FINM 34500/STAT 39000
Problem Set 2 (due January 15)
Reading: Notes through Section 2.8. 2.1 and 2.2 should be review, they were covered in FINM 34000. The material in the small font including all of Section 2.5, is optional.

Exercise 1 *Let B_t be a standard Brownian motion. Find the following probabilities. If you cannot give the answer precisely give it up to at least three decimal places using a table of the normal distribution.*

1. $\mathbb{P}\{B_2 \geq 1.5\}$
2. $\mathbb{P}\{B_1 \leq 1, B_2 - B_1 > 1\}$
3. $\mathbb{P}(E)$ *where E is the event that the path stays below the line $y = 3$ up to time $t = 6$.*
4. $\mathbb{P}\{B_3 \geq 0 \mid B_6 \geq 0\}$.
5. $\mathbb{P}\{B_1^2 \leq 2, (B_3 - B_1)^2 \geq 2, B_1(B_3 - B_1) \geq 0\}$.

Exercise 2 *Suppose B_t is a standard Brownian motion, $\lambda \in \mathbb{R}$, and let $\mathcal{F}_t$ be its corresponding filtration. Let*

$$M_t = e^{\lambda B_t - (\lambda^2/2)t}.$$

1. *Show that M_t is a martingale with respect to $\mathcal{F}_t$. In other words, show that if $s < t$, then*

   $$E(M_t \mid \mathcal{F}_s) = M_s.$$

   You may use the following fact. If $X \sim N(0,1)$, then the moment generating function of X is given by

   $$m(t) = \mathbb{E}\left[e^{tX}\right] = e^{t^2/2}. \tag{1}$$

2. *Find $\mathbb{E}[M_{18}]$ and $\mathbb{E}[M_{60} - 2M_5]$.*

Exercise 3 *Textbook, Exercise 2.6 (You may wish to use* (1) *when doing part 4.)*

Exercise 4 *Prove Theorem 2.6.3 from the notes. In other words, show that if B_t is a standard Brownian motion, $a > 0$, and*

$$Y_t = a^{-1/2}\, B_{at},$$

then Y_t is a standard Brownian motion. (You need to show that Y_t satisfies the four properties that a standard Brownian motion satisfies.)

Exercise 5 *Write a program that will sample from a standard Brownian motion using step size $\Delta t = 1/400$.*

- *Make a graph of* $B_t, 0 \leq t \leq 3$ *from your simulation for one trial.*
- *Repeat the simulation* 1000 *times to give estimates for the following:*

$$\mathbb{P}\{\max_{0 \leq t \leq 3} B_t > 1\},$$

$$\mathbb{P}\{B_{1.5} > 0, B_3 > 0\}.$$

Find the exact values of these probabilities and compare your simulations with the exact values. You may use any computer language and/or package that you choose.

Exercise 1 *Let B_t be a standard Brownian motion. Find the following probabilities. If you cannot give the answer precisely give it up to at least three decimal places using a table of the normal distribution.*

1. $\mathbb{P}\{B_2 \geq 1.5\}$
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3. $\mathbb{P}(E)$ *where E is the event that the path stays below the line $y = 3$ up to time $t = 6$.*
4. $\mathbb{P}\{B_3 \geq 0 \mid B_6 \geq 0\}$.
5. $\mathbb{P}\{B_1^2 \leq 2, (B_3 - B_1)^2 \geq 2, B_1(B_3 - B_1) \geq 0\}$.

$\because$ Bt is a standard Brownian motion

$$B_t = \frac{1}{\sqrt{a}} B_{at} \Rightarrow B_{at} = \sqrt{a} B_t$$

1. $P\{B_2 \geq 1.5\} = P\{\sqrt{2} B_1 \geq 1.5\}$

$= P\{B_1 \geq \frac{1.5}{\sqrt{2}}\}$

$= 1 - P\{B_1 \leq \frac{1.5}{\sqrt{2}}\} = 1 - P(Z < 1.061) = 1 - 0.8554 = 0.1446$

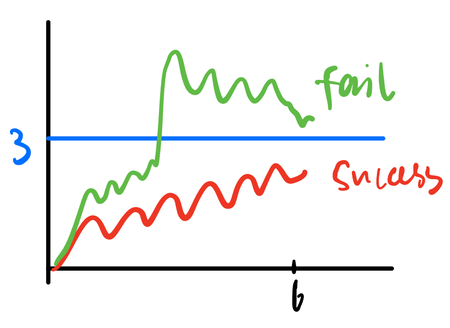
2. $P\{B_1 \leq 1, B_2 - B_1 > 1\} = P\{B_1 \leq 1\} \cdot P\{B_2 - B_1 > 1\}$

$= P\{B_1 \leq 1\} P\{B_{2-1} > 1\}$

$= P\{B_1 \leq 1\} P\{B_1 > 1\}$

$= 0.8413 \times (1 - 0.8413) = 0.1335$

3.



$P\{B_t \geq a\} = \frac{1}{2} P\{T_a \leq t\}$

$\therefore$ For P(path stay below 3 up to time 6)

$$= 1 - 2P\{B_6 \geq 3\}$$

$$= 1 - 2P\{\sqrt{6}B_1 \geq 3\}$$

$$= 1 - 2P\{B_1 \geq \frac{3}{\sqrt{6}}\} = 1 - 2[1 - P\{B_1 \leq \frac{3}{\sqrt{6}}\}]$$

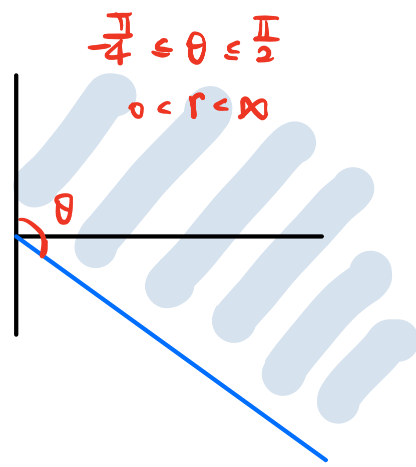
$$= 2P\{B_1 \leq 1.225\} - 1 = 0.7795$$

4. $P\{B_3 \geq 0 \mid B_6 \geq 0\}$

$$= P\{B_3 \geq 0, B_6 \geq 0\} / P\{B_6 \geq 0\}$$

$$= P\{B_1 \geq 0, B_2 \geq 0\} / P\{\sqrt{6}B_1 \geq 0\}$$

$$= \int_0^\infty \underbrace{P\{B_2 \geq 0 \mid B_1 \geq x\}}_{= P\{B_2 - B_1 \geq -x\}} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \;/ P\{B_1 \geq 0\}$$



$$= \int_0^\infty \int_{-x}^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \;/ \frac{1}{2}$$

$$= \int_0^\infty \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2\pi} e^{-r^2/2} r \, d\theta \, dr \cdot 2$$

$$= \int_0^\infty \frac{1}{2\pi} e^{-r^2/2} \cdot r \left(\frac{\pi}{2} - \left(-\frac{\pi}{4}\right)\right) dr \cdot 2$$

$$= \int_0^\infty \frac{1}{2\pi} e^{-r^2/2} r \left(\frac{3}{4}\pi\right) dr \cdot 2$$

$$= \frac{3}{2}\pi \cdot \lim_{r \to \infty} -\frac{1}{2\pi} \cdot e^{-r^2/2} - \frac{3}{2}\pi \left(-\frac{1}{2\pi} e^{-0/2}\right)$$

$$= 0 + \frac{3}{2}\pi \cdot \frac{1}{2\pi} = \frac{3}{4}$$

5. $P\{B_1^2 \le 2,\ (B_3-B_1)^2 \ge 2,\ B_1(B_3-B_1) \ge 0\}$

$= P\{B_1^2 \le 2,\ B_2^2 \ge 2,\ B_1 B_2 \ge 0\}$

$$\begin{cases} -\sqrt{2} \le B_1 \le \sqrt{2} \\ B_2 \ge \sqrt{2} \text{ or } B_2 \le -\sqrt{2} \end{cases}$$

when $B_2 \ge \sqrt{2}$, $0 \le B_1 \le \sqrt{2}$

when $B_2 \le -\sqrt{2}$, $-\sqrt{2} \le B_1 \le 0$

$= P\{B_2 \ge \sqrt{2},\ 0 \le B_1 \le \sqrt{2}\} + P\{B_2 \le -\sqrt{2},\ -\sqrt{2} \le B_1 \le 0\}$

$= P\{\sqrt{2} B_1 \ge \sqrt{2}\} \cdot [P\{B_1 \le \sqrt{2}\} - P\{B_1 \le 0\}] +$

$P\{\sqrt{2} B_1 \le -\sqrt{2}\}\ [P\{B_1 \le 0\} - P\{B_1 \le -\sqrt{2}\}]$

$= (1-0.8413)\cdot[0.9215 - 0.5] + 0.1587 \cdot [0.5 - 0.0786]$

$= 0.1338$

Exercise 2 *Suppose B_t is a standard Brownian motion, $\lambda \in \mathbb{R}$, and let $\mathcal{F}_t$ be its corresponding filtration. Let*

$$M_t = e^{\lambda B_t - (\lambda^2/2)t}.$$

1. *Show that M_t is a martingale with respect to $\mathcal{F}_t$. In other words, show that if $s < t$, then*
$$E(M_t \mid \mathcal{F}_s) = M_s.$$
You may use the following fact. If $X \sim N(0,1)$, then the moment generating function of X is given by
$$m(t) = \mathbb{E}\left[e^{tX}\right] = e^{t^2/2}. \tag{1}$$

2. *Find $\mathbb{E}[M_{18}]$ and $\mathbb{E}[M_{60} - 2M_5]$.*

1. $E[M_t | F_s] = E[e^{\lambda B_t - (\lambda^2/2)t} | F_s]$

$= E[e^{\lambda[B_s + (B_t - B_s)] - (\lambda^2/2)[s + (t-s)]} | F_s]$

$= e^{\lambda B_s - \lambda^2/2 \cdot s} \cdot E[e^{\lambda(B_t - B_s) - (\lambda^2/2)(t-s)} | F_s]$

$= M_s \cdot E[e^{\lambda[B_t - B_s]}] \cdot E[e^{-(\lambda^2/2)(t-s)}]$

$= M_s \cdot e^{-(\lambda^2/2)(t-s)} \cdot E[e^{\lambda[\frac{B_t - B_s}{\sqrt{t-s}}] \cdot \sqrt{t-s}}]$

$\because B_t - B_s \sim N(0, t-s)$

$\therefore \frac{B_t - B_s}{\sqrt{t-s}} \sim N(0,1)$

$\therefore$ according to moment generating function

$= M_s \; e^{-(\lambda^2/2)(t-s)} \cdot (e^{\lambda^2/2 (t-s)})$

$= M_s \; e^{-(\lambda^2/2)(t-s) + (\lambda^2/2)(t-s)} = M_s$

$\therefore M_t$ is a martingale

2. $E[M_{18}] = E[M_0] = E[e^{\lambda B_0 - (\lambda^2/2) \cdot 0}]$

$= E[e^{\lambda \cdot B_0}] = e^{\lambda E(B_0)} = 1$

$E[M_{60} - 2M_5] = E[M_{60}] - 2E[M_5]$

$= E[M_0] - 2E[M_0] = 1 - 2 = -1$

Exercise 2.6. Let B_t be a standard Brownian motion and let $\{\mathcal{F}_t\}$ denote the usual filtration. Suppose $s < t$. Compute the following.

1. $E[B_t^2 \mid \mathcal{F}_s]$
2. $E[B_t^3 \mid \mathcal{F}_s]$
3. $E[B_t^4 \mid \mathcal{F}_s]$
4. $E[e^{4B_t - 2} \mid \mathcal{F}_s]$

1. $E[B_t^2 \mid F_s]$

$= E[(B_s + (B_t - B_s))^2 \mid F_s]$

$= E[B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2 \mid F_s]$

$= B_s^2 + 2B_s E[B_t - B_s] + E[(B_t - B_s)^2]$

$= B_s^2 + [\mathrm{var}(B_t - B_s) + (E(B_t - B_s))^2]$

$= B_s^2 + (t-s) + 0 = B_s^2 + (t-s)$

2. $E[B_t^3 \mid F_s]$

$= E[(B_s + (B_t - B_s))^3 \mid F_s]$

$= E[B_s^3 + 3B_s^2(B_t - B_s) + 3B_s(B_t - B_s)^2 + (B_t - B_s)^3 \mid F_s]$

$= B_s^3 + 3B_s^2 E(B_t - B_s) + 3B_s E((B_t - B_s)^2) + E((B_t - B_s)^3)$

$= B_s^3 + 3B_s^2 \cdot 0 + 3B_s(t-s) + E[(B_t - B_s)^3]$

$= E[(\sqrt{t-s}\, Z)^3] = (t-s)^{\frac{3}{2}} E[Z^3] = 0$

third moment of standard normal $= 0$

$= B_s^3 + 3B_s(t-s)$

3. $E[B_t^4 | F_s]$

$= E[(B_s + (B_t - B_s))^4 | F_s]$

$= E[B_s^4 + 3B_s^3(B_t - B_s) + 6B_s^2(B_t - B_s)^2 + 3B_s(B_t - B_s)^3 + (B_t - B_s)^4 | F_s]$

$= B_s^4 + 3B_s^3 E(B_t - B_s) + 6B_s^2 E[(B_t - B_s)^2] + 3B_s E[(B_t - B_s)^3] + E[(B_t - B_s)^4]$

$\Downarrow$

$E[(\sqrt{t-s}\, Z)^4] = (t-s)^2 E[Z^4] = 3(t-s)^2$

forth moment of standard normal $= 3$

$= B_s^4 + 6B_s^2(t-s) + 3(t-s)^2$

4. $E[e^{4B_t - 2} | F_s]$

$= E[e^{4[B_s + (B_t - B_s)) - 2} | F_s]$

$= e^{4B_s - 2} \cdot E[e^{4(B_t - B_s)}]$

$= e^{4B_s - 2} \cdot E[e^{\frac{4(B_t - B_s)}{\sqrt{t-s}} \sqrt{t-s}}]$ $\quad \because \frac{(B_t - B_s)}{\sqrt{t-s}} \sim N(0,1)$

$= e^{4B_s - 2} \cdot e^{4^2/2 \cdot (t-s)}$

$= e^{4B_s - 2 + 8(t-s)}$

Exercise 4 *Prove Theorem 2.6.3 from the notes. In other words, show that if B_t is a standard Brownian motion, $a > 0$, and*

$$Y_t = a^{-1/2} B_{at},$$

then Y_t is a standard Brownian motion. (You need to show that Y_t satisfies the four properties that a standard Brownian motion satisfies.)

1. $Y_0 = \frac{1}{\sqrt{a}} B_{a \cdot 0} = \frac{1}{\sqrt{a}} B_0$

$\because B_0 = 0 \quad \therefore Y_0 = 0$

2. $Y_t - Y_s = \frac{1}{\sqrt{a}} B_{at} - \frac{1}{\sqrt{a}} B_{as}$

$= \frac{1}{\sqrt{a}}(B_{at} - B_{as}) = \frac{1}{\sqrt{a}}(B_{a(t-s)})$

$E[Y_t - Y_s] = \frac{1}{\sqrt{a}} E[B_{at} - B_{as}] = 0$

$Var[Y_t - Y_s] = Var(\frac{1}{\sqrt{a}} B_{a(t-s)}) = (\frac{1}{\sqrt{a}})^2 \cdot a(t-s) = \frac{1}{a} a(t-s) = t-s$

it follows stationary / normal increments

3. $Y_t - Y_s = \frac{1}{\sqrt{a}}(B_{at} - B_{as})$

$\because$ $B_t - B_s$ has independent increment

$B_{at} - B_{as}$ also has independent increment

$\therefore$ $Y_t - Y_s = \frac{1}{\sqrt{a}}(B_{at} - B_{as})$ has independent increment

4. $\because$ B_t has continuous path

$Y_t = \frac{1}{\sqrt{a}} B_{at}$ is a scaling and time-change of B_t,

$\therefore$ Y_t is also a continuous path

$\therefore$ Y_t satisfies all four properties. so it is a standard Brownian Motion

Exercise 5 *Write a program that will sample from a standard Brownian motion using step size $\Delta t = 1/400$.*

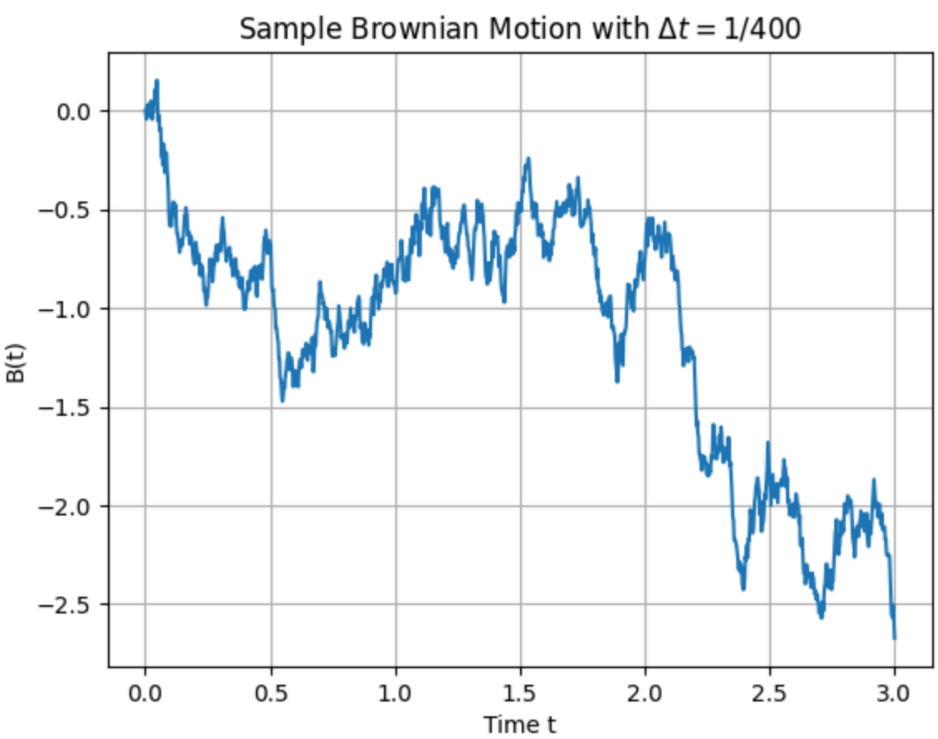
- *Make a graph of $B_t, 0 \le t \le 3$ from your simulation for one trial.*
- *Repeat the simulation 1000 times to give estimates for the following:*

$$\mathbb{P}\{\max_{0 \le t \le 3} B_t > 1\},$$

$$\mathbb{P}\{B_{1.5} > 0, B_3 > 0\}.$$

Find the exact values of these probabilities and compare your simulations with the exact values. You may use any computer language and/or package that you choose.

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
delta_t = 1/400  # Step size
T = 3  # Total time
N = int(T / delta_t)  # Number of steps

# Initialize the Brownian motion
B = np.zeros(N + 1)

# Generate the steps of the Brownian motion
for i in range(1, N + 1):
    # Increment with a normally distributed step
    B[i] = B[i-1] + np.random.normal(0, np.sqrt(delta_t))

# Time vector for plotting
t = np.linspace(0, T, N + 1)

# Plot the Brownian motion
plt.plot(t, B)
plt.title('Sample Brownian Motion with $\Delta t = 1/400$')
plt.xlabel('Time t')
plt.ylabel('B(t)')
plt.grid(True)
plt.show()
```



```python
# Number of simulations
num_simulations = 1000

# Counters for the probabilities
count_max_Bt_greater_than_1 = 0
count_B1_5_and_B3_positive = 0

# Simulate the Brownian motion 1000 times
for _ in range(num_simulations):
    B = np.zeros(N + 1)
    for i in range(1, N + 1):
        B[i] = B[i-1] + np.random.normal(0, np.sqrt(delta_t))

    # Check the conditions
    if np.max(B) > 1:
        count_max_Bt_greater_than_1 += 1
    if B[int(1.5/delta_t)] > 0 and B[N] > 0:  # B at t=1.5 and t=3
        count_B1_5_and_B3_positive += 1

# Estimate probabilities
prob_max_Bt_greater_than_1 = count_max_Bt_greater_than_1 / num_simulations
prob_B1_5_and_B3_positive = count_B1_5_and_B3_positive / num_simulations

prob_max_Bt_greater_than_1, prob_B1_5_and_B3_positive
```

```
(0.534, 0.362)
```

In simulation $P(\max_{0\le t\le 3} B_t > 1) = 0.534$

$P(B_{1.5} > 0, B_3 > 0) = 0.362$

$$P\{\max_{0\le t\le 3} B_t > 1\} = 2P\{B_3 > 1\}$$

$$= 2P\{\sqrt{3}B_1 > 1\}$$

$$= 2P\{B_1 > \frac{1}{\sqrt{3}}\} = 2(1 - P(Z < 0.58))$$

$$= 2(1 - 0.71904) = 0.562$$

$$P\{B_{1.5} > 0, B_3 > 0\}$$

$$= P(B_1 > 0, B_2 > 0)$$

$$= \frac{3}{8} = 0.375$$

Both of these values are similar to the simulations