FINM 34000, Autumn 2023
Lectures 8 and 9

Reading: Notes, Sections 6.1 – 6.3 (we will not cover 6.4)

Exercise 1 *Suppose X_t is a Poisson process with parameter $\lambda = 3$. Find the following.*

1. $\mathbb{P}\{X_3 = 7\}$
   X_3 is Poisson with mean $3\lambda = 9$.
   $$e^{-9}\,\frac{9^7}{7!}.$$

2. $\mathbb{P}\{X_2\,(X_5 - X_2) = 0\}$
   This is the same as the probability that $X_2 = 0$ or $X_5 - X_2 = 0$. Note that these two events are independent with probabilities e^{-6} and e^{-9}, respectively. Using the rule
   $$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B),$$
   and the fact that the events are independent we get
   $$e^{-6} + e^{-9} - e^{-15}.$$

3. *The expected amount of time until $X_t = 6$*
   This is the sum of six random variables each exponential with rate 3. Since each exponential has mean $1/3$ the total expectation $=6\,(1/3) = 2$.

4. $\mathbb{P}\{X_1 = 2 \mid X_3 = 7\}$.
   Using the definition of conditional expectation
   $$\begin{aligned} &= \frac{\mathbb{P}\{X_1 = 2, X_3 = 7\}}{\mathbb{P}\{X_3 = 7\}} \\ &= \frac{\mathbb{P}\{X_1 = 2, X_3 - X_1 = 5\}}{\mathbb{P}\{X_3 = 7\}} \\ &= \frac{e^{-3}\,\frac{3^2}{2!}\,e^{-6}\,\frac{6^5}{5!}}{e^{-9}\,\frac{9^7}{7!}} \end{aligned}$$
   It turns out this is the same as the probability that a binomial random variable with parameters $n = 7, p = 1/3$ equals 2
   $$\binom{7}{3}\,(1/3)^2\,(2/3)^5.$$

Exercise 2 *Suppose that the number of customers arriving at an automobile dealership in Chicago follows a Poisson process with $\lambda = 2$ (time is measured in hours).*

a. *What is the probability that at most two customers arrive in the first hour?*

$$e^{-2} + e^{-2}\, 2 + e^{-2}\, \frac{2^2}{2}$$

b. *Suppose that exactly two customers arrive in the first hour. What is the probability that there will be exactly three customers in the second hour?*

$$\mathbb{P}\{X_2 - X_1 = 3 \mid X_1 = 2\} = \mathbb{P}\{X_2 - X_1 = 3\} = e^{-2}\, \frac{2^3}{3!}.$$

c. *Suppose that exactly four customers arrived in the first two hours. What is the probability that exactly two customers arrived in the first hour?*

$$\mathbb{P}\{X_1 = 2 \mid X_2 = 4\} = \frac{\mathbb{P}\{X_1 = 2, X_2 = 4\}}{\mathbb{P}\{X_2 = 4\}} = \frac{\mathbb{P}\{X_1 = 2, X_2 - X_1 = 2\}}{\mathbb{P}\{X_2 = 4\}}$$

$$\frac{\mathbb{P}\{X_1 = 2\}\, \mathbb{P}\{X_2 - X_1 = 2\}}{\mathbb{P}\{X_2 = 4\}} = \frac{e^{-2}\, \frac{2^2}{2!}\, e^{-2}\, \frac{2^2}{2!}}{e^{-4}\, \frac{4^4}{4!}}$$

d. *An enthusiastic salesperson decides to wait until 10 customers have arrived before going to lunch. What is the expected amount of time she will have to wait?*
The expected time for one customer is $1/2$ ***hence the expected time for 10 customers is*** $10/2 = 5$***.***

e. *Let N denote the number of customers that arrive in the first two hours. Find $\mathbb{E}[N^2]$.*
N ***is a Poisson random variable with mean*** 4 ***and (by looking this up) variance*** 4***. Therefore,***

$$\mathbb{E}[N^2] = \mathbb{E}[N]^2 + \mathrm{Var}[N] = 20.$$

Exercise 3 *Suppose X_t is a continuous time Markov chain with state space $\{0, 1, 2, 3\}$ with rates*

$$\alpha(0,1) = 2,\ \ \alpha(1,2) = 3,\ \ \alpha(2,0) = 1,\ \ \alpha(2,3) = 1,\ \ \alpha(3,1) = 4,$$

with all other rates equal to zero.

1. *Write down the generator* $\mathbf{A}$.

$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 0 & \ \ 1 & \ \ 2 & \ \ 3 \end{array} \\ \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 4 & 0 & -4 \end{pmatrix} \end{array}$$

2. *Is this chain irreducible?*
   Yes

3. *What is the invariant probability?*
   Using $\boldsymbol{\pi}\mathbf{A} = 0$, we get the equations

$$-2\,\pi(0) + \pi(2) = 0$$

$$2\,\pi(0) - 3\,\pi(1) + 4\,\pi(3) = 0$$

$$3\pi(1) - 2\pi(2) = 0$$

$$\pi(2) - 4\,\pi(3) = 0.$$

   This gives

$$\pi(0) = \frac{1}{2}\,\pi(2), \quad \pi(1) = \frac{2}{3}\,\pi(2), \quad \pi(3) = \frac{1}{4}\,\pi(2)$$

   Making it a probability vector gives

$$\boldsymbol{\pi} = \begin{bmatrix} \frac{6}{29} & \frac{8}{29} & \frac{12}{29} & \frac{3}{29} \end{bmatrix}.$$

4. *Suppose we start with $X_0 = 0$. What is the expected amount of time until reaching state 3?*

$$\tilde{\mathbf{A}} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} -2 & 2 & 0 \\ 0 & -3 & 3 \\ 1 & 0 & -2 \end{pmatrix} \end{array}, \quad [-\tilde{\mathbf{A}}]^{-1} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 2/3 & 1 \\ 1/2 & 2/3 & 1 \\ 1/2 & 1/3 & 1 \end{pmatrix} \end{array}, \quad [-\tilde{\mathbf{A}}]^{-1}\,\vec{1} = \begin{array}{c c} \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 8/3 \\ 13/6 \\ 11/6 \end{pmatrix} \end{array}$$

   The answer is $8/3$.

5. *Suppose we start with $X_0 = 0$. What is the expected amount of time until the chain leaves state 0 for the first time?*
   We are leaving at rate 2 so the answer is $1/2$.

6. *Suppose we start with $X_0 = 0$. What is the expected amount of time for the chain to leave 0 and then return to 0 for the first time?*

$$\frac{1}{\pi(0)\,\alpha_0} = \frac{1}{(6/29)\,2} = \frac{29}{12}.$$

Exercise 4 *Answer the same questions as in Exercise 3 with state space $\{0, 1, 2, 3, 4\}$ and rates*

$$\alpha(0,1) = \alpha(1,0) = \alpha(1,2) = \alpha(2,1) = \alpha(2,3) = \alpha(3,2) = \alpha(3,4) = \alpha(4,3) = 1.$$

This is a continuous time version of simple symmetric random walk with reflecting boundary.

$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}.$$

This is irreducible and the invariant probability is the uniform probability

$$\boldsymbol{\pi} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}.$$

We remove the rows and columns from state 3 **to get**

$$\tilde{\mathbf{A}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 4 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

$$\left[-\tilde{\mathbf{A}}\right]^{-1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 4 \end{array} \\ \begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \left[-\tilde{\mathbf{A}}\right]^{-1} \vec{1} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 4 \end{array} \begin{pmatrix} 6 \\ 5 \\ 3 \\ 1 \end{pmatrix}$$

The expected amount of time to go from 0 **to** 3 **is** 6**.**

The expected amount of time to leave state 0 **is** $1/\alpha_0 = 1$**.**

The expected amount of time to leave state 0 **and then return to** 0 **is**

$$\frac{1}{\pi(0)\,\alpha_0} = \frac{1}{(1/5)\,(1)} = 5.$$