FINM 34000, Autumn 2023
Lecture 7

Reading: Notes, rest of Section 5

Exercise 1 *Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with*

$$\mathbb{P}\{X_j = 2\} = \frac{1}{3}, \quad \mathbb{P}\{X_j = \frac{1}{2}\} = \frac{2}{3}.$$

Let $M_0 = 1$ and for $n \geq 1$, $M_n = X_1 X_2 \cdots X_n$.

1. *Show that M_n is a martingale.*
   Note that** $\mathbb{E}[X_j] = 2 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = 1$**.

$$E[M_{n+1} \mid \mathcal{F}_n] = E[M_n\, X_{n+1} \mid \mathcal{F}_n] = M_n\, E[X_{n+1} \mid \mathcal{F}_n] = M_n\, \mathbb{E}[X_{n+1}] = M_n.$$

2. *Explain why M_n satisfies the conditions of the martingale convergence theorem.*
   Since** $M_n \geq 0$**,

$$\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = 1.$$

3. *Let $M_\infty = \lim_{n\to\infty} M_n$. Explain why $M_\infty = 0$. (Hint: there are at least two ways to show this. One is to consider $\log M_n$ and use the law of large numbers. Another is to note that with probability one M_{n+1}/M_n does not converge.)*
   We will discuss both ways.

   For the first note that

$$\mathbb{E}[\log X_j] = \frac{1}{3}\,\log 2 + \frac{2}{3}\,\log(1/2) < 0.$$

If we write** $Y_j = \log X_j$**, then the strong law of large numbers tells us that

$$\frac{Y_1 + \cdots + Y_n}{n} \to \mathbb{E}[Y_j] < 0,$$

which shows that** $Y_1 + \cdots + Y_n \to -\infty$**. Hence

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \exp\{Y_1 + \cdots + Y_n\} = 0.$$

For the other method, we know that** M_n **is converging. If it were converging to a strictly positive number, that would imply (just by the definition of a limit) that

$$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = 1.$$

However, we know that** M_{n+1}/M_n **always takes the value** 2 **or** $1/2$ **which is a contradiction.

4. *Use the optional sampling theorem to determine the probability that M_n ever attains a value as large as 64.*

   For any positive integer r, let T_r be the first time that $M_n = 64$ or $M_n = 2^{-r}$. Then $M_{n\wedge T_r}$ is a bounded martingale and we can use the optional sampling theorem,
$$1 = \mathbb{E}[M_{n\wedge T_r}] = 64\,\mathbb{P}\{T_r = 64\} - 2^{-r}\,\mathbb{P}\{T_r = 2^{-r}\}.$$
   Using $\mathbb{P}\{T_r = 2^{-r}\} = 1 - \mathbb{P}\{T_r = 64\}$, we get
$$\mathbb{P}\{T_r = 64\} = \frac{1+2^{-r}}{64+2^{-r}}.$$
   The probability that one ever reaches 64 can be given as
$$\lim_{r\to\infty} \mathbb{P}\{T_r = 64\} = \lim_{r\to\infty} \frac{1+2^{-r}}{64+2^{-r}} = \frac{1}{64}.$$

5. *Does there exist a $C < \infty$ such that $\mathbb{E}[M_n^2] \le C$ for all n?*

   If this were true we would have $\mathbb{E}[M_0] = \mathbb{E}[M_\infty]$ but we do not have this, so it better not be true!.

   We can check it directly. By independence
$$\mathbb{E}[M_n^2] = \mathbb{E}[X_1^2]\,\mathbb{E}[X_2^2]\cdots\mathbb{E}[X_n^2].$$
   But
$$\mathbb{E}[X_j^2] = 2^2\cdot\frac{1}{3} + 2^{-2}\cdot 23 = \frac{3}{2} > 1$$
   and hence $\mathbb{E}[M_n^2] = (3/2)^n$ and no such C exists.

Exercise 2 *Consider the martingale betting strategy as discussed in Section 5. Let W_n be the "winnings" at time n, which for positive n equals either 1 or $1-2^n$.*

1. *Is W_n a square integrable martingale?*

   In lectures we showed it was a martingale. Since W_n is a bounded random variable (note that $|W_n| \le 2^n$), we have $\mathbb{E}[W_n^2] < \infty$ and hence it is a square integrable martingale.

2. *If $\Delta_n = W_n - W_{n-1}$ what is $\mathbb{E}[\Delta_n^2]$?*

   There are three possible values for $W_n - W_{n-1}$. It equals 0 if there has been a heads on the first $n-1$ flips. Otherwise it is $\pm 2^{n-1}$. Therefore
$$\mathbb{E}[\Delta_n^2] = 2^{-(n-1)}\,[2^{(n-1)}]^2 = 2^{n-1}.$$

3. *What is $\mathbb{E}[W_n^2]$?*
There are two ways to do this. The easiest is to note that

$$\mathbb{P}\{W_n = 1\} = 1 - 2^{-n}, \qquad \mathbb{P}\{W_n = 1 - 2^n\} = 2^{-n}.$$

$$\mathbb{E}[W_n^2] = 1^2\,(1 - 2^{-n}) + (1 - 2^n)^2\,2^{-n} = 2^n - 1$$

The other way is to use the "orthogonality of martingale increments" to see that

$$\mathbb{E}[W_n^2] = \sum_{j=1}^{n} \mathbb{E}[\Delta_j^2] = \sum_{j=1}^{n} 2^{j-1} = 2^n - 1.$$

4. *What is $E(\Delta_n^2 \mid \mathcal{F}_{n-1})$?*

This is just checking the definition. Using part 2 we see that

$$E(\Delta_n^2 \mid \mathcal{F}_{n-1}) = \begin{cases} 0 & \textit{\textbf{if}}\ W_{n-1} = 1 \\ 2^{2(n-1)} & \textit{\textbf{if}}\ W_{n-1} = 1 - 2^n \end{cases}$$

Since W_{n-1} is measurable with respect to $\mathcal{F}_{n-1}$, the right-hand side is also measurable with respect to $\mathcal{F}_{n-1}$.

Exercise 3 *Here are some statements about martingales. Say whether they are always true. If always true give reason (citing a fact from the lecture or notes is fine). If it is not always true give an example to show this. Let $M_n, n = 0, 1, 2, \ldots$ be a martingale with respect to $\{\mathcal{F}_n\}$ with $M_0 = 1$.*

1. *For all positive integers n, $\mathbb{E}[M_n] = 1$.*
Yes, this is a fact from class.

2. *With probability one, the limit*

$$M_\infty := \lim_{n\to\infty} M_n \tag{1}$$

exists and is finite.
No, for example, let M_n be simple random walk starting at 1.

3. *Suppose the limit M_∞ exists as in (1) and is finite. Then $\mathbb{E}[M_\infty] = 1$.*
No. There are many examples. One is the to let $M_n = 1 - W_n$ where W_n is the martingale betting strategy.

4. *Suppose we assume know that with probability one $M_n \geq 0$ for all n? Does this imply that the limit in (1) exists with probability one?*
Yes, we discussed this in class. In this case, $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \mathbb{E}[M_0] = 1$.

5. *If we assume that $M_n \geq 0$ for all n does the answer to part 3 change?*
No, since the example we gave has $M_n \geq 0$ for all n.