# FINM 33000: Homework 5

Due Thursday, November 2, 2023 at 11:59pm

## Problem 1

Let $W$ be a Brownian motion and let

$$Z_t = \exp(W_t^2 - 1), \qquad \text{for } t \geq 0.$$

Write $Z_t$ in terms of drift and diffusion components. You may give your answer using differential notation, i.e. of the form

$$\mathrm{d}Z_t = \underline{\quad\quad}\mathrm{d}t + \underline{\quad\quad}\mathrm{d}W_t,$$

or using integral notation. Give two solutions (a,b):

(a) Let $X_t = W_t$, so $\mathrm{d}X_t = \mathrm{d}W_t$. Then apply Itô to $f(x) = e^{x^2-1}$.

(b) Let $X_t = W_t^2 - 1$, so $\mathrm{d}X_t = \mathrm{d}t + 2W_t\mathrm{d}W_t$. Then apply Itô to $f(x) = e^x$.

(c) Is $Z$ a martingale?

### Comment on part (b)

When $X$ is defined by $X_t = W_t^2 - 1$, how did we know that $\mathrm{d}X_t = \mathrm{d}t + 2W_t\mathrm{d}W_t$.

One way is to use Ito's rule on the function $g(w) = w^2 - 1$, to obtain $\mathrm{d}X_t = \mathrm{d}g(W_t) = g'(W_t)\mathrm{d}W_t + \frac{1}{2}g''(W_t)(\mathrm{d}W_t)^2 = \mathrm{d}t + 2W_t\mathrm{d}W_t$, as claimed.

Alternatively, a second way is to recall that we already calculated $\mathrm{d}(W_t^2)$ in L4.26, so we simply have $\mathrm{d}(W_t^2 - 1) = \mathrm{d}(W_t^2) - 0 = \mathrm{d}t + 2W_t\mathrm{d}W_t$, as claimed.

How do we know that $\mathrm{d}(W_t^2 - 1) = \mathrm{d}(W_t^2) - 0$?

First recall: by L4.10 second point, for any Ito processes $F_t$ and $G_t$ and any constants $a, b$,

$$\boxed{\mathrm{d}(aF_t + bG_t) = a\ \mathrm{d}F_t + b\ \mathrm{d}G_t}$$

and recall L4.10 first point, for the particular case that $F_t$ and $G_t$ are: constant, or $t$, or $W_t$. Specifically, for any constants $a, b, c$, we have

$$\boxed{\mathrm{d}(at + bW_t + c) = a\mathrm{d}t + b\mathrm{d}W_t}$$

and in particular $\mathrm{d}c = 0$ if $c$ is constant.

a) $Z_t = e^{W_t^2-1}$ $\qquad Z_t = f(X_t)$

$f(x) = e^{x^2-1}$ $\qquad f'(x) = e^{x^2-1}\cdot 2x$ $\qquad f''(x) = e^{x^2-1}\cdot 4x^2 + 2e^{x^2-1}$

$X_t = W_t \Rightarrow dX_t = dW_t$

$$dZ_t = df(X_t) = f'(X_t)dX_t + \tfrac{1}{2}f''(X_t)(dX_t)^2$$
$$= e^{W_t^2-1}\cdot 2W_t\cdot dW_t + [2W_t^2 e^{W_t^2-1} + e^{W_t^2-1}](dW_t)^2$$
$$= \underline{2W_t e^{W_t^2-1}}\, dW_t + \underline{[2W_t^2 e^{W_t^2-1} + e^{W_t^2-1}]}\, dt$$

b) Let $X_t = W_t^2 - 1$

$$dX_t = dt + 2W_t\, dW_t$$

$f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$

$$dZ_t = df(X_t) = f'(X_t)dX_t + \tfrac{1}{2}f''(X_t)(dX_t)^2$$
$$= e^{W_t^2-1}[dt + 2W_t\, dW_t] + \tfrac{1}{2}e^{W_t^2-1}[dt + 2W_t\, dW_t]^2$$
$$= e^{W_t^2-1}dt + 2W_t e^{W_t^2-1}dW_t + \tfrac{1}{2}e^{W_t^2-1}\cdot$$
$$[\underbrace{dt^2}_{=0} + 4W_t\underbrace{dt\,dW_t}_{=0} + 4W_t^2\underbrace{dW_t^2}_{=dt}]$$

$$= \underline{[e^{W_t^2-1} + 2W_t^2 e^{W_t^2-1}]}\,dt + \underline{2W_t e^{W_t^2-1}}\,dW_t$$

c). it is not martingale because it contains $dt$, it's Itô process not Itô integral that contains only $dW_t$. so it is not a martingale. drift does not vanish

## Problem 2

Let $W$ be a Brownian motion. Define $X$ by

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

where $\kappa$, $\theta$, $\sigma$, and $X_0$ are all constants.

(a) Write $e^{\kappa T} X_T$ as the sum of a constant, a Riemann integral with respect to $\mathrm{d}t$, and an Itô integral with respect to $\mathrm{d}W_t$, such that both integrands may depend on $t$, but not on $X$.

Hint: As a first step, calculate $\mathrm{d}(e^{\kappa t} X_t) = \ldots$

(b) Find explicit formulas for the mean and variance of $X_T$.

In part (b) you may use the following fact (without providing a proof):

If $\beta_t$ is a *nonrandom* piecewise continuous function of $t$, then

$$\boxed{\int_0^T \beta_t \mathrm{d}W_t \qquad \text{has distribution:} \qquad \text{Normal}\Big(\text{mean } 0,\ \text{variance} \int_0^T \beta_t^2 \mathrm{d}t\Big)}$$

essentially because the sum of independent normals is normal; more specifically because

$$\sum_{n=0}^{N-1} \beta_{t_n} \Delta W_{t_n} \qquad \text{has distribution:} \qquad \text{Normal}\Big(\text{mean } 0,\ \text{variance} \sum_{n=0}^{N-1} \beta_{t_n}^2 \Delta t\Big)$$

for all positive integer $N$, where $\Delta t := T/N$ and $t_n := n\Delta t$ and $\Delta W_{t_n} := W_{t_{n+1}} - W_{t_n}$.

## Problem 3

On next page

a) $f(t,X) = e^{kt}x$

$$d(e^{kt}x) = e^{kt}dX_t + X_t\,de^{kt} + dX_t\,de^{kt} \qquad (L4.32)$$

$$= e^{kt}dX_t + X_t \cdot e^{kt} \cdot k\,dt + 0$$

$$= e^{kt}dX_t + ke^{kt}X_t\,dt$$

$$= e^{kt}[k(\theta - X_t)dt + \sigma dW_t] + ke^{kt}X_t\,dt$$

$$= k\theta e^{kt}dt + \sigma e^{kt}dW_t$$

$$\Rightarrow e^{kT}X_T = e^{k0}X_0 + \int_0^T k\theta e^{kt}dt + \int_0^T \sigma e^{kt}dW_t$$

$$= X_0 + (\theta e^{kT} - \theta) + \int_0^T \sigma e^{kt}dW_t$$

$$= X_0 + \theta(e^{kT}-1) + \int_0^T \sigma e^{kt}dW_t$$

b) If $\beta_t = \sigma e^{kt}$ is a nonrandom piecewise continuous function of $t$, then $\int_0^T \sigma e^{kt}dW_t$ has distribution

$$\sim N\left(0, \int_0^T \sigma^2 e^{2kt}dt = \frac{\sigma^2}{2k}[e^{2kT}-1]\right)$$

$$\therefore \int_0^T \sigma e^{kt}dW_t \sim N\left(0, \frac{\sigma^2}{2k}(e^{2kT}-1)\right)$$

$$\because e^{kT}X_T = X_0 + \theta(e^{kT}-1) + \int_0^T \sigma e^{kt}dW_t$$

$$\therefore e^{kT}X_T \sim N\left(X_0 + \theta(e^{kT}-1), \frac{\sigma^2}{2k}(e^{2kT}-1)\right)$$

$$\therefore X_T \sim N\left(\frac{X_0+\theta(e^{kT}-1)}{e^{kT}}, \frac{\frac{\sigma^2}{2k}(e^{2kT}-1)}{e^{2kT}}\right)$$

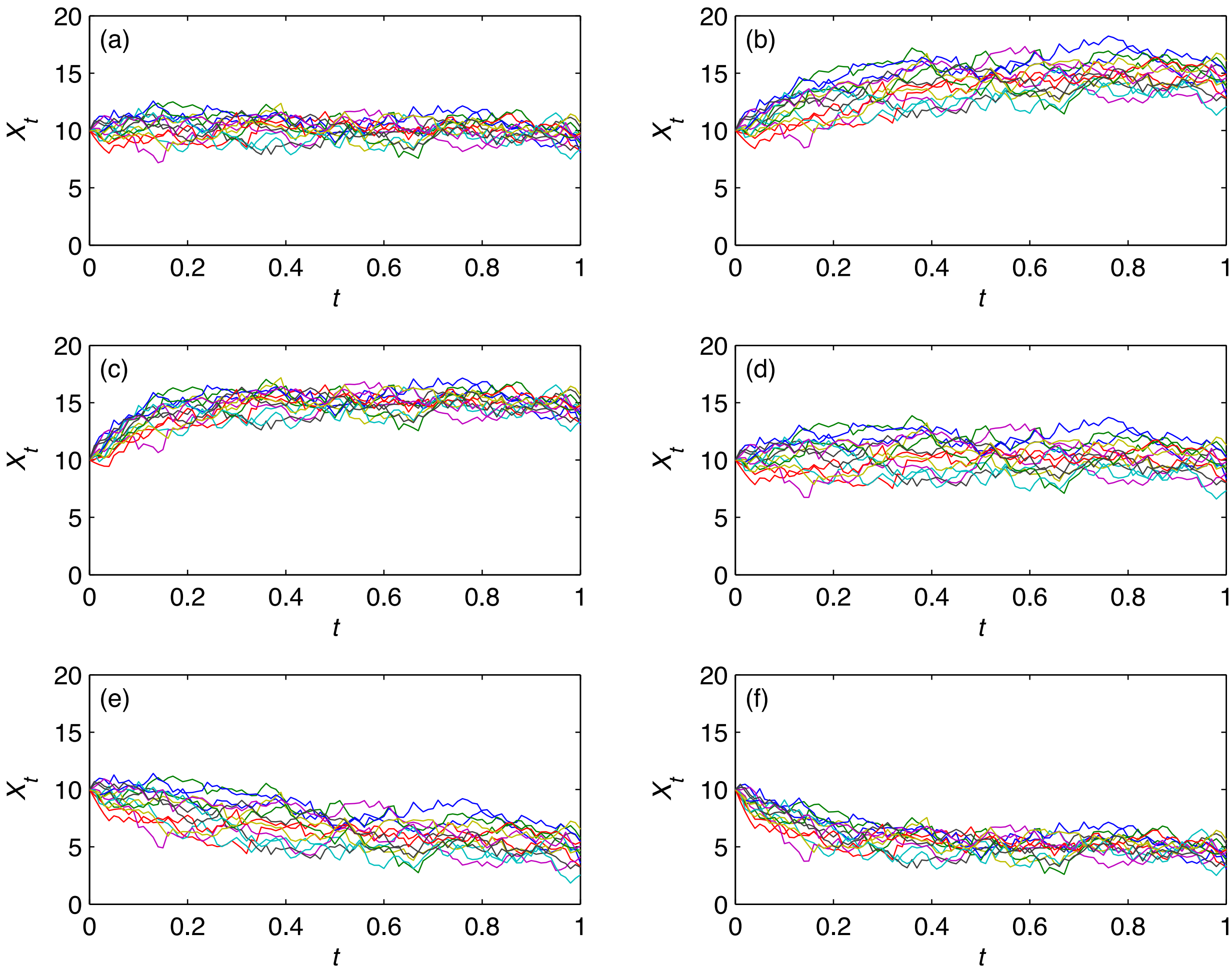
$\mathrm{var}(bX) = b^2\mathrm{var}(X)$

## Problem 3

Here are 6 examples of the dynamics in Problem 2. All have the form

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

where $\sigma = 4$ and $X_0 = 10$ in all cases. For each of the $2 \times 3 = 6$ combinations of choices $\kappa \in \{3, 8\}$ and $\theta \in \{5, 10, 15\}$, I plotted some sample paths.




For each process (a,b,c,d,e,f), state the $\kappa$ and the $\theta$ that I used to generate that process.

No justification necessary.

Hint: In each case the process is *mean-reverting*. Intuitively, $\theta$ is the mean reversion "level" or the "long-term mean"; and $\kappa$ is the mean reversion "rate" or "speed".

$$dX_t = k(\theta - X_t)\,dt + 4\,dW_t$$

| | | |
|---|---|---|
| a | $k = 8$ | $\theta = 10$ |
| b | $k = 3$ | $\theta = 15$ |
| c | $k = 8$ | $\theta = 15$ |
| d | $k = 3$ | $\theta = 10$ |
| e | $k = 3$ | $\theta = 5$ |
| f | $k = 8$ | $\theta = 5$ |