Short Intro to Boltzmann (Kinetic) Equation (Video Series)

Series 1, Title: Lecture 16 -The distribution function, Kinetic theory
Link: https://www.youtube.com/watch?v=FsXPbmrL3nw

Senior Plasma Physics
Lecture 16

Kinetic Theory

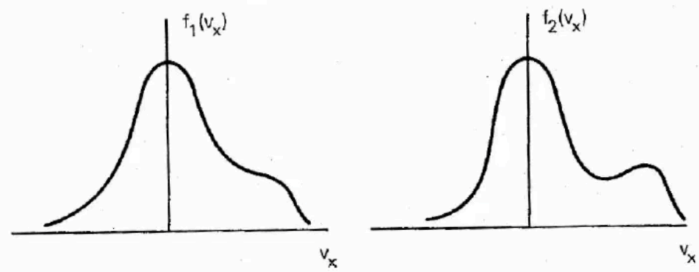
- distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$

position velocity time

The fluid model

- used x, y, z, and t as the independent variables.
- Macroscopic parameters were used to describe a plasma
- electron density, n_e
- pressure, P
- current density, $\boldsymbol{j}$
- Temperature, T
- etc.

Fluid model cannot distinguish between these two distribution functions



Kinetic theory uses the distibution function

$$f(\boldsymbol{r}, \boldsymbol{v}, t)$$

position velocity time

This uses 7 independent parameter: 3 position, 3 velocity, and time.

Could also be written in cylindrical coordinate system

$$\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$$

$$\boldsymbol{v} = v_x\hat{\boldsymbol{i}} + v_y\hat{\boldsymbol{j}} + v_z\hat{\boldsymbol{k}}$$

6 - Dimensional Space

The coordinates $(\boldsymbol{r}, \boldsymbol{v})$ define the particle position in **phase space**.

We now look at the kinetic theory and in particular, the distribution function, which in its general form, is dependent on position, velocity and time.

Kinetic theory deals essentially with the motion of individual particles. However, it is not feasible to track the motion of individual particles so we adopt a statistical approach.

Now, in the fluid model, our plasma also depended on position and time. These were regarded as independent variables. But the fluid model can not really handle the motion of individual particles, so macroscopic parameters were used instead. For example, electron density, pressure, current density, temperature, etc.

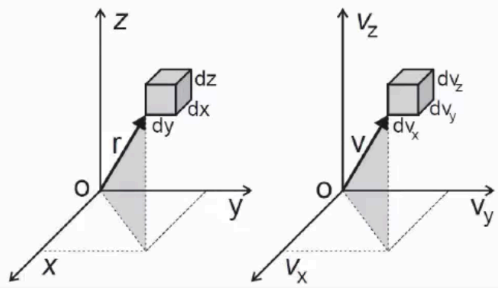
The core part of the kinetic theory is the distribution function. If one can obtain the distribution function, then all the information about a fluid system can be known.

Volume element in phase space

$$\mathrm{d}V = \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{r} = \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z$$

where

$$\mathrm{d}\boldsymbol{r} \equiv \mathrm{d}^3 r \equiv \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$\mathrm{d}\boldsymbol{v} \equiv \mathrm{d}^3 v \equiv \mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z$$



So when we write a volume element in phase space, dV, it is equal to dv * dr.

Note that the phase space is a 6-Dimensional space, so no one can really picture it. So what we often do is that we present two 3-Dimensional spaces.

The meaning of $f(\boldsymbol{r}, \boldsymbol{v}, t)$

$$f(\boldsymbol{r}, \boldsymbol{v}, t)\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{v} = \mathrm{d}N(\boldsymbol{r}, \boldsymbol{v}, t)$$

Number of particles in a volume element $\mathrm{d}V$ in phase space

That means $f(\boldsymbol{r}, \boldsymbol{v}, t)$ is the particle number density in phase space at time t

Particle number density in real space ONLY is

$$n(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z f(\boldsymbol{r}, \boldsymbol{v}, t)$$

$$= \int_{-\infty}^{\infty} f(\boldsymbol{r}, \boldsymbol{v}, t) d\boldsymbol{v} \qquad \text{(shorter notation)}$$

If $\hat{f}(\boldsymbol{r}, \boldsymbol{v}, t)$ is a normalized version of $f(\boldsymbol{r}, \boldsymbol{v}, t)$ then it means

$$\int_{-\infty}^{\infty} \hat{f}(\boldsymbol{r}, \boldsymbol{v}, t) d\boldsymbol{v}\,\mathrm{d}\boldsymbol{r} = 1$$

That means $\hat{f}(\boldsymbol{r}, \boldsymbol{v}, t) d\boldsymbol{v}\,\mathrm{d}\boldsymbol{r}$ is the **probability** of finding a particle in volume element $\mathrm{d}V = \mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{r}$

That means $\hat{f}(\boldsymbol{r}, \boldsymbol{v}, t)$ is the probability per unit volume of phase space. Also known as **Probability Density**

Say, we multiply the distribution function by the 6-dimensional volume element of phase space. This means that this is the number of particles in a volume element dV in phase space.

Let's symbolize that number of particles by dN. So another way of phrasing this is that the distribution function is the particle number density in phase space at time t.

However, say that if we only want the particle number density in real space, then the following integral are carried out, as shown in the left, which essentially eliminates the velocity componen

An example of probability functions

A room contains 14 people whose ages are:

one person aged 14
one person aged 15
three people aged 16
two people aged 22
two people aged 24
five people aged 25.

Let N(j) represent the number of people aged j. So we have

$N(14) = 1$
$N(15) = 1$
$N(16) = 3$
$N(22) = 2$
$N(24) = 2$
$N(25) = 5$

Total number of people

$$N = \sum N(j) = 14$$

Select a person at random. What is the probability that person is aged 16?

$$\text{Probability of age } 16 \equiv P(16) = \frac{N(16)}{N} = \frac{3}{14}$$

In general, $P(j) = \dfrac{N(j)}{N}$

Total probability $= \dfrac{\sum N(j)}{N} = 1$

What is the average (or mean) age?

$$\frac{(14) + (15) + 3(16) + 2(22) + 2(24) + 5(25)}{14} = \frac{294}{14} = 21$$

This is also known as a weighted average.

In general, the average age is written as

$$\bar{j} = \frac{\sum jN(j)}{N} = \sum jP(j)$$

If P(j) and j were continuous quantities then we can rewrite this in terms of an integral.

$$\bar{j} = \int_0^\infty jP(j)dj$$

where $\displaystyle\int_0^\infty P(j)dj = 1$

Back to the distribution function $f(\mathbf{r}, \mathbf{v}, t)$

Similarly, the probability of finding a particle at $\mathbf{r}$ and time t with velocities between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ is

$$\hat{f}(\mathbf{r}, \mathbf{v}, t)d\mathbf{v}$$

So an average speed can be obtain from the following:

$$\bar{v} = \int v\hat{f}(\mathbf{r}, \mathbf{v}, t)d\mathbf{v}$$

Quite often the hat $\wedge$ is left off a normalized distribution function

$$\hat{f}(\mathbf{r}, \mathbf{v}, t) \Longrightarrow f(\mathbf{r}, \mathbf{v}, t)$$

Average speed $\bar{v} = \int vf(\mathbf{r}, \mathbf{v}, t)d\mathbf{v}$

Series 2, Title: Lecture 17 - The Vlasov equation, Collisionless Boltzmann equation, Kinetic theory
Link: https://www.youtube.com/watch?v=u9OgtXxsBNU&pbjreload=10

Senior Plasma Physics
Lecture 17

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The Vlasov equation (collisionless Boltzmann equation)

$$\frac{df(\mathbf{r},\mathbf{v},t)}{dt}$$ Macroscopic forces change the distribution function. e.g. electric field

Recall $f(\mathbf{r},\mathbf{v},t)\,d\mathbf{r}d\mathbf{v}$ Number of particles in a 6-D phase space volume element $d\mathbf{r}d\mathbf{v}$

$$d\mathbf{r}d\mathbf{v} = dx\ dy\ dz\ dv_x\ dv_y\ dv_z$$

The Vlasov equation, is used to obtain the rate of change of the distribution function. This change occurs when there are macroscopic forces on the distribution function.

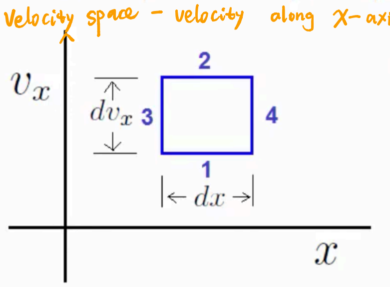
It's hard to obtain the expression for the rate of change of the distribution function in the 6-D phase space, so we will first try to obtain it in the 2-D space, and finally generalize it to the 6-D space.

Need to obtain an equation for $\dfrac{df(\mathbf{r},\mathbf{v},t)}{dt}$

Our volume element has 4 faces, given by the numbers 1-4.

Obtain an equation in 2-D phase space.

Velocity space - velocity along x-axis.



Calculate the change in particle number in this 2-D volume.

= Number of particles (coming out @ t + dt — in @ t)

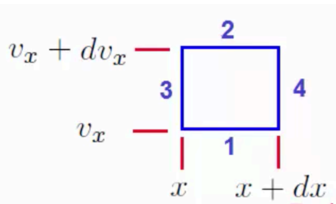
multiply

$$= \big(f(x,v_x,t+dt) - f(x,v_x,t)\big) \times dx\,dv_x$$

$$= df(x,v_x,t)\ dx\,dv_x$$

total change

number of particles (into face 1 - out of face 2)
+
number of particles (into face 3 - out of face 4)



1 into face 1 $\quad df(x,v_x,t)dxdv_x \rightarrow dx\dfrac{dv_x}{dt}dt \rightarrow a_x dxdt$

2 out of face 2 $\quad df(x,v_x+dv_x,t)dxdv_x \rightarrow a_x dxdt$

3 into face 3 $\quad df(x,v_x,t)dxdv_x \rightarrow \dfrac{dx}{dt}dtdv_x \rightarrow v_x dv_x dt$

4 out of face 4 $df(x+dx, v_x, t)dxdv_x \rightarrow v_x dv_x dt$

Total change = 1 − 2 + 3 − 4

$df\,dx\,dv_x$ **=** - $a_x\,df\,dxdt$ **+** - $v_x df dv_x dt$

divide both sides by $dv_x\,dxdt$ and using partials

and rearranging, we finally obtain

partial derivatives

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + a_x \frac{\partial f}{\partial v_x} = 0$$

In 3-D coordinates this can be generalized to

(Also known as the collisionless Boltzmann Equation)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$ **Vlasov equation**

where

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

$$\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$

$$\nabla_r = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

$$\nabla_v = \hat{\mathbf{i}}\frac{\partial}{\partial v_x} + \hat{\mathbf{j}}\frac{\partial}{\partial v_y} + \hat{\mathbf{k}}\frac{\partial}{\partial v_z}$$

Series 3, Title: Lecture 18 - The Vlasov equation, Collisionless Boltzmann equation, Kinetic theory
Link: https://www.youtube.com/watch?v=WGOC7HV-u3Y

Senior Plasma Physics
Lecture 18 - final

Kinetic theory
- The Boltzmann equation

Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_r f + \mathbf{a}\cdot\nabla_v f = 0 \Rightarrow \text{Collisionless}$$

Collisional plasma requires **The Boltzmann equation**

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_r f + \mathbf{a}\cdot\nabla_v f = \left(\frac{\partial f}{\partial t}\right)_c$$

Depends on the type of collisions

Types :
Coulomb, ionization, excitation, momentum transfer, etc

Can be quite a complicated expression

Example

Coulomb collisions

trapped for infinite time

Fully ionized plasma that is infinitely trapped.

Time $\Rightarrow \infty$ $\quad f(t) \Rightarrow f_M$ **Maxwellian**

Collision term
Tau or Krook
collision term : τ

$$\left(\frac{\partial f}{\partial t}\right)_c = \left(\frac{f_M - f}{\tau}\right)$$

mean collision time

Integrates to

$$f(t) = f_M + (f(0) - f_M)\exp(-t/\tau)$$

Not accurate, hardly ever used. e.g, does not include include the mass of different species.

More accurate $\left(\frac{\partial f}{\partial t}\right)_c$ = Fokker - Planck collision term

Beyond an introduction to Plasma Physics

From the previous video, we obtained the Vlasov equation, which shows the evolution of a distribution function, f, in a collision-less plasma. In reality, there is no such thing --- a collision-less plasma.

However, there are timescales where the average time between collisions is large enough for the plasma (during that time) regarded as collision-less. So under those conditions, the Vlasov equation could be quite useful.

But, there are many situations where the mean collision time is too short to use the Vlasov equation. At that situation, we must use the Boltzmann equation.