FINM 34000, Autumn 2023
Lecture 4

Reading: Notes, Section 4.2.

Exercise 1 *Suppose we have a Markov chain with state space $\{0, 1, 2, 3\}$ and transition probabilities*

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 3/4 & 1/4 \end{pmatrix} \end{array}.$$

1. *Suppose that $X_0 = 2$. Find the probability that $X_4 = 3$.*

2. *Suppose that $X_0 = 2$. Find the probability that the following all happen: $X_4 = 3, X_5 = 2, X_6 = 1, X_7 = 1$.*

3. *Find the invariant probability.*

4. *Suppose that $X_0 = 2$. What is the expected amount of time until the chain returns to state 2?*

5. *Suppose that $X_0 = 2$. What is the expected amount of time until the chain reaches state 3?*

Exercise 2 *Consider the infinite Markov chain with state space $\{0, 1, 2, \ldots\}$ that moves according to the following rules.*

- *If the state is currently in state $j > 0$ it moves to state $j - 1$.*
- *If the state is current in state 0, then a Poisson random variable with mean 1 is sampled and one moves to that state. In other words,*

$$p(0, k) = e^{-1} \frac{1}{k!}.$$

Suppose that $X_0 = 0$.

1. *What is the expected number of steps until we return to 0 for the first time?*

2. *Show that this is a positive recurrent chain and give the invariant probability π.*

Exercise 3 *Take a standard* 52 *deck of cards. We will do the following simple "shuffle" of the cards. Choose one of the 51 cards that are not the top card of the deck (uniformly) and move that card to the top of the deck leaving all the other cards in the same order. This is a Markov chain whose state space is the set of 52! orderings (shuffles, permutations) of the deck.*

1. *Is this an irreducible Markov chain?*

2. *Is the transition matrix doubly stochastic? (See Exercise 3 of the previous Problem Set.)*

3. *Suppose we start with a particular ordering of the cards. Assume I do one "shuffle" every second. What is the expected amount of time until the deck is back to the original order?*

Exercise 1 *Suppose we have a Markov chain with state space $\{0,1,2,3\}$ and transition probabilities*

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 3/4 & 1/4 \end{pmatrix} \end{array}.$$

1. *Suppose that $X_0 = 2$. Find the probability that $X_4 = 3$.*

2. *Suppose that $X_0 = 2$. Find the probability that the following all happen: $X_4 = 3, X_5 = 2, X_6 = 1, X_7 = 1$.*

3. *Find the invariant probability.*

4. *Suppose that $X_0 = 2$. What is the expected amount of time until the chain returns to state 2?*

5. *Suppose that $X_0 = 2$. What is the expected amount of time until the chain reaches state 3?*

1. $P\{X_4=3 \mid X_0=2\} = (P^4)_{23} = \frac{17}{128} \approx 0.132$

2. $P\{X_4=3, X_5=2, X_6=1, X_7=1 \mid X_0=2\}$

$= P\{X_4=3 \mid X_0=2\} \times \frac{3}{4} \times \frac{1}{2} \times 0 = 0$

3. it is irreducible

$$\pi P = \pi$$

$$\begin{cases} \frac{1}{2}\pi(0) + \frac{1}{2}\pi(1) + 0 + 0 = \pi(0) \\ \frac{1}{2}\pi(0) + 0 + \frac{1}{2}\pi(2) + 0 = \pi(1) \\ 0 + \frac{1}{2}\pi(1) + 0 + \frac{3}{4}\pi(3) = \pi(2) \\ 0 + 0 + \frac{1}{2}\pi(2) + \frac{1}{4}\pi(3) = \pi(3) \end{cases}$$

$\pi(0) = \pi(1) = \pi(2)$. $\quad \pi(3) = \frac{2}{3}\pi(2) = \frac{2}{3}\pi(0)$

$$\pi = [\ \pi(0)\quad \pi(0)\quad \pi(0)\quad \tfrac{2}{3}\pi(0)\ ]$$

$$(1+1+1+\tfrac{2}{3})\pi(0) = 1$$

$$\pi(0) = \frac{3}{11}$$

$$\pi = [\ \tfrac{3}{11}\quad \tfrac{3}{11}\quad \tfrac{3}{11}\quad \tfrac{2}{11}\ ]$$

4. $\pi(x) = \dfrac{1}{E[T]}$

$$E[T] = \frac{1}{\pi(2)} = \frac{1}{\frac{3}{11}} = \frac{11}{3}$$

5.

$$\begin{array}{c|cccc} & 3 & 0 & 1 & 2 \\ \hline 3 & & & & \\ 0 & & & & \\ 1 & & & Q & \\ 2 & & & & \end{array}$$

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

$$I - Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$M = [I - Q]^{-1} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 6 & 4 & 2 \\ 1 & 4 & 4 & 2 \\ 2 & 2 & 2 & 2 \end{array}$$

$\therefore$ Start from state 2 and reach state 3, the mean passage time is $2+2+2=6$

Exercise 2 *Consider the infinite Markov chain with state space $\{0, 1, 2, \ldots\}$ that moves according to the following rules.*

- *If the state is currently in state $j > 0$ it moves to state $j - 1$.*
- *If the state is current in state 0, then a Poisson random variable with mean 1 is sampled and one moves to that state. In other words,*

$$p(0, k) = e^{-1}\frac{1}{k!}.$$

Suppose that $X_0 = 0$.

1. *What is the expected number of steps until we return to 0 for the first time?*

2. *Show that this is a positive recurrent chain and give the invariant probability π.*

1. $E(T) = E[1 + \sum_{i=1}^{\infty} X_i P_i]$

because it needs one step to return to 0 and plus the step it took from state n to 0 times the probability of reaching state n

$= 1 + 1 = 2$

2. $\because E(T) = \frac{1}{\pi(0)}$

$\therefore \pi(0) = \frac{1}{E(T)} = \frac{1}{2}$

$$\begin{cases} \pi(0) = \pi(1) + \frac{1}{e\,0!}\pi(0) \\ \vdots \\ \pi(n) = \pi(n+1) + \frac{1}{e\,n!}\pi(0) \end{cases}$$

$\therefore$ invariant probability $=$

$$\begin{cases} \pi(n) = \frac{1}{2} - \sum_{i=1}^{n-1} \frac{1}{2e} \cdot \frac{1}{n!} & \text{for } n = 1, 2, 3, \ldots \\ \pi(n) = \frac{1}{2} & \text{for } n = 0 \end{cases}$$

Exercise 3 *Take a standard 52 deck of cards. We will do the following simple "shuffle" of the cards. Choose one of the 51 cards that are not the top card of the deck (uniformly) and move that card to the top of the deck leaving all the other cards in the same order. This is a Markov chain whose state space is the set of 52! orderings (shuffles, permutations) of the deck.*

1. *Is this an irreducible Markov chain?*

2. *Is the transition matrix doubly stochastic? (See Exercise 3 of the previous Problem Set.)*

3. *Suppose we start with a particular ordering of the cards. Assume I do one "shuffle" every second. What is the expected amount of time until the deck is back to the original order?*

1. Yes because you can use shuffle method to finally get the orderings you want. (any permutation)

2. for example if we state at one permutation. there are 51 cards that can be placed on top to change the permutations Therefore, for each permutation at the start, there are exactly 51 permutations for the end with the probability of $\frac{1}{51}$.
For one permutation at the end, there are 51 kinds of shuffles that can get to the final permutation. Thus for each permutation at the end, there are exactly 51 permutation for the start with probability of $\frac{1}{51}$.
Thus, both rows and columns adds up to one $(51 \times \frac{1}{51} = 1)$

3. It is irreducible and according to P63, we can know that the invariant probability is uniformly distributed, which is

$$\pi = \left[\frac{1}{52!} \cdots\cdots \frac{1}{52!}\right]$$

Thus, the expected amount of time to return

$$E(T) = \frac{1}{\pi(x)} = 52! \text{ sec}$$