FINM 32000: Homework 3

Due Friday April 19, 2024 at 11:59pm

Problem 1

Assume that the short rate (the instantaneous spot rate of interest) follows the process

$$\mathrm{d}r_t = \alpha(r_t, t)\mathrm{d}t + \beta(r_t, t)\mathrm{d}W_t$$

where W_t is Brownian motion under risk-neutral probabilities. This framework includes models such as the Vasicek and CIR models, which correspond to particular choices of the functions (α, β), but for part (a), let's leave the α and β as unspecified functions.

(a) Consider an interest rate derivative whose time-T payout has value given by some function $F(r_T)$, and whose time-t price C_t satisfies $C_t = C(r_t, t)$ for some smooth pricing function C.

Apply Ito's rule to find the risk-neutral dynamics of C. Then set its drift equal to rC, to derive a PDE for $C(r, t)$.

Suppose, in particular, that the risk-neutral dynamics of r are given by a Vasicek model

$$\mathrm{d}r_t = \kappa(\theta - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

with parameters $\kappa = 3$, $\theta = 0.05$, $\sigma = 0.03$. Consider a $T = 5$-year discount bond (a zero-coupon bond which pays 1 at maturity T).

(b) Write code to find the time-0 price of bond by applying a standard central-difference explicit finite difference scheme to the PDE in (a). (Therefore C_n^j will be determined by C_{n+1}^{j+1}, C_{n+1}^{j}, and C_{n+1}^{j-1}.)

Complete the code in the file `finm320-24-hw3.ipynb`.

(c) Also write code to price the bond using an explicit *upwind* approximation to $\frac{\partial C}{\partial r}$ instead of the usual central difference. Specifically, for those r_j such that $\kappa(\theta - r_j) \geqslant 0$, approximate $\frac{\partial C}{\partial r}(r_j, t_{n+1})$ using the points C_{n+1}^{j+1} and C_{n+1}^{j}. For those r_j such that $\kappa(\theta - r_j) < 0$, approximate $\frac{\partial C}{\partial r}(r_j, t_{n+1})$ using the points C_{n+1}^{j} and C_{n+1}^{j-1}. (For $\frac{\partial^2 C}{\partial r^2}$, use the usual approximation).

In (b) and (c), to approximate the PDE's rC term, use the values of r and C at node (n, j). (As we said in class, node $(n+1, j)$ would also be a natural choice, but let's choose n instead of $n+1$). At the grid's upper and lower boundaries r_{max} and r_{min}, impose for all $t < T$ the "linearity" boundary conditions

$$C(r_{max}, t) = 2C(r_{max} - \Delta r, t) - C(r_{max} - 2\Delta r, t)$$
$$C(r_{min}, t) = 2C(r_{min} + \Delta r, t) - C(r_{min} + 2\Delta r, t)$$

(This technique can help in some situations where it is not obvious what boundary conditions to use.) Thus, in each column of the grid, first solve for C in the interior nodes; then deal with the top and bottom nodes.

Now let us do some comparison of the central-difference and upwind schemes.

(d) Suppose $f : \mathbb{R} \to \mathbb{R}$ is smooth in some open neighborhood of x. Show that as $h \to 0$,

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = O(h) \quad \text{and} \quad \left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| = O(h^2).$$

using Taylor's theorem. The $O(h)$ means "some function bounded by a constant times h, near $h = 0$." Likewise, $O(h^2)$ means "some function bounded by a constant times h^2, near $h = 0$." Different instances of "O" may mean different functions. The "constants" may depend on x but not h.

(e) For all part (e) calculations: Use the grid spacings $\Delta r = 0.01$ and $\Delta t = 0.01$. Use $r_{max} = 0.35$ and $r_{min} = -0.25$ for the upper and lower boundaries of the grid, respectively.

Run a central-difference calculation and an upwind calculation of the bond price for $r_0 = 0.10$. Which is more accurate? The more accurate of the two solutions should agree, to three significant digits, with the exact bond price in this model: 0.7661. The less accurate of the two solutions will be *very* inaccurate.

(f) Based on your answers to (d) and (e), insert either "greater" or "less" in each blank space in the following rule-of-thumb. No explanation necessary.

> Ignoring stability issues and considering only consistency (i.e. "truncation error," also known as "local discretization error"), the upwind explicit scheme, which uses one-sided spatial differences, discretizes the PDE with ____ accuracy than the standard explicit scheme, which uses central spatial differences.
>
> However, to actually guarantee convergence, the grid spacing must satisfy certain stability constraints. In a PDE exhibiting strong drift, we have just seen that these constraints may allow the upwind scheme ____ freedom in choosing grid spacing, compared to the standard scheme.

(g) The *continuously-compounded yield-to-maturity* of a zero-coupon bond with time-t price P_t and nonrandom face value P_T to be paid at maturity date T is

$$\frac{\log(P_T/P_t)}{T-t}$$

where, as always for us, log denotes natural log, and where $P_T = 1$ according to this problem's assumptions. One way to think of the time-t yield to maturity T is as the average of some type of time-t expectation of the instantaneous spot rates from time t to time T.

Find the yield-to-maturity of a 5-year discount bond, in the case that $r_0 = 0.12$, and in the case that $r_0 = 0.02$. (The "good" results from part (e) may be used here. The "bad" results should not be used, unless you want to fix them by modifying the grid spacings).

Why, intuitively, is the yield for $r_0 = 0.12$ smaller than 0.12, whereas the yield for $r_0 = 0.02$ is greater than 0.02?

Comment: Under these short-rate dynamics, there do exist analytic pricing formulas for bonds. So we do not need finite difference methods to value the simple payoff that we have here. But the finite difference scheme can be modified to handle contracts for which exact pricing formulas do not exist.

Problem 2

Suppose that a non-dividend-paying stock S follows Black-Scholes dynamics, with interest rate r and volatility σ.

(a) Given Δ where $0 < \Delta < 1$, and given $T > 0$, solve for the strike K such that a call option on S with strike K and expiry T has time-0 delta equal to Δ. Express your answer in terms of the inverse cdf N^{-1} of the normal distribution, and any or all of $\Delta, S_0, r, \sigma, T$.

(b) Let $S_0 = 300$ and $T = \frac{1}{12}$ and $\sigma = 0.4$ and $r = 0.01$. Calculate the strikes and the premiums (meaning, the prices) of a 25-delta call and a 75-delta call. This means that $\Delta = 0.25$ and $\Delta = 0.75$ respectively.

A way to get the inverse of the normal cdf in Python is: `from scipy.stats import norm` and then use `norm.ppf()`

If you spend S_0 dollars on stock, you can buy one share, and thereby obtain delta 1.

If you spend the same amount of S_0 dollars on, instead, an option of a given strike and expiry, then you can buy S_0/C_0 contracts, where C_0 is the price of 1 contract. Each contract has delta Δ, so all the contracts together have a total delta

$$\Delta \frac{S_0}{C_0}.$$

This is sometimes denoted by the Greek letter lambda, and sometimes called the "leverage" or "gearing" or "elasticity" of the option. Another way to derive/explain this lambda is that it's the *percentage* change in the option price, per *percentage* change in the underlying.

(c) Calculate the lambdas of the two options in part (b). Which one gives you more leverage?

Comment (no response required from you): This is one of the reasons why, at a given strike, the OTM option (call or put) is usually expected to be more liquid (more activity, narrower spreads) than the ITM option (put or call) at that same strike; see the second of the three reasons below:

- Popularity of option strategies involving OTM options (for example, buying OTM protective puts, and selling OTM covered calls, as you implemented in the IB TWS exercises Fall quarter)
- Popularity of using options to obtain a bullish or bearish exposure to the underlying. More leverage is obtained by using OTM than ITM options. Thus OTM options require less capital commitment to obtain a desired exposure. (To say it another way, OTM options provide more exposure for a given allocation of capital).

- ITM options mainly have higher absolute deltas than OTM options, so ITM options are more exposed (in dollar terms, not percentage terms) to the movements of the underlying. This greater risk is a reason for market-makers to quote wider spreads for the ITM option, compared to narrower spreads for the OTM option.

(Note that this OTM vs. ITM comparision does not compare OTM/ITM vs. *ATM* options, which are particularly active.)

Due to the typically greater liquidity of OTM options, calculations of the implied volatility surface sometimes use *only* the OTM option at each strike (thus, call options at high strikes, put options at low strikes). See for instance Section 11 of Bloomberg's documentation (posted on Canvas) of how they calculate the implied vol surface.