FINM 34000, Autumn 2023
Lecture 1

Reading: Notes, Section 2.

Exercise 1 *Suppose (X, Y) are discrete random variables with joint probabilities*

$$\begin{array}{cc} X\backslash Y & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} .1 & .1 & 0 & .2 \\ .1 & .05 & .05 & 0 \\ .1 & .1 & .1 & .1 \end{pmatrix} \end{array}$$

For example, $\mathbb{P}\{X = 2, Y = 3\} = .05$.

1. *Find the marginal distributions for X and Y.*

2. *Find $\mathbb{E}[X], \mathbb{E}[Y], E(X \mid Y), E(Y \mid X)$ and use these to check directly that*
$$\mathbb{E}[X] = \mathbb{E}[E(X \mid Y)], \quad \mathbb{E}[Y] = \mathbb{E}[E(Y \mid X)].$$

3. *Let A be the event $A = \{Y \text{ is odd}\}$. Which of the following facts hold?*
$$\mathbb{E}[E(X|Y)\, 1_A] = \mathbb{E}[X\, 1_A], \qquad \mathbb{E}[E(Y|X)\, 1_A] = \mathbb{E}[Y\, 1_A].$$

Exercise 2 *Suppose we roll two dice, a red and a green one, and let X be the value on the red die and Y the value on the green die. Let $Z = XY$.*

1. *Find $E[(2X + Y)^2 \mid X]$.*

2. *Find $E[(2X + Y)^2 \mid X, Z]$.*

3. *Let $W = E[Z \mid X]$. What are the possible values for W? Give the distribution of W.*

Exercise 3 *Suppose $X_1, X_2, \ldots$ are independent random variables with*
$$\mathbb{P}\{X_j = 3\} = 1 - \mathbb{P}\{X_j = -1\} = \frac{1}{4}.$$
Let $S_n = X_1 + \cdots + X_n$ and let $\mathcal{F}_n$ denote the information in $X_1, \ldots, X_n$.

1. *Find $\mathbb{E}[X_1], \mathbb{E}[X_1^2], \mathbb{E}[X_1^3]$.*

2. *Find $\mathbb{E}[S_n], \mathbb{E}[S_n^2], \mathbb{E}[S_n^3]$.*

3. *If $m < n$, find*
$$E[S_n \mid \mathcal{F}_m], \quad E[S_n^2 \mid \mathcal{F}_m].$$

4. *If $m < n$, find $E[X_m \mid S_n]$.*