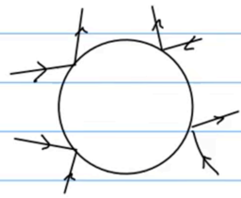
Notes for “fluctuations and the Langevin equation”

Link: https://www.youtube.com/watch?v=J7Df_td7gCc

Fluctuations

Brownian motion:



Light particles collide many times with heavy ones.

Collisions have two effects:

① they cause drag   Drag → drift

② Random kicks on top of the drag.

$$m\dot{v} = -\gamma v + R(t) + F(\vec{r}, t)$$

$-\gamma v$ → Drag force

$R(t)$ → Random force vector, Random kicks

$F(\vec{r}, t)$ → systematic force (or fie[ld]) (like electric fi[eld] caused electric force)

Properties of $R(t)$ (Random force)

(i) $\langle R(t) \rangle = 0$ for all time t.

(ii) About Correlations: $\langle R(t) R(t+\tau) \rangle = \langle R^2 \rangle \delta(t) \Rightarrow$ No Time Correlat[ion]

It's just an approximation, NOT alway true because of the hydrodynamics. But this approximation is valid for heavy particles because they will experience many collisions befo[re] they change their position appreciably. So the timescale at which the heavy particle moves is a lot slower than the correlation time and the fluid.

(iii) $R(t)$ is subject to Gaussian Distribution:

$$P(R) = \frac{1}{\sqrt{2\pi\langle R^2\rangle}} e^{-R^2/2\langle R^2\rangle}$$

Probability of finding a random force of size R

In Computer program, we need to discretize the time:

time steps $t_1, t_2, \cdots$ (equidistance)

$$t_{j+1} - t_j = \Delta t$$

Then $$P(R_1, R_2, \cdots, R_N) = \frac{1}{(2\pi\langle R^2\rangle)^{\frac{N}{2}}} e^{-(R_1^2+R_2^2+\cdots+R_N^2)/2\langle R^2\rangle}$$

The probability density for having a random force R_1 at time t_1, R_2 at time t_2, $\cdots$ R_N at time t_N.

continuum notation

$$\approx \frac{1}{(2\pi\langle R^2\rangle)^{\frac{N}{2}}} e^{-\frac{1}{2\pi\langle R^2\rangle\Delta t}\int_{t_1}^{t_N} R^2(t)\,dt}$$

$$= \frac{1}{(2\pi\langle R^2\rangle)^{\frac{N}{2}}} e^{-\frac{1}{2q}\int_{t_1}^{t_N} R^2(t)\,dt}$$

(We let $q = \pi\langle R^2\rangle\Delta t$)

Since $\langle R(t)R(t+\tau)\rangle = \langle R^2\rangle\delta(t)$,

thus $\Rightarrow \langle R_M R_{M'}\rangle = \langle R^2\rangle\delta_{MM'} = \frac{q}{\pi\Delta t}\delta_{MM'}$

Solution to Langevin Equation.

Consider the situation where the systematic force $F(\vec{r}, t) = 0$ (see above).

$\Rightarrow$ $m\dot{v} = -\gamma v + R(t)$ Langevin Equation

(It's a differential Equation)

for the case of Homogeneous equation, 1D $\Rightarrow R(t) = 0$

$m\dot{v} = -\gamma\tilde{v}$; $\tilde{v} = \tilde{v}_0 e^{-\gamma t/m}$

homogeneous velocity   Homogeneous solution for v

for a particular solution:

Try $v(t) = v_0 e^{-\gamma t/m} f(t)$

(where $f(t)$ depends on time)

Plug into Langevin Equation

$\longrightarrow m(\dot{f} - \gamma \cancel{\frac{f}{m}})\tilde{v}(t) = R(t) - \cancel{\gamma\tilde{v} f}$

$\longrightarrow \dot{f} = \frac{R(t)}{m v_0} e^{\gamma t/m}$

Integrate it

$\longrightarrow f(t) = \frac{1}{m v_0}\int_0^t R(t') e^{\gamma t'/m} dt'$

But $R(t)$ is random force

We can never solve it Analytically

$\Longrightarrow v(t) = v_0 e^{-\gamma t/m} + \frac{1}{m}\int_0^t e^{-\gamma(t-t')/m} R(t') dt'$

(In the beginning, we assume $v(t) = v_0 e^{-\gamma t/m} f(t)$)

Based on the statistical properties of $R(t)$, we can explore the

✓ statistical properties for $\dot{v}(t)$

$\Longrightarrow$ (i) $\langle v(t)\rangle = v_0 e^{-\gamma t/m}$ $\Longleftarrow$ exponentially damped velocity

(since the 2nd term vanishes)

(ii) $$\langle v^2(t)\rangle = v_0^2 e^{-2\gamma t/m} + \frac{1}{m^2}\int_0^t\int_0^t \underbrace{\langle R(t_1)R(t_2)\rangle}_{q\,\delta(t_1-t_2)} e^{-2\gamma t/m} e^{\gamma(t_1+t_2)/m}\, dt_1 dt_2$$

this $\Rightarrow$ t_1 must be equal to t_2

$$= v_0^2 e^{-2\gamma t/m} + \frac{q}{m^2}\int_0^t e^{-2\gamma t/m} e^{-2\gamma t'/m}\, dt'$$

$$= v_0^2 e^{-2\gamma t/m} + \frac{q}{2\gamma m}\Big(1-\underbrace{e^{-2\gamma t/m}}\Big)$$

for long time, this term vanishes

$\Longrightarrow$ for long time t, $\langle v^2(t\to\infty)\rangle = \frac{q}{2\gamma m} = \frac{k_B T}{m}$

$$\Downarrow$$

$$\boxed{q = 2\gamma k_B T}$$

(from the equipartition theorem, if the particle is in a liquid or gas, The velocity squared expectation value should be $\frac{k_B T}{m}$)

Now We go back to Langevin equation, and try to solve it again.

$$m\dot{v} = -\gamma v + R$$

for simplicity, we assume $m = 1$

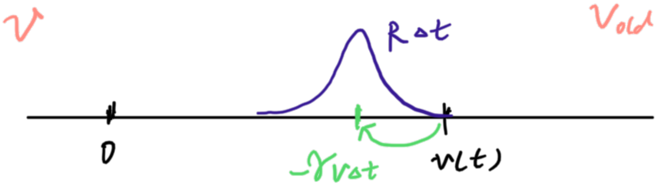
what we want to get is $P(v,t)$.

What we want ...

(The probability of finding a particle with a particular velocity v at a certain time t).

If we know $v(t)$ at time t.

$$\rightarrow \underbrace{v(t+\Delta t)}_{v} = v(t) + a \times \Delta t = \underbrace{v(t)}_{v_{old}} + (-\gamma \underbrace{v}_{v_{old}} + R)\Delta t$$



This $v(t+\Delta t)$ equation hold.

If we know $P(\overset{v_{old}}{v}, t)$ at time t.

$$P(v, t+\Delta t) = \int P(v_{old}, t) \underbrace{\overset{\text{Curly P}}{\mathcal{P}}(R\Delta t)}_{\text{Guassian Distribution}} \delta(v - v_{old} - (-\gamma v + R)\Delta t)\, d(R\Delta t)\, dv_{old}$$

The Probability that we make a jump of our Δt

$$\delta(\lambda x - a) = \frac{1}{|\lambda|}\delta\left(x - \frac{a}{\lambda}\right)$$

$$= \frac{1}{1-\gamma\Delta t} \int P(v + (\gamma v - R)\Delta t, t)\, \mathcal{P}(R\Delta t)\, d(R\Delta t)$$

(Δt is too small, use Taylor series expansion) to the second order of Δt

(Anti-symmetric integrand)

被积函数

$$= \frac{1}{1-\gamma\Delta t} \int \left[P(v,t) + (\gamma v - R)\Delta t \frac{\partial P(v,t)}{\partial v} + \frac{(\gamma v - R)^2 \Delta t^2}{2} \frac{\partial^2 P}{\partial v^2}\right] \mathcal{P}(R\Delta t)\, d(R\Delta t)$$

to order Δt? (数学!) 不懂

Because there is no $(R\Delta t)$ dependence in this $P(v,t)$ term

0 (数学!) (不懂)

The term which is linear in R vanishes

normalized

Because we always make one step

$\Downarrow$

$\int \mathcal{P}(R\Delta t)\, d(R\Delta t)$

$$P(v,t+\Delta t) = P(v,t)(1+\gamma\Delta t) + \gamma v \Delta t \frac{\partial P}{\partial v} + \frac{\gamma^2 v^2 \Delta t^2}{2}\frac{\partial^2 P}{\partial v^2}$$

$$+ \underline{\frac{1}{2}\frac{\partial^2 P}{\partial v^2}\int (R\Delta t)^2 \mathcal{P}(R\Delta t)\, d(R\Delta t)}$$

2nd moment of our gaussian distribution

(we have $\langle R^2\rangle = \frac{q}{\Delta t} = \frac{2\gamma k_B T}{\Delta t}$)

$$\gamma k_B T \Delta t \frac{\partial^2 P}{\partial v^2}$$

$$\Downarrow$$

$$\frac{\partial P(v,t)}{\partial t} = \gamma \frac{\partial}{\partial v}(v P(v,t)) + \gamma k_B T \frac{\partial^2 P(v,t)}{\partial v^2}$$

Fokker Planck Equation

Note: If $m \neq 1$, Fokker Planck Eqn:

$$\frac{\partial P(v,t)}{\partial t} = \frac{\gamma}{m}\frac{\partial}{\partial v}(v P(v,t)) + \frac{\gamma k_B T}{m^2}\frac{\partial^2 P(v,t)}{\partial v^2}$$

Diffusion (Random Walk): $\langle x^2 (t \to \infty)\rangle = 2Dt$

x ↑ Displacement

We try to find $\langle x^2(t)\rangle$

$$x(t) = \int_0^t v(t')\, dt'$$

Recall $v(t) = v_0 e^{-\gamma t/m} + \frac{1}{m}\int_0^t e^{-\gamma(t-t')/m} R(t')\, dt'$

Homogeneous Solution

Particular solution depends on the Random Force $R(t)$

Simpler Method:

$$m\ddot{x} = -\gamma \dot{x} + R(t)$$

Multiply both sides by x:

$$m x \ddot{x} = -\gamma x \dot{x} + R(t)\, x(t)$$

$$\downarrow$$

$$m\left[\frac{d}{dt}(x\dot{x}) - \dot{x}^2\right] = -\gamma x \dot{x} + R(t)\, x(t)$$

Equipartition Theorem: $m\dot{x}^2 = k_B T \longrightarrow$

$$m\frac{d}{dt}(x\dot{x}) + \gamma (x\dot{x}) = k_B T + R(t)\, x(t)$$

$\downarrow$ take expectation values.

$$m\frac{d}{dt}\underbrace{\langle x\dot{x}\rangle}_{y} + \gamma\langle x\dot{x}\rangle = k_B T + \underbrace{\langle R(t)\, x(t)\rangle}_{=0}$$

Because Random Force R and $x(t)$ are uncorrelated

$\langle R(t)\, x(t)\rangle = \langle R(t)\rangle \langle x(t)\rangle$

$\langle R(t)\rangle = 0$

$\therefore \langle R(t)\, x(t)\rangle = 0$

$$\downarrow$$

$$m\frac{v}{dt}y + \gamma y = k_B T$$

Solution: $m\langle x\dot{x}\rangle = C + Ae^{-\gamma t/m}$ $\quad (C = \frac{k_B}{\gamma}$

$t \to \infty$: $\langle x\dot{x}\rangle = \frac{k_BT}{\gamma}$

$t = 0$: $\langle x\dot{x}\rangle = 0 \implies A = -\frac{k_BT}{\gamma}$

Now note that $\langle x\dot{x}\rangle = \frac{1}{2}\frac{d}{dt}\langle x^2\rangle$

$$\longrightarrow \langle x^2\rangle = \frac{2}{m}\int_0^t \frac{k_BT}{\gamma}(1-e^{-\gamma t'/m})\,dt$$

$$= \frac{2k_BT}{\gamma}t + 2\frac{k_BT}{\gamma}\frac{m}{\gamma}(e^{-\gamma t/m}-1)$$

For short time : $(t \to 0)$:

$$\langle x^2\rangle = \frac{2k_BT}{\gamma}\left[t + \frac{m}{\gamma}\left(1-\frac{\gamma t}{m}+\frac{\gamma^2t^2}{2m^2}+\cdots\right)-\frac{m}{\gamma}\right]$$

$$= k_BT\frac{\gamma}{m}t^2$$

$\implies x \propto t$. (at short time (t is very small, particle moves like free particle)

For large time $(t \to \infty)$

$$\langle x^2\rangle = \frac{2k_BT}{\gamma}\left(t-\frac{m}{\gamma}\right) \simeq \frac{2k_BT}{\gamma}t = 2Dt$$

$$\longrightarrow \underline{\underline{D = \frac{k_BT}{\gamma}}}$$

$\gamma = 6\pi\eta a$, η = viscosity

a ↑ particle radius

η ↓ hydrodynamic viscosity

(Stoke's law)

$P(x,t)\,dx$ = probability to find a particle inside $(x, x+dx)$ at time t.

As v is linear in $R(t)$, also x is linear in $R(t)$

$R(t)$ is gaussian $\Rightarrow$ $x(t)$ is Gaussian

We have calculated the width of this gaussian to be $\langle \Delta x^2(t)\rangle = 2Dt$.

Hence $T(\Delta x, t) = \dfrac{1}{\sqrt{4\pi D t}}\, e^{-\Delta x^2/4Dt}$

Transition Probability

The Probability to make a step of size Δx within a time interval t.

From this we can derive an equation for $P(x,t)$ as follows

$$P(x, t+\Delta t) = \int P(x-\Delta x, t)\, T(\Delta x, t)\, dx$$

$$\int \left(P(x,t) - \Delta x \frac{\partial}{\partial x} P(x,t) + \frac{\Delta x^2}{2} \frac{\partial^2 P(x,t)}{\partial x^2} \right) T(\Delta x)\, dx$$

$$= P(x,t) + \frac{\partial^2 P}{2\partial x^2} \int \Delta x^2\, T(\Delta x)\, dx = P(x,t) + D\Delta t \frac{\partial^2 P}{\partial x^2}$$

$$\Rightarrow \frac{\partial P(x,t)}{\partial t} = D \frac{\partial^2 P(x,t)}{\partial x^2}$$

(Diffusion)

D is independent on x

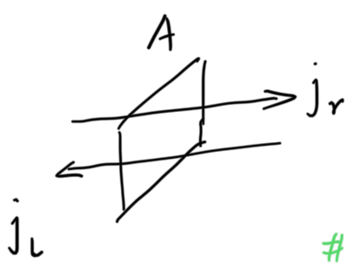
or $$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x} D(x) \frac{\partial P(x,t)}{\partial x}$$ (D is dependent on x in general)

Here we have assumed that $\Delta t \gtrsim \frac{m}{\gamma}$, in order to justify the diffusive limit.

On the other hand $\Delta x \leq \sqrt{Dt}$; $D = \frac{k_B T}{\gamma}$

ok for long enough time.

An alternative derivation for Fick's law:



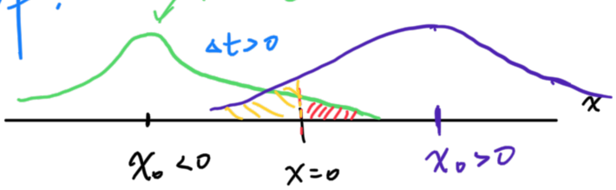
flux : $j(r,t) = \rho(r,t)\, v(r,t)$

particle density (ρ); velocity (v)

\# of particle passing through A (unit Area) per unit time.

$$\frac{\#\text{ of atoms passing through } A \text{ in time } \Delta t}{A\, \Delta t}$$

Proof: Particle distribution



1-D

$$j(x_0<0, t) = \int^{\infty} \frac{e^{-(x-x_0)^2/4D\Delta t}}{\quad} dx$$

$$\int_0 \quad \sqrt{4\pi D\Delta t}$$

$$j(x_0>0,t) = \int_{-\infty}^{0} \frac{e^{-(x-x_0)^2/4D\Delta t}}{\sqrt{4\pi D\Delta t}}\,dx$$

Total:
$$\int_{-\infty}^{0} dx_0 \int_0^{\infty} \rho(x_0,t)\,\frac{e^{-(x-x_0)^2/4D\Delta t}}{\sqrt{4D\Delta t}}\,dx$$

$$-\int_0^{\infty} dx_0 \int_{-\infty}^{0} \rho(x_0,t)\,\frac{e^{-(x-x_0)^2/4D\Delta t}}{\sqrt{4D\Delta t}}\,dx$$

$$\updownarrow$$

$$\int_{-\infty}^{0} dx_0 \int_0^{\infty} \rho(-x_0,t)\,\frac{e^{-(x-x_0)^2/4D\Delta t}}{\sqrt{4D\Delta t}}\,dx$$

Because of the short time assumption, here we can assume that the particles that contribute to the flux is very close to $x=0$.

Then we can apply taylor expansion to $\rho(x,t)$.

$$\rho(x_0,t) = \rho(0,t) + x_0\rho'(0,t)$$

Total: $$= 2\rho'(0,t)\int_{-\infty}^{0} dx_0\, x_0 \int_0^{\infty} \frac{e^{-(x-x_0)^2/4D\Delta t}}{\sqrt{4D\Delta t}}\,dx$$

$$= -\rho'(0,t)\,D\Delta t = j(0,t)\Delta t$$

(by definition)

Therefore ... $\partial\rho(x,t)$ ...

Therefore: $j(x,t) = -D \frac{\partial \rho}{\partial x}$ (1-D)

General: $j(\vec{r},t) = -D \nabla \rho(\vec{r},t)$ Fick's 1st diffus law.

always try to flatten the particle density!

Fick's 1st Law gives us the Diffusive Current.

Consider a volume V with a surface A.

Change in the amount of particles inside V:

$$\int_V [\rho(r, t+\Delta t) - \rho(t)] d^3 r = \int_A j \cdot da \, \Delta t$$

(no sources/sinks) area

$$\Longrightarrow \int_V \frac{\partial \rho}{\partial t} d^3 r = -\int_A j \cdot da = -\int_V \nabla \cdot j \, d^3 r$$

(Divergence Theorem)

$\Downarrow$

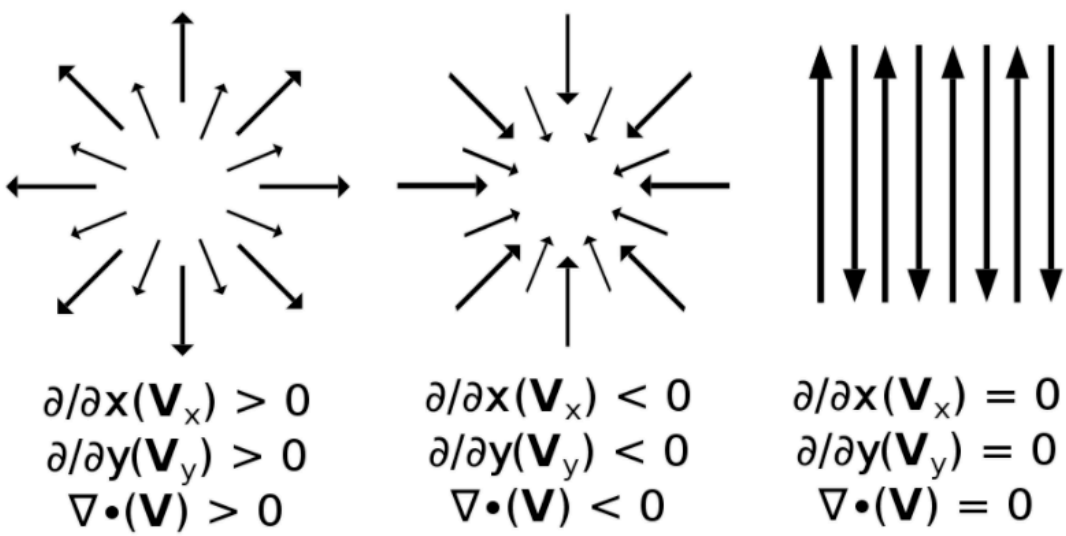
Hence: $\frac{\partial \rho}{\partial t} = -\nabla \cdot j$ Continuity Equation

(Conservation of matter)

Divergence:

Source: https://en.wikipedia.org/wiki/Divergence

In vector calculus, divergence is a **vector operator** that operates on a vector field, producing a scalar field giving the quantity of the vector field's source at each point.



The divergence of different vector fields. The divergence of vectors from point (x,y) equals the sum of the partial derivative-with-respect-to-x of the x-component and the partial derivative-with-respect-to-y of the y-component at that point:

$$\nabla\cdot(\mathbf{V}(x,y)) = \frac{\partial\,\mathbf{V}_x(x,y)}{\partial x} + \frac{\partial\,\mathbf{V}_y(x,y)}{\partial y}$$

Divergence theorem
Source: https://en.wikipedia.org/wiki/Divergence_theorem

In vector calculus, the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem, is a result that

relates the flux of a vector field through a closed surface to the divergence of the field in the volume enclosed.

More precisely, the divergence theorem states that the surface integral of a vector field over a **closed surface**, which is called the flux through the surface, is equal to the volume integral of the divergence over the region inside the surface.

Combining Fick's Law and continuity equation

$(j = -D \nabla \rho)$ $\left(\frac{\partial \rho}{\partial t} = -\nabla \cdot j\right)$

we obtain.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (-D \nabla \rho) = \nabla (D \nabla) \rho$$

Note: $\left(\rho(r,t) \sim P(r,t) \right)$ → probability

derived before

Introduce drift:

→ systematic form force

(i.e. gravity,

$$m\dot{v} = \underbrace{-\gamma v}_{\text{drag force}} + F + \underbrace{R(t)}_{\text{random force}}$$

electric fie

For simplicity, we assume that F is constant in tim and NOT dependent on position.

For F constant:

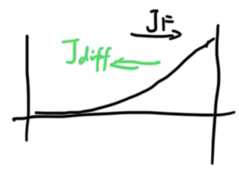
$$m\langle \dot{v} \rangle = -\gamma \langle v \rangle + F = 0 \qquad \text{for } t \gtrsim \frac{m}{\gamma}$$

(if we assume v does not change)

Then $J_F = \rho \langle v \rangle = \rho F/\gamma$

(Drift Flux)

If the system is close



J_{diff} : diffusive

$$J_F = -J_{diff} : \quad D \nabla \rho = \rho \frac{F}{\gamma} \Rightarrow \frac{\nabla \rho}{\rho} = \frac{F}{\gamma D}$$

(gradient of potenti is equal to force)

Therefore $$\nabla \ln \rho = -\nabla U(r)/\gamma D$$

$$\Rightarrow \rho_{eq}(r) = \text{const} \times e^{-U(r)/\gamma D}$$

equilibrium

note: $\gamma D = k_B T$

$$\Downarrow$$

$$\rho_{eq}(r) = C\, e^{-U(r)/k_B T}$$

(The probability of finding a

Previously we have a diffusion equation (particle at a certain positi…)
with No drift force present.

Now, we generalize the diffusion equation to
the case where there is the drift force.

$$P(x, t+\Delta t) = \int P(\underbrace{x-\Delta x}_{x'}, t)\, T(\Delta x, x', \Delta t)\, d\Delta x$$

Transition probability over Δx d… time Δt, but dependent on starting position

$$\langle \Delta x \rangle = \frac{F}{\gamma}\Delta t = \int T(\Delta x, x', \Delta t)\, \Delta x \, d\Delta x$$

(Δx is caused by both diffusion and drift)

diffusion: average out because diffusion caused displacement has no certain direction

$$\langle \Delta x^2 \rangle = 2D\Delta t = \int T(\Delta x, x', \Delta t)\, \Delta x^2 \, d\Delta x$$

Multiply both sides by $\delta(x-x_0)$ and integrate over …

$$\Rightarrow P(x_0, t+\Delta t) = \iint P(x-\Delta x, t)\, T(\Delta x, x-\Delta x, \Delta t)\, \delta(x-x_0)\, d\Delta x\, d\ldots$$

Taylor expansion

(1) → (First order Taylor expansion)

$$\Delta t \frac{\partial P(x_0, t)}{\partial t} = \iint \Big[\Big(-\Delta x \frac{\partial P}{\partial x}\, \delta(x-x_0)\, T(\Delta x, x-\Delta x, t) + P(x,t)\, (-\Delta x) \frac{\partial T(\Delta x, x, t)}{\partial x} \Big] d\ldots$$

$$\left(\frac{d}{dx}\langle \Delta x\rangle = \frac{dF/dx}{\gamma}\Delta t = \int \left(\frac{\partial}{\partial x} T(\Delta x, x-\Delta x, t)\right)\Delta x \, d\Delta x \right)$$

$$\Delta t \frac{\partial P(x_0, t)}{\partial t} = -\int \frac{\partial P}{\partial x}\frac{F}{\gamma}\Delta t\, \delta(x-x_0)dx - \int \frac{P}{\gamma}\frac{\partial F}{\partial x}\delta(x-x_0)\,dx$$

$$= -\frac{1}{\gamma}\frac{\partial}{\partial x_0}\left[P(x_0, t) F(x_0, t)\right]$$

Assume D constant:

2nd order term $\quad D\Delta t \dfrac{\partial^2 P(x_0,t)}{\partial x^2}$

$\rightarrow$ Schmolukowski equation :

(or, Smoluchowski's Convection - diffusion equation

$\rightarrow$ drift force

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(P\frac{F}{\gamma}\right) + \frac{\partial}{\partial x} D \frac{\partial P}{\partial x}$$

diffusion

$$P(x,t) \overset{\text{Proportional}}{\Longleftrightarrow} \rho(x,t)$$

This density distribution d... NOT change with time (equilibrium st...

$$\rightarrow \frac{\partial \rho}{\partial t} = \nabla\left(\frac{\rho F}{\gamma} - D\nabla\rho\right) = 0 \quad \text{for } t\to\infty$$ (Equilibrium

Solution : $\rho = e^{-u(r)/\gamma D}$

Hence $D = k_B T/\gamma$ Einstein Relation

It's a relation between the drag and the diffusion coefficient.

Balance diffusion & drift for electrons:

(i.e. e^- moving in an electric field inside a conduct

$$\underbrace{j_{el}}_{\text{electron current}} = -\underbrace{\rho}_{\text{Charge Density}} \boxed{\frac{F}{\gamma}} = -e \underbrace{j_{diffn}}_{\text{diffusion current}} = -D\frac{\partial \rho}{\partial x} = -D\frac{d}{dx}\big(\underbrace{Ce^{-eV(x)/k}}_{\rho}$$

velocity

Charge Density : $= e \times n$ (n: number density)

$$= \frac{eDE}{k_B T} C e^{-eV(x)/k_B T} = \frac{eDE}{k_B T}\rho$$

Electric Field

We know that $j_{el} = \sigma E$, σ : conductivity

j_{el}: electricity current; E: electric field

Hence : $\sigma = \dfrac{eD\rho}{k_B T} = \dfrac{e^2 D n}{k_B T}$ n : number densi

Ohm's law   Drude Conductivity

Summary Langevin equation & diffusion

(I) Langevin eqn: $m\dot{v} = \underline{-\gamma v} + \underline{R(t)}$

drag Random force

① $\langle R(t)\rangle = 0$;

② $\langle R(t)R(t+\tau)\rangle = \langle R^2\rangle \delta(\tau)$; (No Time Correlatio

③ $P(R) = \frac{1}{\sqrt{2\pi\langle R^2\rangle}} e^{-R^2/2\langle R^2\rangle}$

(II) Properties of the solution to Langevin eqn.

(i) $\langle v(t)\rangle = v_0 e^{-\gamma t/m}$ (The velocity will damp out on average exponentially due to the dra of other part

(ii) $\langle v^2(t)\rangle = v_0 e^{-2\gamma t/m} + \underline{\frac{q}{2\gamma m}(1-e^{-2\gamma t/m})}$

In the end, the particle will keep oscillating due to Random ki

(iii) $t\to\infty$ $\langle v^2(t)\rangle = \frac{q}{2\gamma m} = \frac{k_B T}{m} \to q = 2\gamma k_B T$

(q is defined here in terms of temperature)

(III) Statistical Distribution of velocities. ← Because we have a statist distribution of the Random For

Distribution $P(v,t)$ satisfies the Langevin equation

$$\frac{\partial P}{\partial t} = \gamma v\frac{\partial P}{\partial v} + \gamma k_B T\frac{\partial^2 P}{\partial v^2}$$

(Fokker-Planc Equation

$$\frac{\partial P}{\partial t} = 0 \to P = Ce^{-mv^2/2k_BT}$$

stationery case

This follows the Maxwell Distribut

Evolution of the distribution of the velocities

IV. Look at the Position X

(IV) Look at the

Solution for x: $x = \int^t v\,dt$:

$\langle x^2(t) \rangle = 2Dt$ D: diffusion coefficient

$D = k_B T/\gamma$

$\langle x^2 \rangle$ is linear in time

The most important property for diffusion part.

(V) The distribution of the positions of the particles as a function of time.

Probability density for x satisfying the Langevin equation:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} D(x) \frac{\partial}{\partial x} P \;; \qquad P = P(x,t)$$

In inhomogeneous medium, D is dependent on

In homogeneous medium, D is independent on x

(VI) Fick's Law

$j = D \nabla \rho$ $\rho(r,t)$: particle density $\propto P(r,t)$

concentration

proportional

Continuity eqn: $\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$

Continuity eqn + Fick's law $\Longrightarrow$ Diffusion Equation

$\left(\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} D(x) \frac{\partial P}{\partial x} \right)$

(VII) Introducing drift force F

Solved Langevin equation for $P(x,t)$:

Schmolukowsky equation for P(r,t).
(or Smoluchowski)

Dr

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(P\frac{F}{\gamma}\right) + \frac{\partial}{\partial x} P \frac{\partial}{\partial x} P$$