FINM 34000, Autumn 2023
Lecture 1

Reading: Notes, Section 2.

Exercise 1 *Suppose (X,Y) are discrete random variables with joint probabilities*

$$\begin{array}{cc} X\backslash Y & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} .1 & .1 & 0 & .2 \\ .1 & .05 & .05 & 0 \\ .1 & .1 & .1 & .1 \end{pmatrix} \end{array}$$

For example, $\mathbb{P}\{X=2,Y=3\} = .05$.

1. *Find the marginal distributions for X and Y.*
   The marginal for X is obtained by adding up the rows:

$$\mathbb{P}\{X=1\} = .4, \quad \mathbb{P}\{X=2\} = .2 \quad \mathbb{P}\{X=3\} = .4$$

   and the marginal for Y by adding the columns:

$$\mathbb{P}\{Y=1\} = .3, \quad \mathbb{P}\{Y=2\} = .25, \quad \mathbb{P}\{Y=3\} = .15, \quad \mathbb{P}\{Y=4\} = .3.$$

2. *Find $\mathbb{E}[X], \mathbb{E}[Y], E(X \mid Y), E(Y \mid X)$ and use these to check directly that*

$$\mathbb{E}[X] = \mathbb{E}[E(X \mid Y)], \quad \mathbb{E}[Y] = \mathbb{E}[E(Y \mid X)].$$

$$\mathbb{E}[X] = 1 \cdot .4 + 2 \cdot .2 + 3 \cdot .4 = 2$$
$$\mathbb{E}[Y] = 1 \cdot .3 + 2 \cdot .25 + 3 \cdot .15 + 4 \cdot .3 = 2.45$$

   $E(X|Y)$ ***takes on four values depending on the value of Y.***

$$E(X|Y)(1) = \frac{1 \cdot .1 + 2 \cdot .1 + 3 \cdot .1}{.3} = 2$$

$$E(X|Y)(2) = \frac{1 \cdot .1 + 2 \cdot .05 + 3 \cdot .1}{.25} = 2$$

$$E(X \mid Y)(3) = \frac{1 \cdot 0 + 2 \cdot .05 + 3 \cdot .1}{.15} = \frac{8}{3}$$

$$E(X \mid Y)(4) = \frac{1 \cdot .2 + 3 \cdot .1}{.3} = \frac{5}{3}.$$

$$\mathbb{E}\left[E(X \mid Y)\right] = 2 \cdot .3 + 2 \cdot .25 + \frac{8}{3} \cdot .15 + \frac{5}{3} \cdot .3 = 2.$$

$E(Y \mid X)$ ***takes on three values***

$$E(Y \mid X)(1) = \frac{1 \cdot .1 + 2 \cdot .1 + 3 \cdot 0 + 4 \cdot .2}{.4} = \frac{11}{4}$$

$$E(Y \mid X)(2) = \frac{1 \cdot .1 + 2 \cdot .05 + 3 \cdot .05}{.2} = \frac{7}{4}$$

$$E(Y \mid X)(3) = \frac{1 \cdot .1 + 2 \cdot .1 + 3 \cdot .1 + 4 \cdot .1}{.4} = \frac{5}{2}$$

$$\mathbb{E}[E(Y \mid X)] = .4 \cdot \frac{11}{4} + .2 \cdot \frac{7}{4} + .4 \cdot \frac{5}{2} = 2.45$$

3. *Let A be the event $A = \{Y \text{ is odd}\}$. Which of the following facts hold?*

$$\mathbb{E}[E(X|Y)\, 1_A] = \mathbb{E}[X\, 1_A], \qquad \mathbb{E}[E(Y|X)\, 1_A] = \mathbb{E}[Y\, 1_A].$$

The event A is Y-measurable. Therefore, the first equality holds by definition of conditional expectation. The second does not.

Exercise 2 *Suppose we roll two dice, a red and a green one, and let X be the value on the red die and Y the value on the green die. Let $Z = XY$.*

1. *Find $E[(2X + Y)^2 \mid X]$.*

$$\begin{aligned} &= E[4X^2 \mid X] + E[4XY \mid X] + E[Y^2 \mid X] \\ &= 4X^2 + 4X\, E[Y \mid X] + \mathbb{E}[Y^2] \\ &= 4X^2 + 4X\, \mathbb{E}[Y] + \mathbb{E}[Y^2] \\ &= 4X^2 + 14\, X + \frac{91}{6}. \end{aligned}$$

2. *Find $E[(2X + Y)^2 \mid X, Z]$.*
Note that $Y = Z/X$ and hence Y is measurable with respect to the information in X, Z. So is X and hence

$$E[(2X + Y)^2 \mid X, Z] = (2X + Y)^2.$$

3. *Let $W = E[Z \mid X]$. What are the possible values for W? Give the distribution of W.*
If $X = j$, then $E[Z \mid X] = j\, \mathbb{E}[Y] = (7/2)\, j$. Therefore, there are six possible values for $E[Z \mid X]$,

$$\frac{7}{2}, \quad 2 \cdot \frac{7}{2}, \quad 3 \cdot \frac{7}{2}, \quad 4 \cdot \frac{7}{2}, \quad 5 \cdot \frac{7}{2}, \quad 6 \cdot \frac{7}{2},$$

and each has probability $1/6$.

Exercise 3 *Suppose $X_1, X_2, \ldots$ are independent random variables with*

$$\mathbb{P}\{X_j = 3\} = 1 - \mathbb{P}\{X_j = -1\} = \frac{1}{4}.$$

Let $S_n = X_1 + \cdots + X_n$ and let $\mathcal{F}_n$ denote the information in $X_1, \ldots, X_n$.

1. *Find $\mathbb{E}[X_1], \mathbb{E}[X_1^2], \mathbb{E}[X_1^3]$.*

$$\mathbb{E}[X_1] = 3 \cdot \frac{1}{4} - 1 \cdot \frac{3}{4} = 0, \quad \mathbb{E}[X_1^2] = 3^2 \cdot \frac{1}{4} + (-1)^2 \cdot \frac{3}{4} = 3,$$
$$\mathbb{E}[X_1^3] = 3^3 \cdot \frac{1}{4} + (-1)^3 \cdot \frac{3}{4} = 6.$$

2. *Find $\mathbb{E}[S_n], \mathbb{E}[S_n^2], \mathbb{E}[S_n^3]$.*

$$\mathbb{E}[S_n] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = 0.$$

Since $\mathbb{E}[S_n] = 0$***,***

$$\mathbb{E}[S_n^2] = \mathrm{Var}[S_n] = \mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_n] = \mathbb{E}[X_1^2] + \cdots + \mathbb{E}[X_n^2] = 3n.$$

To compute $\mathbb{E}[S_n^3]$ ***we expand the cube in***

$$\mathbb{E}\left[(X_1 + \cdots + X_n)^3\right].$$

There are n ***terms of the form*** $\mathbb{E}[X_j^3]$ ***each of which contributes*** 6***. The other terms are of the form*** $\mathbb{E}[X_i X_j^2]$ ***or*** $\mathbb{E}[X_i X_j X_k]$ ***where*** i, j, k ***are distinct. But***

$$\mathbb{E}[X_i X_j^2] = \mathbb{E}[X_i]\,\mathbb{E}[X_j^2] = 0, \quad \mathbb{E}[X_1 X_j X_k] = \mathbb{E}[X_o]\,\mathbb{E}[X_j]\,\mathbb{E}[X_k] = 0.$$

Therefore,

$$\mathbb{E}[S_n^3] = 6n.$$

3. *If $m < n$, find*

$$E[S_n \mid \mathcal{F}_m], \quad E[S_n^2 \mid \mathcal{F}_m].$$

These are exactly of the form of the first two examples in Wednesday's lecture.

$$E[S_n \mid \mathcal{F}_m] = S_m + (n-m)\,\mu = S_m, \qquad E[S_n^2 \mid \mathcal{F}_m] = S_m^2 + (n-m)\,\sigma^2 = S_m^2 + 3\,(n-m).$$

4. *If $m < n$, find $E[X_m \mid S_n]$.*
This is like the third example in Wednesday's lecture,

$$E[X_m \mid S_n] = \frac{S_n}{n}.$$