FINM 33000: Homework 7

Due Thursday, November 30, 2023 at 11:59pm

Problem 1

Let r be the constant risk-free interest rate. Suppose that, under risk-neutral probabilities, the time-t-conditional distribution of the random variable Y_T is Normal with some mean M_t (known at time $t < T$) and variance $\sigma^2(T-t)$, where $\sigma > 0$ and T are constants.

(a) Find the time-t price of an option which pays $(Y_T - K)^+$ at time T, where K is a constant.

Hint: $(Y_T - K)^+$ can be rewritten as $(Y_T - M_t)\mathbf{1}_{Y_T > K} + (M_t - K)\mathbf{1}_{Y_T > K}$.

To price the first term, do an explicit integration. To price the second term, write the expectation in terms of the standard normal CDF N. In both terms, M_t can be treated as a constant, because we are conditioning on the information available at time t.

(b) Let $r = 0$ and assume $M_t = Y_t$ (in other words, $\mathbb{E}_t Y_T = Y_t$, which is true if Y is a martingale).

For the option in (a), find its time-t delta and gamma, with respect to Y_t.

(c) Let $r = 0$ and assume $M_t = Y_t$. Assume the option in (a) is at-the-money at time t: $Y_t = K$. Find the option's time-t price. From that price, find its theta, and vega. Vega is defined as the partial derivative of the option pricing function with respect to σ.

Problem 2

Suppose that the bank account B and a non-dividend-paying stock price S have the following dynamics under risk-neutral measure.

$$
\begin{aligned}
\mathrm{d}B_t &= rB_t\mathrm{d}t & B_0 &= 1 \\
\mathrm{d}S_t &= rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t & S_0 &= 216
\end{aligned}
$$

Let $T > 0$, and assume that $\exp(-rT) = 0.96$. Let $K = 250$ and assume that a T-expiry K-strike call on S has time-0 price 34, and a T-expiry K-strike binary call on S has time-0 price 0.44.

The expectation in (d) and probability in (e) are with respect to risk-neutral measure. Compute:

(a) The time-0 price of a T-expiry K-strike binary put on S.

(b) The time-0 price of a T-expiry K-strike put on S.

(c) The time-0 price of a T-expiry K-strike asset-or-nothing call on S.

(d) The time-0 expectation of S_T.

(e) $\mathbb{P}(S_T > K)$.

(f) A portfolio holds $\{B, S\}$ in quantities that vary continuously in time. It is self-financing and its time-T value is $(K - S_T)^+$. How many units of S does it hold at time 0?

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a)

$$Y_T \sim N(M_t,\ 6^2(T-t))$$

$$C_t = e^{-r(T-t)}\, E_t(Y_T-K)^+$$

$$= e^{-r(T-t)}\, E_t\left[(Y_T-M_t)1_{Y_T>K} + (M_t-K)1_{Y_T>K}\right]$$

$$= e^{-r(T-t)}\Big(\int_K^\infty (y-M_t)\frac{1}{\sqrt{2\pi 6^2(T-t)}}\, e^{\frac{-(y-M_t)^2}{26^2(T-t)}}\, dy$$

$$+ (M_t-K)\, P_t(Y_T>K)\Big)$$

$$\boxed{\begin{gathered} -\frac{(y-M_t)^2}{26^2(T-t)} = x \\ dx = -\frac{\cancel{2}(y-M_t)}{\cancel{2}6^2(T-t)}\, dy \end{gathered}}$$

$$= e^{-r(T-t)}\int_K^\infty -6^2(T-t)\frac{1}{\sqrt{2\pi 6^2(T-t)}}\cdot e^x\, dx$$

$$+ e^{-r(T-t)}\cdot(M_t-K)\cdot P_t\left(\frac{Y_t-M_t}{6\sqrt{T-t}} > \frac{K-M_t}{6\sqrt{T-t}}\right)$$

$$= -e^{-r(T-t)} \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}\, e^{-\frac{(y-M_t)^2}{2\sigma^2(T-t)}} \Big|_K^{\infty} + e^{-r(T-t)}(M_t-K)\, N\left(\frac{-(K-M_t)}{\sigma\sqrt{T-t}}\right)$$

$$= e^{-r(T-t)} \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}\, e^{\frac{-(K-M_t)^2}{2\sigma^2(T-t)}} + e^{-r(T-t)}(M_t-K)\, N\left(\frac{M_t-K}{\sigma\sqrt{T-t}}\right)$$

b) when $r=0$, $M_t = Y_t$

$$C_t = \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}\, e^{\frac{-(K-Y_t)^2}{2\sigma^2(T-t)}} + (Y_t-K)\, N\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right)$$

$$\text{delta} = \frac{\partial C}{\partial Y} = \frac{-\cancel{2}(K-Y_t)}{\cancel{2}\sigma^2(T-t)} \cdot \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}\, e^{\frac{-(K-Y_t)^2}{2\sigma^2(T-t)}}$$

$$+ (Y_t-K)\cdot N'\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right) + N\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right)$$

$$= N\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right)$$

$$\text{gamma} = \frac{\partial C}{\partial Y^2} = \frac{\partial\, \text{delta}}{\partial Y}$$

$$= N'(d_1)\frac{\partial d_1}{\partial Y} = \frac{N'(Y_t-K)}{\sigma\sqrt{T-t}}$$

c) When $Y_t = K$, $M_t = Y_t$, $r=0$

$$C_t = e^{-r(T-t)} \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}\, e^{\frac{-(K-M_t)^2}{2\sigma^2(T-t)}} + e^{-r(T-t)}(M_t-K)\, N\left(\frac{M_t-K}{\sigma\sqrt{T-t}}\right)$$

$$= \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}$$

$$\text{Theta} = \frac{\partial c}{\partial t} = \frac{\frac{1}{2}\sigma}{\sqrt{2\pi}\cdot\sqrt{T-t}} = \frac{\sigma}{2\sqrt{2\pi(T-t)}}$$

$$\text{vega} = \frac{\partial c}{\partial \sigma} = \frac{\sqrt{T-t}}{\sqrt{2\pi}} = \sqrt{\frac{T-t}{2\pi}}$$

Problem 2

Suppose that the bank account B and a non-dividend-paying stock price S have the following dynamics under risk-neutral measure.

$$\mathrm{d}B_t = rB_t\mathrm{d}t \qquad\qquad B_0 = 1$$
$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t \qquad\qquad S_0 = 216$$

Let $T > 0$, and assume that $\exp(-rT) = 0.96$. Let $K = 250$ and assume that a T-expiry K-strike call on S has time-0 price 34, and a T-expiry K-strike binary call on S has time-0 price 0.44.

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(d) The time-0 expectation of S_T.

(e) $\mathbb{P}(S_T > K)$.

(f) A portfolio holds $\{B, S\}$ in quantities that vary continuously in time. It is self-financing and its time-T value is $(K - S_T)^+$. How many units of S does it hold at time 0?

a) For binary call and put

binary call + binary put = bond

$\therefore\ C_0 + P_0 = Z_0$

$P_0 = Z_0 - C_0 = e^{-rT} - C_0 = 0.96 - 0.44 = 0.52$

b) For call and put

$C_0 = P_0 + S_0 - KZ_0$

$$C_0 = P_0 + S_0 - K e^{-rT}$$

$$34 = P_0 + 216 - 250 \times e^{-rT}$$

$$34 = P_0 + 216 - 250 \times 0.96$$

$$P_0 = 58$$

c) $$V_T = \begin{cases} S_T & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K \end{cases}$$

asset-or-nothing call = call + K binary call

$$V_0 = 34 + 250 \times 0.44 = 144$$

d) $$S_0 = e^{-rT} E(S_T)$$

$$E(S_T) = \frac{S_0}{e^{-rT}} = \frac{216}{0.96} = 225$$

e) $$P(S_T > K) = N(d_2)$$

$e^{-r(T-t)} N(d_2)$ = K-strike binary call

$$= \frac{C_0}{e^{-r(T-t)}} = \frac{0.44}{0.96} = \frac{11}{24}$$

f)

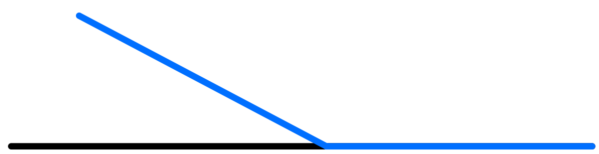
$$C_0 = e^{-r}(F_0)(F_0 N(d_1) - K N(d_2))$$

$$34 = e^{-rT} \cdot [S_0 e^{rT} N(d_1) - K N(d_2)]$$

$$34 = 216 N(d_1) - 240 N(d_2)$$

$$34 = 216 N(d_1) - 240 \cdot \frac{11}{24}$$

$$N(d_1) = \frac{2}{3}$$



since for call and put option

$$C_0 = P_0 + S_0 - KZ_0$$

$$\therefore \frac{\partial C}{\partial S} = \frac{\partial P}{\partial S} + 1$$

delta for call = delta for put + 1

delta for put option = $N(d_1) - 1 = -\frac{1}{3}$

= # of share of S needed to replicate

$\therefore$ $-\frac{1}{3}$ unit of S is hold at time 0