FINM 34000, Autumn 2023
Lecture 2

Reading: Notes, Section 3.

Exercise 1 *Suppose we change the probabilities in simple random walk so that*

$$\mathbb{P}\{X_j = 1\} = 1 - p, \qquad \mathbb{P}\{X_j = -1\} = p,$$

where $1/2 < p < 1$*. Let*

$$q_n = \mathbb{P}\{S_{2n} = 0\}$$

where we start at the origin.

- *Give an exact expression for* q_n*.*

- *Show that*

$$\sum_{n=1}^{\infty} q_n < \infty$$

  and conclude that the random walk does not return to the origin infinitely often.

Exercise 2 *Use the central limit theorem to find*

$$\lim_{n\to\infty} \mathbb{P}\{S_n < \frac{2}{3}\sqrt{n}\}.$$

Do this for both the symmetric simple random walk and the asymmetric random walk in Exercise 1.

Exercise 3 *Let us call* m *an upswing time for (symmetric) simple random walk if* $S_m = S_{m-5} + 5$*, that is, if we have had five consecutive* $+1$ *values. Find the expected number of steps until we have an upswing time. (Hint: a very similar problem was discussed in the August review and you should feel free to consult those notes.)*

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and conclude that the random walk does not return to the origin infinitely often.

$$q_n = P\{S_{2n} = 0\} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!\,n!} \cdot p^n \cdot (1-p)^n$$

when $n \to \infty$, $n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$

$$\therefore q_n = \frac{(2n)!}{n!\,n!} \cdot p^n \cdot (1-p)^n \sim \frac{\sqrt{2\pi}\,(2n)^{2n+\frac{1}{2}} e^{-2n}}{(\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n})^2} \cdot p^n (1-p)^n$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2^{2n+\frac{1}{2}} \cdot \cancel{n^{2n+\frac{1}{2}}} \cancel{e^{-2n}}}{\underbrace{\cancel{n^{2n+1}}}_{\sqrt{n}} \cdot \cancel{e^{-2n}}} \cdot p^n (1-p)^n$$

$$= \frac{2^{2n}}{\sqrt{\pi n}} p^n (1-p)^n$$

$$\lim_{n\to\infty} \left|\frac{q_{n+1}}{q_n}\right| = \lim_{n\to\infty} \frac{\frac{(2(n+1))!}{(n+1)!\,(n+1)!} p^{n+1} (1-p)^{n+1}}{\frac{(2n)!}{n!\,n!} \cdot p^n \cdot (1-p)^n}$$

$$= \lim_{n\to\infty} p(1-p) \cdot \frac{(2n+1)\overset{2}{\cancel{(2n+2)}}}{(n+1)^{\cancel{2}}}$$

$$= \lim_{n\to\infty} p(1-p) \cdot \frac{4n+2}{n+1} = 4p(1-p)$$

$\because \frac{1}{2} < p < 1 \quad \therefore 4p(1-p) < 1$

$\therefore \lim_{n\to\infty} \left|\frac{q_{n+1}}{q_n}\right| < 1 \quad \Rightarrow \quad \sum_{n\geq 1}^{\infty} q_n < \infty$

therefore we can conclude that random walk does not return to origin infinitely often.

Exercise 2 *Use the central limit theorem to find*

$$\lim_{n\to\infty} \mathbb{P}\{S_n < \frac{2}{3}\sqrt{n}\}.$$

Do this for both the symmetric simple random walk and the asymmetric random walk in Exercise 1.

symmetric simple random walk:

$E[X_j] = 0 \qquad Var(X_j) = E[X_j^2] = \frac{1}{2}\cdot 1^2 + \frac{1}{2}\cdot(-1)^2 = 1$

$\therefore Var[S_n] = Var(X_1) + \cdots + Var(X_n) = n \quad , \quad E[S_n] = 0$

$\therefore \frac{S_n}{\sqrt{n}}$ approaches a standard normal

$\lim_{n\to\infty} P(S_n < \frac{2}{3}\sqrt{n}) = \lim_{n\to\infty} P(\frac{S_n}{\sqrt{n}} < \frac{2}{3}) \approx 0.748$

asymmetric random walk:

$E[X_j] = 1\cdot(1-p) + (-1)p = 1-2p$

$Var[X_j] = E[X_j^2] - (E[X_j])^2 = (1-p) + p - (1-2p)^2 = 4p - 4p^2$

$Var[S_n] = 4n(p-p^2)$, $E[S_n] = n(1-2p)$

$\therefore \dfrac{S_n - n(1-2p)}{\sqrt{4n(p-p^2)}}$ approaches standard normal

$$\lim_{n\to\infty} P(S_n < \tfrac{2}{3}\sqrt{n}) = \lim_{n\to\infty} P\left(\frac{S_n - n(1-2p)}{\sqrt{4n(p-p^2)}} < \frac{\frac{2}{3}\sqrt{n} - n(1-2p)}{\sqrt{4n(p-p^2)}}\right)$$

$$= \lim_{n\to\infty} P\left(Z < \frac{\frac{2}{3}\sqrt{n} - n(1-2p)}{\sqrt{4n(p-p^2)}}\right)$$

Let $x = \dfrac{\frac{2}{3}\sqrt{n} - n(1-2p)}{\sqrt{4n(p-p^2)}} = \dfrac{\frac{1}{3}}{\sqrt{p-p^2}} - \dfrac{\sqrt{n}(1-2p)}{\sqrt{4(p-p^2)}}$

$$\begin{cases} p=\frac{1}{2}, \quad x=\frac{2}{3} & \therefore \lim_{n\to\infty} P(Z<\frac{2}{3}) \approx 0.748 \\ p<\frac{1}{2}, \quad 1-2p>0 \to -\frac{\sqrt{n}(1-2p)}{\sqrt{4(p-p^2)}} \to -\infty \text{ as } n\to\infty & \therefore \lim_{n\to\infty} P(Z<x) = P(Z<-\infty) = 0 \\ p>\frac{1}{2} \quad 1-2p<0 \to -\frac{\sqrt{n}(1-2p)}{\sqrt{4(p-p^2)}} \to \infty \text{ as } n\to\infty & \therefore \lim_{n\to\infty} P(Z<x) = P(Z<\infty) = 1 \end{cases}$$

$$\therefore \lim_{n\to\infty} P(S_n < \tfrac{2}{3}\sqrt{n}) = \begin{cases} 0.748 & \text{when } p=\frac{1}{2} \\ 0 & \text{when } p<\frac{1}{2} \\ 1 & \text{when } p>\frac{1}{2} \end{cases}$$

Exercise 3 *Let us call m an upswing time for (symmetric) simple random walk if $S_m = S_{m-5} + 5$, that is, if we have had five consecutive $+1$ values. Find the expected number of steps until we have an upswing time. (Hint: a very similar problem was discussed in the August review and you should feel free to consult those notes.)*

Let $e(j)$ be the expected number of steps until we have j consecutive $+1$ values

In order to get j consecutive $+1$ values, we need to get $(j-1)$ $+1$ values first either take one step to succeed or fail

$$\frac{1}{2}e(j) = e(j-1)+1 \Rightarrow e(j) = 2e(j-1)+2$$

$$e(0)=0, \quad e(1) = 2e(0)+2 = 2$$

$$e(2) = 2e(1)+2 = 6, \quad e(3) = 2e(2)+2 = 14$$

$$e(4) = 2e(3)+2 = 30, \quad e(5) = 2e(4)+2 = 62$$

to get 5 consecutive $+1$ values to an upswing time, the expected number of steps is 62.