Problem 1.

$\Delta = 0.25$, $\Gamma = 0.015$, $C_0 = 2.65$, $S_0 = 90$

dollar value of the stock shares need to be purchased to rebalance a delta hedge

$$= \frac{\partial^2 C}{\partial S^2} \cdot \frac{S_0^2}{100}$$

$$= 0.015 \times \frac{90^2}{100} = 1.215$$

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change in $\Delta = \Gamma \times \Delta S = 0.015 \times 90 \times 1\% = 0.0135$

change in dollar value $= 0.0135 \times 90 = 1.215$

Problem 2

a) let the value in orange box be x

① For central FD

$$\frac{\partial C}{\partial T} = \frac{C(K, T+\delta T) - C(K, T)}{\delta T}$$

$$= \frac{x - 2.55}{0.1}$$

$$\frac{\partial^2 C}{\partial K^2} = \frac{C(K+\Delta K,T) - 2C(K,T) + C(K-\Delta K,T)}{(\Delta K)^2}$$

$$= \frac{2.33 - 2\cdot 2.55 + 2.79}{(0.5)^2} = 0.08$$

$$\frac{\partial C}{\partial K} = \frac{C(K+\Delta K,T) - C(K-\Delta K,T)}{2\cdot\Delta K}$$

$$= \frac{2.33 - 2.79}{2\cdot 0.5} = -0.46.$$

$$\frac{\partial C}{\partial T} = \frac{1}{2} K^2 \sigma^2 \frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}$$

$$\frac{x - 2.55}{0.1} = \frac{1}{2}\cdot 15^2 \cdot 0.35^2 \cdot 0.08 - 0.05\cdot 15\cdot(-0.46)$$

$$x - 2.55 = 0.1\cdot[1.1025 + 0.345]$$

$$x = 2.55 + 0.14475 \approx 2.695$$

b) $$e^{-rT} p(S_{1.1} = 15) = \frac{\partial^2 C}{\partial K^2}$$

$$= \frac{C(K+\Delta K,T) - 2C(K,T) + C(K-\Delta K,T)}{(\Delta K)^2}$$

$$= \frac{2.33 - 2.55\cdot 2 + 2.79}{(0.5)^2} = 0.08$$

$$\therefore p(S_{1.1} = 15) = 0.08\, e^{rT} = 0.08\, e^{0.05\cdot 1.1} \approx 0.0845$$

Problem 4:

a) at $n=1$, when $R=5/6$

$5, 6 > 4.25$, exercise immediately.

when $R=1/2/3/4$

$1,2,3,4 < 4.25$ move to $n=2$

at $n=2$, when $R=4/5/6$

$4,5,6 > 3.5$, exercise immediately.

when $R=1/2/3$

$1,2,3 < 3.5$ move to $n=3$

at $n=3$, when $R=1/2/3/4/5/6$

exercise immediately.

$P(\text{not exercising at } n=1) = \frac{4}{6} = \frac{2}{3}$

$P(\text{not exercising at } n=2) = \frac{3}{6} = \frac{1}{2}$

$\therefore P(\text{exercising time} = 3) = P(\text{not exercising at } n=1) \times P(\text{not exercising at } n=2)$

$= \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

b)

	n=1	n=2	n=3	n=4
R=6	9.5	9.5	9.5	6
R=5	8.5	8.5	8.5	5
R=4	7.75	7.5	7.5	4
R=3	7.75	7	6.5	3
R=2	7.75	7	5.5	2
R=1	7.75	7	4.5	1

$$\text{expected payout} = \frac{9.5+8.5+7.5+6.5+5.5+4.5}{6} = 7$$

n=2

$6+3.5=9.5 > 7$

$5+3.5=8.5 > 7$

$4+3.5=7.5 > 7$

$3+3.5=6.5 < 7$

$2+3.5=5.5 < 7$

$1+3.5=4.5 < 7$

$$\text{expected payout} = \frac{9.5+8.5+7.5+7+7+7}{6} = 7.75$$

$n=1$

$6+3.5=9.5 > 7.75$

$5+3.5=8.5 > 7.75$

$4+3.5=7.5 < 7.75$

$3+3.5=6.5 < 7.75$

$2+3.5=5.5 < 7.75$

$1+3.5=4.5 < 7.75$

$$\therefore \text{time-0 optimized expectation} = \frac{7.75\times 4+8.5+9.5}{6} = \frac{49}{6}$$

$$\approx \boxed{8.167}$$

Problem 5.

	ImpVol(T)	σ(t)	C(T)
t in (0,T_1]		0.340	
T = T_1 = 0.3	0.340		7.419
t in (T_1,T_2]		0.240	
T = T_2 = 0.4	0.318		8.01
t in (T_2,T_3]		0.260	
T = T_3 = 0.6	0.300		9.250

implied volatility $\bar{\sigma}_T = \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)\,dt}$

for $T_1 = 0.3$ $\bar{\sigma}_{0.3} = \sqrt{\frac{1}{0.3}\int_0^{0.3} \sigma^2(t)\,dt}$

$= \sqrt{\frac{1}{0.3}(0.3)\cdot 0.34^2}$

$= 0.34$

According to the bs-call-formula function (shown in code file)

$C(T_1) = 7.419$

According to the implied-volatility function (shown in code file)

$\text{ImpVol}(T_2) = 0.318$

$$\therefore \quad 0.318 = \sqrt{\frac{1}{0.4}\left[\int_0^{0.3} \sigma_{(t)}^2\,dt + \int_{0.3}^{0.4} \sigma_{(t)}^2\,dt\right]}$$

$$0.318 = \sqrt{\frac{1}{0.4}\cdot\left(0.3\cdot 0.342^2 + 0.1\,\sigma_{0.4}^2\right)}$$

$$\sigma_{0.4} = 0.240$$

$$0.300 = \sqrt{\frac{1}{0.6}\left[\int_0^{0.3} \sigma_{(t)}^2\,dt + \int_{0.3}^{0.4} \sigma_{(t)}^2\,dt + \int_{0.4}^{0.6} \sigma_{(t)}^2\,dt\right.}$$

$$0.300 = \sqrt{\frac{1}{0.6}\left(0.3\cdot 0.342^2 + 0.1\cdot 0.240^2 + 0.2\cdot \sigma_{0.6}^2\right)}$$

$$\sigma_{0.6} = 0.260$$

According to the bs-call-formula function (shown in code file)

$C(T_3) = 9.150$