FINM 33000: Homework 4

Due Thursday, October 26, 2023 at 11:59pm

Problem 1

Let S_t, where $t = 0, 1, 2, \ldots, T$, denote the time-t price of a tradeable non-dividend-paying asset.

Let $S_0 = 100$ and let each random increment $S_{t+1} - S_t$ take value $+1$ with physical probability 40%, and value -1 with physical probability 60%, independently of all other increments.

There exists a bond, with constant price 1 at all times.

There exists a call on S, with strike $K = 105$ and expiry at time $T = 12$.

(a) The following argument tries to prove that the no-arbitrage time-0 price of the call is *zero*.

> Consider the trading strategy which holds at each time t the portfolio Θ_t^{rep}, defined as follows. [Recall the convention stated in class: at each time t, the price S_t is revealed, then trading (if any) occurs to change the old holdings $\Theta_{t-1}^{\text{rep}}$ to the new holdings Θ_t^{rep}.]
>
> $$\Theta_t^{\text{rep}} := \begin{cases} (0 \text{ share of asset, } 0 \text{ bonds}) & \text{if } S_t \leq K \\ (1 \text{ share of asset, } -K \text{ bonds}) & \text{if } S_t > K \end{cases}$$
>
> This trading strategy matches the call payoff with probability 1, because either $S_T > K$, in which case the call payoff matches the time-T portfolio value $S_T - K$, or else $S_T \leq K$, in which case the call payoff matches the time-T portfolio value 0.
>
> Therefore, by the law of one price, the no-arbitrage time-0 call price must equal the time-0 value of the replicating portfolio Θ^{rep}, which is *zero*, because $\Theta_0^{\text{rep}} = (0, 0)$ given that $S_0 < K$.

Identify and explain the specific flaw in this "proof".

(b) Find the true time-0 value of the call.

Do not do a 12-step backwards induction, and do not use a computer (unless you want to check your answer).

Although your answer should be explicit, you may leave it unsimplified. You may leave binomial coefficients (numbers of the form: $\binom{n}{k}$, pronounced "n choose k") unsimplified. For example, you may write $\binom{12}{2}$ without actually calculating it.

a) when $S_{t-1} = K$, $S_t = K+1$

$\theta_{t-1} = (0, 0)$, $X_t = (K+1, 1)$

$\therefore \theta_{t-1} \cdot X_t = (0,0)(S_t, 1) = 0$

$\theta_t = (1, -K)$, $X_t = (K+1, 1)$

$\theta_t X_t = (1, -K)(K+1, 1) = 1$

$\therefore \theta_{t-1} X_t \neq \theta_t X_t$

However, the condition for self-financing requires $\theta_{t-1} X_t = \theta_t X_t$

$\therefore$ it is not self-financing, which violates the law of one price. Thus the prove is incorrect.

b) risk-neutral probability is p

$S_t = p(S_t+1) + (1-p)(S_t-1) \Rightarrow p = \frac{1}{2}$

$\because$ for $t=12$, S_{12} would be $100 + n - (12-n) = 2n + 100 - 12$

where n is number of +1 value $\quad \therefore (S_{12}-K)^+ = (2n+100-12-105)^+ = (2n-17)^+$

$$\therefore (S_{12}-K)^+ = \begin{cases} 7, & \text{when } n=12 \\ 5, & \text{when } n=11 \\ 3, & \text{when } n=10 \\ 1, & \text{when } n=9 \\ 0, & \text{when } n=0\text{-}8 \end{cases}$$

$\therefore$ expected discounted payoff is $=$

$$7 \cdot \left(\tfrac{1}{2}\right)^{12}\binom{12}{0} + 5 \cdot \left(\tfrac{1}{2}\right)^{12}\binom{12}{1} + 3\left(\tfrac{1}{2}\right)^{12}\binom{12}{2} + \left(\tfrac{1}{2}\right)^{12}\binom{12}{3}$$

$$= \left(\tfrac{1}{2}\right)^{12}\left[7\binom{12}{0} + 5\binom{12}{1} + 3\binom{12}{2} + 1\binom{12}{3}\right]$$

Problem 2

An asset price S follows a random walk $S_0, S_1, S_2, \ldots$, starting at $S_0 = 2.16$, with step sizes ± 0.01.

Assume that you buy S at time 0 for 2.16 dollars, with a stop-loss level of 2 dollars, and a take-profit level of 3 dollars. Thus you will exit the trade (sell the asset) at the first time that the stock price hits either 2 dollars or 3 dollars (and you will sell for exactly 2 dollar or 3 dollars; ignore spreads, slippage, market impact, fees).

(a) Assume that the steps ± 0.01 are with probability 50% each.

Find the probability that you will exit the trade with a positive profit.

(It can be shown that, with probability 1, you will eventually exit the trade; S cannot stay forever inside the interval between 2 and 3. So the only two possibilities are that you exit at 2 dollars for a loss, or you exit at 3 dollars for a positive profit).

Hint: S is a martingale.

(b) Now assume the asymmetric case that the step sizes $+0.01$ and -0.01 are with probability u and $1-u$ respectively, where $u > 1/2$. Under this assumption, S is not a martingale.

However there is a constant A, where $0 < A < 1$, such that

$$A^{S_t}$$

is a martingale. Solve for A in terms of u.

Hint: Let $M_t = A^{S_t}$. The general condition for M to be a martingale is $\mathbb{E}_t(M_T - M_t) = 0$ for all $T > t$, but in this problem it will be enough if you simply solve

$$\mathbb{E}_t(M_{t+1} - M_t) = 0.$$

(This is enough, because the condition for general $T > t$ can be replaced by $T = t + 1$ using an induction argument.)

Hint: It may be more convenient to use cents instead of dollars, so the step sizes become ± 1.

(c) Under the assumptions of (b), find the probability that you will exit the trade with a profit. You may leave your answer in terms of either A or u.

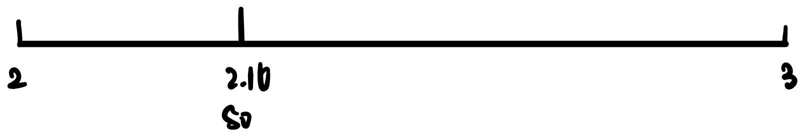
You may assume that the optional stopping theorem applies to the martingale S in part (a), and to the martingale M in part (b), at time 0 and time τ, the exit time.

a)

$S_n = S_0 + X_1 + X_2 + \cdots + X_n \quad \because P(+0.01) = P(-0.01) = \frac{1}{2} \quad \therefore E(X_n) = 0$

$E_t(S_{t+1} - S_t) = E(X_{t+1}) = 0 \Rightarrow E_t(S_t - S_0) = 0$

$\therefore$ S_t is a martingale



$2.16 = p \cdot 3 + (1-p) \cdot 2$

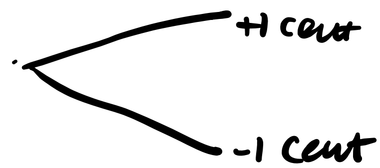
$p = 0.16$

b).

$E_t[M_t - M_0] = 0$

$\Downarrow$

$E_t[M_{t+1} - M_t] = E_t[A^{S_{t+1}} - A^{S_t}] = 0$



$= E_t[A^{S_t + X_{t+1}} - A^{S_t}] = 0$

$E_t[A^{S_t}(A^{X_{t+1}} - 1)] = 0$

$A^{S_t} E_t(A^{X_{t+1}} - 1) = 0$

$E_t(A^{X_{t+1}}) = 1$

$A \cdot u + A^{-1} \cdot (1-u) = 1$

$Au + \frac{1}{A}(1-u) = 1$

$uA^2 - A + (1-u) = 0$

$(uA - 1 + u)(A-1) = 0$

$A = \frac{1-u}{u}$ or $K = 1$

$\because 0 < A < 1 \quad \therefore A = \frac{1-u}{u}$

A^{S_t} is a martingale when $A = \frac{1-u}{u}$

c) $A^{2.16} = E(MT) = p \cdot A^3 + (1-p)A^2$

$$p = \frac{A^{2.16} - A^2}{A^3 - A^2}$$

$$= \frac{\left(\frac{1-u}{u}\right)^{2.16} - \left(\frac{1-u}{u}\right)^2}{\left(\frac{1-u}{u}\right)^3 - \left(\frac{1-u}{u}\right)^2}$$

probability is $\dfrac{\left(\frac{1-u}{u}\right)^{2.16} - \left(\frac{1-u}{u}\right)^2}{\left(\frac{1-u}{u}\right)^3 - \left(\frac{1-u}{u}\right)^2}$