FINM 34000, Autumn 2023
Lecture 7

Reading: Notes, rest of Section 5

Exercise 1 *Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with*

$$\mathbb{P}\{X_j = 2\} = \frac{1}{3}, \quad \mathbb{P}\{X_j = \frac{1}{2}\} = \frac{2}{3}.$$

Let $M_0 = 1$ and for $n \geq 1$, $M_n = X_1 X_2 \cdots X_n$.

1. *Show that M_n is a martingale.*

2. *Explain why M_n satisfies the conditions of the martingale convergence theorem.*

3. *Let $M_\infty = \lim_{n\to\infty} M_n$. Explain why $M_\infty = 0$. (Hint: there are at least two ways to show this. One is to consider $\log M_n$ and use the law of large numbers. Another is to note that with probability one M_{n+1}/M_n does not converge.)*

4. *Use the optional sampling theorem to determine the probability that M_n ever attains a value as large as 64.*

5. *Does there exist a $C < \infty$ such that $\mathbb{E}[M_n^2] \leq C$ for all n?*

Exercise 2 *Consider the martingale betting strategy as discussed in Section 5. Let W_n be the "winnings" at time n, which for positive n equals either 1 or $1 - 2^n$.*

1. *Is W_n a square integrable martingale?*
2. *If $\Delta_n = W_n - W_{n-1}$ what is $\mathbb{E}[\Delta_n^2]$?*
3. *What is $\mathbb{E}[W_n^2]$?*
4. *What is $E(\Delta_n^2 \mid \mathcal{F}_{n-1})$?*

Exercise 3 *Here are some statements about martingales. Say whether they are always true. If always true give reason (citing a fact from the lecture or notes is fine). If it is not always true give an example to show this. Let $M_n, n = 0, 1, 2, \ldots$ be a martingale with respect to $\{\mathcal{F}_n\}$ with $M_0 = 1$.*

1. *For all positive integers n, $\mathbb{E}[M_n] = 1$.*

2. *With probability one, the limit*

$$M_\infty := \lim_{n\to\infty} M_n \tag{1}$$

exists and is finite.

3. *Suppose the limit M_∞ exists as in* (1) *and is finite. Then $\mathbb{E}[M_\infty] = 1$.*

4. *Suppose we assume know that with probability one $M_n \geq 0$ for all n? Does this imply that the limit in* (1) *exists with probability one?*

5. *If we assume that $M_n \geq 0$ for all n does the answer to part 3 change?*

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4. *Use the optional sampling theorem to determine the probability that M_n ever attains a value as large as 64.*

5. *Does there exist a $C < \infty$ such that $\mathbb{E}[M_n^2] \leq C$ for all n?*

1. $E[X_j] = 2\times\frac{1}{3} + \frac{1}{2}\times\frac{2}{3} = 1$

$E[M_{n+1} | F_n] = E[X_1 \cdots X_n \cdot X_{n+1} | F_n]$

$= X_1 \cdot X_2 \cdot \cdots\cdots \cdot X_n \cdot E[X_{n+1}]$

$= X_1 \cdot X_2 \cdot \cdots \cdot X_n \cdot 1 = M_n$

$\therefore$ M_n is a martingale

2. $\because$ M_n is a martingale with respect to F_n

$E[|M_n|] = E[|X_1 X_2 \cdots\cdots X_n|]$

$\because X_1, \cdots, X_n > 0$

$\therefore = E[X_1] \cdot E[X_2] \cdots\cdots E[X_n]$

$= 1 \cdot \cdots\cdots 1 = 1 \leq$ a constant $C < \infty$ for all n

$\therefore$ the conditions are satisfied

3. $E[\log X_j] = \frac{1}{3}\log 2 + \frac{2}{3}\log\frac{1}{2} = -\frac{1}{3}\log 2 < 0$

$\therefore \lim_{n\to\infty} \ln M_n \to -\infty$

according to Law of Large numbers

$M_\infty = \lim_{n\to\infty} M_n = \lim_{n\to\infty} e^{\ln M_n} = 0$

4. $T = \min\{n : M_n = 64 \text{ or } M_n = (\frac{1}{2})^k\}$

$\because$ M_n is a martingale and n is positive integer

$\therefore$ T is a stopping time.

$\because M_\infty = 0 \quad \therefore \lim_{n\to\infty} E\{M_n 1\{T>n\}\} = 0$

$\therefore 1 = E[M_0] = E[M_T] = 64 \cdot P\{M_n = 64\} + 2^{-k}[1 - P\{M_n = 64\}]$

as $k \to \infty$, $2^{-k}[1 - P\{M_n = 64\}] \to 0$

$\Rightarrow P\{M_n = 64\} = \frac{1}{64}$

5. $$E[M_n^2] = E[(X_1 \cdots\cdots X_n)^2]$$
$$= E[X_1^2] \cdots\cdots E[X_n^2]$$
$$E[X_j^2] = 2^2 \cdot \frac{1}{3} + \frac{1}{2}^2 \cdot \frac{2}{3} = \frac{3}{2}$$

$$\therefore E[M_n^2] = \left(\frac{3}{2}\right)^n \to \infty \text{ as } n \to \infty$$

$\therefore$ there does not exist $C < \infty$ ~~that~~ $E[M_n^2] \le C$ for all n

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1. *Is W_n a square integrable martingale?*
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1. $$E[W_n^2] = 1^2 \cdot \left[1 - \left(\tfrac{1}{2}\right)^n\right] + (-2^n + 1)^2 \cdot \left(\tfrac{1}{2}\right)^n$$
$$= 1 - 2^{-n} + (2^{2n} - 2 \cdot 2^n + 1) \cdot 2^{-n}$$
$$= 1 - 2^{-n} + 2^n - 2 + 2^{-n} = 2^n - 1 < \infty \text{ for all } n$$

$\therefore$ W_n is a square integrable martingale

2. $$W_n - W_{n-1} = \begin{cases} 0 & 1 - \left(\tfrac{1}{2}\right)^{n-1} \\ 2^{n-1} & \left(\tfrac{1}{2}\right)^n \end{cases}$$

$$\left| -2^{n-1} \qquad \left(\tfrac{1}{2}\right)^n \right.$$

$$(W_n - W_{n-1})^2 = \begin{cases} 0 & 1-\left(\frac{1}{2}\right)^{n-1} \\ 2^{2n-2} & \left(\frac{1}{2}\right)^{n-1} \end{cases}$$

$$\therefore E[\Delta n^2] = E[(W_n - W_{n-1})^2] = 2^{2n-2} \cdot \left(\tfrac{1}{2}\right)^{n-1} = 2^{n-1}$$

3. $E[W_n^2] = 2^n - 1$ as calculated in Q1

4. $E[\Delta n^2 | F_n]$

$= E[(W_n - W_{n-1})^2 | F_{n-1}]$

$= E[W_n^2 | F_{n-1}] - 2E[W_n W_{n-1} | F_{n-1}] + W_{n-1}^2$

$= 2^n - 1 - 2 \cdot W_{n-1} \cdot 0 + W_{n-1}^2 = 2^n - 1 + W_{n-1}^2$

for $W_{n-1} = 1$, it will stop at $n-1$ round

$$E[\Delta n^2 | F_{n-1}] = \begin{cases} 2^n - 1 + (2^{n-1} - 1)^2 = 2^{2n-2} \\ 0, \quad X_{n-1} \text{ round end} \end{cases}$$

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1. *For all positive integers n, $\mathbb{E}[M_n] = 1$.*

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5. *If we assume that $M_n \geq 0$ for all n does the answer to part 3 change?*

1. Always true

$\because$ M_n is a martingale

$\therefore$ $E[M_{n+1} | F_n] = M_n$

$E[M_n] = E[E[M_n | F_0]] = E[M_0] = 1$ for all n

2. This ask whether it always follows Martingale Convergence Theorem. Not always true

eg $S_n = X_1 + X_2 + \cdots + X_n$ is a simple symmetric random walk $\therefore$ S_n is a martingale

$E[|S_n|] = n$ $\therefore$ $\lim_{n\to\infty} E[|S_n|] \to \infty$ which is not bounded

$\therefore$ doesn't satisfy

3. Not always true

eg. $P\{X_j = 3\} = \frac{1}{4}$, $P\{X_j = \frac{1}{3}\} = \frac{3}{4}$ $M_n = X_1 \cdot X_2 \cdot \cdots \cdot X_n$

M_n is a martingale $\because E[M_{n+1} | F_n] = M_n$

but $E[M_\infty] = E[\lim_{n \to \infty} M_n] = \lim_{n \to \infty} E[M_n] = 1 \neq 0$

$\therefore E[M_\infty] \neq 0$ $\therefore$ doesn't satisfy

4. Always true

$\because M_n \geq 0$ for all n and $E[|M_n|] = 1$ is bounded

according to optional sampling theorem and convergence theorem

$1 = E[M_0] = E[M_\infty]$ $M_\infty = \lim_{n \to \infty} M_n$ exist

5. False, with the same example, the conclusion will not hold