FINM 33000 Practice Final Exam

December 2023

Closed book, closed notes, closed devices

Your Name: ____

General Directions

Results from lecture/homework may be used without justification. Unless otherwise directed, assume frictionless markets and no arbitrage, and assume also the following:

The share price (the price of 1 unit) of any stock is always nonnegative.

All options are European-style (so no early exercise). Calls and puts are standard plain vanilla, unless otherwise noted. Calls and puts have multiplier 1, so each contract pays the positive part of, respectively, underlying minus strike, or strike minus underlying (and not, for instance, 100 times this payoff).

Let P denote the physical probability measure, and let E denote expectation with respect to P.

There exists a bank account with price per unit $B_t = e^{rt}$ where $r > 0$ is constant.

Unless otherwise instructed, in questions asking for numerical answers, the numbers may be left unsimplified, for example $\binom{9}{8} + N(7 - \log(6 \times 5/4) + \exp(3 - 2.1))$ is an acceptable format.

Let $\mathbb{P}$ denote martingale measure (with respect to numeraire B), and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. Let the time-0 filtration be trivial; thus $\mathbb{E}$ and $\mathbb{E}_0$ are the same thing.

Let $\mathbb{R}$ denote the real numbers.

Let N denote the standard normal CDF, and N' denote its derivative $N'(x) = (1/\sqrt{2\pi}) \exp(-x^2/2)$.

Reminder

Here is an expression that may or may not be relevant:

$$\frac{\log(F/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}$$

Use or misuse of this expression is your responsibility. Guidance will not be provided regarding the meaning of the notations in this expression.

Problem 1

Interest rates are zero; the bank account has constant price $B_t = 1$.

Today is November 30, 2022. The group stage of the FIFA World Cup is in progress. Within each group, 4 teams are competing against each other. Based on the outcomes of group play (and a scoring/tiebreaking algrorithm whose details do not concern us), which will be concluded at date T in the future, exactly 2 of the 4 teams in each group will advance to the next stage.

Each team has a contract which pays 1 dollar at time T if that team advances, and pays nothing if that team does not advance. For example, the Switzerland contract pays 1 dollar at time T if Switzerland is one of the two teams from its group that advances to the next stage, and 0 dollars otherwise. Parts (a) and (b) are based on actual data from polymarket.com.

No team is guaranteed to advance yet, and no team has been surely eliminated yet. Aside from that fact, make no assumptions about the distributions and joint distributions of each team advancing; your arbitrages and bounds must be valid irrespective of the probability distributions.

(a) Today's prices of each contract in Group G are as follows. For example, the 59% means that the today's price of the Switzerland contract is 0.59 dollars.

Group G	
Brazil	97%
Switzerland	59%
Serbia	36%
Cameroon	4%

Using some or all of the assets {bank account, Brazil, Switzerland, Serbia, Cameroon} find an arbitrage.

(b) Today's prices of each contract in Group D are as follows. For example, the 9% means that the today's price of the Australia contract is 0.09 dollars.

Group D	
France	96%
Denmark	68%
Tunisia	29%
Australia	9%

Using some or all of the assets {bank account, France, Denmark, Tunisia, Australia} find an arbitrage.

(c) Suppose that in some other group in the FIFA World Cup group stage, teams $\{X, Y, Z, W\}$ are competing, and contracts on two of those teams are availabe:

contract	today's price
Team X	0.92
Team Y	0.60
Team Z	not available to be traded
Team W	not available to be traded

Consider a "parlay" contract, which pays 1 dollar at time T if teams X and Y *both* advance to the next stage, but pays 0 dollars if neither of $\{X, Y\}$, or only one of $\{X, Y\}$ advances.

Find a lower bound on today's price of the parlay contract, and explain why your answer is in fact a lower bound. You do not have to show that your lower bound is tight, but bounds which are not tight will not earn full credit (and a trivial bound will earn almost no credit).

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Problem 2

The interest rate is $r = 0.02$.

A non-dividend paying stock S follows geometric Brownian motion, starting at $S_0 = 38$.

The GBM's volatility and physical drift are not given.

Consider a call and a put option on S, both with strike $K = 40$ and expiry $T = 0.25$

If there is insufficient information to determine any answer, then state that.

(a) The time-0 delta of the *put* is one of the two numbers ± 0.495. Which one is the correct delta of the put? Also find the time-0 delta of the call.

(b) The time-0 gamma of the *put* is one of the two numbers ± 0.035. Which one is the correct gamma of the put? Also find the time-0 gamma of the call.

(c) The time-0 theta of the *call* is one of the two numbers ± 9.71. Which one is the correct theta of the call? Also find the time-0 theta of the put.

(d) The risk-neutral probability that the *call* expires in-the-money is 0.387. Find the risk-neutral probability that the put expires in-the-money.

(e) Find the time-0 price of the call.

All assumptions from previous parts of this problem are in effect here.

(f) Let a, b, c, d, e, f denote the deltas of the following 6 contracts (3 calls and 3 puts) on S with expiriy $T = 0.25$ and strikes $30, 45, 60$:

strike	time-0 call delta	time-0 put delta
30	A	B
45	C	D
60	E	F

Arrange those deltas in increasing order. For example, your answer might look like

$$B < F < C < E < A < D$$

(This ordering in this example is unrelated to the actual answer, it's just to show you the formatting)

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Problem 3

Under risk-neutral measure W is a Brownian motion, and X is some underlying (not necessarily an asset price) with dynamics

$$\mathrm{d}X_t = 3\ \mathrm{d}t + 2\ \mathrm{d}W_t, \qquad X_0 = 1.$$

The interest rate is $r = 0.01$. Let $T = 0.36$.

(a) X_T is normally distributed. Find its mean μ and its variance.

(b) Find the time-0 price of an option which pays $(X_T - \mu)^+$ at time T.

(c) Find the time-0 price of an option which pays 1 at time T if $X_T \geq 2$, and pays 0 if $X_T < 2$.

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Problem 4

Let the bank account B and a non-dividend-paying stock S have physical price dynamics

$$\begin{aligned} \mathrm{d}B_t &= rB_t\mathrm{d}t \\ \mathrm{d}S_t &= \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t \end{aligned}$$

where W is Brownian motion under physical measure.

(a) Let $S_0 = 32$, let $r = 0.04$, let $\mu = 0.05$, and let $\sigma = 0.5$.

Find the time-0 price of an contract which pays at time $T = 0.36$,

$$\begin{cases} 8 \times S_T & \text{if } S_T < 25 \\ 200 & \text{if } 25 \leq S_T \leq 40 \\ 5 \times S_T & \text{if } S_T > 40 \end{cases}$$

(b) In order to replicate the contract in (a) using a continuously-rebalanced self-financing portfolio of only B and S, how many units of the stock S should you hold at time 0? You do not need to report, how many units of B to hold.

Your final answers may be left unsimplified (see instructions on first page), and may involve some or all of the functions N or N' or exp or log, but they should not involve any variables. For example, notations such as "K" or "d_1" or "d_2", should not appear in your final answer, even if you have defined those notations.

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Problem 5

Let the bank account B and a non-dividend-paying stock S have physical price dynamics

$$\mathrm{d}B_t = rB_t\mathrm{d}t$$
$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t$$

where W is Brownian motion under physical measure.

Let L be the value of a self-financing portfolio that maintains 2x leverage on S by dynamically trading S and B. Specifically, the portfolio comprises units of B and S with total time-t value L_t, and the number of units of S that it holds at each time t is $2 \times L_t/S_t$, and the portfolio self-finances.

(a) S and L follow geometric Brownian motion under risk-neutral measure.

What are the drift and volatility of S under risk-neutral measure?

What are the drift and volatility of L under risk-neutral measure?

(b) Let $S_0 = L_0 = 10$, let $r = 0.03$, let $\mu = 0.05$, and let $\sigma = 0.25$.

Find the time-0 price of an option which pays $(K - L_T)^+$ at time T,

where $K = 10$ and $T = 1$.

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