FINM 34000, Autumn 2023
Lectures 8 and 9

Reading: Notes, Sections 6.1 – 6.3 (we will not cover 6.4)

Exercise 1 *Suppose X_t is a Poisson process with parameter $\lambda = 3$. Find the following.*

1. $\mathbb{P}\{X_3 = 7\}$

2. $\mathbb{P}\{X_2\,(X_5 - X_2) = 0\}$

3. *The expected amount of time until* $X_t = 6$

4. $\mathbb{P}\{X_1 = 2 \mid X_3 = 7\}$.

Exercise 2 *Suppose that the number of customers arriving at an automobile dealership in Chicago follows a Poisson process with $\lambda = 2$ (time is measured in hours).*

a. *What is the probability that at most two customers arrive in the first hour?*

b. *Suppose that exactly two customers arrive in the first hour. What is the probability that there will be exactly three customers in the second hour?*

c. *Suppose that exactly four customers arrived in the first two hours. What is the probability that exactly two customers arrived in the first hour?*

d. *An enthusiastic salesperson decides to wait until 10 customers have arrived before going to lunch. What is the expected amount of time she will have to wait?*

e. *Let N denote the number of customers that arrive in the first two hours. Find $\mathbb{E}[N^2]$.*

Exercise 3 *Suppose X_t is a continuous time Markov chain with state space $\{0, 1, 2, 3\}$ with rates*

$$\alpha(0,1) = 2,\ \ \alpha(1,2) = 3,\ \ \alpha(2,0) = 1,\ \ \alpha(2,3) = 1,\ \ \alpha(3,1) = 4,$$

with all other rates equal to zero.

1. *Write down the generator* $\mathbf{A}$.

2. *Is this chain irreducible?*

3. *What is the invariant probability?*

4. *Suppose we start with* $X_0 = 0$. *What is the expected amount of time until reaching state* 3*?*

5. *Suppose we start with* $X_0 = 0$. *What is the expected amount of time until the chain leaves state* 0 *for the first time?*

6. *Suppose we start with* $X_0 = 0$. *What is the expected amount of time for the chain to leave* 0 *and then return to* 0 *for the first time?*

Exercise 4 *Answer the same questions as in Exercise 3 with state space* $\{0, 1, 2, 3, 4\}$ *and rates*

$$\alpha(0, 1) = \alpha(1, 0) = \alpha(1, 2) = \alpha(2, 1) = \alpha(2, 3) = \alpha(3, 2) = \alpha(3, 4) = \alpha(4, 3) = 1.$$

This is a continuous time version of simple symmetric random walk with reflecting boundary.

Exercise 1 *Suppose X_t is a Poisson process with parameter $\lambda = 3$. Find the following.*

1. $\mathbb{P}\{X_3 = 7\}$
2. $\mathbb{P}\{X_2\,(X_5 - X_2) = 0\}$
3. *The expected amount of time until $X_t = 6$*
4. $\mathbb{P}\{X_1 = 2 \mid X_3 = 7\}$.

1. X_3 has mean $= 3 \times 3 = 9$

$$P\{X_3 = 7\} = e^{-9} \cdot \frac{9^7}{7!} \approx 0.117$$

2. $P\{X_2(X_5 - X_2) = 0\}$

$$= P\{X_2 = 0 \text{ or } X_5 - X_2 = 0 \quad \text{or} \quad X_2 = 0 \text{ and } X_5 - X_2 = 0\}$$

$$= P\{X_2 = 0\} + P\{X_5 - X_2 = 0\} - P\{X_2 = 0\} \cdot P\{X_5 - X_2 = 0\}$$

$$= e^{-3\cdot 2} \frac{6^0}{0!} + e^{-3\cdot 3} \frac{9^0}{0!} - e^{-6} \cdot e^{-9} = e^{-6} + e^{-9} \quad e^{-15} \approx 0.0026$$

3. $X_t = 6 \Rightarrow T_1 + T_2 + \cdots + T_6$ where $E[T_j] = \frac{1}{\lambda} = \frac{1}{3}$

$$\therefore \; E[T_1 + \cdots + T_6] = 6 \cdot E[T_j] = 6 \cdot \frac{1}{3} = 2$$

4. $P\{X_1 = 2 \mid X_3 = 7\}$

$$= \frac{P\{X_1 = 2,\, X_3 = 7\}}{P\{X_3 = 7\}}$$

$$= \frac{P(X_3 - X_1 = 5) \cdot P(X_1 = 2)}{P(X_3 = 7)} = \frac{\frac{e^{-2\cdot 3}\, 6^5}{5!} \; \frac{e^{-3}\, 3^2}{2!}}{\frac{e^{-9} \cdot 9^7}{7!}}$$

$$= \frac{7!}{2!\,5!} \cdot \frac{6^5\, 3^2}{9^7} = \frac{7!}{2!\,5!} \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^2 \approx 0.307$$

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e. *Let N denote the number of customers that arrive in the first two hours. Find $\mathbb{E}[N^2]$.*

a. $$P\{X_1 \leq 2\} = P\{X_1 = 0\} + P\{X_1 = 1\} + P\{X_1 = 2\}$$

$$= e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!} = 5e^{-2} \approx 0.677$$

b. $$P\{X_2 - X_1 = 3 \mid X_1 = 2\}$$

$$= P\{X_2 - X_1 = 3\}$$

$$= e^{-2}\frac{2^3}{3!} \approx 0.180$$

c. $$P\{X_1 = 2 \mid X_2 = 4\} = \frac{P\{X_1 = 2, X_2 = 4\}}{P(X_2 = 4)}$$

$$= \frac{P\{X_2 = 4 \mid X_1 = 2\} \cdot P(X_1 = 2)}{P(X_2 = 4)}$$

$$= \frac{P(X_2 - X_1 = 2)\, P(X_1 = 2)}{P(X_2 = 4)} = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

d. $X_t = 10 \Rightarrow T_1 + \cdots + T_{10}$

$E[T_j] = \frac{1}{\lambda} = \frac{1}{2}$

$\therefore E[T_1 + \cdots + T_{10}] = \frac{1}{2} \cdot 10 = 5$

e. N has mean $= 2 \cdot 2 = 4$

$Var(N) = \text{mean} = 4$

$Var(N) = E[N^2] - (E[N])^2$

$\therefore E[N^2] = Var(N) + (E(N))^2 = 4 + 16 = 20$

Exercise 3 *Suppose X_t is a continuous time Markov chain with state space $\{0,1,2,3\}$ with rates*

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with all other rates equal to zero.

1. *Write down the generator* **A**.
2. *Is this chain irreducible?*
3. *What is the invariant probability?*
4. *Suppose we start with $X_0 = 0$. What is the expected amount of time until reaching state 3?*
5. *Suppose we start with $X_0 = 0$. What is the expected amount of time until the chain leaves state 0 for the first time?*
6. *Suppose we start with $X_0 = 0$. What is the expected amount of time for the chain to leave 0 and then return to 0 for the first time?*

1. $A = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -2 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \end{pmatrix} \end{array}$

$$\begin{array}{c} 2 \\ 3 \end{array} \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 4 & 0 & -4 \end{pmatrix}$$

2. the chain is irreducible cuz one can get from any site to any other site with positive probability in some number of steps.

3. $\pi A = 0$

$$-2\pi(0) + 0\cdot\pi(1) + 1\cdot\pi(2) + 0\cdot\pi(3) = 0$$

$$2\pi(0) - 3\pi(1) + 0\pi(2) + 4\pi(3) = 0$$

$$0\pi(0) + 3\pi(1) - 2\pi(2) + 0\pi(3) = 0$$

$$0\pi(0) + 0\pi(1) + \pi(2) - 4\pi(3) = 0$$

$$\pi(2) = 4\pi(3), \quad \pi(2) = \frac{3}{2}\pi(1), \quad \pi(2) = 2\pi(0)$$

$$\pi = \left[\frac{1}{2}\pi(2) \quad \frac{2}{3}\pi(2) \quad \pi(2) \quad \frac{1}{4}\pi(2)\right]$$

$$\left(\frac{1}{2} + \frac{2}{3} + 1 + \frac{1}{4}\right)\pi(2) = 1$$

$$\pi(2) = \frac{12}{29}$$

$$\therefore \pi = \left[\frac{6}{29} \quad \frac{8}{29} \quad \frac{12}{29} \quad \frac{3}{29}\right]$$

4.

$$\tilde{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} -2 & 2 & 0 \\ 0 & -3 & 3 \\ 1 & 0 & -2 \end{pmatrix} \end{array}$$

$$\text{Expected } e = -\tilde{A}^{-1} \cdot \mathbb{1} = \begin{pmatrix} 1 & \frac{2}{3} & 1 \\ \frac{1}{2} & \frac{2}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{array}{c} 0 \\ 1 \\ 2 \end{array}\begin{pmatrix} \frac{8}{3} \\ \frac{13}{6} \\ \frac{11}{6} \end{pmatrix}$$

$\therefore$ expected of time (state at 0, reach 3) is $\frac{8}{3}$

5. expected time that leave 0 $= \frac{1}{\alpha_0} = \frac{1}{2}$

6. expected time return first time

$$= \frac{1}{\alpha_0 \pi(0)} = \frac{1}{2 \cdot \frac{6}{29}} = \frac{29}{12}$$

Exercise 4 *Answer the same questions as in Exercise 3 with state space* $\{0, 1, 2, 3, 4\}$ *and rates*

$$\alpha(0,1) = \alpha(1,0) = \alpha(1,2) = \alpha(2,1) = \alpha(2,3) = \alpha(3,2) = \alpha(3,4) = \alpha(4,3) = 1.$$

This is a continuous time version of simple symmetric random walk with reflecting boundary.

1.

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

2. the chain is irreducible cuz one can get from any site to any other site with positive probability in some muber of steps.

3. $\pi A = 0$

$$\begin{cases} -\pi(0) + \pi(1) = 0 \\ \pi(0) - 2\pi(1) + \pi(2) = 0 \\ \pi(1) - 2\pi(2) + \pi(3) = 0 \\ \pi(2) - 2\pi(3) + \pi(4) = 0 \\ \pi(3) - \pi(4) = 0 \end{cases} \Rightarrow \quad \pi(0) = \pi(1) = \pi(2) = \pi(3) = \pi(4)$$

$$(1+1+1+1+1)\,\pi(0) = 1$$

$$\pi(0) = \frac{1}{5}$$

$$\therefore \ \pi = [\tfrac{1}{5} \ \ \tfrac{1}{5} \ \ \tfrac{1}{5} \ \ \tfrac{1}{5} \ \ \tfrac{1}{5}]$$

4. $$\tilde{A} = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 4 \end{matrix} & \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

$$\text{expected } e = -\tilde{A}^{-1} \mathbb{1} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 4 \end{array} \begin{pmatrix} 6 \\ 5 \\ 3 \\ 1 \end{pmatrix}$$

$\therefore$ expected of time (state at 0, reach 3) is 6

5. expected time that leave 0 $= \frac{1}{\alpha_0} = \frac{1}{1} = 1$

6. expected time return first time

$$= \frac{1}{\alpha_0 \pi(0)} = \frac{1}{1 \cdot \frac{1}{5}} = 5$$