FINM 33000 Midterm

November 8, 2023

Closed book, closed notes, no calculators, no devices

Your Name: Yining (Coco) Ou

Scores

1	2	3	4	5	Total

General Directions

Results from lecture/homework may be used without justification. Unless otherwise directed, assume frictionless markets and no arbitrage, and assume also the following:

The share price (the price of 1 unit) of any stock is always nonnegative.

All options are European-style (so no early exercise). Calls and puts are standard plain vanilla, unless otherwise noted. All strike prices are positive. All assets pay no dividends/coupons. A binary call with strike K pays 1 if the underlying is $\geq K$ at expiration, otherwise it pays 0. A binary put with strike K pays 1 if the underlying is $< K$ at expiration, otherwise it pays 0.

Let T, T_1, T_2, T^*, and K, K_1, K_2, K^* denote positive constants. "Positive" means > 0.

Let P denote the physical probability measure, and let E denote expectation with respect to P.

There exists a bank account with price per unit $B_t = e^{rt}$ where $r > 0$ is constant.

Let $\mathbb{P}$ denote martingale measure (with respect to numeraire B), and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. Let the time-0 filtration be trivial; thus $\mathbb{E}$ and $\mathbb{E}_0$ are the same thing.

Let $\mathbb{R}$ denote the real numbers.

For events F and G, recall that $\mathbb{P}(F|G)$ denotes conditional probability .

Let N denote the standard normal CDF, and N' denote its derivative $N'(x) = (1/\sqrt{2\pi})\exp(-x^2/2)$.

Problem 1

In each part, arbitrage exists.

Find a static arbitrage portfolio, by specifying how many units of each asset to hold. If you do not mention an asset, we will assume that your portfolio holds zero units of that asset. Either type 1 or type 2 arbitrage is acceptable. Do not propose more than one arbitrage portfolio.

The assets listed inside the curly braces { } are the *only* assets available for you to trade.

In all parts of the problem S is the share price of a non-dividend-paying stock.

In every part, the expiry of every option is time $T = 1.0$.

Aside from the conditions on page 1, and any other conditions explicitly stated in each part of the problem, we make no further assumptions about the distribution of the underlying S_T; your arbitrage must be valid regardless of the distribution of S_T.

Assumptions in the introduction (the lines above here) apply to every part of the problem.

Assumptions in individual parts of the problem (a,b,c,d) are specific to that part of the problem, and do not carry over into other parts of the problem.

(a) Asset B is the bank account, with time-0 price $B_0 = 1$, and time-1 price $B_1 = 10/9$.

Asset C is a 11-strike call on S with time-0 price $C_0 = 3.5$.

Asset P is a 11-strike put on S with time-0 price $P_0 = 5.5$.

Assume $S_0 = 8$.

Find an arbitrage, using some or all of the assets $\{B, S, C, P\}$.

(b) Asset B is the bank account, with time-0 price $B_0 = 1$, and time-1 price $B_1 = 10/9$.

Asset C is a 5-strike call on S with time-0 price $C_0 = 8.0$.

Asset D is a 10-strike call on S with time-0 price $D_0 = 7.0$.

Assume $S_0 = 15$.

Find an arbitrage, using some or all of the assets $\{B, S, C, D\}$.

(c) Asset B is the bank account, with time-0 price $B_0 = 1$, and time-1 price $B_1 = 10/9$.

Asset P is a 5-strike put on S with time-0 price $P_0 = 3.0$.

Asset Q is a 10-strike put on S with time-0 price $Q_0 = 4.0$.

Assume $S_0 = 15$.

Find an arbitrage, using some or all of the assets $\{B, S, P, Q\}$.

(d) Asset B is the bank account, with time-0 price $B_0 = 1$, and time-1 price $B_1 = 10/9$.

Asset F is a binary call on S with strike 25 and time-0 price 0.5.

Assume $S_0 = 10$.

Find an arbitrage, using some or all of the assets $\{B, F, S\}$

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~~a) [+1 unit P, +1 unit S, $-\frac{99}{10}$ units B, -1 unit C]~~

$\cancel{V_0 = 5.5 + 8 - 11 \cdot 1 - 3.5 = 13.5 - 11 - 3.5 = -1 < 0}$

$\cancel{\text{when } S_T > K = 11, \quad V_T = S_T - 11 \cdot \frac{10}{9} - (S_T - 11) =}$

$\cancel{V_0 = 5.5 + 8 - \frac{99}{10} \cdot 1 - 3.5 = 10 - 9.9 =}$

$3.5 - 5.5 - 8 + \frac{99}{10}$

$-10 + 9.9 = -0.1$

$< 11 \quad -(11 - S_T) - S_T + \frac{99}{10} \times \frac{10}{9} = 0$

$> 11 \quad (S_T - 11) - S_T + \frac{99}{10} \times \frac{10}{9} = 0$

a) [+1 unit C, -1 unit P, -1 unit S, $+\frac{99}{10}$ unit B]

$$V_0 = 3.5 - 8 - 5.5 + \frac{99}{10} \times 1 = 3.5 - 13.5 + 9.9 = -0.1 < 0$$

when $S_T > K$: $V_T = (S_T - 11) - S_T + \frac{99}{10} \times \frac{10}{9} = -11 + 11 = 0$

when $S_T < K$: $V_T = -(11 - S_T) - S_T + \frac{99}{10} \times \frac{10}{9} = -11 + 11 = 0$

it satisfies the type 2 arbitrage

~~b) [$\frac{1}{2}$ unit D (10-strike call), -1 unit C (5-strike call)]~~

$\cancel{V_0 = \frac{1}{2} \cdot 7.0 - 8 = 3.5 - 8 = -4.5 < 0}$

~~when $S_T < 5$ $V_T =$~~

~~when $5 < S_T < 10$ $V_T = -(S_T - 5)$~~

~~when $S_T > 10$ $V_T = \frac{1}{2}(S_T - 10)$~~

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c. [−1 unit P (5-strike put), $+\frac{1}{2}$ unit Q (10-strike price)]

$$V_0 = -3 + \frac{1}{2}\cdot 4 = -3+2 = -1 < 0$$

when $S_T<5$ $\quad V_T = -(5-S_T) + \frac{1}{2}(10-S_T) = -5+S_T+5-\frac{1}{2}S_T = \frac{1}{2}S_T > 0$

when $5<S_T<10$ $\quad V_T = \frac{1}{2}(10-S_T) > 0$

when $S_T>10$ $\quad V_T = 0$ $\quad$ which satisfies type-2 arbitrage.

b. [+1 unit C (5-strike call), −1 unit S, $+\frac{9}{2}$ unit B]

$$V_0 = 8 - 15 + \frac{9}{2}\times 1 = 8+4.5-15 = -2.5 < 0$$

when $S_T<5$: $\quad V_T = -S_T + \frac{9}{2}\times\frac{10}{9} = 5-S_T > 0$

when $S_T>5$: $\quad V_T = (S_T-5) - S_T + \frac{9}{2}\times\frac{10}{9} = S_T-5-S_T+5 = 0$

which satisfies type-2 arbitrage

d) [+1 unit S, −25 unit F]

$$V_0 = 10 - 25 \times 0.5 = 10 - 12.5 = -2.5 < 0$$

when $S_T<25$: $\quad V_T = S_T > 0$

when $S_T>25$: $\quad V_T = S_T - 25 > 0$

which satisfies type2 arbitrage.

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Problem 2

Assume that discount bonds maturing at T have time-0 price 0.9 per unit.

Assume that T-expiry standard ("plain vanilla") European calls on a random variable S_T are available at the following strikes K and time-0 prices $C_0(K)$. Exactly four basic assets are available: the bond and these three calls, nothing else. The underlying S_T is not available to be traded, and does not have a time-0 price.

As defined in class, these standard European calls pay $(S_T - K)^+$.

K	$C_0(K)$
20.0	6.15
25.0	2.60
30.0	0.80

put + call = Z

A T-expiry K-strike binary or digital *put* on S_T pays at time T either 1 if $S_T < K$, or 0 if $S_T \geqslant K$.

(a) Find upper and lower bounds on the time-0 price of a T-expiry 25-strike binary put on S_T.

(b) Find the time-0 price of a contract (known as a "straddle") which pays $|S_T - 25|$ at time T.

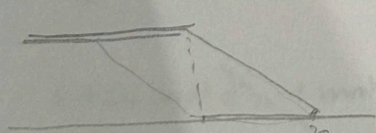
S= 24

(c) Find an upper bound on the time-0 price of a T-expiry 28-strike standard call. (This is a plain vanilla call, not a binary call.)

Do so by using static portfolios to (depending on the question) replicate or superreplicate or subreplicate the derivative contract. Your final answers should be numbers.

In parts (a,c), you do not need to prove that your bounds are tight, but bounds that are not tight will lose some credit. Tight means that they cannot be improved (lowered in the case of an upper bound, raised in the case of a lower bound) without making further assumptions.

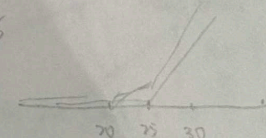
a)



$0.9 - 0.2 \cdot 2.6 + 0.2 \cdot 0.8$

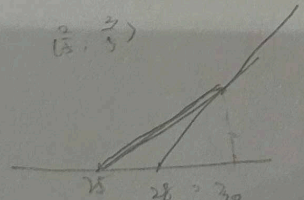
$= 0.9 - 0.52 + 0.16$

$= 1.06 = 0.54$



2.65

$(\frac{2}{5}, \frac{3}{5})$



$(+1Z, -\frac{1}{5}C(25), +\frac{1}{5}C(30))$

$1 - \frac{1}{5}(S_T - 25) + \frac{1}{5}(S_T - 30)$

30 $1 + 5 - 6 = 0$

$0.4 \cdot 2.6 + 0.6 \cdot 0.8$

$= 0.48 + 1.04 = 1.5$ ✓

$\frac{2}{5}(S_T - 25)$

$(+1Z, -\frac{1}{5}C(20), +\frac{1}{5}C(25))$

$1 - \frac{1}{5}(S_T - 20) + \frac{1}{5}(S_T - 25)$

$0.9 - 0.2 \cdot 6.15 + 0.2 \cdot 2.6$

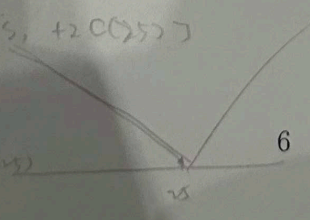
1.23

$0.9 + 0.52 - \cancel{1.23}$

$1.42 - 1.23 = 0.19$

$(25Z, -1S, +2C(25))$

$25 \times 0.9 = 22.5$



$25 - S_T + 2(S_T - 25)$

$S_T - 25$

$25 \times 0.9 - 24 + 2 \cdot 2.6$

$22.5 + 5.2 = 27.7 - 24 = 3.7$

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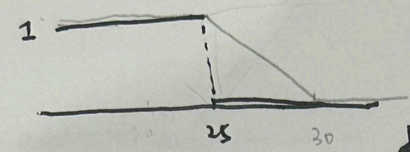
a)

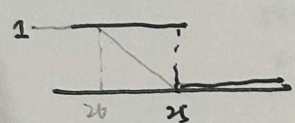


upper bound: [+1 Z, $-\frac{1}{5}$ unit 25-strike call, $+\frac{1}{5}$ unit 30-strike call]

∴ upper bound on time-0 price = $0.9-\frac{1}{5}2.6+\frac{1}{5}0.8$ ~~= 1.7 − 2.6 = −0.9~~

$= 0.9-0.52+0.16 = 0.54$

lower bound



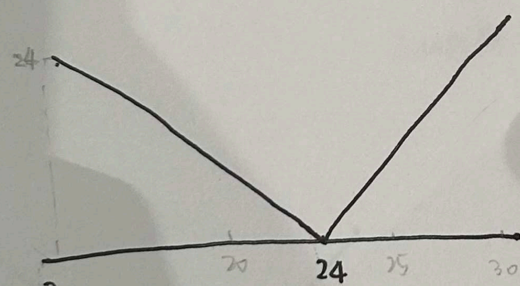
[+1 Z, $-\frac{1}{5}$ unit 20-strike call, $+\frac{1}{5}$ unit 25-strike call]

$0.9-\frac{1}{5}\cdot 6.15+\frac{1}{5}\cdot 2.6$

lower bound on time-0 price = ~~0.9 − 6.15 + 2.6 = 3.5 − 6.15 = −2.65~~

$= 0.9 - 1.23 + 0.52 = 0.19$

b)



let proportion of strike-20 call be x

$20x+25(1-x)=24$

$-5x+25=24$

$x=\frac{1}{5}$

for strike-20 call unit $=\frac{1}{5}\times 2=\frac{2}{5}$

for strike-25 call unit $=(1-\frac{1}{5})\times 2=$

[+24 unit Z, −1 unit S, $+\frac{2}{5}$ unit strike-20 call, $+\frac{8}{5}$ unit strike 25 call]

∴ time-0 price of a contract

$= 24 \times 0.9$

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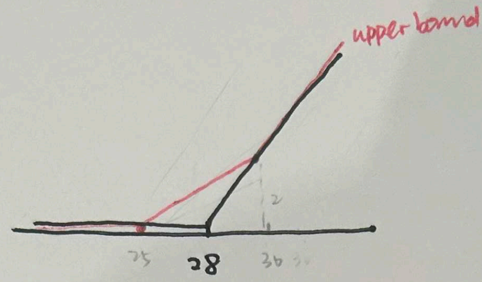
c)



let x be the proportion on strike-25 call

$25x + 30(1-x) = 28$

$30 - 5x = 28$

$5x = 2$

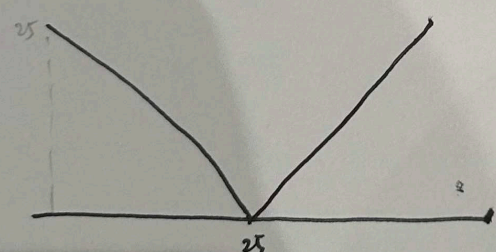
$x = \frac{2}{5}$. proportion on strike-30 call is $\frac{3}{5}$

∴ [$+\frac{2}{5}$ unit strike-25 call, $+\frac{3}{5}$ unit strike-30 call]

∴ upper band $= \frac{2}{5} \times 2.6 + \frac{3}{5} \cdot 0.8$

$= 1.04 + 0.48 = 1.52$

b)



[+25 unit Z, −1 unit S, +2 unit strike-25 call]

time-0 price $= 25 \times 0.9 - 24 + 2 \cdot 2.6$

$= 22.5 - 24 + 5.2 = 3.7$

27.7

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Problem 3

Let S_t, where $t = 0, 1, 2, \ldots, T$, denote the time-t price of a tradeable non-dividend-paying asset.

Let $S_0 = 100$ and let each random increment $S_{t+1} - S_t$ take value $+1$ with physical probability 60%, and value -1 with physical probability 40%, independently of all other increments.

There exists a bond, with constant price 1 at all times.

(a) Find the time-0 value of a $\boxed{\text{put}}$ on S, with strike $K = 90$ and expiry at time $T = 15$.

Do not do a 15-step backwards induction.

Although your answer should be explicit, you may leave it unsimplified. You may leave binomial coefficients (numbers of the form: $\binom{n}{k}$), pronounced "n choose k") unsimplified.

(b) Let τ be the first time that S hits the set consisting of the two price levels $\{96, 108\}$.

You may assume the fact that $\tau < \infty$ with probability 1. (In other words, τ is finite. However, τ is not bounded.)

So S_τ is defined and satisfies $S_\tau = 96$ or $S_\tau = 108$.

Find the risk-neutral probability that $S_\tau = 108$.

Although τ is not bounded, you may assume the fact that the conclusion of the optional stopping theorem is still valid for τ.

$-13+2 = 89$

$-14+1 = 87$

$-15+0 = 85$

a) risk-neutral probability: $0 = +1 \cdot p + (-1)\cdot(1-p) \Rightarrow p = 0.5$

$$S_{15} = \begin{cases} 89 & , 13 \text{ unit } (-1) \text{ move}, 2 \text{ unit } (+1) \text{ move} \\ 87 & , 14 \text{ unit } (-1) \text{ move}, 1 \text{ unit } (+1) \text{ move} \\ 85 & , 15 \text{ unit } (-1) \text{ move}, 0 \text{ unit } (+1) \text{ move} \end{cases}$$

number of

$$\therefore K - S_T = 90 - S_{15} = \begin{cases} 1 & , (-1) \text{ move} = 13 \\ 3 & , (-1) \text{ move} = 14 \\ 5 & , (-1) \text{ move} = 15 \\ 0 & , (-1) \text{ move} = 0, 1, 2, \ldots, 12 \end{cases}$$

$$\therefore S_0 = \text{expected payoff} = 1 \cdot \binom{15}{13} \cdot \left(\tfrac{1}{2}\right)^{15} + 3 \cdot \binom{15}{14} \cdot \left(\tfrac{1}{2}\right)^{15} + 5 \cdot \binom{15}{15} \left(\tfrac{1}{2}\right)^{15} + 0$$

$$= \left(\tfrac{1}{2}\right)^{15} \left[\binom{15}{2} + 3 \cdot \binom{15}{1} + 5 \binom{15}{0}\right]$$

$$= \left(\tfrac{1}{2}\right)^{15} (105 + 15 + 5) = \left(\tfrac{1}{2}\right)^{15} \cdot 125$$

$\frac{15 \times 14}{2}$

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b) because it is a martigale

$$\therefore M_0 = E[M_T]$$

$$100 = 96 \cdot (1-p) + 108 \cdot p$$

$$100 = 96 - 96p + 108p$$

$$12p = 4$$

$$p = \frac{1}{3}$$

the risk-neutral probability that $S_T = 108$ is $\frac{1}{3}$

Problem 4

Emma and Simona play a tennis match, which consists of "sets": a first set, then a second set, and then possibly a third set.

The possible outcomes of each set are that either Emma wins the set or Simona wins the set.

If a player wins both the first set and the second set, then that player wins the match, which ends without playing a third set.

If the players split the first two sets (each player winning one set), then a third set is played, and whoever wins the third set wins the match.

Interest rates are zero; units of the bank account have constant price 1 dollar.

Immediately before nth set (where $n = 1, 2$, and, if the third set is played, 3), you can buy or sell, at a price of 0.4 dollars per unit, arbitrary quantities of "Simona wins set n" contracts, which pay you +1 dollar if Simona wins set n but pay you 0 if Emma wins set n. (Note that the payoff if Simona loses is zero, not a negative payoff.)

Assume that Simona has a physical probability of 50% of winning each game, independently of all other games.

(a) Find M_0, the no-arbitrage time-0 price of a "Simona wins match" contract which pays 100 dollars if Simona wins the match, or 0 dollars if Simona loses the match.

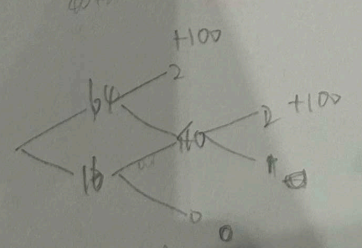
(b) Consider a different contract, C, which at the end of the match pays (the total number of sets won by Simona)×100 dollars. You may assume that C has time-0 price $C_0 = 99.2$; and time-1 price $C_1 = 164$ if Simona wins the first set, or $C_1 = 56$ if Simona loses the first set.

Suppose we want to replicate the C contract using a self-financing trading strategy in only two assets: the bank account B, and the M ("Simona wins match") contract. You are not allowed to use the single-set ("Simona wins set n") contracts.

In order to replicate the C contract, how many units of M contracts should you hold at the beginning (immediately before set 1)?

If the number of units is negative, you must either say "short" or use a negative sign, otherwise we will assume you intend to hold a positive number of units of the M contract.

You do not need to report how many units of the bank account your replicating portfolio holds.



4

$\frac{108}{48} = \frac{27}{12} = \frac{9}{4}$

64×0.4+16×0.6

9.6+256 = 35.2

16×9 = 144

8.8×9

79.2

36+20=56

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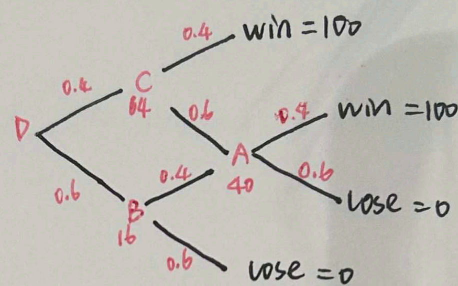
a) risk-neutral probability $0.4 = p \cdot 1 + 0 \cdot (1-p)$

$p = 0.4$

the diagram shows Simona wins or not



$A = 100 \times 0.4 + 0 \cdot 0.6 = 40$

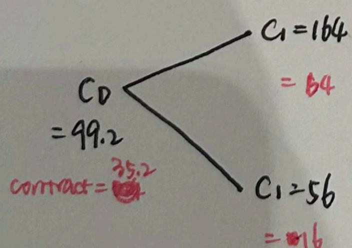
$B = 40 \cdot 0.4 + 0 \cdot 0.6 = 16$

$C = 0.4 \times 100 + 0.6 \times 40 = 40 + 24 = 64$

$D = 0.4 \times 64 + 0.6 \times 16 = 25.6 + 9.6 = 35.2$

$\therefore M_0$ = no-arbitrage time-0 price of "Simona wins match" is \$35.2

b)



to use "Simona wins match n" to replicate C contract.

$\frac{164-56}{64-16} = \frac{9}{4}$ units of M contract

and the portfolio would be $(+\frac{9}{4} M, +20 B)$

$$\frac{\partial f}{\partial \cancel{t}} d\cancel{t} + \frac{\partial f}{\partial x} dx$$

$$+\cancel{\frac{1}{2}\frac{\partial^2 f}{\partial t}(dt)^2} + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x}(dx)^2 + \cancel{\frac{\partial^2 f}{\partial t \partial x} dt\, dx}$$

$(dt)^2=0$

Problem 5

Let W be a Brownian motion. Define X by

$$\mathrm{d}X_t = -6X_t\mathrm{d}t + 4\mathrm{d}W_t, \qquad X_0 = 3$$

For all $t \geq 0$ let

$$Z_t = e^{at}X_t$$

where a is a constant.

(a) Write Z_t explicitly in terms of drift and diffusion components. You may use either differential notation

$$\mathrm{d}Z_t = \_\_\_\mathrm{d}t + \_\_\_\mathrm{d}W_t,$$

or integral notation. Your answers may depend on a and X_t.

(b) For which value of a is Z a martingale?

(c) Find the expectation of X_2.

Hint: One approach would be to first find the expectation of Z_2.

$$\frac{\partial f}{\partial t} = ae^{at}X_t$$

$$\frac{\partial f}{\partial x} = e^{at}$$

$$\frac{\partial^2 f}{\partial x^2} = 0$$

a). $dX_t = -6X_t dt + 4dW_t$

$$dZ_t = d(e^{at}\cdot X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

$$= ae^{at}X_t\,dt + e^{at}(-6X_t dt + 4dW_t)$$

$$= ae^{at}X_t\,dt - e^{at}6X_t\,dt + 4e^{at}dW_t$$

$$= (a-6)e^{at}X_t\,dt + 4e^{at}dW_t$$

b). when Z is a merthgale, $dX_t = \mu dt + 6dW_t$ where μ should be 0

$$\therefore\ (a-6)e^{at}X_t = 0$$

$$a-6=0$$

$$a=6$$

c). $Z_t = Z_0 + \int_0^t (a-6)e^{as}X_s\,ds + \int_0^t 4e^{as}dW_s$

when $a=6$

$Z_t = Z_0 + \int_0^t 4e^{as}dW_s$

because now Z is a marthgale

$E(Z_2) = Z_0 = e^{a\cdot 0}X_0 = X_0 = 3$