FINM 34000, Autumn 2023
Lecture 5

Reading: Notes, Section 5.1.

Exercise 1 *Suppose $S_n = X_1 + \cdots + X_n$ is simple symmetric random walk in one dimension. Let $\mathcal{F}_n$ denote the information in $X_1, X_2, \ldots, X_n$. For each of the following say if the process is a martingale, submartingale, or supermartingale (it can be more than one and it might be none of these) with respect to $\mathcal{F}_n$. Give reasons (citing a fact from lecture or notes is a sufficient reason).*

1. $M_n = S_n$

2. $M_n = S_n^2$

3. $M_n = S_n^3$

4. $M_n = 2^{S_n}$.

5. $M_n = S_n/n$

6. $M_n = S_{n+1}\, S_n$.

7.
$$M_n = 0 + X_1\, X_2 + X_2\, X_3 + \cdots + X_{n-1}\, X_n = \sum_{j=1}^{n} X_{j-1}\, X_j.$$

Exercise 2 *This exercise concerns Polya's urn and has a computing/simulation component. Let us start with one red and one green ball as in the lecture and let M_n be the fraction of red balls at the nth stage.*

1. *Show that the distribution of M_n is uniform on the set*
$$\left\{ \frac{1}{n+2}, \frac{2}{n+2}, \ldots, \frac{n+1}{n+2} \right\}.$$
(Use mathematical induction, that is, note that it is obviously true for $n = 0$ and show that if it is true for n then it is true for $n + 1$.)

2. *Write a short program that will simulate this urn. Each time you run the program note the fraction of red balls after 2000 draws and after 4000 draws. Compare the two fractions. Then, repeat this thirty times.*