FINM 32000 Final Exam

May 22, 2024

More detailed instructions: see Canvas announcement.
Open book, open notes, open computers. No collaboration.
No audio devices (airpods/earbuds/headphones/speakers). Cite all sources.
Problems (and problem parts) may have different weights. Late penalty 1% per minute

All numerical answers in all problems should be reported to at least 3 significant digits (3 digits not counting leading zeros), and to at least 2 digits after the decimal point – whichever of these two rules produces more digits.

Problem 1

At time 0, a non-dividend paying stock has price $S_0 = 90$, and a call option has price 2.65, delta 0.25, and gamma 0.015.

Approximate the dollar value of the stock shares that need to be purchased/sold in order to *rebalance* (*update*) a delta hedge, per 1 percent upward movement in the stock price (from the 90 price). The words "rebalance/update" mean that this is not asking for the total dollar value of the stock in the delta hedge, it is asking for the *change* in that dollar value due to revising that hedge as a result of a +1% movement in the stock price.

Assume the hedge is hedging against a short position in the call, so it is trying to replicate a long position in the call.

A positive answer will be understood to mean that stock needs to be purchased; a negative answer will be understood to mean that stock needs to be sold.

Problem 2

You are solving the PDE

$$\frac{\partial C}{\partial T} = \frac{1}{2}K^2\sigma^2\frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}$$

to compute the function $C(K,T)$. Let $r = 0.05$ and $\sigma = 0.35$.

Suppose that you have already computed finite difference approximations $\texttt{C}(K,T)$ for $T = 1.1$, displayed in this table:

	T=1.1	T=1.2
K=15.5	2.33	
K=15.0	2.55	
K=14.5	2.79	

(Notation: C is exact solution, $\texttt{C}$ is the FD approximate solution)

(a) Calculate the FD approximation $\texttt{C}(15.0, 1.2)$ in terms of $\texttt{C}(15.5, 1.1)$ and $\texttt{C}(15.0, 1.1)$ and $\texttt{C}(14.5, 1.1)$ using an explicit FD scheme. Thus, you will fill the orange cell using the three yellow values from the $T = 1.1$ column. Your final answer should be a number.

Use the standard approximation of $\frac{\partial^2 C}{\partial K^2}$ at location $(K = 15.0, T = 1.1)$.

Use a central (two-sided) finite difference approximation of $\frac{\partial C}{\partial K}$ at $(K = 15.0, T = 1.1)$

Use a one-sided finite difference approximation of $\frac{\partial C}{\partial T}$ at location $(K = 15.0, T = 1.1)$.

You are working *from left to right* in the grid, meaning that you already have the values at $T = 1.1$ and you want to solve at $T = 1.2$. This does not affect how the $\frac{\partial C}{\partial T}$ approximation is constructed; it merely affects which terms in that approximation are known or unknown.

The financial interpretation of C is not needed in part (a).

(b) The function $C(K,T)$ has a financial interpretation: it is the time-0 price of a call option, on a non-dividend paying stock S, as a function of strike K and expiry T. In this context, the pricing date 0 and the stock price S_0 do not vary. This grid prices *many different contracts* (rather than a single contract at many different times and many different stock price levels).

Given the time-0 prices of options at expiry $T = 1.1$ as shown in yellow, find an approximation to the probability density function of S_T (equivalently, $S_{1.1}$) at the price level 15.0.

Problem 3

Let S be the price of a non-dividend paying stock which follows Geometric Brownian motion with volatility 70% and $S_0 = 10$. The interest rate is 0.02.

Let $S_{0.5}$ and $S_{1.0}$ denote S at time 0.5 and time 1.0 respectively.

Find the time 0 price of a contract which pays at time 1

$$\left(\frac{S_{0.5} + S_{1.0}}{2} - 12\right)^+$$

using *conditional* Monte Carlo, conditioning on $S_{0.5}$.

Use 100000 simulations. Each simulation should use only 1 pseudo-random normal (to simulate $S_{0.5}$). Do not use numerical integration (quadrature).

Report a price and standard error (estimated standard deviation of your estimate).

Submit an `ipynb` file (no new `ipynb` is provided to you today).

Hint: A special case is needed for the scenario where $S_{0.5} \geqslant 24$. In that case, the conditional expected payout is approximately $S_{0.5} - 12$, but to get full credit, that needs to be modified slightly.

Problem 4

A fair die, by definition, generates independent rolls that are uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$.

Part (b) has more weight than part (a).

(a) A fair die is rolled 3 times.

You receive the dollar amount shown on just *one* of those three rolls – the first or second or third roll – it's your choice, but if you want the nth roll, you must *exercise* that choice immediately after seeing the nth roll, before seeing the $(n + 1)$th roll.

Then to maximize your expected payout (with no discounting), you can induct backward in a grid that shows (for each $n = 1, 2, 3$ and each possible outcome $R = 1, 2, 3, 4, 5, 6$) the optimized time-n expectation conditional on not exercising prior to the nth roll, and seeing that the nth roll is R (on which you can choose to exercise or not).

This grid is equivalent to a tree, with 6 branches from each node, in which we apply the usual backward induction.

	n=1	n=2	n=3
R=6	6	6	6
R=5	5	5	5
R=4	4.25	4	4
R=3	4.25	3.5	3
R=2	4.25	3.5	2
R=1	4.25	3.5	1

The time-0 optimized expectation is then $(6 + 5 + 4.25 \times 4)/6 \approx 4.67$.

Suppose you play the unique strategy that attains this maximal expected payoff. Find the probability that your exercise time is time 3 (the time immediately after the third roll is revealed). In other words, this is the probability that you wait until the very end to exercise.

(b) A fair die is rolled 4 times.

You get to choose any *two* of those 4 rolls and receive the *total* dollar amount shown on those two rolls combined. Thus, you can exercise twice – but each exercise must be done immediately after seeing whatever roll you wish to claim, before seeing the next roll.

The two exercises must be in two different rounds/turns of the game. The footnote[1] has an example.

Complete the 13 blanks in this table that shows, for each n and R, the optimized time-n expectation of the *total* payoff conditional on not exercising (not even once) prior to the nth roll, and observing that the nth roll is R (on which you can choose to exercise or not).

	n=1	n=2	n=3	n=4
R=6			9.5	6
R=5			8.5	5
R=4			7.5	4
R=3			6.5	3
R=2			5.5	2
R=1			4.5	1

Time-0 optimized expectation: $\boxed{}$

If you are doubtful about the numbers in the $n = 4$ column, see footnote[2].

[1] For example, suppose you roll a 6 in the first round. You are not allowed to exercise/claim twice in round 1; you may exercise at most once in that round. Then your second exercise must be in rounds 2 or 3 or 4. If it turns out that the second exercise happens to be on a new roll of 6, that would be allowable; it's ok to claim an equal numerical value twice, you just can't claim twice in the same round/turn.

[2] The numbers provided for you in the $n = 4$ column are small, because all numbers in this table are conditional on never previously exercising (not even once), and the last column is the (sad) case in which the player has hypothetically never exercised in rounds 1-3, and has thus forfeited an exercise opportunity, as only 1 exercise opportunity remains. That $n = 4$ column does not directly help you calculate the $n = 1$ and $n = 2$ columns, so feel free to ignore it.

Problem 5

Assume zero interest rates. $\boxed{\text{Let } T_1 = 0.3, \text{ let } T_2 = 0.4, \text{ let } T_3 = 0.6}$.

Suppose that a non-dividend-paying stock has dynamics

$$dS_t = \sigma(t) S_t dW_t, \qquad S_0 = 100$$

where W is Brownian motion under risk-neutral probabilities, and where the time-dependent but *non-random* instantaneous or local volatility function $\sigma : [0, T_3] \to \mathbb{R}$ is a step function, constant within each interval $(0, T_1]$, $(T_1, T_2]$, and $(T_2, T_3]$.

For any $T \in [0, T_3]$, let $C(T)$ be the time-0 price, and ImpVol(T) be the time-0 Black-Scholes implied volatility, of a European call option on S with strike 100 and expiration T.

Fill in the 6 blank spaces of the following table. You may, but are not required to, write Python code to solve this problem. In any case, please show your calculations.

	ImpVol(T)	σ(t)	C(T)
t in $(0,T_1]$		0.340	
$T = T_1$			
t in $(T_1,T_2]$			
$T = T_2$			8.01
t in $(T_2,T_3]$			
$T = T_3$	0.300		