FINM 34000, Autumn 2023
Lecture 3

Reading: Notes, Section 4.1.

Note: you may use any program to do the matrix calculations. This is true for future problem sets as well.

Exercise 1 *Suppose X_n is a Markov chain with state space $S = \{A, B, C\}$ and transition matrix*

$$\mathbf{P} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \\ 1/4 & 1/2 & 1/4 \end{pmatrix} \end{array}.$$

1. *Find the long range fraction of time that the chain spends in the three states A, B, C.* ***We need find the invariant probability. One could do it by computer but we will do it by hand. The equation $\boldsymbol{\pi}\,\mathbf{P} = \boldsymbol{\pi}$ gives us***

$$\pi_A = \frac{1}{2}\,\pi_A + \frac{1}{3}\,\pi_B + \frac{1}{4}\,\pi_C$$

$$\pi_B = \frac{1}{4}\,\pi_A + \frac{1}{3}\,\pi_B + \frac{1}{2}\,\pi_C,$$

$$\pi_C = \frac{1}{4}\,\pi_A + \frac{1}{3}\,\pi_B + \frac{1}{4}\,\pi_C.$$

The third equation gives

$$\pi_C = \frac{1}{3}\,\pi_A + \frac{4}{9}\,\pi_B,$$

and then the second equation becomes

$$\pi_B = \frac{1}{4}\,\pi_A + \frac{1}{3}\,\pi_B + \frac{1}{2}\,\left[\frac{1}{3}\,\pi_A + \frac{4}{9}\pi_B\right], \qquad \pi_B = \frac{15}{16}\,\pi_A.$$

$$\pi_C = \frac{3}{4}\,\pi_A.$$

Imposing the condition $\pi_A + \pi_B + \pi_C = 1$ gives

$$\pi_A = \frac{16}{43}, \qquad \pi_B = \frac{15}{43}, \qquad \pi_C = \frac{12}{43}.$$

2. *Suppose an investor gains \$10 when the chain is in state B, \$5 when the chain is in state C, and loses \$5 when the chain is in state A. Then the long range earnings after n steps is proportional to $c\,n$ for some constant c. What is c?*
   In the long run the expected gain per step is

$$c = -5\pi_A + 10\,\pi_B + 5\,\pi_C = -\frac{80}{43} + \frac{150}{43} + \frac{60}{43} = \frac{130}{43}$$

3. *Suppose that $X_0 = A$. Find*

$$\mathbb{P}\{X_4 = B \mid X_3 = C\}, \qquad \mathbb{P}\{X_3 = C \mid X_4 = B\}.$$

The first is easy and follows immediately from the homogeneous Markov property

$$\mathbb{P}\{X_4 = B \mid X_3 = C\} = p(C, B) = \frac{1}{2}.$$

The second is harder and we should go back to the definition of conditional probability

$$\mathbb{P}\{X_3 = C \mid X_4 = B\} = \frac{\mathbb{P}\{X_3 = C, X_4 = B\}}{\mathbb{P}\{X_4 = B\}} = \frac{\mathbb{P}\{X_3 = C\}\,\mathbb{P}\{X_4 = B \mid X_3 = C\}}{\mathbb{P}\{X_4 = B\}}.$$

With aid of computer we see that

$$\mathbf{P}^3 = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} .3767 & .3455 & .2778 \\ .3704 & .3495 & .2801 \\ .3681 & .3524 & .2795 \end{pmatrix} \end{array}. \qquad \mathbf{P}^4 = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} .3730 & .3482 & .2778 \\ .3717 & .3492 & .2791 \\ .3714 & .3492 & .2794 \end{pmatrix} \end{array}.$$

Reading off from this (using the A row) we get

$$\mathbb{P}\{X_3 = C\} = .2778, \qquad \mathbb{P}\{X_4 = B\} = .3482, \quad \mathbb{P}\{X_4 = B \mid X_3 = C\} = \frac{1}{2}.$$

The answer is

$$\frac{.2778\,(1/2)}{.3482} = .399.$$

Exercise 2 *Consider the following Markov chain with state space $\{0, 1, 2, 3\}$.*

- *When the chain is in state j, then $3 - j$ fair dice are rolled. Let k be the number of these dice that come up either 5 or 6. Then we move to state k. In particular, if we are in state 3 we always move to state 0.*

1. *Write down the transition matrix for this chain.*
This is an exercise in the binomial distribution. The probability of success is the probability to get a 5 or a 6 which is $1/3$***.***

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \;\;0\;\; & \;\;\;1\;\;\; & \;\;\;2\;\;\; & \;\;\;3\;\; \end{array} \\ \begin{pmatrix} 8/27 & 12/27 & 6/27 & 1/27 \\ 4/9 & 4/9 & 1/9 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

2. *Give the invariant probability for the chain.*
The equations are

$$\pi_0 = \frac{8}{27}\,\pi_0 + \frac{4}{9}\,\pi_1 + \frac{2}{3}\,\pi_2 + \pi_3$$
$$\pi_1 = \frac{12}{27}\,\pi_0 + \frac{4}{9}\,\pi_1 + \frac{1}{3}\,\pi_2$$
$$\pi_2 = \frac{6}{27}\,\pi_0 + \frac{1}{9}\,\pi_1$$
$$\pi_3 = \frac{1}{27}\,\pi_0.$$

Substituting the fourth equation into the first gives

$$\pi_0 = \frac{2}{3}\,\pi_1 + \pi_2,$$

and plugging this into the third equation gives

$$\pi_2 = \frac{6}{27}\,\left[\frac{2}{3}\,\pi_1 + \pi_2\right] + \frac{1}{9}\,\pi_1, \qquad \pi_2 = \frac{1}{3}\,\pi_1.$$

$$\pi_0 = \pi_1, \qquad \pi_3 = \frac{1}{27}\,\pi_1.$$

Imposing the condition $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$***, gives***

$$\pi_0 = \frac{27}{64}, \quad \pi_1 = \frac{27}{64}, \quad \pi_2 = \frac{9}{64}, \quad \pi_3 = \frac{1}{64}.$$

3. *Suppose we start at state* 0*. What is the probability that after three steps we are in state* 3*?*
This is the $(0,3)$ ***entry of*** $\mathbf{P}^3$***.***

4. *Suppose we start at state* 0*. What is the expected number of steps until we return to state* 0*?*

$$\frac{1}{\pi_0} = \frac{64}{27}.$$

Exercise 3 *Suppose X_n is a finite irreducible Markov chain whose transition matrix is doubly stochastic, that is, both the rows and the columns add up to one. Show that the uniform distribution is the invariant probability distribution for the chain.*

This is straightforward if one thinks about it. Suppose there are k states. If the columns of the transition matrix P add up to one and $\boldsymbol{\pi} = (\frac{1}{k}, \frac{1}{k}, \cdots, \frac{1}{k})$, it is immediate that

$$\boldsymbol{\pi}\,\mathbf{P} = \boldsymbol{\pi}.$$