**FINM 34000, Autumn 2023**
**Lecture 5**

**Reading**: Notes, Section 5.1.

**Exercise 1** *Suppose $S_n = X_1 + \cdots + X_n$ is simple symmetric random walk in one dimension. Let $\mathcal{F}_n$ denote the information in $X_1, X_2, \ldots, X_n$. For each of the following say if the process is a martingale, submartingale, or supermartingale (it can be more than one and it might be none of these) with respect to $\mathcal{F}_n$. Give reasons (citing a fact from lecture or notes is a sufficient reason).*

1. $M_n = S_n$
2. $M_n = S_n^2$
3. $M_n = S_n^3$
4. $M_n = 2^{S_n}$.
5. $M_n = S_n/n$
6. $M_n = S_{n+1}\, S_n$.
7. $$M_n = 0 + X_1\, X_2 + X_2\, X_3 + \cdots + X_{n-1}\, X_n = \sum_{j=1}^{n} X_{j-1}\, X_j.$$

**Exercise 2** *This exercise concerns Polya's urn and has a computing/simulation component. Let us start with one red and one green ball as in the lecture and let $M_n$ be the fraction of red balls at the $n$th stage.*

1. *Show that the distribution of $M_n$ is uniform on the set*
$$\left\{\frac{1}{n+2}, \frac{2}{n+2}, \ldots, \frac{n+1}{n+2}\right\}.$$
*(Use mathematical induction, that is, note that it is obviously true for $n = 0$ and show that if it is true for $n$ then it is true for $n + 1$.)*
2. *Write a short program that will simulate this urn. Each time you run the program note the fraction of red balls after 2000 draws and after 4000 draws. Compare the two fractions. Then, repeat this thirty times.*

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7. $$M_n = 0 + X_1 X_2 + X_2 X_3 + \cdots + X_{n-1} X_n = \sum X_{j-1} X_j.$$

1. $E[X_j] = \frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = 0$

$\therefore E[M_n] = E[S_n] = E[X_1 + X_2 + \cdots + X_n] = n \cdot E[X_0] = 0$

$\therefore E[|M_n|] = 0 < \infty$

$E[M_{n+1} | F_n] = E[S_{n+1} | F]$

$= E[S_n + X_{n+1} | F_n]$

$= E[X_{n+1}] + E[S_n | F_n]$

$= 0 + S_n = S_n = M_n$

$\therefore$ it is martingale

2. $Var(X_j) = E(X_j^2) - (E(X_j))^2 = \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot (-1)^2 - 0 = 1$

$\therefore Var(S_n) = n Var(X_0) = n$

$E[M_n] = E[S_n^2] = Var[S_n] + (E[S_n])^2 = n + 0^2 = n$

$E[|M_n|] = n < \infty$

$$E[M_{n+1}|F_n] = E[(S_n + X_{n+1})^2 | F_n]$$

$$= E[S_n^2 | F_n] + 2E[S_n X_{n+1} | F_n] + E[X_{n+1}^2 | F_n]$$

$$= S_n^2 + 2S_n E(X_{n+1}) + E(X_{n+1}^2)$$

$$= S_n^2 + 2 \cdot S_0 \cdot 0 + 1 = S_n^2 + 1 \geq M_n$$

$\therefore$ $\underline{E[M_{n+1}|F_n] \geq M_n}$

$\therefore$ it is a submartingale

3. $E[X_j^3] = \frac{1}{2}(1)^3 + \frac{1}{2}(-1)^3 = 0$

$E[M_n] = E[S_n^3] = E[\underbrace{X_1^3 + \cdots + X_n^3}_{=0} + \text{rest}] = 0 < \infty$

$\therefore$ $\underline{E[|M_n|] = 0 < \infty}$ rest is 0 because term $E(X_j) = 0$

$E[M_{n+1}|F_n] = E[(S_n + X_{n+1})^3 | F_n]$

$= E[S_n^3 | F_n] + 3S_n E(X_{n+1}^2 | F_n] + 3S_n^2 E(X_{n+1} | F_n) + E(X_{n+1}^3 | F_n]$

$= S_n^3 + 3S_n \cdot 1 + 3S_n^2 \cdot 0 + 0 = S_n^3 + 3S_n$

It is $\begin{cases} \text{submartigale} & \text{if } S_m > 0 \\ \text{martingle} & \text{if } S_m = 0 \\ \text{supermartingale} & \text{if } S_m < 0 \end{cases}$

or since $S_n$ could be negative, positive, or zero, it is not any of these.

4. $E(2^{X_j}) = 2^1 \cdot \frac{1}{2} + 2^{-1} \cdot \frac{1}{2} = \frac{5}{4}$

$$E[M_n] = E[2^{S_n}] = E[2^{(X_1 + \cdots + X_n)}]$$
$$= E[2^{X_1} 2^{X_2} \cdots 2^{X_n}]$$
$$= E(2^{X_1}) E(2^{X_2}) \cdots E(2^{X_n})$$
$$= \left(\frac{5}{4}\right)^n < \infty$$

$$E[M_{n+1} | F_n] = E[2^{S_n + X_{n+1}} | F_n]$$
$$= E[2^{S_n} | F_n] \cdot E[2^{X_{n+1}} | F_n]$$
$$= \frac{5}{4} E[2^{S_n} | F_n] = \frac{5}{4} M_n \geq M_n$$

$\therefore$ it is submartingale

5. $E[M_n] = E[S_n / n] = 0$ $\quad \therefore E[|M_n|] = 0 < \infty$

$$E[M_{n+1} | F_n] = E\left[\frac{S_n + X_{n+1}}{n+1} \middle| F_n\right]$$
$$= E\left[\frac{S_n}{n+1} \middle| F_n\right] + E\left[\frac{X_{n+1}}{n} \middle| F_n\right]$$
$$= \frac{S_n}{n+1} + 0 = \frac{S_n}{n+1} \leq \frac{S_n}{n} = M_n$$

it is supermartingale

6. $\because$ $S_{n+1}$ is not $F_n$-measurable

$\therefore$ $M_n = S_{n+1} S_n$ is not any of these

7. $E[M_n] = E[\sum_{j=1}^{n} X_{j-1}X_j]$

$= E[X_1X_2 + \cdots + X_{n-1}X_n]$

$= E[X_1]E[X_2] + \cdots + E[X_{n-1}]E[X_n] = 0$

$\therefore \underline{E[|M_n|] = 0 < \infty}$

$E[M_{n+1}|F_n] = E[\sum_{1}^{n} X_{j-1}X_j + X_nX_{n+1} | F_m]$

$= E[\sum_{1}^{n} X_{j-1}X_j | F_n] + E[X_nX_{n+1} | F_n]$

$= \sum_{1}^{n} X_{j-1}X_j + X_nE[X_{n+1}] = \sum_{1}^{n} X_{j-1}X_j = M_n$

$\therefore$ M is martingale

**Exercise 2** *This exercise concerns Polya's urn and has a computing/simulation component. Let us start with one red and one green ball as in the lecture and let $M_n$ be the fraction of red balls at the nth stage.*

1. *Show that the distribution of $M_n$ is uniform on the set*

$$\left\{ \frac{1}{n+2}, \frac{2}{n+2}, \ldots, \frac{n+1}{n+2} \right\}.$$

*(Use mathematical induction, that is, note that it is obviously true for $n = 0$ and show that if it is true for n then it is true for $n + 1$.)*

2. *Write a short program that will simulate this urn. Each time you run the program note the fraction of red balls after 2000 draws and after 4000 draws. Compare the two fractions. Then, repeat this thirty times.*

1. $M_n = \frac{R_n}{G_n + R_n} = \frac{R_n}{n+2}$

when $n=0$

$M_0$ has 1 green ball and 1 red ball

$\therefore$ $M_0$ is uniform on the set $\{\frac{1}{n+2}\} = \frac{1}{2}$

assume it is true for $n=k$ that

$M_n$ is uniform on the set

$\{\frac{1}{n+2}, \frac{2}{n+2}, \ldots\ldots, \frac{n+1}{n+2}\}$ with the probability $\frac{1}{n+1}$

when $n=k+1$

$$P(M_{n+1} = \frac{k+1}{n+3}) = \frac{1}{n+1} \times \frac{k}{n+2} + \frac{1}{n+1} \cdot \frac{(n+2)-(k+1)}{n+2}$$

$$= \frac{k+n-k+1}{(n+1)(n+2)} = \frac{1}{n+2}$$

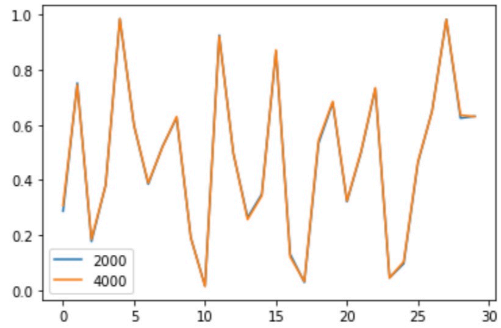
which means for each k, the probability is $\frac{1}{(n+1)+1} = \frac{1}{n+2}$

thus the statement is true for $n=k+1$ when $n=k$ is true

thus we can prove that $M_n$ follows uniform distribution

k+1.

2. Two fractions converges for large n, and two results are close to 0.5




```python
import random
import matplotlib.pyplot as plt
```

```python
result_2000 = []
result_4000 = []
for j in range(30):
    R = 1
    G = 1
    for i in range(4050):
        values = [0, 1]
        probabilities = [R/(R+G), G/(R+G)]  # 30% chance of 0, 70% chance of 1

        random_choice = random.choices(values, probabilities)[0]
        if random_choice == 0:
            R += 1
        else:
            G += 1
        M = R / (i + 2)

        if i == 1999:
            #print("For 2000 draws, the fraction is " + str(M))
            result_2000.append(M)
        if i == 3999:
            #print("For 4000 draws, the fraction is " + str(M))
            result_4000.append(M)

plt.plot(result_2000)
plt.plot(result_4000)
plt.legend([2000,4000])
plt.show
```