FINM 34000, Autumn 2023
Lecture 3

Reading: Notes, Section 4.1.

Note: you may use any program to do the matrix calculations. This is true for future problem sets as well.

Exercise 1 *Suppose X_n is a Markov chain with state space $S = \{A, B, C\}$ and transition matrix*

$$\mathbf{P} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \\ 1/4 & 1/2 & 1/4 \end{pmatrix} \end{array}.$$

1. *Find the long range fraction of time that the chain spends in the three states A, B, C.*

2. *Suppose an investor gains \$10 when the chain is in state B, \$5 when the chain is in state C, and loses \$5 when the chain is in state A. Then the long range earnings after n steps is proportional to cn for some constant c. What is c?*

3. *Suppose that $X_0 = A$. Find*

$$\mathbb{P}\{X_4 = B \mid X_3 = C\}, \qquad \mathbb{P}\{X_3 = C \mid X_4 = B\}.$$

Exercise 2 *Consider the following Markov chain with state space $\{0, 1, 2, 3\}$.*

- *When the chain is in state j, then $3 - j$ fair dice are rolled. Let k be the number of these dice that come up either 5 or 6. Then we move to state k. In particular, if we are in state 3 we always move to state 0.*

1. *Write down the transition matrix for this chain.*

2. *Give the invariant probability for the chain.*

3. *Suppose we start at state 0. What is the probability that after three steps we are in state 3?*

4. *Suppose we start at state 0. What is the expected number of steps until we return to state 0?*

Exercise 3 *Suppose X_n is a finite irreducible Markov chain whose transition matrix is doubly stochastic, that is, both the rows and the columns add up to one. Show that the uniform distribution is the invariant probability distribution for the chain.*

Exercise 1 *Suppose X_n is a Markov chain with state space $S = \{A, B, C\}$ and transition matrix*

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$$\mathbb{P}\{X_4 = B \mid X_3 = C\}, \quad \mathbb{P}\{X_3 = C \mid X_4 = B\}.$$

1. matrix P is irreducible and aperiodic

$$\pi P = \pi$$

$$\begin{cases} \frac{1}{2}\pi(A) + \frac{1}{3}\pi(B) + \frac{1}{4}\pi(C) = \pi(A) \\ \frac{1}{4}\pi(A) + \frac{1}{3}\pi(B) + \frac{1}{2}\pi(C) = \pi(B) \\ \frac{1}{4}\pi(A) + \frac{1}{3}\pi(B) + \frac{1}{4}\pi(C) = \pi(C) \end{cases}$$

$$\pi(B) = \frac{15}{16}\pi(A), \quad \pi(C) = \frac{4}{5}\pi(B) = \frac{4}{5} \times \frac{15}{16}\pi(A) = \frac{3}{4}\pi(A)$$

$\therefore$ invariant probability:

$$[\pi(A) \quad \frac{15}{16}\pi(A) \quad \frac{3}{4}\pi(A)]$$

$$(1 + \frac{15}{16} + \frac{3}{4})\pi(A) = 1$$

$$\frac{43}{16}\pi(A) = 1$$

$$\pi(A) = \frac{16}{43}$$

$$\therefore \ \pi = \left[\frac{16}{43} \quad \frac{15}{43} \quad \frac{12}{43} \right]$$

Long fraction of time spent in

$$A = \frac{43}{16}$$

$$B = \frac{43}{15}$$

$$C = \frac{43}{12}$$

2. $$\frac{15}{43} \times 10 + \frac{12}{43} \times 5 - \frac{16}{43} \times 5 = \frac{130}{43}$$

expected earning of one step $= \frac{130}{43}$

$\therefore$ Total expected earning of n step $= \frac{130}{43} n$

$$\therefore \ C = \frac{130}{43}$$

3. $\underset{0}{A}$ $\underset{1}{\_}$ $\underset{2}{\_}$ $\underset{3}{C}$

$$P\{X_3 = C\} = \overset{CC}{\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}} + \overset{CB}{\frac{1}{4} \times \frac{1}{2} \times \frac{1}{3}} + \overset{CA}{\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}}$$

$$+ \overset{BC}{\frac{1}{4} \times \frac{1}{3} \times \frac{1}{4}} + \overset{BB}{\frac{1}{4} \times \frac{1}{3} \times \frac{1}{3}} + \overset{BA}{\frac{1}{4} \times \frac{1}{3} \times \frac{1}{4}}$$

$$+ \overset{AC}{\frac{1}{2} \times \frac{1}{4} \times \frac{1}{4}} + \overset{AB}{\frac{1}{2} \times \frac{1}{4} \times \frac{1}{3}} + \overset{AA}{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{4}} = \frac{5}{18}$$

OR $P\{X_3 = C\} = P\{X_3 = C \mid X_0 = A\} = (P^3)_{AC} = \frac{5}{18}$

$$P\{X_4 = B \mid X_3 = C\} = \frac{P(X_3 = C, X_4 = B)}{P(X_3 = C)} = \frac{\frac{5}{18} \times \frac{1}{2}}{\frac{5}{18}} = \frac{1}{2}$$

$\underset{0}{A}$ $\underset{1}{\_}$ $\underset{2}{\_}$ $\underset{3}{\_}$ $\underset{4}{B}$

$$P\{X_4=B\} = \overset{AA}{\frac{1}{2}\times\frac{1}{2}}\times\left(\overset{A}{\frac{1}{2}\times\frac{1}{4}}+\overset{B}{\frac{1}{4}\times\frac{1}{3}}+\overset{C}{\frac{1}{4}\times\frac{1}{2}}\right)+\overset{AB}{\frac{1}{2}\times\frac{1}{4}}\times\left(\overset{A}{\frac{1}{2}\times\frac{1}{4}}+\overset{B}{\frac{1}{3}\times\frac{1}{3}}+\overset{C}{\frac{1}{3}\times\frac{1}{2}}\right)+\overset{AC}{\frac{1}{2}\times\frac{1}{4}}\times\left(\overset{A}{\frac{1}{4}\times\frac{1}{4}}+\overset{B}{\frac{1}{2}\times\frac{1}{3}}+\overset{C}{\frac{1}{4}\times\frac{1}{2}}\right)$$

$$+\overset{BA}{\frac{1}{4}\times\frac{1}{3}}\times\left(\overset{A}{\frac{1}{2}\times\frac{1}{4}}+\overset{B}{\frac{1}{4}\times\frac{1}{3}}+\overset{C}{\frac{1}{4}\times\frac{1}{2}}\right)+\overset{BB}{\frac{1}{4}\times\frac{1}{3}}\times\left(\overset{A}{\frac{1}{2}\times\frac{1}{4}}+\overset{B}{\frac{1}{3}\times\frac{1}{3}}+\overset{C}{\frac{1}{3}\times\frac{1}{2}}\right)+\overset{BC}{\frac{1}{4}\times\frac{1}{3}}\times\left(\overset{A}{\frac{1}{4}\times\frac{1}{4}}+\overset{B}{\frac{1}{2}\times\frac{1}{3}}+\overset{C}{\frac{1}{4}\times\frac{1}{2}}\right)$$

$$+\overset{CA}{\frac{1}{4}\times\frac{1}{4}}\times\left(\overset{A}{\frac{1}{2}\times\frac{1}{4}}+\overset{B}{\frac{1}{4}\times\frac{1}{3}}+\overset{C}{\frac{1}{4}\times\frac{1}{2}}\right)+\overset{CB}{\frac{1}{4}\times\frac{1}{2}}\times\left(\overset{A}{\frac{1}{2}\times\frac{1}{4}}+\overset{B}{\frac{1}{3}\times\frac{1}{3}}+\overset{C}{\frac{1}{3}\times\frac{1}{2}}\right)+\overset{CC}{\frac{1}{4}\times\frac{1}{4}}\times\left(\overset{A}{\frac{1}{4}\times\frac{1}{4}}+\overset{B}{\frac{1}{2}\times\frac{1}{3}}+\overset{C}{\frac{1}{4}\times\frac{1}{2}}\right)$$

$$=\frac{2407}{6912}$$

OR $$P\{X_4=B\} = P\{X_4=B \mid X_0=A\} = (P^4)_{AB} = \frac{2407}{6912}$$

$$P\{X_3=C \mid X_4=B\} = \frac{P(X_3=C, X_4=B)}{P(X_4=B)} = \frac{\frac{5}{18}\times\frac{1}{2}}{\frac{2407}{6912}} \approx 0.4$$

Exercise 2 *Consider the following Markov chain with state space* $\{0, 1, 2, 3\}$.

- *When the chain is in state* j, *then* $3-j$ *fair dice are rolled. Let* k *be the number of these dice that come up either 5 or 6. Then we move to state* k. *In particular, if we are in state 3 we always move to state 0.*

1. *Write down the transition matrix for this chain.*

2. *Give the invariant probability for the chain.*

3. *Suppose we start at state 0. What is the probability that after three steps we are in state 3?*

4. *Suppose we start at state 0. What is the expected number of steps until we return to state 0?*

1. $p(3,0)=1, \quad p(3,1)=p(3,2)=p(3,3)=0$

if start in 0, $j=0$, $3-0=3$ dices rolled

$$P(k=0) = \binom{3}{0}\cdot\left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$P(k=1) = \binom{3}{1}\left(\frac{2}{3}\right)^2\cdot\frac{1}{3} = \frac{12}{27}$$

$$P(k=2) = \binom{3}{2}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 = \frac{6}{27}$$

$$P(k=3) = \binom{3}{3}\left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

if start in 1, j=1. 3-1=2 dices rolled

$P(K=0) = \binom{2}{0}\cdot(\frac{2}{3})^2 = \frac{4}{9}$

$P(K=1) = \binom{2}{1}(\frac{1}{3})\cdot(\frac{2}{3}) = \frac{4}{9}$

$P(K=2) = \binom{2}{2}(\frac{1}{3})^2 = \frac{1}{9}$

if start in 2, j=2, 3-2=1 dices rolled

$P(K=0) = \frac{2}{3}$

$P(K=1) = \frac{1}{3}$

$$\therefore P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{8}{27} & \frac{12}{27} & \frac{6}{27} & \frac{1}{27} \\ 1 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ 2 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{array}$$

2. $\because$ P is irreducible and aperiodic

$\therefore \pi P = \pi$

$\therefore \frac{8}{27}\pi(0) + \frac{4}{9}\pi(1) + \frac{2}{3}\pi(2) + \pi(3) = \pi(0)$

$\frac{12}{27}\pi(0) + \frac{4}{9}\pi(1) + \frac{1}{3}\pi(2) + 0 = \pi(1)$

$\frac{6}{27}\pi(0) + \frac{1}{9}\pi(1) + 0 + 0 = \pi(2)$

$\frac{1}{27}\pi(0) + 0 + 0 + 0 = \pi(3)$

$\pi(3) = \frac{1}{27}\pi(0)$ $\quad\quad$ $\pi(1) = \pi(0)$ $\quad\quad$ $\pi(2) = \frac{1}{3}\pi(0)$

$\therefore$ invariant probability =

$$\left[\ \pi(0) \quad \pi(0) \quad \tfrac{1}{3}\pi(0) \quad \tfrac{1}{27}\pi(0)\ \right]$$

$$\left(1+1+\tfrac{1}{3}+\tfrac{1}{27}\right)\times\pi(0)=1$$

$$\pi(0)=\frac{27}{64}$$

$$\therefore \pi = \left[\ \frac{27}{64} \quad \frac{27}{64} \quad \frac{9}{64} \quad \frac{1}{64}\ \right]$$

3.

$$\frac{0}{0} \qquad \frac{\ }{1} \qquad \frac{0}{2} \qquad \frac{3}{3}$$

step3 (X_2) must be 0 because only H reaches state 3

$$P\{X_3=3\} = \frac{8}{27}\times\frac{8}{27}\times\frac{1}{27} + \frac{12}{27}\times\frac{4}{9}\times\frac{1}{27} + \frac{6}{27}\times\frac{2}{3}\times\frac{1}{27} + \frac{1}{27}\times1\times\frac{1}{27}$$

$$= \frac{343}{19683}$$

OR $P\{X_3=3\} = P\{X_3=3 \mid X_0=0\} = (P^3)_{03} = \dfrac{343}{19683} \approx 0.017$

4.

$$\pi(x) = \frac{1}{E(T)}$$

$$E(T) = \frac{1}{\pi(0)} = \frac{64}{27}$$

Exercise 3 *Suppose X_n is a finite irreducible Markov chain whose transition matrix is doubly stochastic, that is, both the rows and the columns add up to one. Show that the uniform distribution is the invariant probability distribution for the chain.*

$$\pi P = \pi$$

$$\pi(y) = \sum_x \pi(x) p(x,y)$$

assume $\pi(x) = a$ for all $x \in [0, n]$

$\therefore \pi(0) = \pi(1) = \cdots = \pi(n) = 1$

$\therefore \pi(y) = \sum_x p(x,y) \cdot a$

since it is doublely stochastic

$\sum_x p(x,y) = 1 \Rightarrow \pi(y) = a$

Thus, $\pi(x) = a$ is a valid solution to the first equation

$\because$ sum of all $\pi(y) = 1$

$(n+1) \cdot a = 1$

$a = \frac{1}{n+1}$

$\therefore \pi = [\frac{1}{n+1} \cdots\cdots \frac{1}{n+1}]$ is the invariant probability distribution for the chain