FINM 34000, Autumn 2023
Lecture 6

Reading: Notes, Section 5.2, 5.3.

Exercise 1 *Suppose $S_n = X_1 + \cdots + X_n$ is simple symmetric random walk in one dimension. Let $\mathcal{F}_n$ denote the information in $X_1, X_2, \ldots, X_n$. State which of the following are stopping times for the random walk. Give reasons.*

1. *T is the first time n such that $S_n < 0$.*
   Yes, this is an example in class.

2. *T is the first time n that*
$$\frac{S_n}{n} > S_1.$$
   Yes, you need only see $S_1, \ldots, S_n$ to determine if $T = n$.

3. *T is the first time n that*
$$S_{n+1} > S_n.$$
   No, since you must see S_{n+1} to determine if $T = n$.

4. *Let τ be the first time m that $S_m \geq 4$ and let T be the first time n after τ that $S_n \leq -5$.*
   Yes, you need only see $S_1, \ldots, S_n$ to determine if $T = n$.

Exercise 2 *In this exercise, we consider simple, asymmetric independent random variables with*
$$\mathbb{P}\{X_j = 1\} = 1 - \mathbb{P}\{X_j = -1\} = q, \qquad 0 < q < \frac{1}{2}.$$
Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_n$ denote the information contained in $X_1, \ldots, X_n$.

1. *Which of these is S_n: martingale, submartingale, supermartingale (more than one answer is possible)?*
   We showed in class that
$$E[S_{n+1} \mid \mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1}] = S_n + 2q - 1.$$
   Since $2q - 1 < 0$, this is a supermartingale.

2. *For which values of r is $M_n = S_n + rn$ a martingale?*
$$E[M_{n+1} \mid \mathcal{F}_n] = E[S_{n+1} \mid \mathcal{F}_n] + r(n+1) = S_n + (2q-1) + r(n+1) = M_n + [2q - 1 + r].$$
   It is a martingale for $r = 1 - 2q$.

3. *Let $\theta = (1-q)/q$ and let*

$$M_n = \theta^{S_n}.$$

Show that M_n is a martingale.

$$\begin{aligned} E[M_{n+1} \mid \mathcal{F}_n] &= E[\theta^{S_n + X_{n+1}} \mid \mathcal{F}_n] \\ &= E[\theta^{S_n}\, \theta^{X_{n+1}} \mid \mathcal{F}_n] \\ &= \theta^{S_n}\, E[\theta^{X_{n+1}} \mid \mathcal{F}_n] \\ &= M_n\, \mathbb{E}[\theta^{X_{n+1}}]. \end{aligned}$$

Note that

$$\mathbb{E}[\theta^{X_{n+1}}] = \frac{1-q}{q} \cdot q + \frac{1}{\frac{1-q}{q}} \cdot (1-q) = 1.$$

Hence this is a martingale.

4. *Let a, b be positive integers, and*

$$T_{a,b} = \min\{j : S_j = b \ or \ S_j = -a\}.$$

Use the optional sampling theorem to determine

$$\mathbb{P}\left\{S_{T_{a,b}} = b\right\}.$$

We will write $T = T_{a,b}$ and $p = \mathbb{P}\left\{S_{T_{a,b}} = b\right\}$. Note that $M_{n \wedge T}$ is a martingale that is bounded (values lie between θ^{-a} and θ^b). Hence the conditions of the optimal sampling theorem hold and we get

$$\mathbb{E}[M_T] = \mathbb{E}[M_0] = 1.$$

But,

$$\mathbb{E}[M_T] = p\,\theta^b + (1-p)\,\theta^{-a}.$$

Therefore

$$p = \frac{1-\theta^{-a}}{\theta^b - \theta^{-a}} = \frac{\theta^a - 1}{\theta^{a+b} - 1}.$$

5. *Let $T_b = T_{\infty,b}$. Find*

$$\mathbb{P}\{T_b < \infty\}.$$

If we think about it, the probability that we never reach b is the limit as a goes to infinity of the probability to reach b before reaching $-a$.

$$\mathbb{P}\{T_b = \infty\} = \lim_{a \to \infty} \mathbb{P}\{S_{T_{a,b}} = b\} = \lim_{a \to \infty} \frac{\theta^a - 1}{\theta^{a+b} - 1} = \theta^{-b} = \left(\frac{q}{1-q}\right)^b.$$

Exercise 3 *Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with*

$$\mathbb{P}\{X_j = 1\} = q, \quad \mathbb{P}\{X_j = -1\} = 1 - q.$$

Let $S_0 = 0$ and for $n \geq 1$, $S_n = X_1 + X_2 + \cdots + X_n$. Let $Y_n = e^{S_n}$.

1. *For which value of q is Y_n a martingale?*

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[e^{S_{n+1}} \mid \mathcal{F}_n] = \mathbb{E}[e^{S_n}\, e^{X_{n+1}} \mid \mathcal{F}_n] = e^{S_n}\, \mathbb{E}[e^{X_{n+1}} \mid \mathcal{F}_n] = e^{S_n}\, \mathbb{E}[e^{X_{n+1}}].$$

Hence we need $\mathbb{E}[e^{X_{n+1}}] = 1$, that is,

$$1 = q\, e + (1 - q)\, \frac{1}{e}.$$

Solving for q we get

$$q = \frac{1 - \frac{1}{e}}{e - \frac{1}{e}} = \frac{e - 1}{e^2 - 1} = \frac{1}{e + 1}.$$

2. *For the remaining parts of this exercise assume q takes the value from part 1. Use the optional sampling theorem to determine the probability that Y_n ever attains a value greater than 100.*
The smallest integer k with $e^k \geq 100$ is $k = 5$ and hence this is the same thing as asking if S_n ever obtains the value 5. This is the same as the final part of the last problem with $q = 1/(e+1)$, $\theta = e$, and $b = 5$. The answer is e^{-5}.

3. *Does there exist a $C < \infty$ such that $\mathbb{E}[Y_n^2] \leq C$ for all n?*

$$\mathbb{E}[Y_n^2] = \mathbb{E}[e^{2(X_1 + \cdots + X_n)}] = \mathbb{E}[e^{2X_1}]\, \mathbb{E}[e^{2X_2}] \cdots \mathbb{E}[e^{2X_n}].$$

Also,

$$\mathbb{E}[e^{2X_1}] = \frac{1}{e+1}\, e^2 + \frac{e}{e+1}\, e^{-2} = \frac{e^2 + e^{-1}}{e + 1} > 1,$$

Therefore,

$$\mathbb{E}[Y_n^2] = \left[\frac{e^2 + e^{-1}}{e + 1}\right]^n$$

and this goes to infinity as $n \to \infty$. Therefore no such C exists.