FINM 34000, Autumn 2023
Lecture 1

Reading: Notes, Section 2.

Exercise 1 *Suppose (X, Y) are discrete random variables with joint probabilities*

$$\begin{array}{cc} X\backslash Y & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} .1 & .1 & 0 & .2 \\ .1 & .05 & .05 & 0 \\ .1 & .1 & .1 & .1 \end{pmatrix} \end{array}$$

For example, $\mathbb{P}\{X = 2, Y = 3\} = .05$.

1. *Find the marginal distributions for X and Y.*
2. *Find $\mathbb{E}[X], \mathbb{E}[Y], E(X \mid Y), E(Y \mid X)$ and use these to check directly that*
$$\mathbb{E}[X] = \mathbb{E}[E(X \mid Y)], \quad \mathbb{E}[Y] = \mathbb{E}[E(Y \mid X)].$$
3. *Let A be the event $A = \{Y \text{ is odd}\}$. Which of the following facts hold?*
$$\mathbb{E}[E(X|Y)\, 1_A] = \mathbb{E}[X\, 1_A], \qquad \mathbb{E}[E(Y|X)\, 1_A] = \mathbb{E}[Y\, 1_A].$$

Exercise 2 *Suppose we roll two dice, a red and a green one, and let X be the value on the red die and Y the value on the green die. Let $Z = XY$.*

1. *Find $E[(2X + Y)^2 \mid X]$.*
2. *Find $E[(2X + Y)^2 \mid X, Z]$.*
3. *Let $W = E[Z \mid X]$. What are the possible values for W? Give the distribution of W.*

Exercise 3 *Suppose $X_1, X_2, \ldots$ are independent random variables with*

$$\mathbb{P}\{X_j = 3\} = 1 - \mathbb{P}\{X_j = -1\} = \frac{1}{4}.$$

Let $S_n = X_1 + \cdots + X_n$ and let $\mathcal{F}_n$ denote the information in $X_1, \ldots, X_n$.

1. *Find $\mathbb{E}[X_1], \mathbb{E}[X_1^2], \mathbb{E}[X_1^3]$.*
2. *Find $\mathbb{E}[S_n], \mathbb{E}[S_n^2], \mathbb{E}[S_n^3]$.*
3. *If $m < n$, find*
$$E[S_n \mid \mathcal{F}_m], \quad E[S_n^2 \mid \mathcal{F}_m].$$
4. *If $m < n$, find $E[X_m \mid S_n]$.*

Exercise 1 *Suppose (X, Y) are discrete random variables with joint probabilities*

$$
\begin{array}{c c}
X \backslash Y & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} .1 & .1 & 0 & .2 \\ .1 & .05 & .05 & 0 \\ .1 & .1 & .1 & .1 \end{pmatrix}
\end{array}
$$

For example, $\mathbb{P}\{X = 2, Y = 3\} = .05$.

1. *Find the marginal distributions for X and Y.*

2. *Find $\mathbb{E}[X], \mathbb{E}[Y], E(X \mid Y), E(Y \mid X)$ and use these to check directly that*
$$\mathbb{E}[X] = \mathbb{E}[E(X \mid Y)], \quad \mathbb{E}[Y] = \mathbb{E}[E(Y \mid X)].$$

3. *Let A be the event $A = \{Y \text{ is odd}\}$. Which of the following facts hold?*
$$\mathbb{E}[E(X|Y)\, 1_A] = \mathbb{E}[X\, 1_A], \qquad \mathbb{E}[E(Y|X)\, 1_A] = \mathbb{E}[Y\, 1_A].$$

1. $P(X=1) = 0.1+0.1+0+0.2 = 0.4$ $\quad$ $P(Y=1) = 0.1+0.1+0.1 = 0.3$

$P(X=2) = 0.1+0.05+0.05+0 = 0.2$ $\quad$ $P(Y=2) = 0.1+0.05+0.1 = 0.25$

$P(X=3) = 0.1+0.1+0.1+0.1 = 0.4$ $\quad$ $P(Y=3) = 0+0.05+0.1 = 0.15$

$P(Y=4) = 0.2+0+0.1 = 0.3$

2. $E(X) = 1\times 0.4 + 2\times 0.2 + 3\times 0.4 = 2$

$E(Y) = 1\times 0.3 + 2\times 0.25 + 3\times 0.15 + 4\times 0.3 = 2.45$

$E(X|Y) = E(X|Y=1) + E(X|Y=2) + E(X|Y=3) + E(X|Y=4)$

$= (1\times\frac{1}{3} + 2\times\frac{1}{3} + 3\times\frac{1}{3}) + (1\times\frac{10}{25} + 2\times\frac{5}{25} + 3\times\frac{10}{25})$

$+ (1\times\frac{0}{15} + 2\times\frac{5}{15} + 3\times\frac{10}{15}) + (1\times\frac{2}{3} + 3\times\frac{1}{3})$

$= 2 + 2 + \frac{8}{3} + \frac{5}{3} = \frac{25}{3}$

$E[E(X|Y)] = E(X|Y=1)\times P(Y=1) + E(X|Y=2)\cdot P(Y=2) + E(X|Y=3)\cdot P(Y=3) +$

$E(X|Y=4)\cdot P(Y=4)$

$$=\left(1\times\frac{1}{3}+2\times\frac{1}{3}+3\times\frac{1}{3}\right)\cdot 0.3+\left(1\times\frac{10}{25}+2\times\frac{5}{25}+3\times\frac{10}{25}\right)\cdot 0.25$$

$$+\left(1\times\frac{0}{15}+2\times\frac{5}{15}+3\times\frac{10}{15}\right)\cdot 0.15+\left(1\times\frac{2}{3}+3\times\frac{1}{3}\right)\cdot 0.3$$

$$=2\times 0.3+2\times 0.25+\frac{8}{3}\times 0.15+\frac{5}{3}\times 0.3=2=E(X)$$

$$\therefore E(X)=E(E(X|Y))$$

$$E(Y|X)=E(Y|X=1)+E(Y|X=2)+E(Y|X=3)$$

$$=\left(1\times\frac{1}{4}+2\times\frac{1}{4}+4\times\frac{2}{4}\right)+\left(1\times\frac{1}{2}+2\times\frac{1}{4}+2\times\frac{1}{4}\right)+\left(1\times\frac{1}{4}+2\times\frac{1}{4}+3\times\frac{1}{4}+4\times\frac{1}{4}\right)$$

$$=\frac{11}{4}+\frac{7}{4}+\frac{5}{2}=7$$

$$E[E(Y|X)]=\frac{11}{4}\times P(X=1)+\frac{7}{4}\cdot P(X=2)+\frac{5}{2}\cdot P(X=3)$$

$$=\frac{11}{4}\times 0.4+\frac{7}{4}\times 0.2+\frac{5}{2}\times 0.4=2.45=E(Y)$$

$$\therefore E(Y)=E(E(Y|X))$$

3. According to the concept

$$E[Y|A]=E[E[Y|F_n]|A] \text{ if } A \text{ is } F_n\text{-measurable}$$

In this case $\because$ A is Y-measurable event

$\therefore E[X|A]=E[E[X|Y]|A]$ the first fact holds

$$E[X|A]=1\times 0.1+2\times(0.1+0.05)+3\times(0.1+0.1)=1$$

$E[E[X|Y]|A]=2\times 0.3+\frac{8}{3}\times 0.15=1$ which holds the fact

Exercise 2 *Suppose we roll two dice, a red and a green one, and let X be the value on the red die and Y the value on the green die. Let $Z = XY$.*

1. *Find $E[(2X+Y)^2 \mid X]$.*

2. *Find $E[(2X+Y)^2 \mid X, Z]$.*

3. *Let $W = E[Z \mid X]$. What are the possible values for W? Give the distribution of W.*

1. $E[(2X+Y)^2 \mid X]$

$= E[4X^2 \mid X] + E[4XY \mid X] + E[Y^2 \mid X]$

$4X^2$ is X-measurable

$= 4X^2 + 4X\,E[Y \mid X] + E[Y^2 \mid X]$

Y is independent of X

$= 4X^2 + 4X\,E(Y) + E(Y^2)$

$E(Y) = E(X) = 3.5$

$E(Y^2) = E(X^2) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)$

$= \frac{91}{6}$

$= 4X^2 + 14X + \frac{91}{6}$

2. $E[(2X+Y)^2 \mid X, Z]$

$\because (2X+Y)^2$ is X, Z-measurable

$\therefore = (2X+Y)^2$

$= 4X^2 + 4XY + Y^2$

$= 4X^2 + 4Z + \left(\frac{Z}{X}\right)^2$

3. $W = E[Z \mid X] = E[XY \mid X] = X\,E[Y \mid X] = X\,E(Y)$

$E[Y] = \frac{1}{6} \times (1+2+3+4+5+6) = \frac{7}{2}$

$\therefore W = \frac{7}{2}X$

possible values: $W = 1 \times \frac{7}{2} = 3.5$

$= 2 \times \frac{7}{2} = 7$

$= 3 \times \frac{7}{2} = 10.5$

$= 4 \times \frac{7}{2} = 14$

$= 5 \times \frac{7}{2} = 17.5$

$= 6 \times \frac{7}{2} = 21$

$W =$	3.5	7	10.5	14	17.5	21
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Exercise 3 *Suppose $X_1, X_2, \dots$ are independent random variables with*

$$\mathbb{P}\{X_j = 3\} = 1 - \mathbb{P}\{X_j = -1\} = \frac{1}{4}.$$

Let $S_n = X_1 + \cdots + X_n$ and let $\mathcal{F}_n$ denote the information in $X_1, \dots, X_n$.

1. *Find $\mathbb{E}[X_1], \mathbb{E}[X_1^2], \mathbb{E}[X_1^3]$.*
2. *Find $\mathbb{E}[S_n], \mathbb{E}[S_n^2], \mathbb{E}[S_n^3]$.*
3. *If $m < n$, find*
$$E[S_n \mid \mathcal{F}_m], \quad E[S_n^2 \mid \mathcal{F}_m].$$
4. *If $m < n$, find $E[X_m \mid S_n]$.*

1. $P(X_j = -1) = \frac{3}{4}$, $P(X_j = 3) = \frac{1}{4}$

$E[X_1] = -1 \times \frac{3}{4} + 3 \times \frac{1}{4} = 0$

$E[X_1^2] = (-1)^2 \cdot \frac{3}{4} + 3^2 \cdot \frac{1}{4} = 3$

$E[X_1^3] = (-1)^3 \cdot \frac{3}{4} + 3^3 \cdot \frac{1}{4} = 6$

2. $E[S_n] = E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E(X_n)$

$\because E(X_1) = E(X_2) = \cdots = E(X_n)$

$\therefore E(S_n) = n \cdot E(X_1) = 0$

$E(S_n^2) = Var(S_n) + (E(S_n))^2 = Var(S_n)$

$= Var(X_1 + \cdots + X_n)$ $\because$ independent

$= Var(X_1) + \cdots + Var(X_n)$

$Var(X_1) = Var(X_2) = \cdots = Var(X_n) = E(X_1^2) - (E(X_1))^2 = 3$

$\therefore E(S_n^2) = n \cdot Var(X_1) = 3n$

$E(S_n^3) = E((X_1 + \cdots + X_n)(X_1 + \cdots + X_n)(X_1 + \cdots + X_n))$

$$= E[X_1^3 + \cdots + X_n^3 + \text{rest}]$$

since $X_1, \cdots, X_n$ are independent, all others can be converted to $E(X_j^2)E(X_j)$ or $E(X_i)E(X_j)E(X_k)$ $\because E(X_j)=0$

$\therefore$ rest $= 0$

$$= E[X_1^3] + \cdots + E(X_n^3) = 6n$$

3. $E(S_n \mid F_m)$

$$= E[(S_m + (S_n - S_m)) \mid F_m]$$

$$= E(S_m \mid F_m) + E(S_n - S_m \mid F_m)$$

$$= S_m + E(S_n - S_m)$$

$$= S_m + E(X_1)\cdot(n-m) = S_m + 0\cdot(n-m) = S_m$$

$E(S_n^2 \mid F_m)$

$$= E[(S_m + (S_n - S_m))^2 \mid F_m]$$

$$= E[S_m^2 + 2S_m(S_n - S_m) + (S_n - S_m)^2 \mid F_m]$$

$$= E[S_m^2 \mid F_m] + 2S_m E[S_n - S_m \mid F_m] + E[(S_n - S_m)^2 \mid F_m]$$

$$= S_m^2 + 2S_m \cdot 0 + E[(S_n - S_m)^2]$$

$= S_m^2 + Var(S_n - S_m) = S_m^2 + (n-m)\cdot 3$

4. $E[X_m \mid S_n]$

$\because E(X_1 \mid S_n) + E(X_2 \mid S_n) + \cdots + E(X_n \mid S_n)$

$= E(X_1 + \cdots + X_n \mid S_n) = E(S_n \mid S_n) = S_n$

$\because E(X_1 \mid S_n) = E(X_2 \mid S_n) = \cdots = E(X_m \mid S_n) = \cdots = E(X_n \mid S_n)$

$\therefore E[X_m \mid S_n] = \frac{S_n}{n}$