

Landau Ginzburg theory of Phase Transitions

Link: <https://www.youtube.com/watch?v=yaBBBLV39Vs>

Landau Theory

Ising Model (There are only 2 phases, which are 2 magnetic phases).

Partition function for the Ising Model:

$$Z = \sum_{\{S_i = \pm 1\}} e^{J \sum_{\langle i,j \rangle} S_i S_j + B \sum_i S_i}$$

sum for all the possible spin configurations.

S_i : Spin Values, +1 or -1

Extra Interaction caused by external field (This field helps one favors one kind of spin)

Interaction between nearest neighbours

$$J = \beta K, \quad B = \beta h, \quad \beta = \frac{1}{k_B T}$$

coupling constants $\leftrightarrow J$

external field

K is some coupling constant

m : Average

$$\text{Then, } m = \frac{1}{N} \sum_i S_i$$

(average spin)

Magnetisation

If all S_i are +1, then $m = +1$;

If all S_i are -1, then $m = -1$;

Thus, $-1 \leq m \leq 1$

$$Z = \sum_m Z_{B=0}(m, J) e^{BmN}$$

A new way to solve out the partition function: Instead of solving over each individual spin configuration, we collect all the configurations which have a particular value of m . If we perform the sum over those configurations we call the result Z and we evaluated at $B=0$ and then as a function of m & J .

$$Z_{B=0}(m, J)$$

\uparrow
 m means that we have summed over all the configurations for which the spins have an average magnetization m .

\downarrow We have excluded B from this sum because for all these configurations we can replace the sum over i ($B \sum_i S_i$) by $N \times m$.
 \curvearrowright $(m = \frac{1}{N} \sum_i S_i)$

If we have the partition function, we can then calculate the free energy: (The difference between F & $F(m)$ is that F is a free energy which is calculated from a partition function which is a sum over all configurations, whereas $F(m)$ is determined from a partition sum over only those configs which has a magnetisation m . Therefore, in an Ising model you would sum over all configs of the spins, with the restriction that $\sum_i S_i = Nm$ N being the number of sites.)

$$F = -k_B T \ln Z$$

$$F(m) = -k_B T \ln Z_{B=0}(m, J) - k_B T B m N$$

$$\Rightarrow \frac{F(m)}{N} = -h m + f(m, J) \quad (B = \beta h = \frac{h}{k_B T})$$

$F(m)$ is This is an intrinsic quantity, intensive quantity.

an extensive quantity.

Consider the fact that we are close to a critical phase transition:
(which we are interested)

When it is close to a critical phase transition.

m is small \rightarrow Expand f in terms of m .
(Taylor Expansion)

Since $f(m, J)$ is calculated at $B=0$, there is no bias to favor the positive or negative magnetization.

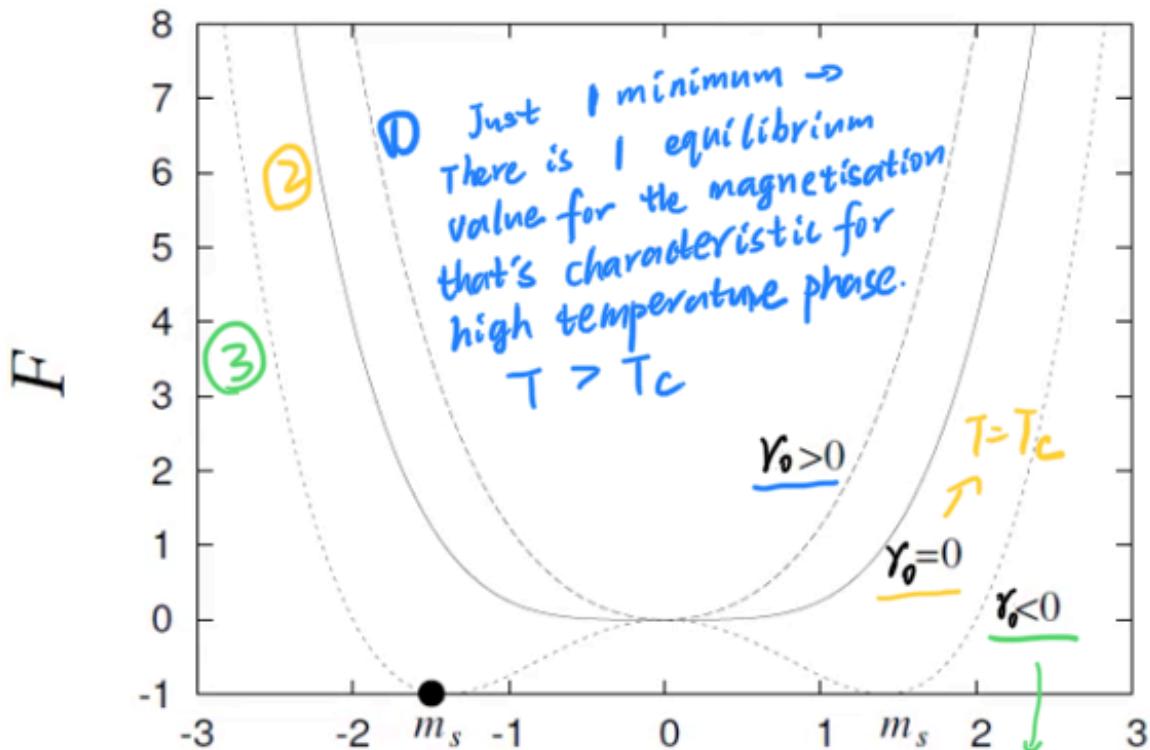
In fact, $f(m, J)$ is an even function of the magnetization. So if we do a Taylor expansion, we only expect even powers of m to survive.

$$\rightarrow \frac{F(m)}{N} = -hm + q + \gamma_0 m^2 + \kappa m^4$$

Since q is just a constant, it does NOT affect much in our equation, we just leave it out.

$$\text{Thus, } \frac{F(m)}{N} = -hm + \gamma_0 m^2 + \kappa m^4$$

$$\text{If } h=0 : f(m) = \gamma_0 m^2 + \kappa m^4$$



$T < T_c \iff$ for $\gamma_0 < 0$, there are 2 minima which correspond to the low temperature phase.

Assuming γ varies linearly with the temperature,

$$\gamma_0 = \gamma t, \quad t = \frac{T - T_c}{T_c}$$

↑
t: reduced temperature.

The above picture gives us an idea of what happens close to a phase transition.

We see that the curve is approaching from

below ($\gamma_0 < 0$) to ($\gamma_0 = 0$), this curve with

.

the 2 minima will resemble more and more to this ($T=T_c$, $\gamma_0=0$) curve which has only 1 minimum. This means that these 2 minima are getting closer and closer together.

For the ($T=T_c$, $\gamma_0=0$) curve, we see that the (bottom) basin in which m moves is very

wide which means that there was not too much free energy cost associated with a substantial variation of the magnetization.

In this case, it's very easy to have large fluctuations. \Rightarrow We know that close to the critical point, the fluctuations increase.

Fluctuations are strong close to the critical point!

The critical exponents for the Ising Model.

$$\frac{F}{N} = \gamma_0 m^2 + \mu m^4 ; \frac{dF}{dm} = 0 \rightarrow$$

$$2\gamma_0 m + 4\mu m^3 = 0 \rightarrow m^2 = -\frac{\gamma_0}{2\mu} \text{ or } m = 0$$

if $\gamma_0 < 0$, $\rightarrow T < T_c$, m corresponds to the 2 minima in the $\gamma_0 < 0$ curve.

For $T > T_c$, $m=0$ is the stable minimum.
(It's a local maximum if $T > T_c$)

We want to know the critical exponent which describes how m vanishes when we approach the critical temperature (This expression: $m^2 = -\frac{r_0}{2u}$ gives us the answer.)

$$m = \sqrt{\frac{-rt}{2(u_0 + u_1 t + \dots)}} \sim t^{\frac{1}{2}}$$

The critical exponent β

$$\beta = \frac{1}{2}$$

($r_0 = rt$, we already know)

For the critical exponent of α , describes the divergence of the specific heat vanishes

$$C_h \sim \frac{\partial^2 F}{\partial T^2}, \quad \text{specific heat} \quad r m^2 \sim t^2, \quad \text{hence } C_h \sim \text{const} \quad \alpha = 0$$

Exponents for the magnetic susceptibility: γ (with $\gamma \approx 1$)

$$\chi \sim |T - T_c|^{-\gamma} \quad \gamma = 1 \quad (\text{magnetic susceptibility})$$

$$m_{T=T_c} \sim h^{1/\delta} \quad \delta = 3$$

Landau theory \Rightarrow mean field critical exponents

for magnetization, how it vanishes as a function of an applied field at the critical temperature described by a critical exponent δ ($\delta = 3$)

Mean field critical exponent

Landau theory: An expansion of the free energy in terms of the magnetization we always find the critical exponents from the

mean field theory.

Landau theory just takes the magnetization as above as a parameter and the free energy gives rise to the mean field critical exponents

Landau - Ginzburg Theory

(In Landau theory, we assume that everywhere throughout the lattice there was some average magnetization which was m)

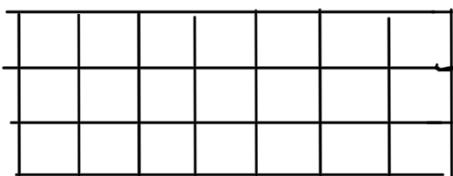
In Landau - Ginzburg theory, we allow for that m to fluctuate. To do so, we assume

different values at different places. So, we replace m by $\underline{m(r)}$ Dependent on position.

$$m \rightarrow \underline{m(r)}$$

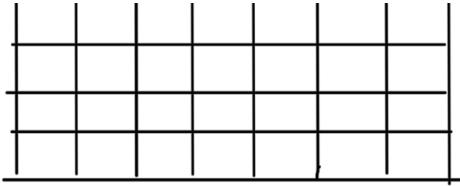
(We are NOT going back to the description in terms of the individual spins (± 1)).

Actually, $m(r)$ is a mesoscopic average *
of the spin.)



$$m \rightarrow \underline{m(r)}$$

Each box's size: $\ll L$



↑
 contains many copies
 (Suppose we have a system of
 size L that we are going to
 divide it into boxes)

On the other hand, the individual spins, they are so close that we can fit many many individual spins within each box.

Within each box, we can define the average value of the spin, and it will be almost

a continuous number — A real number that depends on the position.

But we are NOT going to look at the system so fine-grained that we reach the

lattice constant of the original spin model.

We stay away from that so that we contain (in each box) many many spins.

Thus, the Hamiltonian for the entire system in terms of the coarse-grained spins :

* (The Hamiltonian is a sum of the free energy in each box)

k: weight
$$H = \int_T [k m(r)^2 + \dots + m^4(r)] dr$$

factor $H = \int L(r) \cdot m(r) + \mu_0 \nabla \cdot m(r) + \mu_0 \chi(r) m(r)$
 An external field
 that depends on r .
 (we assume that we allow
 for the magnetic field be non-homogeneous)

Local Hamiltonians:
 $\mathcal{H}_0(m(r))$

→ An extra contribution to the energy, which resides
 in the fact that we might want 2 boxes have a very different
 magnetization. We will have domain walls between

2 neighbouring boxes. Suppose the magnetization
 of 1 box is $+1$, and the neighbouring box's
 magnetization is -1 , (extreme case). (this means
 that we have a sharp boundary between these
 2 boxes). As a result, we have to pay
 some free energy price.

This more refined description allows us to
 calculate an object like correlation function:

$$g(r, r') = \langle m(r) m(r') \rangle - \langle m(r) \rangle \langle m(r') \rangle$$

we take 2 positions in
 the system: r & r'

If the 2 positions are
 very close, (r is close to r')
 Then $\langle m(r) m(r') \rangle$ will

be a positive result here
 in the ferromagnetic
 moment since each spin

i.e. if we have a system
 in the high temperature phase,
 then the average value of
 the magnetization is zero.
 $\langle m(r) \rangle = 0$, $\langle m(r') \rangle = 0$

But for case of long
 distance between r & r' ,

locally wants to have,
like, neighbours.

we expect that the
spins are no longer
correlated.

$$\langle m(r) \rangle = \frac{\int_{-\infty}^{\infty} e^{-\beta H} m(\underline{r}) \frac{\pi}{r} dm(\underline{r})}{\int_{-\infty}^{\infty} e^{-\beta H} \frac{\pi}{r} dm(\underline{r})}$$

Normalization $\rightarrow \int_{-\infty}^{\infty} e^{-\beta H} \frac{\pi}{r} dm(\underline{r})$
This is noted as D_m

(In fact, the value of Hamiltonian H decreases rapidly with increasing value of m).

To find the correlation function, it's useful to introduce a field which is inhomogeneous and it lives just at one position that is at the origin. (value of h)

$$h(\underline{r}) = h \delta(\underline{r}) \quad (\delta(\underline{r}): \text{Delta function})$$

$$\langle m(\underline{r}) \rangle_h = \frac{\int e^{-\beta(H_0 + hm(0))} m(\underline{r}) D_m}{\int e^{-\beta(H_0 + hm(0))} D_m}$$

(So if we now calculate the magnetization that will depend on the value of the field that we have applied at the origin and therefore we use h here as a subscript)

$\rightarrow H_0$ is the Hamiltonian without any external field
The external field has been made explicit and extracted here.

$$\frac{d}{dh} \langle m(\underline{r}) \rangle_{h=0} = -\beta \frac{\int e^{-\beta H_0} m(0) m(\underline{r}) D_m}{\int e^{-\beta H_0} D_m}$$

$$\frac{\langle e^{\beta H_0} m(r) D_m \rangle e^{-\beta H_0} m(0) V_m}{(\int e^{-\beta H_0} D_m)^2}$$

+ β

Expectation Value:
 $\langle m(0) m(r) \rangle$

If we look at the definition
 of the correlation function:
 $g(r, r') = \langle m(r) m(r') \rangle - \langle m(r) \rangle \langle m(r') \rangle$.

$$\frac{d}{dh} \langle m(r) \rangle_{h=0} = -\beta g(r)$$

↓

2 expectation values:
 $\langle m(r) \rangle$ & $\langle m(0) \rangle$

So if we apply a small magnetic field at the origin
 and then we take the derivative of the magnetization
 throughout the lattice with respect to that field, we
 obtain the correlation function with an extra factor
 of $-\beta$. This result is a rather general result.

↓
 This is called the fluctuation response
 theorem.

Go back to the partition function.

$$Z = \int e^{-\beta H} Dm$$

and ask ourselves which configurations of m
 give a major contribution to this partition
 function

Saddle Point: $\int^{\Lambda f} d^d r \approx e^{\Lambda f_{\max}}$, Λ large

(Saddle Point Approximation)

If we have an integral of an exponential and the exponent assumes to be very big values. Then that integral is usually dominated by the Maxima of the exponent. The reason is that if you just change f here (in the above equation) a little bit because Λ (Capital of λ) is assumed to be large.

The idea now is to generalize this approximation to the problem we are dealing with: → we try to find those fields $m(r)$ which give us the optimal contribution to the exponent. (

$$Z = \int e^{-\beta H} Dm$$

In order to find those dominant contributions, we usually use the variational condition
(变分条件)

Use $H(m(r) + \delta m(r)) - H(m(r)) = 0$ (Variational)

1st order in δm

If we have an optimum of this Hamiltonian we evaluate it for a certain field $m(r)$ which we vary a little bit (from $m(r)$ → $m(r) + \delta m(r)$). Then the difference between the varied Hamiltonian and the unvaried Hamiltonian that should vanish the first order in the variation $\delta m(r)$.

$$\int [k(\nabla m + \nabla \delta m)^2 + h(r)(m + \delta m) + r(m + \delta m)^2 + u(m + \delta m)^4] d^d r \\ = \int [k(\nabla m)^2 + h(r)m + rm^2 + um^4] d^d r$$

This parameter r is just a parameter which is just the prefactor in the Taylor expansion of the Hamiltonian of the m . This parameter r is different from the position r in $h(r)$

(where $h(r)$ is the only position r in this equation).

The above equation is approximately

$$\approx \int [2k\nabla m \nabla \delta m h(r) \delta m + 2rm \delta m + 4um^3 \delta m] d^d r$$

(keep only the terms linear in δm)

(we neglect $(\nabla \delta m)^2$, etc. We threw away all the higher powers of (δm))

Green's Theorem is used to move the gradient here from δm to the m .

Then the equation becomes:

$$= \int [-2k\nabla^2 m + h(r) + 2rm + 4um^3] \delta m d^d r$$

(If we want the above integration : $\int [-2k\nabla^2 m + h(r) + 2rm + 4um^3] \delta m d^d r = 0$ to be equal to 0 ($= 0$))

for all small variations of δm .

Then the only possible way is that the term

$$[-2k\nabla^2 m + h(\underline{r}) + 2rm + 4\mu m^3] = 0 \quad (\text{always be } 0)$$

Now let's make a little pause here and realize what we are doing here.

Previously, ($h(\underline{r}) = h\delta(\underline{r})$), we have seen that applying a small field at the origin, that gives us magnetization ($\langle m(\underline{r}) \rangle_h = \frac{\int e^{-\beta(H_0 + hm(0))} m(\underline{r}) Dm}{\int e^{-\beta(H_0 + hm(0))} Dm}$)

that depends on the field.

And if we take the derivative of that magnetization ($\frac{d}{dh} \langle m(\underline{r}) \rangle_{h=0}$), we obtain the correlation function ($\frac{d}{dh} \langle m(\underline{r}) \rangle_{h=0} = -\beta g(\underline{r})$)

Fluctuation-response theorem

Now we are doing now is to calculate this magnetization ($\langle m(\underline{r}) \rangle_h$) as a function of

the field. And we do that using the saddle point approximation so we find the optimum contributions that give a dominant contribution and for the partition function ($Z = \int e^{-\beta H} Dm$),

111 . . . 11 . . . 11 . . .

We'll take that m from the eqn: $-2k\nabla^2m + h(r) + 2rm + 4um^3 = 0$

We take the derivative with respect to the field. We restore again the $h\delta(r)$ and that tells us what the correlation function ($g(r)$) actually is. \Rightarrow That is our goal here.

$$-2k\nabla^2m + h\delta(r) + \underline{2rm} + 4um^3 = 0$$

Solve for m .

(Because the field h is very small, we'll use the perturbation theory with respect to the small parameter h .)

if $h=0$ and $T > T_c$ ($r > 0$): $m_0 = 0$

if $h=0$ and $T < T_c$ ($r < 0$): $m_0^2 = -\frac{r}{2u}$

The solution would be just like m is a constant throughout the lattice (as shown above)

However, when we switch on the small field $h(r)$

$$m(r) = m_0 + h\varphi(r)$$

\hookrightarrow We do linear perturbation here).

And the equations for $\varphi(r)$ is what we need to solve.

We collect all the terms that are linear in h

$$\dots - \nabla^2m - 2rm - 4um^3 - \nabla^2(h\varphi) - h\nabla^2m - h\delta(r) - 2r\varphi - 4u(m+h\varphi)^3 - \dots$$

$$T > T_c \quad (m_0 = 0) \rightarrow -\nabla^2 \underline{\psi} + \delta(r) + 2r\underline{\psi} = 0$$

plug into the eqn: $-2k\nabla^2 m + h\delta(r) + 2rm + 4um^3 = 0$

$$\rightarrow \nabla^2 \underline{\psi} - \frac{r}{k} \underline{\psi} = \frac{\delta(r)}{k}$$

This r is the parameter
this r is the position.

$$T < T_c \quad (m_0 = \sqrt{\frac{r}{2u}}) \rightarrow -2k\nabla^2 \underline{\psi} + \delta(r) + 2r\underline{\psi}$$

An extra term as compared to
 $T > T_c$, because $m_0 \neq 0$

(Apart from terms which vanish
because m_0 satisfies this equation,
those terms which do NOT scale with H , will
vanish. And the terms that scale linearly
with H , have been collected here).

As a short summary,

$$\text{for } T > T_c, \quad \nabla^2 \underline{\psi} - \frac{r}{k} \underline{\psi} = \frac{\delta(r)}{k}$$

$$\text{for } T < T_c, \quad \nabla^2 \underline{\psi} + \frac{2r}{k} \underline{\psi} = \frac{\delta(r)}{k}$$

(since $m_0 = \sqrt{\frac{r}{2u}}$)

} 2 Helmholtz
type of
equations

$$\left(\nabla^2 - \frac{1}{\xi^2} \right) \underline{\psi}(r) = A S(r)$$

gradient squared

length squared

(Correlation Length)

Helmholtz
Equation

1. evanescence length 2. asymptotic solutions.

This equation has a asymptotic solution
(渐近解)

Dimension d: $\varphi(\underline{r}) \sim |\underline{r}|^{\frac{1-D}{2}} e^{-|\underline{r}|/\xi}$ for $|\underline{r}| \gg \xi$ large r

$$\sim |\underline{r}|^{2-D} e^{-|\underline{r}|/\xi} \text{ for } |\underline{r}| \ll \xi \quad \text{small r}$$

ξ : correlation length.

$$\xi^2 \sim \frac{k}{r} \sim \frac{1}{t} \quad \xi \sim \frac{1}{\sqrt{t}} \sim t^{-\nu}$$

(since $r \propto t$)

$$(\nu = \frac{1}{2})$$

\uparrow
is the critical exponent.

(The critical exponent for the correlation length
is $= \frac{1}{2}$)

$$m(\underline{r}) = m_0 + h \varphi(\underline{r}) ; g(\underline{r}) = \frac{\partial m}{\partial h}$$

(if we switch on a small field at the origin) \downarrow

$$g(\underline{r}) \sim \varphi(\underline{r})$$

We have a magnetization which varies in space.

(Constant m_0 + a small perturbation h caused by the field)

We have also seen that the correlation function is given as $g(\underline{r}) = \frac{\partial m}{\partial h}$.

So we see immediately that a correlation function is just $\varphi(\underline{r})$ (That's the solution to our Helmholtz equation).

Short Summary:

The typical situation we are in
is the length scales that are shorter
than ℓ_3 .

(a) $T_c : \ell_3 \rightarrow \infty \rightarrow |\underline{r}| \ll \ell_3$

$$\frac{e^{-\underline{r}/\ell_3}}{\gamma^{d-2}} = g(\underline{r}) = \frac{1}{\gamma^{d-2+n}}$$

It's a length, not parameter

since $|\underline{r}| \ll \ell_3$, $e^{-\underline{r}/\ell_3}$ can be neglected,
thus $n = 0$

Summary :

Landau - Ginzburg

$$\alpha = 0$$

$$\beta = \frac{1}{2}$$

$$\gamma = 1$$

$$\nu = \frac{1}{2}$$

Experiment for 3D

$$\alpha = 0.1$$

$$\beta = 0.3 - 0.4$$

$$\gamma \approx 1.25$$

$$\nu \approx 0.7$$

Ginzburg Criterion :

To what extent is it justified to replace $m(\underline{r})$ by m_0 ?

Going back to the place where we derived the saddle point equations



Use $H(m(r) + \delta m(r)) - H(m(r)) = 0$. (variational).

We took the Hamiltonian and expanded it to first order in δm . Then we integrated over space and required that the variation would be 0. → 1st order Expansion

$$\int [k(\nabla m + \nabla \delta m)^2 + h(r)(m + \delta m) + r(m + \delta m)^2 + u(m + \delta m)^4] d^d r$$

We have neglected the 2nd order terms

$$= \int [k(\nabla m)^2 + h(r)m + rm^2 + um^4] d^d r$$

$$\approx \int [2k \nabla m \nabla \delta m + h(r)\delta m + 2rm\delta m + 4um^3\delta m] d^d r$$

$$= \int (-2k \nabla^2 m + h(r) + 2rm + 4um^3) \delta m d^d r = 0$$

Question is now: How important are these 2nd order terms?

$\int \{ [\nabla(\delta m(r))]^2 + r[\delta m(r)]^2 + b u m_0^2 (\delta m(r))^2 \} d^d r$

There is the m_0^2 term here since that's the point around which we did the 2nd order expansion

These higher ordered terms can be worked out straightforwardly and we find this above expression which is indeed quadratic in the fluctuation m .

So if we want to access the validity of the neglect of the 2nd order contributions, we should check whether

$$\int \{ [\nabla(\delta m(r))]^2 + r[\delta m(r)]^2 + b u m_0^2 (\delta m(r))^2 \} d^d r$$

m_0 : found from the saddle point approximation.

$$<< \int (r m_0^2 + u m_0^4) d^d r$$

the zero-th order expansion

This criterion is satisfied when we choose $\delta m(\underline{r})$ such that $\langle \delta m(\underline{r})^2 \rangle \ll m_0^2$ --- Ginzburg Criterion.

Ginzburg Criterion tells us how justified the Saddle point approximation in the Landau - Ginzburg Hamiltonian is.

In order to analyze this criterion, we define the magnetization on the entire lattice :

$$M = \frac{1}{V} \int m(\underline{r}) d^d r \quad \rightarrow V M = \int m(\underline{r}) d^d r$$

Volume

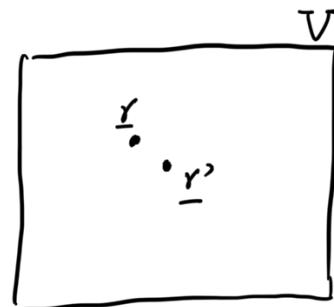
$$V^2 \langle \delta M^2 \rangle = \left\langle \int m(\underline{r}) d^d r \int m(\underline{r}') d^d r' \right\rangle - \left\langle \int m(\underline{r}) d^d r \right\rangle \left\langle \int m(\underline{r}') d^d r' \right\rangle$$

We are interested in the fluctuations of the local magnetization, so we should start by looking at the fluctuations of the global magnetization.

$$= \int \underbrace{\left[\langle m(\underline{r}) m(\underline{r}') \rangle - \langle m(\underline{r}) \rangle \langle m(\underline{r}') \rangle \right]}_{g(\underline{r}, \underline{r}')} d^3 r d^3 r' \quad V \gg \xi^d$$

$$= \int g(\underline{r}, \underline{r}') d^d r d^d r'$$

$$\underline{R} = \frac{\underline{r} + \underline{r}'}{2}, \quad \underline{\Sigma} = \underline{r} - \underline{r}'$$



If we are close to, but not exactly at the critical point, the correlation length (in length) is finite. In the thermodynamic limit we can take the V arbitrarily large. So we take the volume V a lot larger than ξ^d . ($V \gg \xi^d$) or ξ is a lot smaller than the linear size of the domain.

This means that the vast majority of contributions to this integral ($\int g(\underline{r}, \underline{r}') d^d r d^d r'$) arise from terms where \underline{r} & \underline{r}' are both far from the boundary of the system and far from the boundary means a lot farther than the correlation length.

In this case, if the dominant contribution comes from points who do not feel the boundary of the system, it makes sense to transfer the integration variables \underline{r} & \underline{r}' to new integration variables one for the center of mass ($\underline{R} = \frac{\underline{r} + \underline{r}'}{2}$) and one for the relative coordinates ($\underline{s} = \underline{r} - \underline{r}'$).

And the correlation function $g(r, r')$ is then assumed to depend only on \underline{s} , which is $s = r - r'$. Also because of the rotational symmetry we do not see the structure of the individual lattice points anymore. It only depends on the size of s .

So this implies that the integral of $g(\underline{r}, \underline{r}')$ can

now be written as

$$g(\underline{r}, \underline{r}') \rightarrow g(\underline{s})$$

$$V \int g(\underline{s}) d^d s \Leftarrow \int g(\underline{r}, \underline{r}') d^d r d^d r'$$

arise from an integral over
the center of mass coordinate because
we can put this pair $(\underline{r}, \underline{r}')$ anywhere
in the volume

So using radial coordinates, we can rewrite
the integral as:

We see that $g(s)$ decays exponentially and before \underline{s} . $g(s) = \frac{e^{-s/\xi}}{s^{d-2}}$

So for s a lot smaller than ξ , the form $\frac{1}{s^{d-2}}$ dominates in $g(s)$.

$$V \int g(\underline{s}) d^d s = V C \int g(s) s^{d-1} ds$$

C : some factor which is 2π or 4π or something like
that. It's a factor of order 1. Denoted as C here.

$$\text{Thus, } V C \int g(s) s^{d-1} ds \approx V C \int_0^\xi \frac{1}{s^{d-2}} s^{d-1} ds$$

C is left out since
it's a constant and is
of order 1.

$$= V \xi^2$$

$$= V^2 \langle \delta M^2 \rangle$$

Recalling what we were calculating
(It was $V^2 \langle \delta M^2 \rangle$).

$$\text{Thus, } \langle \delta M^2 \rangle = \frac{\xi^2}{V}$$

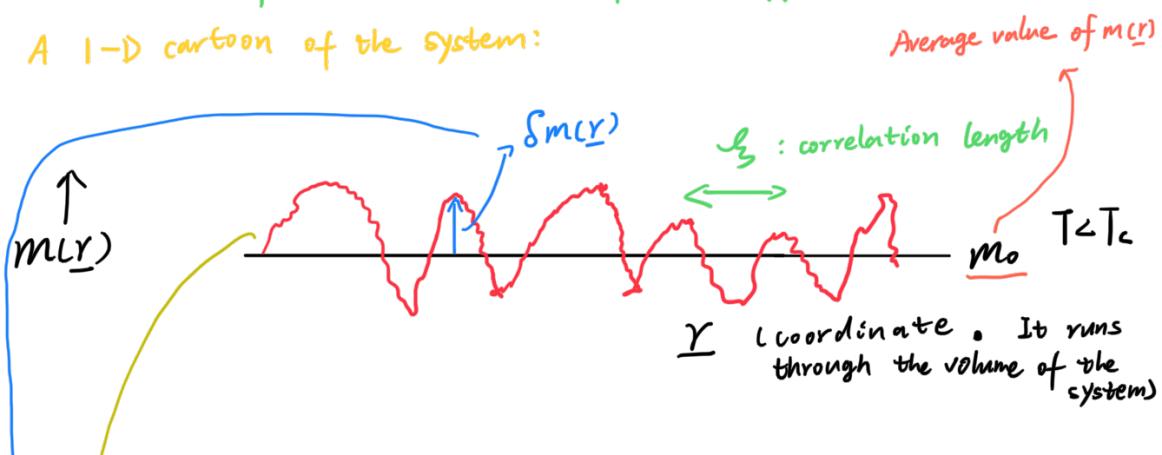
Recall that M is the average value of the
local magnetization $M = \frac{1}{V} \int m(\underline{r}) d^d r$

$$\delta m(\underline{r}) = m(\underline{r}) - m_0$$

And what we are interested in is in fact the fluctuation of that local magnetization which is the actual deviation of the actual value of the magnetization from the average value which is m_0 .

from the saddle point approximation

A 1-D cartoon of the system:



The magnetization will have different values. It will sometimes be larger than m_0 , sometimes smaller than m_0 . So we can see such erratic (不稳定的) behavior. Sometimes there are longer patches (长J). The size of the domains in which the magnetization has only one sign with respect to m_0 where δm has only one sign.

The size of those domains is typically given by the correlation length ξ (ξ : correlation length. It's the length over which the magnetic moments influence each other effectively. So it's the size of these patches).

So it means that this $\langle \delta M^2 \rangle$, (we know that $\langle \delta M^2 \rangle = \frac{\ell_s^2}{V}$ from above), is the result of an average over fluctuations that we see over the lattice. So the picture which emerges is that we have a volume consisting of different patches and within each patch, we have an independent value of the magnetization with the typical deviation which is the same in each patch but it can be positive or negative. The size of the patches is ℓ_s^d .

Patches have a volume of ℓ_s^d .

$$\text{The number of the patches : } N = \frac{V}{\ell_s^d}$$

And we know that the average of all the values of the magnetization and all these patches are typically smaller than the value in each patch itself and it scales by $\frac{1}{\sqrt{N}}$. (Volume per patch)

$$\delta M \sim \frac{\delta m}{\sqrt{N}} = \sqrt{\frac{\langle (\delta m(r))^2 \rangle}{V/\ell_s^d}}$$

$$\rightarrow \langle (\delta m(r))^2 \rangle = \delta M^2 \frac{V}{\ell_s^d} = \ell_s^{2-d} = t^{\frac{d-2}{2}}$$

(From previous analysis, we know that $\delta M^2 = \ell_s^2/V$)

(Using the fact that $\ell_s = \frac{1}{t}$)

(t is the reduced temperature.)

The requirement for neglecting the fluctuations is that

$$\langle (\delta m(r))^2 \rangle = t^{\frac{d-2}{2}} \ll m_0^2 \sim t$$

We can find that

$t^{\frac{d-2}{2}}$ should be much

smaller than t itself.

(when we approach
 T_c , m_0^2 will take
 the value of t)

$$\frac{\langle (\delta m(r))^2 \rangle}{m_0^2} = t^{\frac{d-2}{2}-1} = t^{\frac{d-4}{2}} \ll 1$$

In this case, we can neglect the fluctuations with respect to the average value of m .

This value should be small if the Ginzburg criterion is justified

This relationship holds indeed true for d greater than 4. ($d > 4$)

So for $d > 4$, we can expect that the mean field theory gives us correct results and the deviations that we previously have seen for $d = 3$, they arise from an unjustified approximation in which we have neglected all the fluctuations.

