling dimensions y & z ($\lambda = l^y$, $y < 0$; $\mu = l^z$, $z > 0$)

Now all the relations that we have derived, they all depend either on z or on ν, or on both. But they NEVER depend on y.

For example, we have $\nu = \frac{1}{z}$. (No y inside)

$\alpha = 2 - d\nu$, $\beta = (d - \nu)\nu$. (No y again)

$\beta(\delta - 1) = \gamma$

Thus,

z & ν tell us something about the eigenvalues of the renormalization transformation close to the critical point.

In other words, we could express all the critical exponents in terms of just 2 numbers (z & ν). These 2 numbers are called as the relevant scaling dimensions.

$y < 0 \longleftrightarrow$ Irrelevant scaling dimension

Now, in order to get more information about a specific model, we need to perform an explicit renormalization transformation (which is very difficult).