

FINM 34000, Autumn 2023

Lecture 4

Reading: Notes, Section 4.2.

Exercise 1 Suppose we have a Markov chain with state space $\{0, 1, 2, 3\}$ and transition probabilities

$$\begin{array}{c} \begin{array}{cccc} & 0 & 1 & 2 & 3 \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 3/4 & 1/4 \end{pmatrix} \end{array} \end{array}.$$

1. Suppose that $X_0 = 2$. Find the probability that $X_4 = 3$.
2. Suppose that $X_0 = 2$. Find the probability that the following all happen: $X_4 = 3, X_5 = 2, X_6 = 1, X_7 = 1$.
3. Find the invariant probability.
4. Suppose that $X_0 = 2$. What is the expected amount of time until the chain returns to state 2?
5. Suppose that $X_0 = 2$. What is the expected amount of time until the chain reaches state 3?

Exercise 2 Consider the infinite Markov chain with state space $\{0, 1, 2, \dots\}$ that moves according to the following rules.

- If the state is currently in state $j > 0$ it moves to state $j - 1$.
- If the state is current in state 0, then a Poisson random variable with mean 1 is sampled and one moves to that state. In other words,

$$p(0, k) = e^{-1} \frac{1}{k!}.$$

Suppose that $X_0 = 0$.

1. What is the expected number of steps until we return to 0 for the first time?

2. Show that this is a positive recurrent chain and give the invariant probability π .

Exercise 3 Take a standard 52 deck of cards. We will do the following simple “shuffle” of the cards. Choose one of the 51 cards that are not the top card of the deck (uniformly) and move that card to the top of the deck leaving all the other cards in the same order. This is a Markov chain whose state space is the set of $52!$ orderings (shuffles, permutations) of the deck.

1. Is this an irreducible Markov chain?
2. Is the transition matrix doubly stochastic? (See Exercise 3 of the previous Problem Set.)
3. Suppose we start with a particular ordering of the cards. Assume I do one “shuffle” every second. What is the expected amount of time until the deck is back to the original order?