FINM 33000: Homework 1

Due Thursday October 5, 2023 at 11:59pm

Problem 1

In each part, construct an arbitrage using some or all of the available assets: Specify a static portfolio that meets the definition of arbitrage (which you are to verify, using either the type-1 or type-2 condition). You can specify the portfolio's assets in any order; just be clear and consistent about what order you have chosen. For example, you could say (3 units of A, 1 unit of B, -5 units of C) or (1 unit of B, -5 units of C, 3 units of A), but not just $(3, 1, -5)$ nor just $(1, -5, 3)$ without further specification. In this course, "log" with no subscript always denotes natural log.

Let $T > 0$. Wherever "Z" appears, it denotes a discount bond with maturity T.

Wherever "S" appears, it denotes a non-dividend-paying stock. Assume that $S_T \geq 0$. Unless otherwise directed, make no further assumptions about the distribution of S_T; your arbitrage must be valid regardless of the distribution of S_T.

(a) Exactly two assets are available: two bank accounts B and B^* with $B_0 = B_0^* = 1$ and $B_T = e^{rT}$ and $B_T^* = e^{r^*T}$ where $r < r^*$.

(b) Exactly three assets are available: A discount bond Z with $Z_0 = 0.9$; the stock S with $S_0 = 100.0$; and C, a European call on S with strike 110, expiry T, and time-0 price $C_0 = 0.50$.

(c) Exactly three assets are available: the stock S with $S_0 = 100.0$; a contract G that pays $\min(S_T, 110)$ at time T, and has time-0 price $G_0 = 85$; and C, a European call on S with strike 110, expiry T, and time-0 price $C_0 = 20$.

(d) Exactly four assets are available: A discount bond Z with $Z_0 = 0.9$; and three calls (all on the same underlying. The underlying is not available for you to trade). The calls have expiry T, strike $K \in \{20.0, 22.5, 25.0\}$, and time-0 price $C_0(K)$, where

$$C_0(20.0) = 6.40,\ C_0(22.5) = 3.10,\ C_0(25.0) = 1.00.$$

(e) Assume that $S_T > 0$ and $\mathsf{P}(S_T \neq 100) > 0$. Exactly 3 assets are available:

a contract X that pays $X_T := -2\log(S_T/100)$ at time T and has time-0 price $X_0 = 0.2$,

a forward contract Y on the underlying S, with delivery price 100 and delivery date T, and time-0 price $Y_0 = -10$,

and a discount bond Z with maturity T and time-0 price $Z_0 = 0.9$.

Problem 2

Interest rates are zero; the bank account has price $B_t = 1$ at all times.

Trump is the Republican Party's candidate in the US Presidential election.

Biden is the Democratic Party's candidate in the US Presidential election.

Assume that exactly one of these two candidates {Trump the Republican, Biden the Democrat} will win US Presidential election.

According to the rules of the election, the winner depends on who wins in each *state* of the US. You do not need to know the full rules that determine the US Presidential election winner from the state winners. You need to know only the information provided below.

Assume that the current time is 7:30am on Thursday the 5th of November 2020 – which we will designate as "time 0".

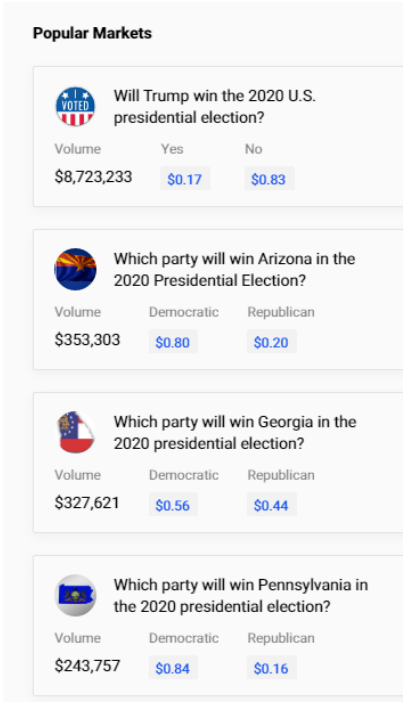
Election Day was two days ago. The winner of the election in many of the states of the US has been already determined, but in a few US states, finishing the vote count will require some time.

Assume that the winner has not yet been determined in only 3 US states:

Arizona (AZ), Georgia (GA), and Pennsylvania (PA)

Assume that, by some specified time $T > 0$, in each of those 3 states, exactly one of the two candidates will be revealed to be the winner: Trump the Republican, or Biden the Democrat.

There are contracts frictionlessly available at time 0 for the following prices in blue color: (Ignore the dollar amounts in black color listed under "Volume"; they are not relevant to this question).



(This is actual data from `polymarket.com` on 2020 November 5.)

Let us assign names to the above 8 contracts, and to make sure there is no confusion, let us write these contract names in the same ordering as in the above table.

`US.Trump`	`US.Biden`
`AZ.Biden`	`AZ.Trump`
`GA.Biden`	`GA.Trump`
`PA.Biden`	`PA.Trump`

(Be careful – Biden is listed on the right-hand side for the US election in the top row, but listed on the left-hand side in each individual state in the other three rows).

Each contract pays 1 dollar at time T if the referenced candidate wins the referenced state (or the US, for the contracts in the top row), and pays zero otherwise.

So, for example, the `GA.Trump` contract has a time-0 price of 0.44 dollars. It pays 1 dollar if Trump wins in Georgia, and pays 0 if Biden wins in Georgia.

Assume that **Trump wins the US Presidential election if and only if he wins all three states: Arizona, Georgia, Pennsylvania.**

Then arbitrage exists.

(a) Find an arbitrage that uses some or all of those 8 listed contracts, but does not use the bank account. In part (a), short-selling (holding negative quantities of an asset) is allowed for any of those 8 listed contracts.

(b) Find an arbitrage that uses some or all of those 8 listed contracts, but does not involve short-selling of any of those 8 listed contracts. In part (b) you are allowed to hold the bank account, in positive or negative quantities.

You may use either the type-1 or type-2 definition of arbitrage. Make no further assumptions regarding the probability distribution of the outcomes in the 3 states; your arbitrage must be valid, irrespective of the probability distributions of the outcomes in those 3 states (and irrespective of the joint distributions of those outcomes).

Problem 1

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Let $T > 0$. Wherever "Z" appears, it denotes a discount bond with maturity T.

Wherever "S" appears, it denotes a non-dividend-paying stock. Assume that $S_T \geq 0$. Unless otherwise directed, make no further assumptions about the distribution of S_T; your arbitrage must be valid regardless of the distribution of S_T.

(a) Exactly two assets are available: two bank accounts B and B^* with $B_0 = B_0^* = 1$ and $B_T = e^{rT}$ and $B_T^* = e^{r^*T}$ where $r < r^*$.

(-1 unit of B, 1 unit of B^*)

$V_0 = 0$, $V_T = e^{r^*T} - e^{rT} > 0$ since $r < r^*$ type 1 arbitrage

(b) Exactly three assets are available: A discount bond Z with $Z_0 = 0.9$; the stock S with $S_0 = 100.0$; and C, a European call on S with strike 110, expiry T, and time-0 price $C_0 = 0.50$.

(110 unit of Z, -1 unit of S, 1 unit of C)

$V_0 = 110 \times 0.9 + 0.5 - 100 = -0.5 < 0$

when $S_T \geq K$

$V_T = 110 - S_T + (S_T - 110) = 0$

when $S_T < K$

$V_T = 110 - S_T > 0$ type 2 arbitrage

(c) Exactly three assets are available: the stock S with $S_0 = 100.0$; a contract G that pays $\min(S_T, 110)$ at time T, and has time-0 price $G_0 = 85$; and C, a European call on S with strike 110, expiry T, and time-0 price $C_0 = 20$.

(1 unit of S, -1 unit of G, -1 unit of C)

$V_0 = 100 - 85 - 20 = -5 < 0$

when $S_T > 110$: $V_T = S_T - 110 - (S_T - 110) = 0$

when $S_T < 110$: $V_T = S_T - S_T = 0$ type 2 arbitrage

(d) Exactly four assets are available: A discount bond Z with $Z_0 = 0.9$; and three calls (all on the same underlying. The underlying is not available for you to trade). The calls have expiry T, strike $K \in \{20.0, 22.5, 25.0\}$, and time-0 price $C_0(K)$, where

$$C_0(20.0) = 6.40,\ C_0(22.5) = 3.10,\ C_0(25.0) = 1.00.$$

(2.5 unit of Z, -1 unit of strike-20 C, 1 unit of strike-22.5 C)

$V_0 = 2.5 \times 0.9 + 2.1 - 6.4 = -1.05 < 0$

when $S_T > 22.5$: $V_T = 2.5 + (S_T - 22.5) - (S_T - 20) = 0$

when $20 < S_T < 22.5$, $V_T = 2.5 + (S_T - 20) > 0$

when $S_T < 20$, $V_T = 2.5 > 0$ type 2 arbitrage

(e) Assume that $S_T > 0$ and $\mathsf{P}(S_T \neq 100) > 0$. Exactly 3 assets are available:
a contract X that pays $X_T := -2\log(S_T/100)$ at time T and has time-0 price $X_0 = 0.2$,
a forward contract Y on the underlying S, with delivery price 100 and delivery date T, and time-0 price $Y_0 = -10$,
and a discount bond Z with maturity T and time-0 price $Z_0 = 0.9$.

(50 unit of X, 1 unit of Y)

$V_0 = 50 \times 0.2 + (-10) \times 1 = 0$

when $S_T \geq 100$, $V_T = -2\log\left(\frac{S_T}{100}\right) + (S_T - 100)$

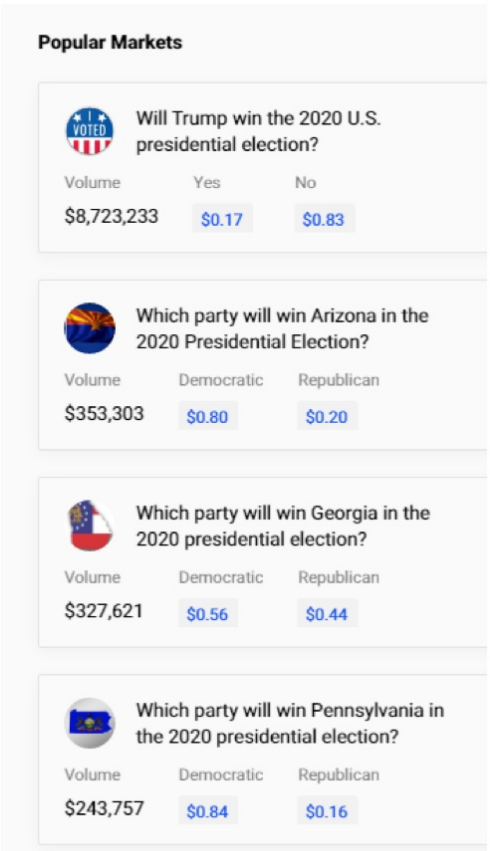
$= S_T - 100 - 100 \log\left(\frac{S_T}{100}\right)$

$\geq S_T - 100 + 100 - S_T = 0$

$(S_T - 100)$ increases faster than the decrease in $100 \log\left(\frac{S_T}{100}\right)$

when $S_T < 100$, $V_T = -2 \log\left(\frac{S_T}{100}\right) \cdot 50 > 0$

$\therefore S_T < 100$ Type 1 arbitrage



(a) Find an arbitrage that uses some or all of those 8 listed contracts, but does not use the bank account. In part (a), short-selling (holding negative quantities of an asset) is allowed for any of those 8 listed contracts.

(b) Find an arbitrage that uses some or all of those 8 listed contracts, but does not involve short-selling of any of those 8 listed contracts. In part (b) you are allowed to hold the bank account, in positive or negative quantities.

a. (1 unit US.Biden, −1 unit PA.Biden)

$V_0 = 0.83 - 0.84 = -0.01 < 0$

- when Trump wins 3 states: $V_T = 0 + 0 = 0 \geq 0$
- when Biden wins 2/3 including PA: $V_T = 1 - 1 = 0 \geq 0$
- when Biden wins excluding PA: $V_T = 1 - 0 = 1 > 0$

∴ type 2 arbitrage

b. (1 unit US.Biden, 1 unit PA.Trump, −1 borrow from Bank)

$V_0 = 0.83 + 0.16 - 1 = -0.01 < 0$

- when Trump wins 3 states: $V_T = 0 + 1 - 1 = 0 \geq 0$
- when Biden wins 2/3 including PA: $V_T = 1 + 0 - 1 = 0 \geq 0$
- when Biden wins excluding PA: $V_T = 1 + 1 - 1 = 1 > 0$

∴ type 2 arbitrage