FINM 33000: Homework 6

Due Thursday, November 16, 2023 at 11:59pm

General advice

Do not re-invent the wheel. Try to apply conclusions that have already been derived.

Problem 1

As an employee of Stark Industries, you participate in the employee stock purchase plan. The share price of Stark today is $S_0 = 40$. One year from today, the company will deduct \$3000 from your paycheck, and in exchange you will acquire shares of Stark at the below-market price of $\min(30, 0.75 \times S_1)$ per share (a 25% discount to the price today or to the price in a year, whichever is lower). The entire \$3000 will be spent on Stark shares (including fractional shares if necessary). There is no risk that Stark will default on this plan, and no risk that you will leave the company in the next year.

Stark stock follows a geometric Brownian motion with $\sigma = 0.22$. It pays no dividends. The interest rate on the bank account is $r = 0.05$. Markets are frictionless.

If this question asks you for a value or a position, your final answer must be a *number*. When you state a position, we will assume it is a long position, unless you either use a "-" sign or say "short".

(a) The stock purchase plan (consisting of the debit of \$3000 and the receipt of shares) is an asset to you. Find its time-0 value.

(b) The value of the plan will fluctuate as Stark stock fluctuates. Suppose you don't want to be subject to this risk. What time-0 position, in vanilla options, should you take, in order to hedge perfectly that risk? (Assume that European calls and puts on Stark are available at any strikes and expiries that you desire.) Does this position need to be rebalanced over the course of the year?

(c) Now suppose that options on Stark are not available. What time-0 position in Stark stock should you take, in order to hedge perfectly that risk? Does this position need to be rebalanced over the course of the year?

By "hedge perfectly", I mean that the stock purchase plan together with your proposed hedge (a self-financing portfolio strategy) should have a total time-1 value which is non-random.

a) $\text{payoff} = \frac{3000}{\min(30, 0.75 S_1)} \times S_1 - 3000$

$$= \max\left(\frac{3000}{30}, \frac{3000}{0.75 S_1}\right) \cdot S_1 - 3000$$

$$= \max(100 S_1, 4000) - 3000$$

$$= 100 \max(S_1, 40) - 3000$$

$$= 100 \max(S_1 - 40, 0) + 4000 - 3000$$

$$= 100 \max(S_1 - 40, 0) + 1000$$

it is equivalent to 100 calls with strike price 40 and 1000 dollar.

Black-Scholes formula:

$$C^{BS}(S, t) = e^{-r(T-t)} (F N(d_1) - k N(d_2))$$

where $F = S e^{r(T-t)}$

$$d_{1,2} = d_{+-} = \frac{\log(F/k)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}$$

$T-t = 1$, $\sigma = 0.22$, $r = 0.05$, $k = 40$

$$\therefore F = S_0 e^{r(T-t)} = 40 \cdot e^{0.05}$$

$$d_{1,2} = \frac{\log(40 \cdot e^{0.05}/40)}{0.22 \cdot \sqrt{1}} \pm \frac{0.22\sqrt{1}}{2}$$

$$= \frac{0.05}{0.22} \pm 0.11 \Rightarrow d_1 = 0.3373, \ d_2 = 0.1173$$

$$C^{BS}(S_0, t) = e^{-r(T-t)} (F N(d_1) - k N(d_2))$$

$= e^{-0.05}(40e^{0.05}N(0.3373) - 40N(0.1173))$

$= 40N(0.3373) - 40e^{-0.05}N(0.1173)$

$= 4.4811$

b) payoff $= 100\max(S_1 - 40, 0) + 1000$

in order to perfectly hedge the risk,

we should short 100 calls with strike price 100.

this position does not need to be balanced

c) Delta = $N(d_1)$ = number of shares of S needed to replicate one option $= N(0.3373) \approx 0.6321$

in order to replicate 100 call option, we need $100 \times 0.6321 = 63.21$ shares.

Since option price don't change one-to-one with stock price the position must be rebalanced at every point

Problem 2

Suppose that S is the value of one share of a non-dividend-paying stock.

Let r be the constant interest rate on the bank account $B_t = e^{rt}$.

Suppose that there exists a dynamic portfolio strategy, consisting of shares of the stock, and units of the bank account (and no other assets), that satisfies the following three conditions:

First, the portfolio's total time-t value is a stochastic process L_t.

Second, at every time-t it holds

$$\beta \frac{L_t}{S_t}$$

shares of stock. Here $\beta > 1$ is a constant. So we are constructing the portfolio in such a way that a 1 percent move in S always causes a β percent move in the portfolio value L.

Third, the portfolio is self-financing.

(a) According to the first and second conditions, how many units of the bank account does the portfolio hold at time t?

(b) Suppose that S follows geometric Brownian motion

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

where μ and $\sigma > 0$ are constants, and W is a Brownian motion under physical probabilities.

Using the self-financing condition, show that L is also a geometric Brownian motion.

(Still working under physical probability measure) What are its drift and volatility?

This L is a model of a *leveraged* trading strategy with *leverage ratio* β.

a) portfolio $\left\langle \begin{matrix} S \\ B \end{matrix} \right.$, let unit of bank account to be x

$$L_t = \beta \cdot \frac{L_t}{S_t} \cdot S_t + x \cdot B_t$$

$$L_t = \beta L_t + x \cdot B_t$$

$$x = \frac{(1-\beta)L_t}{B_t} = \frac{(1-\beta)L_t}{e^{rt}}$$

b) $\because$ the portfolio is self-financing

$$dV_t = \theta_t \cdot dX_t$$

$$\therefore \ dL_t = \beta \cdot \frac{L_t}{S_t} \cdot dS_t + x \, dB_t$$

$\because$ S follows geometric Brownian motion

$$dB_t = rB_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$\therefore \ dL_t = \beta \frac{L_t}{S_t} [\mu S_t dt + \sigma S_t dW_t] + x r B_t dt$$

$$= \beta \mu L_t dt + \beta L_t \sigma dW_t + (1-\beta) \frac{L_t}{B_t} B_t r dt$$

$$= (\beta \mu L_t + (1-\beta) L_t r) dt + \beta \sigma L_t dW_t$$

$$= \underline{(\mu\beta + r - r\beta)} L_t dt + \underline{\beta\sigma} L_t dW_t$$

thus, L_t is also a geometric brownian motion

drift $= \mu\beta + r - r\beta$

volatility $= \beta\sigma$

Problem 3

There exist a bank account $B_t = e^{rt}$ and a non-dividend-paying stock S. For (a) we impose no model on the S dynamics.

(a) State put-call parity for the time-t prices of a call and put on S, where the call and put have the same strike K and same expiry T. (In class we had $t = 0$; now we have general $t < T$).

In parts (b,c,d) work under the Black-Scholes model as defined in L5.
The time-t price of a put on S, with strike K and expiry T can be shown to be

$$Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1)$$

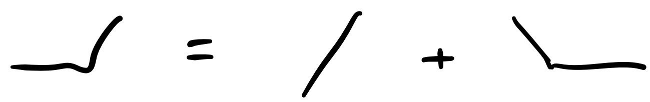
(b) Derive this formula from the call formula, using put-call parity. Hint: what is $N(x)+N(-x)$.

(c) Find a formula for the time-t delta of that put.

(d) Find a formula for the time-t gamma of that put.

In parts (c) and (d) you may either use put-call parity, or differentiate the (b) answer.

a)



$$C_0(K,T) = P_0(K,T) + S_0 - KZ_0(T)$$

$$Z_0(T) = e^{-rT}$$

at time T:

$$C_T = P_T + S_T - K$$

at time t:

$$C_t = P_t + S_t - Ke^{-rT} \cdot e^{rt}$$

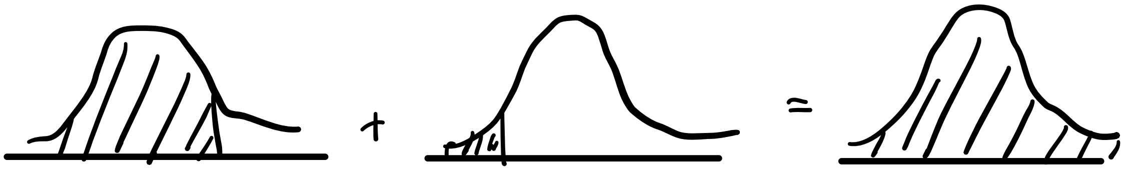
$$C_t = P_t + S_t - Ke^{-r(T-t)}$$

b) $$P_t = C_t - S_t + Ke^{-r(T-t)}$$

$$= e^{-r(T-t)}(S_te^{r(T-t)}N(d_1) - KN(d_2)) - S_t + Ke^{-r(T-t)}$$

$$= S_t N(d_1) - Ke^{-r(T-t)} N(d_2) - S_t + Ke^{-r(T-t)}$$

$$\because \quad N(x) + N(-x) = 1$$



$$= S_t[1 - N(-d_1)] - Ke^{-r(T-t)}[1 - N(-d_2)] - S_t + Ke^{-r(T-t)}$$

$$= \cancel{S_t} - S_t N(d_1) - \cancel{Ke^{-r(T-t)}} + Ke^{-r(T-t)} N(-d_2) - \cancel{S_t} + \cancel{Ke^{-r(T-t)}}$$

$$= Ke^{-r(T-t)} N(-d_2) - S_t N(d_1)$$

c) $$P_t = C_t - S_t + Ke^{-r(T-t)}$$

$$= C_t - (S_t - Ke^{-r(T-t)})$$

$\therefore$ delta of put = delta of call - delta of forward contract

for call, delta $= N(d_1)$

how much the forward contract moves with S

$$= \frac{\partial S_t - Ke^{-r(T-t)}}{\partial S_t} = 1$$

$$= N(d_1) - 1$$

d) $$P_t = C_t - S_t + Ke^{-r(T-t)}$$

$$= C_t - (S_t - Ke^{-r(T-t)})$$

· gamma of put = gamma of call - gamma of forward contract

$\partial^2(S_t - Ke^{-r(T-t)})$

$$\frac{[\text{illegible}]}{\partial S_t^2} = 0$$

$$= \frac{N'(d_1)}{S_t \sigma \sqrt{T-t}} - 0$$

$$= \frac{N'(d_1)}{S_t \sigma \sqrt{T-t}}$$