FINM 34500/STAT 39000
Winter 2024
Problem Set 1 (due January 8)

Reading: Chapter 1 through Section 1.6 (we will not cover 1.7). Much of this material should be review since it was covered in FINM 34000. Indeed, much of Sections 2 and 5 of the FINM 34000 notes were cut-and-pasted from the text for this course. The first lecture will discuss the discrete stochastic integral and give a couple of problems on the material for this chapter, but your other job for the first week is to check that you know the rest of the material in the chapter. Next week we will be moving to Chapter 2.

Exercise 1 *Suppose $X_1, X_2, \ldots$ are independent random variables with*

$$\mathbb{P}\{X_j = 1\} = \frac{1}{4}, \quad \mathbb{P}\{X_j = 0\} = \frac{1}{4} \quad \mathbb{P}\{X_j = -1\} = \frac{1}{2}.$$

Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_n$ denote the information contained in $X_1, \ldots, X_n$.

1. *Which of these is S_n: martingale, submartingale, supermartingale (more than one answer is possible)?*

2. *For which values of r is $Y_n := S_n - rn$ a martingale?*

3. *For what $u > 1$ is it true that if $M_n := u^{S_n}$, then M_n is a martingale? For the remainder of this exercise, use this value of u.*

4. *Is $\{M_n\}$ a square integrable martingale?*

5. *Find*
$$\mathbb{E}\left[\sum_{j=1}^{20} (M_j - M_{j-1})^2\right].$$

6. *Find*
$$E[S_{20}^2 \mid \mathcal{F}_5] \quad E[M_{20}^2 \mid \mathcal{F}_5].$$

Exercise 2 *Suppose $X_1, X_2, \ldots$ are independent random variables each with a $N(0,1)$ distribution. Let $\mathcal{F}_n$ denote the information in $X_1, \ldots, X_n$, and let $A_1, A_2, \ldots$ be a sequence of random variables such that for each j, A_j is $\mathcal{F}_{j-1}$-measurable. Let $W_0 = 0$ and for $n > 0$,*

$$W_n = \sum_{j=1}^{n} A_j \, X_j.$$

1. *Suppose that $A_j = \sqrt{j}$. Explain why W_n has a mean zero normal distribution. What is $\mathbb{E}[W_n^2]$?*

2. *Suppose that* $A_1 = 1$ *and for* $n > 1$,

$$A_n = 1 \qquad \textit{if } A_{n-1} = 1 \textit{ and } W_{n-1} \leq 0,$$

and $A_n = 0$ *otherwise. Show that for all* $n > 1$, W_n *does not have a normal distribution. (Hint: consider* $\mathbb{E}[W_n]$ *and* $\mathbb{P}\{W_n > 0\}$*.)*

3. *Suppose that* $A_1 = 1$ *and* $A_n = W_{n-1}$ *for all* $n \geq 2$. *Let* T *be the first* n *such that* $W_n > 0$. *Explain why this is a stopping time* T *with respect to the filtration such that with probability one* $T < \infty$ *and* $W_T > 0$.

4. *Let* $M_n = W_{n \wedge T}$ *where* $n \wedge T = \min\{n, T\}$, *Is* M_n *a martingale with respect to the filtration?*

5. *Does the optional sampling theorem hold for* $\{M_n\}$ *and* T, *that is, is it true that* $\mathbb{E}[M_T] = \mathbb{E}[M_0]$*?*

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1. $E[X_j] = \frac{1}{4} \times 1 + \frac{1}{4} \times 0 + \frac{1}{2} \times (-1) = -\frac{1}{4}$

$E[S_{n+1} | F_n] = E[S_n + X_{n+1} | F_n]$

$= E[S_n | F_n] + E[X_{n+1}]$

$= S_n - \frac{1}{4} \leq S_n$

$\therefore$ S_n is supermartingale

2. $Y_n = S_n - rn$

For a martingale $E[Y_{n+1} | F_n] = Y_n$

$E[Y_{n+1} | F_n] = E[S_{n+1} - r(n+1) | F_n]$

$= E[S_n - rn + X_{n+1} - r | F_n]$

$= E[S_n - rn | F_n] + E[X_{n+1} - r | F_n]$

$= Y_n + E[X_{n+1} - r] \quad = Y_n$

$E[X_{n+1}] = r$

$-\frac{1}{4} = r$

3. $M_n = u^{S_n}$

$E[u^{X_i}] = \frac{1}{4} \cdot u^1 + \frac{1}{4} \cdot u^0 + \frac{1}{2} \cdot u^{-1} = \frac{1}{4}u + \frac{1}{4} + \frac{1}{2u}$

For a martingale $E[M_{n+1} | F_n] = M_n$

$E[M_{n+1} | F_n] = E[u^{S_n + X_{n+1}} | F_n]$

$= E[u^{S_n} \cdot u^{X_{n+1}} | F_n]$

$= M_n \cdot E[u^{X_{n+1}}] = M_n$

$\frac{1}{4}u + \frac{1}{4} + \frac{1}{2u} = 1$

$u^2 + u + 2 = 4u$

$u^2 - 3u + 2 = 0$

$(u-1)(u-2) = 0$

$u = 1$ or $u = 2$ since $u > 1$

$\therefore u = 2$

4. $E[4^{X_j}] = \frac{1}{4} \cdot 4^1 + \frac{1}{4} \cdot 4^0 + \frac{1}{2} \cdot 4^{-1} = 1 + \frac{1}{4} + \frac{1}{8} = \frac{11}{8}$

$E[M_n^2] = E[2^{S_n} \cdot 2^{S_n}]$

$= E[2^{S_n + S_n}] = E[2^{2S_n}]$

$= E[4^{S_n}]$

$= E[4^{(X_1 + \cdots + X_n)}]$

$= E[4^{X_1}] \cdots E[4^{X_n}] = (\frac{11}{8})^n < \infty$

$\therefore$ it is a square integrable martingale

5. For a square integrable martingale

$$E(M_n)^2 = E(M_0^2) + \sum_{j=1}^{n} E[(M_j - M_{j-1})^2]$$

$$\therefore E\left[\sum_{j=1}^{20}(M_j - M_{j-1})^2\right] = E[M_{20}^2] - E[M_0^2]$$

$$= \left(\frac{11}{8}\right)^{20} - 1$$

6. $E(X_j^2) = \frac{1}{4}\cdot 1^2 + \frac{1}{4}\cdot 0^2 + \frac{1}{2}\cdot(-1)^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$Var(X_j) = E(X_j^2) - (E(X_j))^2 = \frac{3}{4} - (\frac{1}{4})^2 = \frac{11}{16}$

$$E[S_{20}^2 \mid F_5] = E[(S_5 + (S_{20} - S_5))^2 \mid F_5]$$

$$= E[S_5^2 + 2S_5\cdot(S_{20} - S_5) + (S_{20} - S_5)^2 \mid F_5]$$

$$= S_5^2 + 2S_5 \cdot E[S_{20} - S_5] + E((S_{20} - S_5)^2)$$

$$= S_5^2 + 2S_5 \cdot 15E[X_j] + [Var(S_{20} - S_5) + (E(S_{20} - S_5))^2]$$

$$= S_5^2 + 2\cdot S_5\cdot(-\frac{15}{4}) + 15\cdot\frac{11}{16} + (-\frac{15}{4})^2$$

$$= S_5^2 - \frac{15}{2}S_5 + \frac{195}{8}$$

M_n is a martingale: $E(M_n) = E(M_0)$

$E(2^{X_j}) = \frac{1}{4}\cdot 2 + \frac{1}{4} + \frac{1}{2}\cdot 2 = 1$, $E[(2^{X_j})^2] = E[4^{X_j}] = \frac{11}{8}$

$Var(2^{X_j}) = E(4^{X_j}) - (E(2^{X_j}))^2 = \frac{3}{8} = Var(M_i)$

$$E(M_{20}^2 \mid F_5) = E[2^{2(S_5 + X_6 + \cdots + X_{20})} \mid F_5]$$

$$= E[2^{2S_5} \mid F_5]\cdot E[2^{2(X_6 + \cdots + X_{20})} \mid F_5]$$

$$= M_5 \cdot (E[4^{X_j}])^{15}$$

$$= M_5 \left(\frac{11}{8}\right)^{15}$$

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and $A_n = 0$ otherwise. Show that for all $n > 1$, W_n does not have a normal distribution. (Hint: consider $\mathbb{E}[W_n]$ and $\mathbb{P}\{W_n > 0\}$.)

3. *Suppose that $A_1 = 1$ and $A_n = W_{n-1}$ for all $n \geq 2$. Let T be the first n such that $W_n > 0$. Explain why this is a stopping time T with respect to the filtration such that with probability one $T < \infty$ and $W_T > 0$.*

4. *Let $M_n = W_{n \wedge T}$ where $n \wedge T = \min\{n, T\}$, Is M_n a martingale with respect to the filtration?*

5. *Does the optional sampling theorem hold for $\{M_n\}$ and T, that is, is it true that $\mathbb{E}[M_T] = \mathbb{E}[M_0]$?*

1. $E[W_n] = E[\sum_{j=1}^{n} A_j X_j] = \sum_{j=1}^{n} E[\sqrt{j} X_j] = \sum_{j=1}^{n} \sqrt{j} E[X_j] = \sum_{j=1}^{n} \sqrt{j} \cdot 0 = 0$

According to Linearity, Wn is also a normal distribution with 0 mean,

$E[W_n^2] = E[\sum_{j=1}^{n} A_j^2 X_j^2] = \sum_{j=1}^{n} E[j X_j^2] = \sum_{j=1}^{n} j E[X_j^2] = \sum_{j=1}^{n} j = \frac{(1+n) \cdot n}{2}$

2. $A_1 = 1$

$n > 1$, $A_n = \begin{cases} 1, & A_{n-1} = 1 \ \& \ W_{n-1} \leq 0 \\ 0, & \text{otherwise} \end{cases}$

when $A_n = 0$, $E[W_n] = 0$

when $A_n = 1$. $E[W_n] = E[\sum_{j=1}^{n} X_j) = \sum_{j=1}^{n} E[X_j] = 0$

$E[W_n] = E[W_0] = 0$ $\therefore$ Wn is a martingale

$\therefore$ If it has a normal distribution, $P\{W_n > 0\} = 0.5$.

$$P\{W_n>0\} = P\{W_n>0 \mid X_1>0\} P(X_1>0) + P\{W_n>0 \mid X_1 \le 0\} P(X_1 \le 0)$$

① when $X_1 > 0$ & $A_1 = 1$, $W_1 = A_1 X_1 = X_1 > 0 \Rightarrow A_2 = 0$, $W_2 = X_1 + 0 = X_1 > 0$

$\rightarrow A_3 = A_4 \cdots = A_n = 0$ and $W_1 = W_2 = \cdots\cdots = W_n = X_1 > 0$

$$\therefore P\{W_n > 0 \mid X_1 > 0\} = 1$$

② when $X_1 \le 0$ & $A_1 = 1$, $W_1 = A_1 X_1 = X_1 \le 0 \Rightarrow A_2 = 1$, $W_2 = X_1 + X_1 X_2$

$\therefore \quad 0 < P\{W_n > 0 \mid X_1 \le 0\} < 1$

$\therefore P\{W_n > 0\} = \frac{1}{2} \times 1 + \frac{1}{2} \times P\{W_n > 0 \mid X_1 \le 0\} > \frac{1}{2}$

$\therefore$ it is not symmetric $\therefore$ it does not have a normal distribution

3. $A_1 = 1$, $A_n = W_{n-1}$ for all $n \geq 2$

$$W_n = \sum_{j=1}^{n} A_j X_j = W_1 X_1 + W_1 X_2 + W_2 X_3 + \cdots + W_{n-1} X_n = W_{n-1} + W_{n-1} X_n$$

$T = \min\{W_n > 0\}$ is a positive integer

Also, For all n the event is F_n-measurable

$\therefore T$ is a stopping time

$$P(T=\infty) = P(W_n \le 0 \text{ for every } n)$$

$$= P(W_1 \le 0)\, P(W_2 \le 0 \mid W_1 \le 0)\, P(W_3 \le 0 \mid W_2 \le 0, W_1 \le 0) \cdots\cdots$$

$$= \prod_{j=1}^{\infty} P(W_j \le 0 \mid W_{j-1} \le 0)$$

$$= \prod_{j=1}^{\infty} P(W_{j-1} X_j \le 0 \mid W_{j-1} \le 0)$$

$$= \prod_{j=1}^{\infty} P(X_j > 0) = \left(\frac{1}{2}\right)^{\infty} = 0$$

$\therefore$ T is a stopping time wrt filtration that $P\{T<\infty \text{ and } W_T>0\}=1$

4. $M_n = W_{n\wedge T}$, $n\wedge T = \min\{n, T\}$

$$E[W_{n+1} \mid F_n] = E[\sum_{j=1}^{n+1} A_j X_j \mid F_n]$$

$$= E[\sum_{j=1}^{n} A_j X_j + W_n X_{n+1} \mid F_n]$$

$$= E[\sum_{j=1}^{n} A_j X_j \mid F_n] + W_n E[X_{n+1}]$$

$$= W_n + W_n \cdot 0 = W_n$$

$\therefore$ W_n is a martingale.

According to optional sampling theorem I, since T is a stopping time and W_n is a martingale with respect to $\{F_n\}$, $\therefore$ $M_n = W_{n\wedge T}$ is also a martingale.

5. $E[M_0] = E[W_0] = E[A_0 X_0] = A_0 E[X_0] = 0$

$E[M_T] = E[W_T] > 0$ because $P\{W_T > 0\} = 1$

$\therefore$ $E[M_0] \neq E[M_T]$

Also $E[|W_{n\wedge T}|^2]$ does not exist $C<\infty$ such that

$$E[|W_{n\wedge T}|^2] \le C$$

$\therefore$ $E[M_0] \neq E[M_T]$