FINM 34000, Autumn 2023
Lecture 6

Reading: Notes, Section 5.2, 5.3.

Exercise 1 *Suppose $S_n = X_1 + \cdots + X_n$ is simple symmetric random walk in one dimension. Let $\mathcal{F}_n$ denote the information in $X_1, X_2, \ldots, X_n$. State which of the following are stopping times for the random walk. Give reasons.*

1. *T is the first time n such that $S_n < 0$.*

2. *T is the first time n that*
$$\frac{S_n}{n} > S_1.$$

3. *T is the first time n that*
$$S_{n+1} > S_n.$$

4. *Let τ be the first time m that $S_m \geq 4$ and let T be the first time n after τ that $S_n \leq -5$.*

Exercise 2 *In this exercise, we consider simple, asymmetric independent random variables with*
$$\mathbb{P}\{X_j = 1\} = 1 - \mathbb{P}\{X_j = -1\} = q, \qquad 0 < q < \frac{1}{2}.$$
Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_n$ denote the information contained in $X_1, \ldots, X_n$.

1. *Which of these is S_n: martingale, submartingale, supermartingale (more than one answer is possible)?*

2. *For which values of r is $M_n = S_n + rn$ a martingale?*

3. *Let $\theta = (1-q)/q$ and let*
$$M_n = \theta^{S_n}.$$
Show that M_n is a martingale.

4. *Let a, b be positive integers, and*
$$T_{a,b} = \min\{j : S_j = b \text{ or } S_j = -a\}.$$
Use the optional sampling theorem to determine
$$\mathbb{P}\left\{S_{T_{a,b}} = b\right\}.$$

5. *Let* $T_b = T_{-\infty,b}$. *Find*

$$\mathbb{P}\{T_b < \infty\}.$$

Exercise 3 *Let* $X_1, X_2, \ldots$ *be independent, identically distributed random variables with*

$$\mathbb{P}\{X_j = 1\} = q, \quad \mathbb{P}\{X_j = -1\} = 1 - q.$$

Let $S_0 = 0$ *and for* $n \geq 1$, $S_n = X_1 + X_2 + \cdots + X_n$. *Let* $Y_n = e^{S_n}$.

1. *For which value of* q *is* Y_n *a martingale?*

2. *For the remaining parts of this exercise assume* q *takes the value from part 1. Use the optional sampling theorem to determine the probability that* Y_n *ever attains a value greater than* 100.

3. *Does there exist a* $C < \infty$ *such that* $\mathbb{E}[Y_n^2] \leq C$ *for all* n*?*