**FINM 34000, Autumn 2023**
**Lecture 4**

**Reading**: Notes, Section 4.2.

**Exercise 1** *Suppose we have a Markov chain with state space $\{0, 1, 2, 3\}$ and transition probabilities*

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 3/4 & 1/4 \end{pmatrix} \end{array}.$$

1. *Suppose that $X_0 = 2$. Find the probability that $X_4 = 3$.*
   ***Using a computer we see that***

$$\mathbf{P}^4 = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} .375 & .25 & .2813 & .0938 \\ ..25 & .4063 & .1094 & .2344 \\ .2813 & .1094 & .4766 & .1328 \\ .1406 & .3516 & .1992 & .3095 \end{pmatrix} \end{array}.$$

   ***We need the*** $(2, 3)$ ***entry —- .1328***

2. *Suppose that $X_0 = 2$. Find the probability that the following all happen: $X_4 = 3, X_5 = 2, X_6 = 1, X_7 = 1$.*
   ***Using the Markov property (and time homogeneity),***

$$\mathbb{P}\{X_4 = 3, X_5 = 2, X_6 = 1, X_7 = 1\} = \mathbb{P}\{X_4 = 3\}\, p(3, 2)\, p(2, 1)\, p(1, 1) = 0$$

   ***since*** $p(1, 1) = 0$***.***

3. *Find the invariant probability.*
   ***We will do this by hand. The equations are***

$$\pi_0 = \frac{1}{2}\,\pi_0 + \frac{1}{2}\,\pi_1,$$

$$\pi_1 = \frac{1}{2}\,\pi_0 + \frac{1}{2}\,\pi_2,$$

$$\pi_2 = \frac{1}{2}\,\pi_1 + \frac{3}{4}\,\pi_3,$$

$$\pi_3 = \frac{1}{2}\,\pi_2 + \frac{1}{4}\pi_3.$$

***The first and last equations give***

$$\pi_0 = \pi_1, \qquad \pi_3 = \frac{2}{3}\pi_2$$

***and plugging into the second gives***

$$\pi_2 = \pi_1$$

***Imposing the condition that the sum must equal one gives the answer***

$$\pi(0) = \pi(1) = \pi(2) = \frac{3}{11}, \quad \pi(3) = \frac{2}{11}.$$

4. *Suppose that $X_0 = 2$. What is the expected amount of time until the chain returns to state 2?*

$$\frac{1}{\pi(2)} = \frac{11}{3}.$$

5. *Suppose that $X_0 = 2$. What is the expected amount of time until the chain reaches state 3?*
***We use the matrix***

$$\mathbf{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} \end{array}.$$

$$\mathbf{I} - \mathbf{Q} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}$$

$$\mathbf{M} = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 6 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 2 \end{pmatrix} \end{array}.$$

***The expected number of steps to reach state*** 3 ***starting at state*** 2 ***is the sum of the entries in the row associated to state*** 2***,***

$$2 + 2 + 2 = 6.$$

**Exercise 2** *Consider the infinite Markov chain with state space $\{0, 1, 2, \ldots\}$ that moves according to the following rules.*

- *If the state is currently in state $j > 0$ it moves to state $j - 1$.*

- *If the state is current in state 0, then a Poisson random variable with mean 1 is sampled and one moves to that state. In other words,*

$$p(0,k) = e^{-1}\,\frac{1}{k!}.$$

*Suppose that $X_0 = 0$.*

1. *What is the expected number of steps until we return to 0 for the first time?*
   ***If the first move from 0 is to $k$, there will be $k$ more steps until we return which would make the total number of steps $k+1$ (counting the first step. We do this directly***

   $$\mathbb{E}[T] = \sum_{k=0}^{\infty} \mathbb{P}\{T_1 = k\}\,\mathbb{E}[T \mid X_1 = k] = \sum_{k=0}^{\infty}(1+k)\,p(0,k) = 1 + 1 = 2.$$

   ***The way to see the last equality (if you do not want to check the infinite sum) is to remember that***

   $$\sum_{k=0}^{\infty} k\,\mathbb{P}\{X_1 = k\}$$

   ***is the expected value of a Poisson distribution which in this case is 1.***

2. *Show that this is a positive recurrent chain and give the invariant probability $\pi$.*
   ***Since the expected time to return is finite, it is positive recurrent and we know that***

   $$\pi(0) = \frac{1}{2}.$$

   ***For $k > 0$, we get the equation,***

   $$\pi(k) = e^{-1}\,\frac{1^k}{k!}\,\pi(0) + \pi(k+1) = \frac{1}{2}\,e^{-2}\,\frac{2^k}{k!} + \pi(k+1).$$

   ***This may look a little daunting, but a little staring will convince you that***

   $$\pi(k) = \frac{1}{2}\sum_{j=k}^{\infty} e^{-1}\,\frac{1}{k!}.$$

   ***satisfies the equation. (Don't worry too much if you did not figure this out.)***

**Exercise 3** *Take a standard 52 deck of cards. We will do the following simple "shuffle" of the cards. Choose one of the 51 cards that are not the top card of the deck (uniformly) and move that card to the top of the deck leaving all the other cards in the same order. This is a Markov chain whose state space is the set of 52! orderings (shuffles, permutations) of the deck.*

1. *Is this an irreducible Markov chain?*
   *If you think for a minute you can see that one can get any ordering of the cards by a sequence of moves as described. Yes, it is irreducible.*

2. *Is the transition matrix doubly stochastic? (See Exercise 3 of the previous Problem Set.)*
   ***If you look at it, for any particular ordering there are exactly 51 orderings from which I can get to that order in one move. This says that the columns of the matrix have*** $51$ ***entries equal to*** $1/51$ ***and all others are 0. Therefore it is doubly stochastic.***

3. *Suppose we start with a particular ordering of the cards. Assume I do one "shuffle" every second. What is the expected amount of time until the deck is back to the original order?*
   ***Since it is doubly stochastic, the invariant probability is the uniform distribution which gives probability*** $1/52!$ ***to each ordering. Therefore the expected time to return is one over this which is*** $52!$***.*** $52!$ ***seconds is about*** $8.06 \times 10^{67}$ ***seconds. The number of seconds in a (non-leap) year is 31,536,000.***