FINM 34000, Autumn 2023
Lecture 2

Reading: Notes, Section 3.

Exercise 1 *Suppose we change the probabilities in simple random walk so that*

$$\mathbb{P}\{X_j = 1\} = 1 - p, \qquad \mathbb{P}\{X_j = -1\} = p,$$

where $1/2 < p < 1$. *Let*

$$q_n = \mathbb{P}\{S_{2n} = 0\}$$

where we start at the origin.

- *Give an exact expression for* q_n.

$$\binom{2n}{n} p^n \, (1-p)^n.$$

- *Show that*

$$\sum_{n=1}^{\infty} q_n < \infty$$

  and conclude that the random walk does not return to the origin infinitely often.
  Note that

$$q_n = (4p(1-p))^n \left[\binom{2n}{n} (1/2)^n \, (1/2)^n \right] \leq (4p(1-p))^n \, .$$

  This is because the quantity in the square brackets is the probability that the symmetric ($p = 1/2$ ***) random walk as at the origin and hence is no more than*** 1***. Also*** $p(1-p) < 1/4$***. (This is a calculus exercise, the function***

$$f(p) = p(1-p), \qquad 0 \leq p \leq 1$$

  obtains its maximum at $p = 1/2$***.) The sum is bounded by a geometric series that is finite.***

Exercise 2 *Use the central limit theorem to find*

$$\lim_{n \to \infty} \mathbb{P}\{S_n < \frac{2}{3} \sqrt{n}\}.$$

Do this for both the symmetric simple random walk and the asymmetric random walk in Exercise **??**.
For the simple symmetric random walk, $\mathbb{E}[X_j] = 0, \mathrm{Var}[X_j] = 1$ ***and***

$$\lim_{n \to \infty} \mathbb{P}\{S_n < \frac{2}{3} \sqrt{n}\} = \lim_{n \to \infty} \mathbb{P}\{\frac{S_n}{\sqrt{n}} < \frac{2}{3}\} = \Phi(2/3).$$

For the asymmetric walk,

$$\mathbb{E}[X_j] = (1-p) - p = 1-2p, \qquad \mathbb{E}[X_j^2] = (1-p) + p = 1, \qquad \mathrm{Var}[X_n] = \sigma^2$$

where $\sigma^2 = 1 - (1-2p)^2$***.***

$$\lim_{n\to\infty} \mathbb{P}\{S_n < \frac{2}{3}\sqrt{n}\} = \lim_{n\to\infty} \mathbb{P}\{\frac{S_n - n(1-2p)}{\sigma\sqrt{n}} < \frac{\frac{2}{3}\sqrt{n} - n(1-2p)}{\sigma\sqrt{n}}\}$$

Since $1-2p<1$***, we see that***

$$\lim_{n\to\infty} \frac{\frac{2}{3}\sqrt{n} - n(1-2p)}{\sigma\sqrt{n}} = -\infty,$$

and the probability equals one.

Exercise 3 *Let us call* m *an upswing time for (symmetric) simple random walk if* $S_m = S_{m-5} + 5$*, that is, if we have had five consecutive* $+1$ *values. Find the expected number of steps until we have an upswing time. (Hint: a very similar problem was discussed in the August review and you should feel free to consult those notes.)*
This is the same as the "Time until a pattern appears" in coin-tossing in Section 1.5.4, page 20 of the August review notes. We refer you there to the solution which is

$$e(5) = 2^{5+1} - 2 = 62.$$