FINM 34000, Autumn 2023
Lecture 5

Reading: Notes, Section 5.1.

Exercise 1 *Suppose $S_n = X_1 + \cdots + X_n$ is simple symmetric random walk in one dimension. Let $\mathcal{F}_n$ denote the information in $X_1, X_2, \ldots, X_n$. For each of the following say if the process is a martingale, submartingale, or supermartingale (it can be more than one and it might be none of these) with respect to $\mathcal{F}_n$. Give reasons (citing a fact from lecture or notes is a sufficient reason).*

1. $M_n = S_n$
   We showed this was a martingale in class and hence is also a submartingale and a supermartingale.

2. $M_n = S_n^2$
   As was shown in class,

$$E[S_{n+1}^2 \mid \mathcal{F}_n] = E[(S_n + X_{n+1})^2 \mid \mathcal{F}_n] = S_n^2 + 1.$$

   This is a submartingale only.

3. $M_n = S_n^3$
   Note that

$$\begin{aligned} E[(S_{n+1} - S_n)^3 \mid \mathcal{F}_n] &= E[S_n^3 \, F_n] + 3\, E[S_n^2 \, X_{n+1} \mid \mathcal{F}_n] + 3\, \mathbb{E}[S_n \, X_{n+1}^2 \mid \mathcal{F}_n] + E[X_n^3 \mid \mathcal{F}_n] \\ &= S_n^3 + 3\, S_n^2 \, \mathbb{E}[X_{n+1}] + 3\, S_n \, \mathbb{E}[X_{n+1}^2] + \mathbb{E}[X_n^3] \\ &= S_n^3 + 3\, S_n. \end{aligned}$$

   Since S_n takes on both positive and negative values this does not satisfy the conditions for any of martingale, submartingale, supermartingale.

4. $M_n = 2^{S_n}$.

$$\begin{aligned} E[M_{n+1} \mid \mathcal{F}_n] = E[2^{S_{n+1}} \mid \mathcal{F}_n] &= E[2^{S_n + X_{n+1}} \mid \mathcal{F}_n] \\ &= E[2^{S_n} \, e^{X_{n+1}} \mid \mathcal{F}_n] \\ &= 2^{S_n} \, \mathbb{E}[2^{X_{n+1}}] \\ &= M_n \, \frac{2 + 2^{-1}}{2}. \end{aligned}$$

   Since M_n is nonnegative and $(2 + \frac{1}{2})/2 > 1$, we see that this is a submartingale (but not a martingale or a supermartingale).

5. $M_n = S_n/n$
   Note that

$$E(M_{n+1} \mid \mathcal{F}_n) = E\left(\frac{S_{n+1}}{n+1} \mid \mathcal{F}_n\right) = \frac{1}{n+1} E[S_{n+1} \mid \mathcal{F}_n] = \frac{1}{n+1} S_n = \frac{n}{n+1} M_n.$$

   Since M_n ***could be positive or negative we see that this is not any of sub-martingale, martingale, or supermartingale.***

6. $M_n = S_{n+1} S_n$.
   Note that M_n ***is not*** $\mathcal{F}_n$***-measurable so it is none of these.***

7.

$$M_n = 0 + X_1 X_2 + X_2 X_3 + \cdots + X_{n-1} X_n = \sum_{j=1}^{n} S_{j-1} X_j.$$

   This is an example of a discrete stochastic integral and hence it is a martingale (and hence also a submartingale and a supermartingale).

Exercise 2 *This exercise concerns Polya's urn and has a computing/simulation component. Let us start with one red and one green ball as in the lecture and let* M_n *be the fraction of red balls at the* n*th stage.*

1. *Show that the distribution of* M_n *is uniform on the set*

$$\left\{\frac{1}{n+2}, \frac{2}{n+2}, \ldots, \frac{n+1}{n+2}\right\}.$$

   (Use mathematical induction, that is, note that it is obviously true for $n = 0$ *and show that if it is true for* n *then it is true for* $n + 1$*.)*

   For $n = 0$ ***this is obvious. Assume true for*** n***. Note that***

$$\begin{aligned}
\mathbb{P}\{M_{n+1} = \frac{k}{n+3}\} &= \mathbb{P}\{M_n = \frac{k-1}{n+2}, M_{n+1} = \frac{k}{n+3}\} + \mathbb{P}\{M_n = \frac{k}{n+2}, M_{n+1} = \frac{k}{n+3}\} \\
&= \mathbb{P}\{M_n = \frac{k-1}{n+2}\} \, \mathbb{P}\{M_{n+1} = \frac{k}{n+3} \mid M_n = \frac{k-1}{n+2}\} \\
&\quad + \mathbb{P}\{M_n = \frac{k}{n+2}\} \, \mathbb{P}\{M_{n+1} = \frac{k}{n+3} \mid M_n = \frac{k}{n+2}\} \\
&= \frac{1}{n+1} \, \frac{k-1}{n+2} + \frac{1}{n+1} \, \frac{(n+2)-k}{n+2} \\
&= \frac{1}{n+2}
\end{aligned}$$

2. *Write a short program that will simulate this urn. Each time you run the program note the fraction of red balls after* 2000 *draws and after* 4000 *draws. Compare the two fractions. Then, repeat this thirty times.*

   One should observe that the fraction after 2000 draws is random (roughly uniform on $[0, 1]$***) but the fraction after 4000 draws is very close to the fraction after 2000 draws.***