FINM 32000: Homework 4

Due Friday April 26, 2024 at 11:59pm

Problem 1

Let S be a futures price. Assume that under risk-neutral probability measure, S has *CEV* dynamics

$$\mathrm{d}S_t = \sigma S_t^{1+\alpha}\mathrm{d}W_t, \qquad S_0 = 100$$

with constants σ, α. The superscript on S_t is an exponent (power). The interest rate on the bank account is r. The S here is a futures price, so it has drift coefficient 0. (But prices of European options on S still have drift coefficient r).

(a) Let $C(S_t, t)$ be the time-t no-arbitrage price of an European put on S, with strike K and expiry T. Write down a PDE, with terminal condition, for $C(S, t)$. Leave your answer in terms of r, σ, α, K, T.

(b) Let $r = 0.05$, $\sigma = 3$, $\alpha = -0.5$. Use Crank-Nicolson to find the time-0 price of an *American* put on S with strike $K = 100$ and expiry $T = 0.25$. Partial code is provided in the `ipynb` file. You may use the boundary conditions implemented in the function

`FD_CrankNicolson_Engine.price_put_CEV`.

At the low-S boundary, it assumes the put value equals intrinsic value (exercise value). At the high-S boundary, it approximates the put value as zero.

You may use the FD grid given in the `ipynb` file.

(c) Compute numerically the time-0 delta and gamma of the put in (b).

(d) Using exactly the same `FD_CrankNicolson.price_put_CEV` function as in (b) – meaning that you can change the input passed into the function, but cannot change the function's code – find the time-0 price of the American put in (b), but assuming *Black-Scholes* dynamics for S with volatility 0.30 and interest rate 0.05 and $S_0 = 100$.

Problem 2

Playing no-limit hold 'em (in a cash game, not a tournament), Jamie is all-in on the turn, and Patrik has called. Given the 8 cards that have been revealed (4 pocket, 4 on the board), Patrik is ahead, but Jamie does have 10 outs. There is no chance of a tie. If you are not familiar with poker, you can ignore all of the above, and just start reading at the next sentence:

In other words: there are 44 unrevealed cards, of which 10 would make Jamie the better hand, and the other 34 would make Patrik the better hand.

At this stage, the usual procedure is that one more card will be revealed. If the card is one of the 10 that favor Jamie, then Patrik collects 0% of the money in the pot. If it is one of the 34 that favor Patrik, then Patrik collects 100% of the money in the pot. (The dollar value of the pot has been finalized; it does not matter in this problem, what value it is.)

(a) Find the expectation and standard deviation of the fraction of the pot that Patrik will collect, when the last card (the "river") is dealt in the usual way.

(b) Suppose that, before the river is dealt, Jamie and Patrik agree to "run it three times," with replacement.[1] This means that the last card will be dealt three times, with replacement and reshuffling after each deal of that card. For each of the three iterations of the river card, whoever wins with that card gets one-third of the pot. Therefore Patrik will win either 0%, or 1/3, or 2/3, or 100% of the full pot, depending on whether 0, 1, 2, or 3 of the cards to be dealt turn out to favor Patrik.

Find the expectation and standard deviation of the fraction of the pot that Patrik will collect.

(c) Same question as (b) but *without* replacement after each deal of the river card. Is the standard deviation larger or smaller than the standard deviation in part (b), and does that make sense?

Hint for standard deviation in (c): One approach is to calculate the probability distribution of Patrik's winnings. For instance the probability that Patrik wins exactly 2/3 of the full pot is

$$\frac{\binom{34}{2}\binom{10}{1}}{\binom{44}{3}} \approx 42.36\%$$

and do likewise for the other outcomes. Then use the definition of variance.

Hint for expectation calculation in (b) and (c): $\mathbb{E}(X+Y+Z) = \mathbb{E}X + \mathbb{E}Y + \mathbb{E}Z$.

[1] Replacement is not actually done in practice. Running it twice, or three times, or four times, is conventionally done *without* replacement. So part (c) is more realistic.