

# FINM 32000: Homework 5

Due Friday May 3, 2024 at 11:59pm

## Problem 1

Let  $r$  be the constant interest rate. Let  $0 < T_1 < T_2$ .

- (a) Let  $F_t$  be the time- $t$  forward price for  $T_2$ -delivery of some arbitrary underlying  $S$ , not necessarily tradeable. Recall from FINM 33000, that a *forward price* is *not the same thing* as the *value of a forward contract*. By definition of the time- $t$  forward price  $F_t$  for  $T_2$ -delivery:

a forward contract paying  $S_{T_2} - F_t$  at time  $T_2$  has time- $t$  value 0.

Let  $f_t$  be the time- $t$  value of a  $T_2$ -forward contract on the same underlying, but with some delivery price  $K$  (not necessarily equal to  $F_t$ ).

Express  $f_t$  in terms of  $K$  and  $F_t$  and a discount factor.

Hint: consider a portfolio long one  $(K, T_2)$ -forward contract and short one  $(F_t, T_2)$ -forward contract. The portfolio has (in terms of  $f_t$ ) what value at time  $t$ ? The portfolio pays how much at expiration?

- (b) If  $S$  is a *stock* paying no dividends, the forward price must be  $F_t = S_t e^{r(T_2-t)}$ ; otherwise, arbitrage would exist.

If, say,  $F_t > S_t e^{r(T_2-t)}$ , then arbitrage would exist: at time  $t$ , borrow  $S_t$  dollars, buy the stock, and short the forward (with delivery price  $F_t$  and time- $t$  value 0). At time  $T_2$ , deliver the stock, and receive  $F_t$ , which is more than enough to cover your accumulated debt of  $S_t e^{r(T_2-t)}$  dollars.

However, if  $S$  is the spot price of a barrel of crude oil (so, for all  $t$ , the time- $t$  price for time- $t$  delivery is  $S_t$  per barrel), then this argument fails. Explain briefly (one or two sentences, no math) why *this specific arbitrage* does not apply to crude oil, by specifically pinpointing, in the quote above, why we cannot simply replace “stock” with “crude oil”.

Hint: Consider practical complications.

So we need more assumptions to relate  $F_t$  and  $S_t$  (here and in (c,d,e,f,g), the  $S$  denotes spot crude oil, and  $F_t$  denotes the time- $t$  forward price for  $T_2$ -delivery crude oil). One approach is to model the risk-neutral dynamics of  $S$ . Assume that  $S$  satisfies

$$\begin{aligned} S_t &= \exp(X_t) \\ dX_t &= \kappa(\alpha - X_t)dt + \sigma dW_t. \end{aligned}$$

where  $W$  is Brownian motion, under risk-neutral measure.

Then, since  $r$  is constant and  $\mathbb{E}_t(e^{-r(T_2-t)}(S_{T_2} - F_t))$  must be 0, one can calculate

$$F_t = \mathbb{E}_t(S_{T_2}) = \exp \left[ e^{-\kappa(T_2-t)} \log S_t + (1 - e^{-\kappa(T_2-t)})\alpha + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa(T_2-t)}) \right],$$

where  $\mathbb{E}_t$  is time- $t$  conditional expectation. Suppose  $\kappa = 0.472$ ,  $\alpha = 4.4$ ,  $\sigma = 0.368$ ,  $r = 0.05$ , and the time-0 spot price is  $S_0 = 106.9$ .

Let  $C$  be the time-0 price of a  $K$ -strike  $T_1$ -expiry European call on  $F$ . So this call pays  $(F_{T_1} - K)^+$ . Let the call option have strike  $K = 103.2$  and expiration  $T_1 = 0.5$ . Let the forward have delivery date  $T_2 = 0.75$ . See the `ipynb` file.

- (c) Estimate  $C(S_0)$  using Monte Carlo simulation of  $S$  with 100 timesteps on  $[0, T_1]$ . Choose the number of paths large enough that the standard error [the sample standard deviation, divided by the square root of the number of paths] is less than 0.05. Report the standard error. Don't use any variance reduction technique.
- (d) Estimate  $\partial C / \partial S$  by using Monte Carlo simulation to calculate  $(C(S_0 + 0.01) - C(S_0)) / 0.01$ . For the  $C(S_0 + 0.01)$  calculation, *reuse* the same normal random variables which you generated for the  $C(S_0)$  calculation. (Do not re-generate random variables to compute  $C(S_0 + 0.01)$ )
- (e) Calculate analytically  $\partial f_0 / \partial S$ , where  $f_0$  is the time-0 value of a position long one **forward contract** on a barrel of crude oil, with delivery date  $T_2$  and some fixed delivery price  $K$ .
- (f) Suppose you want to hedge a position short one call (so your hedge portfolio should replicate a position long one call), by continuously rebalancing a position in  $T_2$ -delivery forward contracts. Your hedge portfolio at time 0 should be long how many forward contracts? Your final answer should be a number.

The delivery price  $K$  of the forward contracts is irrelevant to the answer here; it would affect only how many units of the bank account to carry in the portfolio (which I am not asking you to compute).

- (g) Consider the following “purchase agreement” contract. The holder of this contract receives time- $T_2$  delivery of  $\theta$  barrels of crude oil, and pays, at time  $T_2$ , a delivery price of  $K$  dollars per barrel. The  $\theta$  is chosen at time  $T_1$  by the holder of the purchase agreement, subject to the restriction that  $4000 \leq \theta \leq 5000$ ; in particular,  $\theta = 0$  is not a valid choice, because the contract is a commitment to purchase at least 4000 barrels. Using your answer to (c), without running any new simulations, find the time-0 value of this contract.

Here  $K, T_1, T_2$  have the same values as on the previous page.

Hint: Assume the holder acts optimally; thus  $\theta$  is either 4000 or 5000, depending on  $F_{T_1}$ .

## Problem 2

Suppose that a non-dividend-paying stock has dynamics

$$dS_t = rS_t dt + \sigma(t)S_t dW_t \quad (1)$$

where  $W$  is Brownian motion under risk-neutral probabilities, and where the time-dependent but non-random volatility function  $\sigma : [0, T] \rightarrow \mathbb{R}$  is piecewise continuous and sufficiently integrable. L2.13 shows that this particular type of local volatility function  $\sigma$  (to be specific: the type of  $\sigma$  function that depends on  $t$  but does not depend on  $S$ , nor on anything else that is random) has an explicit relationship with the Black-Scholes implied volatility  $\sigma_{imp}$ .

- (a) Are the dynamics (1) capable of generating a non-constant (with respect to  $T$ ) term-structure of implied volatility? Are they capable of generating an implied volatility skew (non-constant with respect to  $K$ )? Explain briefly.
- (b) Let  $S_0 = 100$  and  $r = 0.05$ . At time 0, you observe the prices of at-the-money (this means  $K = 100$ ) European calls at 0.1-year, 0.2-year, and 0.5-year expiries to be 5.25, 7.25, and 9.5, respectively. First find the Black-Scholes implied volatilities of the three options. Then find (calibrate) a time-varying local volatility function  $\sigma : [0, 0.5] \rightarrow \mathbb{R}$  consistent with these option prices. A step function suffices (but other answers are also acceptable).
- (c) Consistently with your local volatility function  $\sigma$  from part (b), find the time-0 price of an at-the-money European call with expiry 0.4, and find the time-0.1 implied volatility of that call (the European call with expiry  $T = 0.4$ ). Do not use a tree or finite difference or Monte Carlo calculation.