FINM 33000: Homework 2

Due Thursday, October 12, 2023, 11:59pm

Notation: "Lx.y" refers to Lecture x, Slide y.

Problem 1

Assume that discount bonds maturing at T have time-0 price 0.95 per unit. Assume that T-expiry standard ("plain vanilla") European calls on a random variable S_T are available at the following strikes K and time-0 prices $C_0(K)$. Exactly six basic assets are available: the bond and these five calls, nothing else. As defined in class, these standard European calls pay $(S_T - K)^+$.

K	$C_0(K)$
20.0	6.15
22.5	4.15
25.0	2.60
27.5	1.50
30.0	0.80

A T-expiry K-strike *binary* or *digital* call on S_T pays at time T either 1 if $S_T \geqslant K$, or 0 if $S_T < K$. A T-expiry K-strike binary or digital *put* on S_T pays at time T either 1 if $S_T < K$, or 0 if $S_T \geqslant K$. (This specification of which inequalities are strict is not universal.)

(a) Find upper and lower bounds on the time-0 price of a T-expiry 22.5-strike binary call on S_T.

(b) Find upper and lower bounds on the time-0 price of a T-expiry 22.5-strike binary put on S_T.

(c) Find the time-0 price of a contract which pays $\max(2.5,\ S_T - 22.5)$ at time T.

(d) Find an upper bound on the time-0 price of a T-expiry 28-strike standard call. (This is a plain vanilla call, not a binary call.)

Do so by using static portfolios to (depending on the question) replicate or superreplicate or subreplicate the derivative contract. Your final answers should be numbers.

In parts (a,b,d), try to find tight bounds (but you do not need to prove that they are tight). Tight means that they cannot be improved (lowered in the case of an upper bound, raised in the case of a lower bound) without making further assumptions.

a. ① super-replicating:

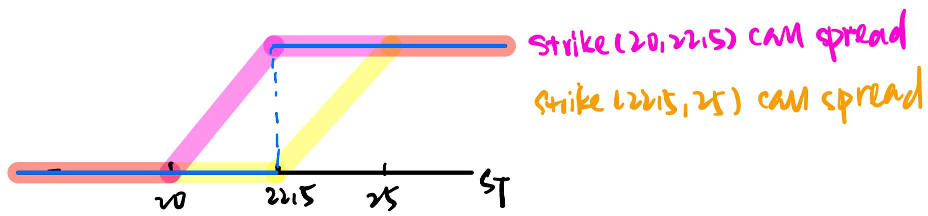
It is bounded above by $\frac{1}{2.5}$ units of call spread with strike (20, 22.5)

$\therefore$ upper bound $= \frac{1}{2.5} \times (6.15 - 4.15) = 0.8$

② sub-replicating:

It is bounded below by $\frac{1}{2.5}$ units of call spread with strike (22.5, 25)

lower bound $= \frac{1}{2.5} \times (4.15 - 2.6) = 0.62$

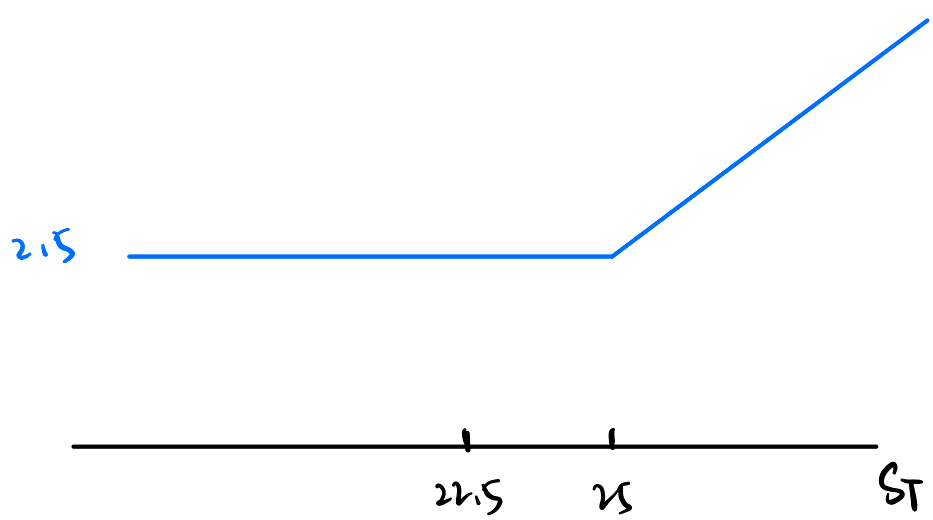


b. binary put + binary call = bond

$\therefore$ upper bound $= 0.95 - 0.62 = 0.33$

lower bound $= 0.95 - 0.8 = 0.15$

c.



contract = 2.5 units of discount bond + 1 unit of 25-strike call option

$C_0 = 2.5 \times 0.95 + 2.60 = 4.975$

d.

$$\begin{cases} 27.5x + 30y = 28 \\ x + y = 1 \end{cases} \Rightarrow \begin{cases} x = 0.8 \\ y = 0.2 \end{cases}$$

super-replicating:

$$C_0 = 0.8 \times 1.5 + 0.2 \times 0.8 = 1.36$$

Discussion of Problem 1

In general, when you are given a set of basic assets and asked to *price* or *value* a "contingent claim" or a "derivative" contract, this means to assign a price to the derivative in such a way that a frictionless "extended market" consisting of the basic assets together with the derivative contract [at your proposed price] would admit no arbitrage.

If the derivative can be perfectly replicated by the basic assets, then we showed in lecture that the unique no-arbitrage price of the derivative is the value of the replicating portfolio. For example, this was the case in L1.21, where the derivative was a forward contract, and the basic assets were the underlying S and the bond Z.

If the derivative cannot be perfectly replicated by the basic assets, then there is a *range* of prices that could be assigned to the derivative, all of which would be consistent with no-arbitrage. For example, this was the case in L1.27, where the derivative was a call, and the basic assets were the underlying non-dividend-paying stock S and the bond Z, and we considered only static ("buy-and-hold") portfolios. We could not exactly replicate the call payoff and find a unique price. Instead, what we did, as shown in the diagram, was to *superreplicate* the call payoff using S_T, and to *subreplicate* the call payoff using $S_T - K$ (and also using the zero payoff). By no-arbitrage, we concluded that the time-0 value of the call is *bounded above* by the superreplicating portfolio's time-0 value, and *bounded below* by the subreplicating portfolios' time-0 values. So there was not a unique price, but there was a range of no-arbitrage prices that had upper and lower bounds, expressible in terms of the prices of the basic assets.

In problem 1, the basic assets are the five (standard) calls and the bond. The derivative in question depends on what part of the problem you are doing. For the derivative in Problem 1(c), perfect replication is possible. For the derivatives in Problem 1(a,b,d), perfect replication is impossible (unless you make further assumptions – such as assuming that S_T must take values in some particular set, or assuming a model for the dynamics of S and doing dynamic trading at dates intermediate between 0 and T – but I am not allowing you to make such extra assumptions); so in these cases 1(a,b,d), you should find a static "superreplicating" or "dominating" porfolio to obtain an upper bound, and a static "subreplicating" or "dominated" portfolio to obtain a lower bound. Suggestions: Draw a payoff diagram, like the ones we did in class. Include both the derivative's payoff and your [super/sub]-replicating portfolio's payoff (as we did in class), to make sure that your proposed portfolio really does [super/sub]-replicate [tightly].

Problem 2

(a) Let $f : (0, \infty) \to \mathbb{R}$ be twice continuously differentiable. Prove that for any $K_* > 0$ and any $s > 0$,

$$f(s) = f(K_*) + f'(K_*)(s - K_*) + \int_0^{K_*} f''(K)(K - s)^+ \mathrm{d}K + \int_{K_*}^{\infty} f''(K)(s - K)^+ \mathrm{d}K.$$

Hint: First consider the case $s > K_*$. One of the two integrals will vanish. Simplify the non-vanishing integral by changing a limit of integration in a way that allows you to delete the $^+$ in the integrand. Then integrate by parts.

(b) Part (a) indicates how to replicate the time-T payoff $f(S_T)$ using a static portfolio of T-maturity bonds, forwards, puts, and calls on S_T. In practice it may not be *exactly* implementable, because it uses a mixture of calls and puts across a continuum of strikes from 0 to ∞, whereas in practice one can hold calls and puts at only a discrete set of strikes.

Let S be some positive price process. Suppose you want to replicate approximately the payoff $-2 \times \log(S_T)$, by making use of puts and calls with strikes at, and only at, all the positive integer multiples of 5. (So we are ignoring the fact that in practice you cannot acquire options at infinitely many strikes, but we are taking account of the fact that options are listed only at certain discrete strikes.)

Let us choose, say, $K_* = 1960$. (In theory one can choose the K_* arbitrarily; in practice one typically chooses K_* close to S_0.)

You will hold 0 calls with strike 1950. Approximately how many puts with strike 1950 should you hold, according to part (a)? You will also hold puts at other strikes, but I am asking you only about the 1950 strike. Give at least 3 significant digits (digits after the leading zeros).

As a first step, approximate the appropriate integral by a Riemann sum.

Discussion of Problem 2

- Why are we interested in replicating and/or pricing a contract with payoff = constant$\times \log S_T$?

  Answer: Such "log contracts" help us to understand the CBOE's VIX index and to create *variance swaps*.

- Your portfolio will include options not only at strike 1950, but also at all other available strikes: puts at all strikes below the designated K_*, and calls at all strikes above the designated K_*.

  What about the strike 1960 exactly equal to K_* – should you hold puts or calls of that strike?

  Answer: For the strike that is exactly equal to K_*, one approach is to hold *both* puts and calls of that strike. In that case, the number of puts and calls would each be cut in half, for that particular strike. That is the approach that the CBOE has chosen. (Another approach would be to avoid this issue by choosing K_* to be different from any listed strike, for example taking $K_* = 1962.5$.)

a. when $S > K_*$, $(K-S)^+ = 0$ $(S-K)^+ = S-K$

$$f(K_*) + f'(K_*)(S-K_*) + \int_0^{K_*} f''(K)(K-S)^+ dK + \int_{K_*}^{S} f''(K)(S-K)^+ dK$$

$$= f(K_*) + f'(K_*)(S-K_*) + \int_{K_*}^{S} f''(K)S - f''(K)\cdot K \, dK$$

$u=-K$ $u'=-1$

$v'=f''(K)$ $v=f'(K)$

$$= f(K_*) + f'(K_*)(S-K_*) + f'(K)\cdot S\Big|_{K_*}^{S} - Kf'(K)\Big|_{K_*}^{S} + \int_{K_*}^{S} f'(K)\,dK$$

$$= f(K_*) + f'(K_*)(S-K_*) + f'(S)\cdot S - f'(K_*)S - Sf'(S) +$$

$$K_*f(K_*) + f(K)\Big|_{K_*}^{S}$$

$$= f(\cancel{K_*}) + f'(K_*)\cancel{(S}-K_*) + f'(K_*)(\cancel{K_*}-S) - f(\cancel{K_*}) + f'(\cancel{S})S - \cancel{f'}(S)S + f(S)$$

$$= f(S)$$

when $S < K_*$, $(K-S)^+ = K-S$ $(S-K)^+ = 0$

$$f(K_*) + f'(K_*)(S-K_*) + \int_S^{K_*} f''(K)(K-S)^+ dK + \int_{K_*}^{S} f''(K)(S-K)^+ dK$$

$$= f(K_*) + f'(K_*)(S-K_*) + \int_S^{K_*} f''(K)K - f''(K)S \, dK$$

$u=K$ $u'=1$

$v'=f''(K)$ $v=f'(K)$

$$= f(K_*) + f'(K_*)(S-K_*) + Kf'(K)\Big|_S^{K_*} - \int f'(K)\,dK - f'(K)S\Big|_S^{K_*}$$

$$= f(\cancel{K_*}) + f'(K_*)\cancel{(}S-K_*) + K_*\cancel{f'}(K_*) - S\cancel{f'}(S) - f(\cancel{K_*}) + f(S) - \cancel{f'}(K_*)S + \cancel{f'}(S)S$$

$$= f(S)$$

b. the payoff of put with strike 1950 is $(1950-S)^+$

the approximation is:

$$\int_0^{K^*} f''(K)(K-S)^+ \, dK$$

$f(S) = -2\log(S)$

$f(K) = -2\log(K)$, $f'(K) = \frac{-2}{K}$, $f''(K) = \frac{2}{K^2}$

$$\int_0^{K^*} f''(K)(K-S)^+ \, dK = \int_0^{K^*} \frac{2}{K^2}(K-S)^+ \, dK$$

Applying Riemann sum:

$$\sum_{K_n < K^*} \frac{2}{K_n^2}(K_n - S)^+ \Delta K$$

$K_n = 1950$ here $\Delta K = 5$

$\therefore$ units of put $= \frac{2}{K_n^2} \times \Delta K = 5 \cdot \frac{2}{1950} \approx 2.63 \times 10^{-6}$

long 2.63×10^{-6} units of put