FINM 33000: Homework 3

Due Thursday, October 19, 2023 at 11:59pm

Problem 1

Consider a one-period market with time points 0 and T, and three outcomes $\Omega = \{\omega_u, \omega_m, \omega_d\}$ where each outcome has nonzero physical probability. The market has three assets: bank account B, stock S, and option C, where

$$B_0 = 1, \qquad B_T(\omega_u) = B_T(\omega_m) = B_T(\omega_d) = 1.2$$

and

$$S_0 = 145, \qquad S_T(\omega_u) = 240,\ S_T(\omega_m) = 120,\ S_T(\omega_d) = 60$$

and

$$C_0 = 10, \qquad C_T(\omega_u) = 0,\ C_T(\omega_m) = 30,\ C_T(\omega_d) = 30.$$

(a) Recall our basic definition of an equivalent martingale measure: it is a probability measure, equivalent to the physical probability measure, such that each asset price divided by B is a martingale.

Find all equivalent martingale measures. (A probability measure on a finite space can be specified by giving the probabilities of individual outcomes. So solve for *all* (p_u, p_m, p_d), strictly positive and summing to 1, such that S/B and C/B are martingales.)

If there is more than one martingale measure, then the Second Fundamental Theorem says that the market is incomplete. If there is only one martingale measure, then the Second Fundamental Theorem says that the market is complete. Is the market $\{B, S, C\}$ complete?

(b) Suppose you want to replicate the payoff X_T where

$$X_T(\omega_u) = 120,\ X_T(\omega_m) = 60,\ X_T(\omega_d) = 0$$

Can this be done using a portfolio of B, S, C?

If so, then find a replicating portfolio.

If not, then say why not. Note that "the market $\{B, S, C\}$ is incomplete" (even if true) is not adequate justification, because even in an incomplete market, *some* payoffs can be replicated.

(c) If the answer to (b) is yes, then find the no-arbitrage time-0 price of payoff X_T in *two* ways: use the replicating portfolio from part (b), and use the pricing probabilities from part (a).

Problem 2

Redo all parts of Problem 1, with the following changes to the definition of C:

$$C_0 = 11, \qquad C_T(\omega_u) = 0,\ C_T(\omega_m) = 24,\ C_T(\omega_d) = 36.$$

Problem 3

The White Sox play the Cubs in Major League Baseball's World Series, a sequence of baseball games. In each game, one team wins, the other team loses. The first team to win four games wins the Series; at that point the Series ends.

You want to place a \$1000 even-money bet on the White Sox to win the Series. This means that your objective is to win \$1000 if the White Sox win the Series, and to lose \$1000 otherwise.

Unfortunately, your bookie/casino does not accept bets on the Series. However, your bookie *does* accept even-money bets on each particular game. This means that immediately before each game n that is played (where $n = 1, 2, 3, 4$, and possibly $5, 6, 7$), he is ready to sell you, at zero price, arbitrary quantities of "White Sox in Game n" bets, which pay you $+1$ dollar if the White Sox win game n and -1 dollar if the Cubs win game n; and he is also ready to sell you, at zero price, arbitrary quantities of "Cubs in Game n" bets, which pay you -1 dollar if the White Sox win game n and $+1$ dollar if the Cubs win game n. Your bookie provides these services frictionlessly. (Note that in practice, when you place a bet with a casino/bookmaker, your stake typically must be paid up front, instead of being incorporated into the payoff; but here we are doing the latter for simplicity.)

Interest rates are zero; there exists a bank account with constant price 1 dollar per unit.

Assume that the White Sox have physical probability 55% of winning each game, independently of all other games.

In order to achieve your objective, what bet should you place on Game 1? (Specify the amount and the team.) Explain.

Problem 1

Consider a one-period market with time points 0 and T, and three outcomes $\Omega = \{\omega_u, \omega_m, \omega_d\}$ where each outcome has nonzero physical probability. The market has three assets: bank account B, stock S, and option C, where

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$$S_0 = 145, \qquad S_T(\omega_u) = 240,\ S_T(\omega_m) = 120,\ S_T(\omega_d) = 60$$

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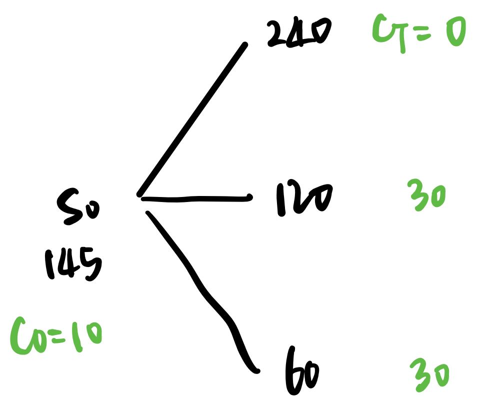
$$C_0 = 10, \qquad C_T(\omega_u) = 0,\ C_T(\omega_m) = 30,\ C_T(\omega_d) = 30.$$

(a) Recall our basic definition of an equivalent martingale measure: it is a probability measure, equivalent to the physical probability measure, such that each asset price divided by B is a martingale.

Find all equivalent martingale measures. (A probability measure on a finite space can be specified by giving the probabilities of individual outcomes. So solve for *all* (p_u, p_m, p_d), strictly positive and summing to 1, such that S/B and C/B are martingales.)

If there is more than one martingale measure, then the Second Fundamental Theorem says that the market is incomplete. If there is only one martingale measure, then the Second Fundamental Theorem says that the market is complete. Is the market $\{B, S, C\}$ complete?



$$\left\{\begin{array}{l} \dfrac{S_0}{B_0} = \dfrac{S_T}{B_T} \\ p_u + p_m + p_d = 1 \\ \dfrac{C_0}{B_0} = \dfrac{C_T}{B_T} \end{array}\right. \Rightarrow \left\{\begin{array}{l} 145 = \dfrac{240}{1.2} \cdot p_u + \dfrac{120}{1.2} p_m + \dfrac{60}{1.2} p_d \\ p_u + p_m + p_d = 1 \\ 10 = 0 + \dfrac{30}{1.2} p_m + \dfrac{30}{1.2} p_d \end{array}\right.$$

$$\begin{cases} 200\,p_u + 100\,p_m + 50\,p_d = 145 \\ p_m + p_d = 0.4 \\ p_u + p_m + p_d = 1 \end{cases} \Rightarrow \begin{cases} p_u = 0.6 \\ p_m = 0.1 \\ p_d = 0.3 \end{cases}$$

There is only one unique martingale, so the market $\{B, S, C\}$ is complete

(b) Suppose you want to replicate the payoff X_T where

$$X_T(\omega_u) = 120,\ X_T(\omega_m) = 60,\ X_T(\omega_d) = 0$$

Can this be done using a portfolio of B, S, C?

If so, then find a replicating portfolio.

If not, then say why not. Note that "the market $\{B, S, C\}$ is incomplete" (even if true) is not adequate justification, because even in an incomplete market, *some* payoffs can be replicated.

let units of B be x, units of S be y, units of C be z

$$\begin{cases} 1.2x + 240y + 0 \cdot z = 120 \\ 1.2x + 120y + 30z = 60 \\ 1.2x + 60y + 30z = 0 \end{cases} \rightarrow \begin{cases} x = -100 \\ y = 1 \\ z = 2 \end{cases}$$

-100 units of B, 1 unit of S, 2 unit of C

(c) If the answer to (b) is yes, then find the no-arbitrage time-0 price of payoff X_T in *two* ways: use the replicating portfolio from part (b), and use the pricing probabilities from part (a).

method I: $X_0 = 0.6 \times \frac{120}{1.2} + 0.1 \times \frac{60}{1.2} + 0.3 \times 0 = 65$

method 2: $X_0 = 145 \times 1 - 100 + 2 \times 10 = 65$

Problem 2

Redo all parts of Problem 1, with the following changes to the definition of C:

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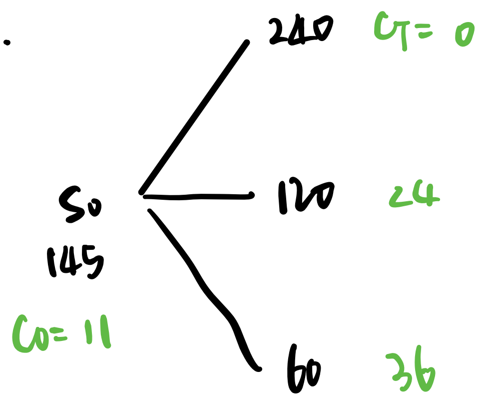
1.



$$\begin{cases} \dfrac{S_0}{B_0} = \dfrac{S_T}{B_T} \\ p_u + p_m + p_d = 1 \\ \dfrac{C_0}{B_0} = \dfrac{C_T}{B_T} \end{cases} \Rightarrow \begin{cases} 145 = \dfrac{240}{1.2} \cdot p_u + \dfrac{120}{1.2} p_m + \dfrac{60}{1.2} p_d \\ p_u + p_m + p_d = 1 \\ 11 = 0 + \dfrac{24}{1.2} p_m + \dfrac{36}{1.2} p_d \end{cases}$$

$$\Rightarrow \begin{cases} 200 p_u + 100 p_m + 50 p_d = 145 & (1) \\ 20 p_m + 30 p_d = 11 & (2) \\ p_u + p_m + p_d = 1 & (3) \end{cases}$$

$\Rightarrow$ (1) + 5 × (2) which is linear relationship to (3), there are infinite solution to the problem which indicates that the market is incomplete.

2. let units of B be x, units of S be y, units of C be z

$$\begin{cases} 1.2x + 240y + 0 \cdot z = 120 \\ 1.2x + 120y + 24z = 60 \\ 1.2x + 60y + 36z = 0 \end{cases} \rightarrow \begin{array}{c} -240y - 180 + 240y = 120 \\ \text{no solution} \end{array}$$

$\therefore$ there is no replicating portfolio

Problem 3

The White Sox play the Cubs in Major League Baseball's World Series, a sequence of baseball games. In each game, one team wins, the other team loses. The first team to win four games wins the Series; at that point the Series ends.

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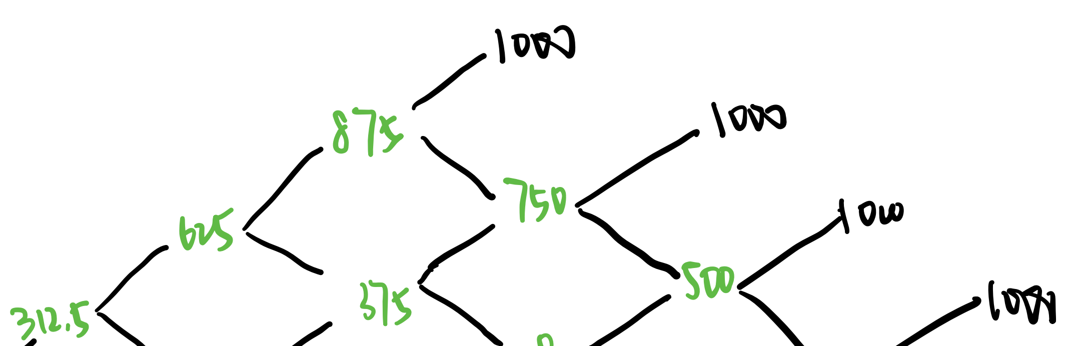

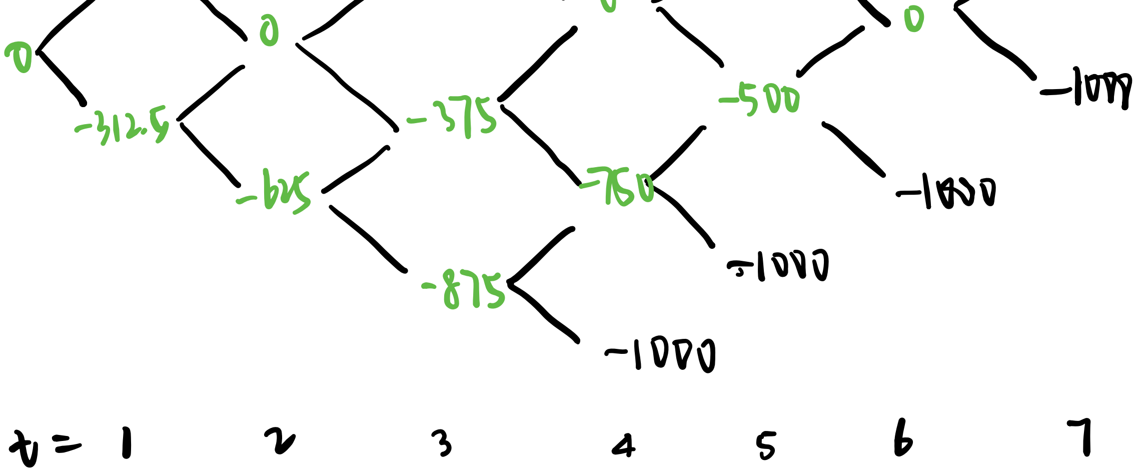

therefore, for the first game. we should bet 312.5 units of White Sox in Game 1.