FINM 33000 Final Exam

December 2023

Closed book, closed notes, closed devices, no calculators

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This exam has 5 questions, not all equally weighted.

General Directions

Results from lecture/homework may be used without justification. Unless otherwise directed, assume frictionless markets and no arbitrage, and assume also the following:

The share price (the price of 1 unit) of any stock is always nonnegative.

All options are European-style (so no early exercise). Calls and puts are standard plain vanilla, unless otherwise noted. Calls and puts have multiplier 1, so each contract pays the positive part of, respectively, underlying minus strike, or strike minus underlying (and not, for instance, 100 times this payoff).

Let P denote the physical probability measure, and let E denote expectation with respect to P.

There exists a bank account with price per unit $B_t = e^{rt}$ where $r > 0$ is constant.

Unless otherwise instructed, in questions asking for numerical answers, the numbers may be left unsimplified, for example $\binom{9}{8} + e^{3-2.1}(7 - \log(6 \times 5/4))$ is an acceptable format.

A final answer that includes the standard normal CDF N will lose credit, unless it is impossible to simplify to remove the N. Hint: Simplification to a form that does not include N is possible on all final answers in the exam, except within *one* of the five problems.

Let $\mathbb{P}$ denote martingale measure (with respect to numeraire B), and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. Let the time-0 filtration be trivial; thus $\mathbb{E}$ and $\mathbb{E}_0$ are the same thing.

Let $\mathbb{R}$ denote the real numbers.

Let N denote the standard normal CDF, and N' denote its derivative $N'(x) = (1/\sqrt{2\pi})\exp(-x^2/2)$.

Reminder

Here is an expression that may or may not be relevant:

$$\frac{\log(F/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}$$

Use or misuse of this expression is your responsibility. Guidance will not be provided regarding the meaning of the notations in this expression.

Problem 1

Time 0 is today, December 6, 2023. The US Presidential election is currently in the *nomination* stage. Candidates are competing to become *nominees* (winners of the nominations), who are revealed at some time $T > 0$. (After time T, the nominees will then proceed to the next stage, known as the general election, but that is not relevant for us here.) Two of the candidates competing to try to win nominations (and thus become nominees) are Biden and Trump.

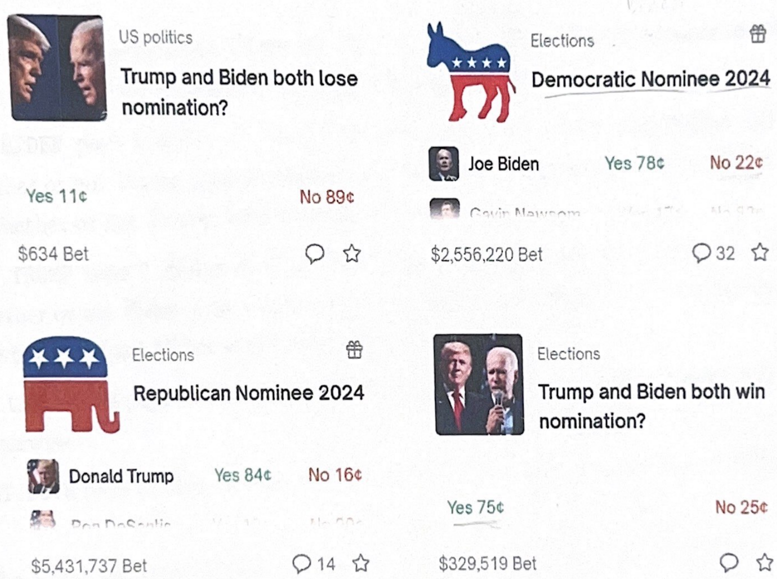
Interest rates are zero; there exists a bank account BANK with price 1 at all times. Contracts

$$\texttt{NEITHER}, \texttt{BIDEN}, \texttt{TRUMP}, \texttt{BOTH}$$

are frictionlessly available, respectively, at time-0 prices

$$0.11, 0.78, 0.84, 0.75$$

as shown in the actual data from polymarket.com on December 6, 2023:



BIDEN pays 1 dollar at time T if Biden wins a nomination (irrespective of whether or not Trump wins a nomination). It pays 0 if Biden does not win a nomination (irrespective of whether or not Trump wins a nomination).

TRUMP pays 1 dollar at time T if Trump wins a nomination (irrespective of whether or not Biden wins a nomination). It pays 0 if Trump does not win a nomination (irrespective of whether or not Biden wins a nomination)

NEITHER pays 1 dollar at time T if neither Biden nor Trump win nominations, otherwise 0.

BOTH pays 1 dollar at time T if both Biden and Trump win nominations, otherwise 0.

(a) Find an arbitrage, using some or all of the five assets {BANK, NEITHER, BIDEN, TRUMP, BOTH}.

In part (a), the "No" contracts in the screenshot are not available. The 4 "Yes" contracts, and the bank account, are available in (a).

(b) Suppose that in part (a) you found an arbitrage (a BANK, b NEITHER, c BIDEN, d TRUMP, e BOTH), where (a, b, c, d, e) are the quantities (how many units) of each asset.

Find another arbitrage, using some or all of the five assets

{BANK, AT.LEAST.ONE, NOT.BIDEN, NOT.TRUMP, NOT.BOTH}

For part (b), the four contracts

AT.LEAST.ONE, NOT.BIDEN, NOT.TRUMP, NOT.BOTH

are frictionlessly available, respectively, at time-0 prices

0.08

$$0.89, 0.22, 0.16, 0.25$$

0.09

as shown in the screenshot (these are the "No" contracts). These contracts are defined to be the opposite of the contracts in part (a). In other words:

0.03

0.22 NOT.BIDEN pays 1 dollar at time T if Biden does not win a nomination (irrespective of whether or not Trump wins a nomination). It pays 0 if Biden wins a nomination (irrespective of whether or not Trump wins a nomination). Biden X ② ③

0.16 NOT.TRUMP pays 1 dollar at time T if Trump does not win a nomination (irrespective of whether or not Biden wins a nomination). It pays 0 if Trump wins a nomination (irrespective of whether or not Biden wins a nomination) Trump X ② ④

0.89 AT.LEAST.ONE pays 1 dollar at time T if at least one of Biden and/or Trump win nominations, otherwise 0. exclude Biden Trump X ① ③ ④

0.25 NOT.BOTH pays 1 dollar at time T if not both of Biden and Trump win nominations, otherwise 0. ~~Biden Trump X~~ exclude Biden Trump V ② ③ ④

The "Yes" contracts in part (a) are not available in part (b).

You may express your answers using explicit numbers (quantities) of contracts to hold, or you may express your answers in terms of the notations (a, b, c, d, e). If you are not sure about your answer to part (a), then the latter is recommended: give your answers in terms of (a, b, c, d, e).

Part (a) of this problem will be worth more credit than part (b). In both parts of this problem, ignore the Gavin Newsom and Ron DeSantis contracts that are partially visible in the screenshot.

Make no assumption about the probability distributions of the outcomes for Trump and Biden. Your arbitrages in (a) and (b) must be valid, irrespective of the distributions (and irrespective of the joint distributions) of the outcomes for Trump and Biden.

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a) (+1 Neither, +1 Biden, +1 Trump, −1 Both, −1 Bank)

$V_0 = 0.11 + 0.78 + 0.84 - 0.75 - 1 = 0.95 + 0.78 - 0.75 - 1 = -0.02 < 0$

$$V_T = \begin{cases} 0+1+1-1-1=0 & \text{when both win} \\ 1+0+0-0-1=0 & \text{when both lose} \\ 0+1+0+0-1=0 & \text{when Biden wins only} \\ 0+0+1+0-1=0 & \text{when } \cancel{\text{Biden}} \text{ Trump wins only} \end{cases}$$

∴ it is type-2 arbitrage

b) (+1 Not.Biden, +1 Not.Trump, +6 AtLeast one, +1 NotBoth, −6 Bank)

$V_0 = 0.22 + 0.16 + 0.25 + 6 \times 0.89 - 6 = 0.63 + 6(1-0.11) - 6 = -0.03 < 1$

$$V_T = \begin{cases} 6-6=0 & \text{when Trump win, Biden win} \\ 1+0+6+1-6=2 & \text{when Trump win, Biden lose} \\ 0+1+6 & \text{when Trump lose, Biden win} \end{cases}$$

b) (−1 At.Least One, −1 Not.Biden, −1 Not.Trump, +1 Not Both, +1 Bank)

$V_0 = -0.89 - 0.22 - 0.16 + 0.25 + 1 = -0.02 < 0$

$$V_T = \begin{cases} -1 & \text{when both win} \\ 1 & \\ 1 & \Big\} \text{ one win} \\ 1 & \text{both lose} \end{cases}$$

type 2 arbitrage

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b) [±1 Not.Biden, ±1 Not.Both, ∓3 Not Trump]

$V_0 = \pm 0.22 \pm 0.25 \mp 3 \times 0.16 = 0.47 - 0.48 = -0.01 < 0$

$V_T = \begin{cases} -3 & \text{when Trump win, Biden win} \end{cases}$

[+1 Not.Biden, +2 At.Least.One, -2 Bank]

$V_0 = 0.22 + 2 \times 0.89 - 2 = 0$

$$V_T = \begin{cases} 0 & \text{when Trump win Biden win} \\ 0 & \text{Trump lose Biden win} \\ 2 & \text{Trump win Biden lose} \\ 2 & \text{Trump lose Biden lose.} \end{cases}$$

type II - arbitrage

Problem 2

Suppose that the bank account B and a non-dividend-paying stock price S have the following dynamics under risk-neutral measure.

$$dB_t = rB_t dt \qquad B_0 = 1$$

$$dS_t = rS_t dt + \sigma S_t dW_t \qquad S_0 = 42\,.$$

Let $r = 0.04$. Assume that a 1.0-expiry 50-strike call on S has at time-0 a **delta**, and a **risk-neutral probability of expiring in-the-money** of 0.38 and 0.27, not necessarily in that order.

(a) Which one of $\{0.27, 0.38\}$ is the delta, and which one is the risk-neutral probability of finishing in-the-money?

(b) Find the time-0 price of the 1.0-expiry 50-strike $\boxed{\text{call}}$ on S.

(c) Find the time-0 price of a 1.0-expiry 50-strike $\boxed{\text{binary put}}$ on S.

(d) Find the strike K^* such that the K^*-strike call and K^*-strike put have the same time-0 price. The put and call are on S and have the same strike K^* and same expiry $T = 1.0$.

(e) Let a, b, c, d denote the deltas of the following 4 calls with expiries in $\{0.5, 1.0\}$ and strikes in $\{30, 60\}$:

strike	Time-0 call delta, expiry 0.5	Time-0 call delta, expiry 1.0
30	a	b
60	c	d

Arrange those deltas in increasing order. For example, your answer might look like

$$d < c < b < a$$

$Z(0,1)$

(This ordering in this example is unrelated to the actual answer, it's just to show you the formatting)

$$\text{delta} = N(d_1) = N\left(\frac{\log(S_0 e^{r(T-t)}/K)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}\right)$$

e)

when $T\uparrow$, $\frac{\log(S_0 e^{r(T-t)}/K)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$ increases $\therefore N(d_1)$ increases

when $K\uparrow$, $\frac{\log(S_0 e^{r(T-t)}/K)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$ decreases $\therefore N(d_1)$ decrease

~~when $K\times 2$, $T\div 2$~~

$\cancel{N\left(\frac{\log(S_0 e^{r\frac{T}{2}}/(2K))}{\sigma\frac{T}{2}} + \frac{\sigma\frac{T}{2}}{2}\right)}$

when $K=30$, $T=0.5$ $\quad N(d_1) = \frac{\sqrt{2}\, r \log(\frac{50}{30})}{\sigma} + \frac{\sqrt{2}\sigma}{2}$

6

when $K=60$ $T=1$ $\quad N(d_1) = \frac{r \log(\frac{50}{60})}{\sigma} + \frac{\sigma}{2}$

$c < a < d < b$

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a) $\text{delta} = N(d_1) = N\left(\frac{\log F/k}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}\right)$

risk-neutral probability of expiring in the money $= N(d_2) = N\left(\frac{\log F/k}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}\right)$

$\because N(d_1) > N(d_2)$

$\therefore \text{delta} = 0.38$

risk-neutral prob in money $= N(d_2) = 0.27$

b) $C = e^{-r(T-t)}\,\mathbb{E}(S_T - 50)^+$

$= e^{-r(T-t)}\,\mathbb{E}(S_T - 50)1_{S_T > 50}$

$= e^{-r(T-t)}\int_{50}^{\infty}(S_T - 50)\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\,e^{-(S_T-5}$

$C^{BS} = SN(d_1) - Ke^{-r(T-t)}N(d_2)$

$C_0 = S_0 N(d_1) - Ke^{-rT}N(d_2)$

$= 42 \cdot 0.38 - 50e^{-0.04} \cdot 0.27$

c). binary put + binary call $= Z_0 = e^{-rT}$

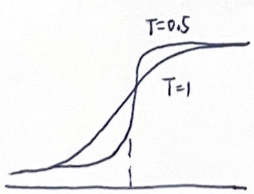
time-0 binary call price $= e^{-rT} \cdot N(d_2)$

$= e^{-0.04} \cdot 0.27$

$\therefore$ time-0 binary put price $= e^{-rT}$ − binary call time-0 price

$= e^{-0.04} - e^{-0.04} \cdot 0.27 = e^{-0.04} \cdot 0.73$

e)



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d)

binary call price + binary put price $= e^{-rT}$

$\therefore$ binary call price = binary put price $= \frac{1}{2} e^{-0.04}$

binary call price $= \frac{1}{2} e^{-0.04} = e^{-0.04} \cdot N(d_2)$

$$0.5 = N\left(\frac{\log F/K}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}\right)$$

this means that $\frac{\log F/K}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2} = 0$ because $Z \sim N(0,1)$

$P(Z<0) = N(d_2) = 0.5$

$$\therefore \frac{\log S_0 e^{rT}/K^*}{\sigma\sqrt{T-t}} = \frac{\sigma\sqrt{T-t}}{2}$$

$$\frac{\log 42 \cdot e^{0.04}/K^*}{\sigma} = \frac{\sigma}{2}$$

$$= \frac{\sigma^2}{2}$$

d)

$C_t = P_t + S_t - Ke^{-r(T-t)}$

$C_0 = P_0 + S_0 - Ke^{-rT}$

~~strike call price =~~

strike call price at time-0 $= S_0 N(d_1) - K^* e^{-rT} N(d_2)$

strike put price at time-0 $= S_0 N(d_1) - K^* e^{-rT} N(d_2) - S_0 + Ke^{-rT}$

$\cancel{= S_0(1-N(-d_1)) - K^* e^{-rT}(1-N(-d_2)) - S_0 + Ke^{-rT}}$

$\cancel{= -S_0 N(-d_1) + K^* e^{-rT} N(-d_2)}$

$\because$ strike call price = strike put price

$\therefore S_0 N(d_1) - K^* e^{-rT} N(d_2) = S_0 N(d_1) - K^* e^{-rT} N(d_2) - S_0 + K^* e^{-rT}$

$$S_0 = K^* e^{-rT}$$

$$32 = K^* e^{-0.04}$$

$$K^* = \frac{32}{e^{-0.04}}$$

Problem 3

Under $\boxed{\text{risk-neutral measure}}$ W is a Brownian motion, and X is some underlying (not necessarily an asset price) with dynamics

$$dX_t = 0.4\, dt + 2t\, dW_t, \qquad X_0 = 1.$$

The interest rate is $r = 0.01$. Let $T = 3$.

(a) X_T is normally distributed. Find its mean M and its variance V.

You may leave your answers to (b) and (c) in terms of M and/or V.

(b) Find the time-0 price of an option which pays $(X_T - M)^+$ at time T.

(c) Find the time-0 price of an option which pays at time T:

$$\begin{cases} 0 \text{ dollars} & \text{if } X_T < 2 \\ 10 \text{ dollars} & \text{if } 2 \le X_T < 3 \\ 30 \text{ dollars} & \text{if } 3 \le X_T. \end{cases}$$

a).

$dX_t = 0.4dt + 2t\, dW_t$

$X_t = X_0 + \int_0^t 0.4\, ds + \int_0^t 2s\, dW_s$

$= X_0 + 0.4t + 2t\, dW_t$

$\therefore$ mean $M = X_0 + 0.4T = 1 + 0.4 \cdot 3 = 1 + 1.2 = 2.2$

variance $V = \int_0^T \beta^2 dt = \int_0^T 4t^2 dt = \frac{4}{3}t^3 \Big|_0^T = \frac{4}{3} \cdot T^3 = \frac{4}{3} \cdot 27 = 36$

b).

$C = e^{-r(T-t)} E(X_T - M)^+$

$C = e^{-r(T-t)} E[(X_T - M) 1_{X_T > M}]$

$= e^{-r(T-t)} \int_M^\infty (y - M) \frac{1}{\sqrt{2\pi \sigma^2 (T-t)}} e^{\frac{-(y-M)^2}{2\sigma^2(T-t)}} dy$

let $x = y - M$

$dx = dy$

$= e^{-r(T-t)} \int_{M-M}^\infty x \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{x^2}{2\sigma^2(T-t)}} dx$

$= e^{-r(T-t)} \frac{-\sigma\sqrt{T-t}}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2\sigma^2(T-t)}} \Big|_0^\infty$

$= 0 + e^{-r(T-t)} \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}$

$\therefore C_0 = e^{-rT} \frac{\sigma\sqrt{T}}{2\pi} = e^{-0.01 \cdot 3} \frac{6 \cdot \sqrt{3}}{\sqrt{2\pi}}$

$= e^{-0.03} \frac{6\sqrt{3}}{\sqrt{2\pi}}$

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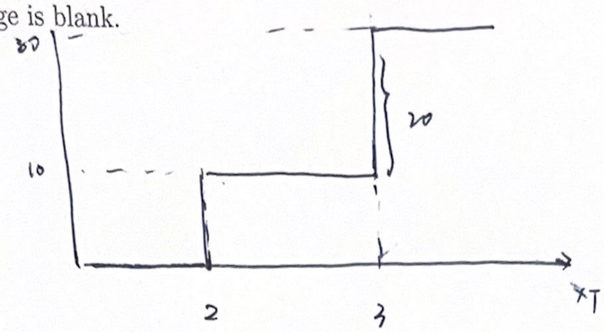
c)



time-0 price of option = 10 · binary call with strike 2 + 20 binary call with strike 3

time-0 binary call with strike 2 = $e^{-rT} \cdot N(d_2)$

$$= e^{-0.03} \cdot N\left(\frac{\log S_0 e^{rT}/K}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2} \right)$$

$$= e^{-0.03} N\left(\frac{\log 1 \cdot e^{0.03}/2}{6\sqrt{3}} - \frac{6\sqrt{3}}{2} \right)$$

$$= e^{-0.03} N\left(\frac{\log(e^{0.03}/2)}{6\sqrt{3}} - 3\sqrt{3} \right)$$

time-0 binary call with strike 3 = $e^{-rT} N(d_2)$

$$= e^{-0.03} N\left(\frac{\log(e^{0.03}/3)}{6\sqrt{3}} - 3\sqrt{3} \right)$$

$\therefore$ time-0 price of option $= 10 e^{-0.03} N\left(\frac{\log(e^{0.03}/2)}{6\sqrt{3}} - 3\sqrt{3} \right)$

$$+ 20 e^{-0.03} N\left(\frac{\log(e^{0.03}/3)}{6\sqrt{3}} - 3\sqrt{3} \right)$$

Problem 4

The interest rate on the bank account B is $r = 0.02$.

A non-dividend-paying stock S follows geometric Brownian motion starting at $S_0 = 33$.

On the next page is a table of time-0 prices and deltas of calls on S at various strikes. One of the "delta" columns, and one of the "price" columns, are entirely FAKE. One of the delta columns, and one of the price columns, are entirely valid. If you need a delta or price, use the real delta column or the real price column, not the false columns. All expiries are $T = 0.5$.

For all $a \le b$, all $x \in \mathbb{R}$, the clamp function $\text{Clamp}(x, a, b)$ is defined by

$$\text{Clamp}(x,a,b) := \min(\max(x,a),b) = \begin{cases} a & \text{if } x < a \\ x & \text{if } a \le x \le b \\ b & \text{if } b < x \end{cases}$$

(a) In order to replicate a contract paying at time T

$$\text{Clamp}(\overset{x}{2S_T},\ \overset{a}{44},\ \overset{b}{70})$$

using a continuously-rebalanced self-financing portfolio of only B and S, how many units of the stock S should you hold at time 0? You do not need to report, how many units of B to hold.

(b) Find the time-0 price of a contract which pays at time T

$$\text{Clamp}(|S_T - 40| - 4,\ 0,\ 5)$$

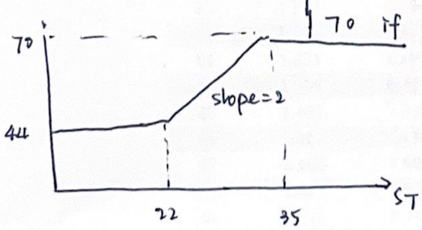
where $|S_T - 40|$ denotes the absolute value of $S_T - 40$. $c(K_1) - c(K_2) \le (K_2 - K_1) \cdot Z_0$

Hint: This payoff is 0 if $36 \le S_T \le 44$.

delta $= N(d_1) = N\left(\frac{\log F/K}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}\right)$ when $K\uparrow$ $N(d_1)\downarrow$ delta $\downarrow$. use first one

for call price when $K\uparrow$, the payoff decreases $\therefore$ call price $\downarrow$, use first one

a). $\text{clamp}(2S_T, 44, 70) = \begin{cases} \overset{44}{\cancel{2S_T}} & \text{if } 2S_T \le 44 \\ 2S_T & \text{if } 44 \le 2S_T \le 70 \\ 70 & \text{if } 2S_T > 70 \end{cases} \Rightarrow \begin{cases} 44 & \text{if } S_T < 22 \\ 2S_T & \text{if } 22 < S_T < 35 \\ 70 & \text{if } S_T > 35 \end{cases}$

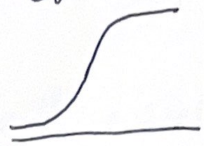


$\therefore$ it can be replicated by $44 Z_0 + 2C(22) - 2C(35)$

next page

call price

call delta



K↑ N(d1)↓ ↓

Strike	Call Price?	Call Price?	Call Delta?	Call Delta?
20	14.609	0.772	0.882	0.107
21	13.898	0.808	0.864	0.111
22	13.216	0.845	0.845	0.116
23	12.563	0.885	0.826	0.120
24	11.938	0.926	0.806	0.125
25	11.341	0.970	0.786	0.130
26	10.771	1.016	0.765	0.135
27	10.227	1.064	0.744	0.140
28	9.710	1.115	0.723	0.146
29	9.217	1.169	0.702	0.152
30	8.748	1.225	0.680	0.158
31	8.303	1.284	0.659	0.164
32	7.880	1.346	0.639	0.171
33	7.478	1.412	0.618	0.178
34	7.096	1.481	0.598	0.185
35	6.735	1.554	0.578	0.192
36	6.391	1.631	0.558	0.200
37	6.066	1.712	0.539	0.208
38	5.757	1.797	0.520	0.216
39	5.465	1.887	0.502	0.225
40	5.187	1.981	0.484	0.234
41	4.925	2.081	0.467	0.243
42	4.676	2.186	0.450	0.253
43	4.440	2.297	0.434	0.263
44	4.217	2.414	0.418	0.273
45	4.005	2.538	0.402	0.284
46	3.805	2.668	0.387	0.296
47	3.615	2.805	0.373	0.307
48	3.435	2.950	0.359	0.320
49	3.265	3.103	0.345	0.332
50	3.103	3.265	0.332	0.345
51	2.950	3.435	0.320	0.359
52	2.805	3.615	0.307	0.373
53	2.668	3.805	0.296	0.387
54	2.538	4.005	0.284	0.402
55	2.414	4.217	0.273	0.418
56	2.297	4.440	0.263	0.434
57	2.186	4.676	0.253	0.450
58	2.081	4.925	0.243	0.467
59	1.981	5.187	0.234	0.484
60	1.887	5.465	0.225	0.502
61	1.797	5.757	0.216	0.520
62	1.712	6.066	0.208	0.539
63	1.631	6.391	0.200	0.558
64	1.554	6.735	0.192	0.578
65	1.481	7.096	0.185	0.598
66	1.412	7.478	0.178	0.618
67	1.346	7.880	0.171	0.639
68	1.284	8.303	0.164	0.659
69	1.225	8.748	0.158	0.680
70	1.169	9.217	0.152	0.702

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$\therefore$ portfolio price at $t=0$ $= 44e^{-rT} + 2C_0(\text{strike } 22)$

$- 2C_0(\text{strike } 35)$

~~units hold for stock S~~

$\cancel{= \text{delta} = N(d_1) = \frac{\partial \text{portfolio}}{\partial S} = 2\left[N\left(\frac{\log S_0 e^{rT}/22}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right)\right] - 2\left[N\left(\frac{\log S_0 e^{rT}/35}{\sigma\sqrt{T}}\right.\right.}$

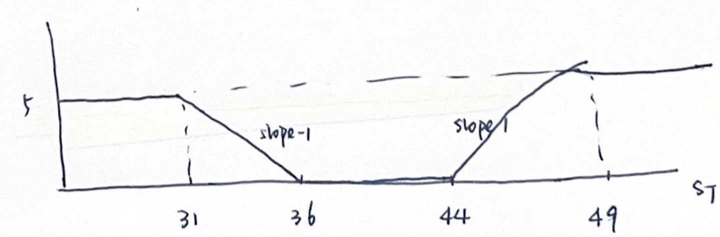
units hold for stock S $= \text{delta} = N(d_1) = \frac{\partial \text{portfolio}}{\partial S}$

$= 2 \cdot$ delta with strike 22 $- 2 \cdot$ delta with strike 35

$= 2 \cdot 0.845 - 2 \cdot 0.578 = 2 \cdot (0.845 - 0.578) = 2 \cdot 0.267 = 0.534$

b) $\text{clamp}(|S_T - 40| - 4, 0, 5) = \begin{cases} 0 & \text{if } |S_T - 40| - 4 < 0 \\ |S_T - 40| - 4 & \text{if } 0 \le |S_T - 40| - 4 \le 5 \\ 5 & \text{if } |S_T - 40| - 4 > 5 \end{cases}$

$= \begin{cases} |S_T - 40| - 4 & \text{if } 31 \le S_T \le 36,\ 44 \le S_T \le 49 \\ 5 & \text{if } S_T < 31,\ S_T > 49 \\ 0 & \text{if } 36 < S_T < 44 \end{cases}$



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267

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the portfolio can be replicated by

$(5Z, -S, +C(\text{strike } 36), +C(\text{strike } 44), -C(\text{strike } 49))$

$\therefore$ the time-0 price

$$= 5 \cdot e^{-rT} - S_0 + C_0(\text{strike } 36) + C_0(\text{strike } 44) - C_0(\text{strike } 49)$$

$$= 5 \cdot e^{-0.02 \cdot 0.5} - 33 + 6.391 + 4.217 - 3.625$$

$$= 5e^{-0.01} - 33 + 6.391 + 4.217 - 3.625$$

Problem 5

Let the bank account B and a non-dividend-paying stock S have physical price dynamics

$$dB_t = rB_t dt$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad S_0 = 20$$

where $\mu = 0.05$ and $\sigma > 0$ and W is Brownian motion under $\boxed{\text{physical measure.}}$

A 25-strike 1.0-expiry call option on S has time-0 price 0.90.

Answer the following questions about other contracts. All of the above assumptions apply in all parts.

(a) Assume $r = 0.03$. Find the price of a 25-strike 1.0-expiry put option on S.

(b) Assume $r = 0$. Find the price of a 16-strike 1.0-expiry put option on S.

(c) Assume $r = 0$.

Let L be the value of a self-financing portfolio that maintains 3x leverage on S by dynamically trading S and B. Specifically, the portfolio comprises units of B and S with total time-t value L_t, and the number of units of S that it holds at each time t is $3 \times L_t/S_t$, and the portfolio self-finances. Assume $L_0 = 20$.

A 25-strike T-expiry call option on L has price 0.90. Find T.

Your final answers should not involve σ (which you are not given).

a).

$$\text{call} = \text{put} + S - KZ$$

$$\therefore\ C_0 = P_0 + S_0 - KZ_0$$

$$P_0 = C_0 - S_0 + Ke^{-rT} = 0.9 - 20 + 25 \cdot e^{-0.03 \cdot 1} = 0.9 - 20 + 25 \cdot e^{-0.03}$$

b)

$$\text{call} = \text{put} + S - KZ$$

$$P_0 = C_0 - S_0 + KZ_0$$

$$= C_0 - S_0 + Ke^{-rT} \ \ldots$$

$$= [S_0 N(d_1) - Ke^{-rT} N(d_2)] - S_0 + Ke^{-rT}$$

when $r = 0$

$$= 20 N\left(\frac{\log 20/16}{6} + \frac{6}{2}\right) - 16 N\left(\frac{\log 20/16}{6} - \frac{6}{2}\right) - 20 + 16$$

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C7. $L_t = 3 \cdot \frac{L_t}{S_t} \cdot S_t + b_t B_t$

$L_t = 3L_t + b_t B_t$

$b_t = \frac{-2L_t}{B_t}$

$\therefore L_t = 3\frac{L_t}{S_t} S_t - 2\frac{L_t}{B_t} B_t$

$\because$ self-financing

$dL_t = 3\frac{L_t}{S_t} dS_t - 2\frac{L_t}{B_t} dB_t$

$= 3\frac{L_t}{S_t}(\mu S_t dt + \sigma S_t dW_t) - 2\frac{L_t}{B_t}(rB_t dt)$

$= 3L_t \mu dt + 3\sigma L_t dW_t - 2rL_t dt$

$\because r = 0$

$= (3\mu) L_t dt + (3\sigma) L_t dW_t$

$C = L_0 N(d_1) - KN(d_2) = 0.9$

$\because L_t = 3S_t$

$C(K=25)$ on $S = 0.9$ with $T=1$

$\therefore C(K=25)$ on $L = 0.9$ with $T=3$.