**FINM 34000, Autumn 2023**
**Lecture 6**

**Reading**: Notes, Section 5.2, 5.3.

**Exercise 1** *Suppose $S_n = X_1 + \cdots + X_n$ is simple symmetric random walk in one dimension. Let $\mathcal{F}_n$ denote the information in $X_1, X_2, \ldots, X_n$. State which of the following are stopping times for the random walk. Give reasons.*

1. *$T$ is the first time $n$ such that $S_n < 0$.*

2. *$T$ is the first time $n$ that*
$$\frac{S_n}{n} > S_1.$$

3. *$T$ is the first time $n$ that*
$$S_{n+1} > S_n.$$

4. *Let $\tau$ be the first time $m$ that $S_m \geq 4$ and let $T$ be the first time $n$ after $\tau$ that $S_n \leq -5$.*

**Exercise 2** *In this exercise, we consider simple, asymmetric independent random variables with*
$$\mathbb{P}\{X_j = 1\} = 1 - \mathbb{P}\{X_j = -1\} = q, \qquad 0 < q < \frac{1}{2}.$$
*Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_n$ denote the information contained in $X_1, \ldots, X_n$.*

1. *Which of these is $S_n$: martingale, submartingale, supermartingale (more than one answer is possible)?*

2. *For which values of $r$ is $M_n = S_n + rn$ a martingale?*

3. *Let $\theta = (1-q)/q$ and let*
$$M_n = \theta^{S_n}.$$
*Show that $M_n$ is a martingale.*

4. *Let $a, b$ be positive integers, and*
$$T_{a,b} = \min\{j : S_j = b \text{ or } S_j = -a\}.$$
*Use the optional sampling theorem to determine*
$$\mathbb{P}\left\{S_{T_{a,b}} = b\right\}.$$

5. *Let* $T_b = T_{-\infty,b}$. *Find*

$$\mathbb{P}\{T_b < \infty\}.$$

**Exercise 3** *Let* $X_1, X_2, \ldots$ *be independent, identically distributed random variables with*

$$\mathbb{P}\{X_j = 1\} = q, \quad \mathbb{P}\{X_j = -1\} = 1 - q.$$

*Let* $S_0 = 0$ *and for* $n \geq 1$, $S_n = X_1 + X_2 + \cdots + X_n$. *Let* $Y_n = e^{S_n}$.

1. *For which value of* $q$ *is* $Y_n$ *a martingale?*

2. *For the remaining parts of this exercise assume* $q$ *takes the value from part 1. Use the optional sampling theorem to determine the probability that* $Y_n$ *ever attains a value greater than* 100.

3. *Does there exist a* $C < \infty$ *such that* $\mathbb{E}[Y_n^2] \leq C$ *for all* $n$*?*

**Exercise 1** *Suppose $S_n = X_1 + \cdots + X_n$ is simple symmetric random walk in one dimension. Let $\mathcal{F}_n$ denote the information in $X_1, X_2, \ldots, X_n$. State which of the following are stopping times for the random walk. Give reasons.*

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1. $T = \min\{S_n < 0\}$ is nonnegative value ( $S_0 = 0$, but $S_n < 0$ must be $n > 0$ ) , also, for each event $T = n$ is $F_n$-measurable,

$\therefore$ it is a stopping time

2. $T = \min\{n : \frac{S_n}{n} > S_1\}$ is a positive integer, also $S_n$ & $S_1$ are $F_n$-measurable for all $n$ $\therefore$ it is a stopping time

3. $T = \min\{n : S_{n+1} > S_n\}$ $\because$ $S_{n+1}$ is not $F_n$-measurable

$\therefore$ $T$ is not a stopping time

4. $\tau = \min\{m : S_m \geq 4\}$

$T = \min\{n > \tau, S_n \leq -5\}$

$\because$ $\tau$ and $T$ are positive integer $\because$ $\tau < n$, $\therefore$ $S_m$ and $S_n$ are both $F_n$-measurable for all $n$

$\therefore$ $T$ is a stopping time, and $\tau$ is a stopping time

**Exercise 2** *In this exercise, we consider simple, asymmetric independent random variables with*

$$\mathbb{P}\{X_j = 1\} = 1 - \mathbb{P}\{X_j = -1\} = q, \qquad 0 < q < \frac{1}{2}.$$

*Let* $S_0 = 0$ *and* $S_n = X_1 + \cdots + X_n$. *Let* $\mathcal{F}_n$ *denote the information contained in* $X_1, \ldots, X_n$.

1. *Which of these is* $S_n$*: martingale, submartingale, supermartingale (more than one answer is possible)?*

2. *For which values of* $r$ *is* $M_n = S_n + rn$ *a martingale?*

3. *Let* $\theta = (1-q)/q$ *and let*

$$M_n = \theta^{S_n}.$$

*Show that* $M_n$ *is a martingale.*

4. *Let* $a, b$ *be positive integers, and*

$$T_{a,b} = \min\{j : S_j = b \text{ or } S_j = -a\}.$$

*Use the optional sampling theorem to determine*

$$\mathbb{P}\left\{S_{T_{a,b}} = b\right\}.$$

5. *Let* $T_b = T_{-\infty,b}$. *Find*

$$\mathbb{P}\{T_b < \infty\}.$$

1. $E[X_j] = 1 \cdot q + (-1) \cdot (1-q) = 2q-1$

$E[S_n] = nE[X_j] = n(2q-1)$

$E[S_{n+1} \mid F_n] = E[X_{n+1} + S_n \mid F_n]$

$= E[X_{n+1}] + S_n$

$= n(2q-1) + 2q-1 = S_n + (2q-1)$

$\because 0 < q < \frac{1}{2} \quad \therefore (2q-1) < 0$

$\therefore E[S_{n+1} \mid F_n] < S_n$

$\therefore$ it is a supermartingale

2. $M_n = S_n + rn$

$E[M_{n+1} \mid F_n] = E[S_{n+1} + r(n+1) \mid F_n]$

$= E[X_{n+1} + r(n+1) + S_n \mid F_n]$

$= S_n + E[X_{n+1}] + r(n+1)$

$= n(2q-1) + 2q-1 + r(n+1)$

$= S_n + 2q-1 + r(n+1)$

in order for $M_n$ to be martingale

$S_n + 2q-1 + r(n+1) = M_n = S_n + rn$

$2q-1+r = 0$

$r = 1-2q$

if $r = 1-2q$, $M_n$ is a martingale

3. $E[(\frac{1-q}{q})^{X_j}] = (\frac{1-q}{q})^1 \cdot q + (\frac{1-q}{q})^{-1} \cdot (1-q) = 1$

$M_n = \theta^{S_n} = (\frac{1-q}{q})^{S_n}$

$E[M_{n+1} | F_n] = E[((\frac{1-q}{q})^{S_n + X_{n+1}} | F_n]$

$= E[(\frac{1-q}{q})^{S_n} | F_n] E[(\frac{1-q}{q})^{X_{n+1}}]$

$= M_n \cdot 1 = M_n$

$\therefore$ $M_n$ is a martingale, and also supermartingale

and submartingale

4. $\because$ a, b are positive integer

$\therefore$ $M_n = (\frac{1-q}{q})^{S_n}$ is also a martingale

$\because$ $-a \leq S_n \leq b$ for all n

$\therefore E[|M_{n \wedge T}|^2] \leq C$ is satisfied

$\therefore$ According to Optional Sampling Theorem

$$E[M_T] = E[M_0]$$

$$1 = \left(\frac{1-q}{q}\right)^0 = E[M_0] = \left(\frac{1-q}{q}\right)^{-a} P\{S_T = -a\} + \left(\frac{1-q}{q}\right)^b P\{S_T = b\}$$

$$= \left(\frac{1-q}{q}\right)^{-a} [1 - P\{S_T = b\}] + \left(\frac{1-q}{q}\right)^b P\{S_T = b\}$$

$$P\{S_T = b\} = \frac{1-\theta^{-a}}{\theta^b - \theta^{-a}} \quad \text{where } \theta = \frac{1-q}{q}$$

5. By definition

$$T_{-\infty, b} = \min\{j : S_j = -\infty \text{ or } S_j = b\}$$

$$= \min\{j : S_j = b\}$$

$$P\{T_b = \infty\}$$

$$= \lim_{j \to \infty} \{S_{T_b} = -j\}$$

$$= \lim_{j \to \infty} \left(1 - \frac{1-\theta^{-j}}{\theta^b - \theta^{-j}}\right) = 1 - \frac{1}{\theta^b}$$

$$\therefore P\{T_b < \infty\} = 1 - P\{T_b = \infty\} = \frac{1}{\theta^b} \quad \text{where } \theta = \frac{1-q}{q}$$

**Exercise 3** *Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with*

$$\mathbb{P}\{X_j = 1\} = q, \quad \mathbb{P}\{X_j = -1\} = 1 - q.$$

*Let $S_0 = 0$ and for $n \geq 1$, $S_n = X_1 + X_2 + \cdots + X_n$. Let $Y_n = e^{S_n}$.*

1. *For which value of $q$ is $Y_n$ a martingale?*
2. *For the remaining parts of this exercise assume $q$ takes the value from part 1. Use the optional sampling theorem to determine the probability that $Y_n$ ever attains a value greater than 100.*
3. *Does there exist a $C < \infty$ such that $\mathbb{E}[Y_n^2] \leq C$ for all $n$?*

1. $E[e^{X_j}] = e^1 \cdot q + e^{-1}(1-q) = qe + \frac{1-q}{e}$

$E[Y_{n+1} | F_n]$

$= E[e^{S_n + X_{n+1}} | F_n]$

$= Y_n \cdot E[e^{X_{n+1}} | F_n] = Y_n \cdot (qe + \frac{1-q}{e})$

Let it be $Y_n$

$$qe + \frac{1-q}{e} = 1$$

$$q = \frac{1}{e+1}$$

2. $E[e^{X_j}] = \frac{e}{e+1} + \frac{1-\frac{1}{e+1}}{e} = \frac{e+1}{e+1} = 1$

Let $T = \min\{n : Y_n = 100\}$

$\because$ $Y_n$ is a martingale and $n$ is positive $\therefore$ $T$ is a stopping time

$\because$ $M_n = Y_{n \wedge T}$ is also a martingale

$\therefore$ $E[|M_{n\wedge T}|^2] \leq C$ is satisfied

For $Y_n > 100$, $S_n > \ln 100 = 4.6$ $\therefore S_n \geq 5$

$\therefore$ $1 = E(M_0) = E(M_T) = e^5 \cdot P\{M_T = e^5\}$

$\therefore$ $P\{Y_n > 100\} = \frac{1}{e^5}$

3. $Y_n^2 = [e^{S_n}]^2 = e^{2S_n}$

$$E[Y_n^2] = E[e^{2S_n}] = E[e^{2X_j}]^n$$

$$E[e^{2X_j}] = e^2 \cdot \frac{1}{e+1} + e^{-2} \cdot \left(1 - \frac{1}{e+1}\right)$$

$$= \frac{e^2}{e+1} + \frac{1}{e(e+1)}$$

$$= \frac{e^3+1}{e(e+1)}$$

$$\therefore E[Y_n^2] = \left[\frac{e^3+1}{e(e+1)}\right]^n$$

$$\lim_{n\to\infty} \left(\frac{e^3+1}{e(e+1)}\right)^n \to \infty$$

$\therefore$ There don't exist $C<\infty$ that $E[Y_n^2] \le C$ for all $n$.