FINM 32000 Practice Final Exam

University of Chicago
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Open book, open notes, open computers. No collaboration.
No audio devices (airpods/earbuds/headphones/speakers). Cite all sources.
5 problems, of which Problem 3 has the highest weight.

Problem 1

All options in this problem are European-style. Let $C(K)$ denote the time-0 price of a T-expiry K-strike vanilla call on a non-dividend-paying stock S.

Assume that the distribution of the stock price S_T has a probability density function p and a cumulative distribution function F, and that $p(100) = 0.15$ and $F(100) = 0.5$.

Assume that $C(100) = 8$.

Assume constant interest rate r, with $e^{-rT} = 0.8$ in all parts.

(a) Using a second-order Taylor expansion, approximate $C(99.5)$.

(b) Approximate the price of a binary put with strike 101 and expiry T.

(c) Approximate $\mathbb{P}(99 < S_T < 101)$.

Problem 2

You are solving the PDE

$$\frac{\partial C}{\partial t} + 0.75\,\frac{\partial^2 C}{\partial x^2} = 0$$

to compute the function $C(x,t)$. Suppose that you have already computed finite difference approximations $\mathtt{C}(x,t)$ for $t = 0.1$ and $t = 0.2$, displayed in this table:

	t=0	t=0.1	t=0.2
X=15.5		1.16	
X=15.0		1.05	1.01
X=14.5		1.04	

(Notation: C is exact solution, $\mathtt{C}$ is the FD approximate solution)

(a) Calculate the FD approximation $\mathtt{C}(15.0, 0)$ in terms of $\mathtt{C}(15.5, 0.1)$ and $\mathtt{C}(15.0, 0.1)$ and $\mathtt{C}(14.5, 0.1)$ using the standard explicit FD scheme. Thus, you will fill the orange cell using the three yellow values from the $t = 0.1$ column. Your final answer should be a number.

(b) Consider a different FD scheme:

Calculate the FD approximation $\mathtt{C}(15.0, 0)$ in terms of $\mathtt{C}(15.5, 0.1)$ and $\mathtt{C}(15.0, 0.1)$ and $\mathtt{C}(14.5, 0.1)$ and $\mathtt{C}(15.0, 0.2)$ using the same approximation of $\frac{\partial^2 C}{\partial x^2}$ as in the standard explicit scheme; but to approximate $\frac{\partial C}{\partial t}$, construct the finite difference using $\mathtt{C}(15.0, 0)$ and $\mathtt{C}(15, 0.2)$, in a way that is accurate (no proof needed) to $O(\Delta t)^2$ at the point $(15.0, 0.1)$. Thus, you will fill the orange cell using the four yellow values. Your final answer should be a number.

In (b), do not use $\mathtt{C}(15.0, 0.1)$ in your approximation of $\partial C/\partial t$.

(c) For these *specific* grid spacings of $\Delta x = 0.5$ and $\Delta t = 0.1$, and the *specific* PDE given above; but for *general* values of $\mathtt{C} \in \mathbb{R}$ in the yellow cells (*not* necessarily the specific values 1.01, 1.04, 1.05, 1.16):

Does the scheme in (a) guarantee the following stability property?

$$|\mathtt{C}(15.0, 0)| \leqslant \max(|\mathtt{C}(15.5, 0.1)|, |\mathtt{C}(15.0, 0.1)|, |\mathtt{C}(14.5, 0.1)|)$$

Does the scheme in (b) guarantee the following stability property?

$$|\mathtt{C}(15.0, 0)| \leqslant \max(|\mathtt{C}(15.5, 0.1)|, |\mathtt{C}(15.0, 0.1)|, |\mathtt{C}(14.5, 0.1)|, |\mathtt{C}(15.0, 0.2)|)$$

Again, "guarantee" means that the inequality holds for all choices of the quantities on the right-hand side, not just the specific numbers printed in the yellow above.

Hint: $\mathtt{C}(15.0, 0)$ is a weighted combination of nearby cells. Look at the weights in (a) and (b)

Problem 3

Let S be the price of a non-dividend paying stock which follows Geometric Brownian motion with volatility 70% and $S_0 = 10$. The interest rate is 0.02.

Using Monte Carlo, find the time 0 price of a contract which pays at time 1

$$\left(\frac{S_{0.5} + S_{1.0}}{2} - 12\right)^+$$

where $S_{0.5}$ and $S_{1.0}$ denote S at time 0.5 and time 1.0 respectively.

(a) Use ordinary Monte Carlo

Each simulated payoff should use 2, and only 2, pseudo-random normals. (If you were to use conditional Monte Carlo, then only 1 pseudo-random normal would be needed per simulation, but you are not asked to do that). Do not use numerical integration (quadrature).

Report a price and standard error (estimated standard deviation of your estimate).

(b) Use a plain vanilla European call as a control variate. You are to choose the parameters (strike, expiry) of the control contract, such that the control variate has at least 90% correlation (in the Pearson sense, using, for instance, `numpy.corrcoef`) with the given payoff.

There is no extra credit for achieving 98% correlation compared to 92% correlation.

Report a price and standard error (estimated standard deviation of your estimate).

Complete the code in the `ipynb` file. Use 100000 simulations in each part.

Problem 4

A 4 card deck contains 2 red and 2 black cards, with all permutations equally likely. You draw cards without replacement, winning 1 dollar for each black card you see, and losing 1 dollar for each red card you see. You are allowed to quit at any time (including at time 0, quitting before seeing any cards, for a net profit of 0) – but you are *not* always allowed to continue. Specifically:

(a) Suppose that you are forced to stop playing immediately, if your total profit ever reaches -1, a net loss of 1 dollar. (Think of this as your broker issuing a margin call, which you do not meet, and thus your broker closes out your position. You protest and plead, saying that the position will surely move in your favor soon, but your broker is unswayed. You do not get to see any more cards, and your final profit is -1.) To be specific, if the first card is Red, or if the first three cards are Black-Red-Red, then you are forced to quit as soon as that happens.

If you don't play at all, you get zero profit. If you play (meaning, you draw the first card, and possibly more if it is optimal to do so), what is your expected optimized profit?

Should you play?

(b) Same questions as part (a), except here assume that the profit level that triggers the forced exit is -2 (negative 2 dollars). At a total profit of -1 your broker allows you to keep playing, but at -2 you get shut down, and you are then stuck with a total profit of -2.

Problem 5

Let $0 < T_1 < T_2 < T_3$. Assume zero interest rates.

Suppose that a non-dividend-paying stock has dynamics

$$\mathrm{d}S_t = \sigma(t) S_t \mathrm{d}W_t, \qquad S_0 = 100 \tag{1}$$

where W is Brownian motion under risk-neutral probabilities, and where the time-dependent but *non-random* instantaneous or local volatility function $\sigma : [0, T_3] \to \mathbb{R}$ is a step function, constant within each interval $(0, T_1]$, $(T_1, T_2]$, and $(T_2, T_3]$.

For any $T \in [0, T_3]$, let $C(T)$ be the time-0 price, and $\mathrm{ImpVol}(T)$ be the time-0 Black-Scholes implied volatility, of a European call option on S with strike 100 and expiration T.

You may, but are not required to, write Python code to solve these problems. In any case, please show your calculations, and report all results to at least three significant digits (three digits, not counting leading zeros).

(a) Let $T_1 = 0.10$, let $T_2 = 0.25$, let $T_3 = 0.50$. Fill in the 6 blank spaces of the following table.

	ImpVol(T)	σ(t)	C(T)
t in (0,T_1]			
T = T_1	0.250		
t in (T_1,T_2]		0.320	
T = T_2			
t in (T_2,T_3]			
T = T_3			6.20

(b) Again let $T_1 = 0.10$, let $T_2 = 0.25$, let $T_3 = 0.50$.

In this part, there is not a unique way to fill in the 3 blank spaces, so let us specify which one of the many solutions to choose:

Fill in the 3 blank spaces of the following table such that ImpVol(T_2) is *as large as possible*, given ImpVol(T_1) and ImpVol(T_3).

	ImpVol(T)	σ(t)
$T = T_1$	0.250	
t in $(T_1,T_2]$		
$T = T_2$		
t in $(T_2,T_3]$		
$T = T_3$	0.360	