FINM 34000, Autumn 2023
Lecture 7

Reading: Notes, rest of Section 5

Exercise 1 *Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with*

$$\mathbb{P}\{X_j = 2\} = \frac{1}{3}, \quad \mathbb{P}\{X_j = \frac{1}{2}\} = \frac{2}{3}.$$

Let $M_0 = 1$ and for $n \geq 1$, $M_n = X_1 X_2 \cdots X_n$.

1. *Show that M_n is a martingale.*

2. *Explain why M_n satisfies the conditions of the martingale convergence theorem.*

3. *Let $M_\infty = \lim_{n\to\infty} M_n$. Explain why $M_\infty = 0$. (Hint: there are at least two ways to show this. One is to consider $\log M_n$ and use the law of large numbers. Another is to note that with probability one M_{n+1}/M_n does not converge.)*

4. *Use the optional sampling theorem to determine the probability that M_n ever attains a value as large as 64.*

5. *Does there exist a $C < \infty$ such that $\mathbb{E}[M_n^2] \leq C$ for all n?*

Exercise 2 *Consider the martingale betting strategy as discussed in Section 5. Let W_n be the "winnings" at time n, which for positive n equals either 1 or $1 - 2^n$.*

1. *Is W_n a square integrable martingale?*

2. *If $\Delta_n = W_n - W_{n-1}$ what is $\mathbb{E}[\Delta_n^2]$?*

3. *What is $\mathbb{E}[W_n^2]$?*

4. *What is $E(\Delta_n^2 \mid \mathcal{F}_{n-1})$?*

Exercise 3 *Here are some statements about martingales. Say whether they are always true. If always true give reason (citing a fact from the lecture or notes is fine). If it is not always true give an example to show this. Let $M_n, n = 0, 1, 2, \ldots$ be a martingale with respect to $\{\mathcal{F}_n\}$ with $M_0 = 1$.*

1. *For all positive integers n, $\mathbb{E}[M_n] = 1$.*

2. *With probability one, the limit*

$$M_\infty := \lim_{n \to \infty} M_n \tag{1}$$

exists and is finite.

3. *Suppose the limit M_∞ exists as in (1) and is finite. Then $\mathbb{E}[M_\infty] = 1$.*

4. *Suppose we assume know that with probability one $M_n \geq 0$ for all n? Does this imply that the limit in (1) exists with probability one?*

5. *If we assume that $M_n \geq 0$ for all n does the answer to part 3 change?*