FINM 34500/STAT 39000
Winter 2024
Problem Set (due January 22)

Reading: Sections 2.8 — 2.10.

Exercise 1 *Let $f(t)$ be a continuous function for $0 \leq t \leq 1$ and let*

$$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \right|^{3/2}.$$

What is Q

1. *If $f(t) = t^3$?*
2. *If $f(t) = 3\, B_t$ where B_t is a standard Brownian motion.*

Exercise 2 *Suppose B_t, W_t are independent standard Brownian motions.*

1. *Let $Y_t = B_t + 4\, W_t - 2t$. Show that Y_t is a (one-dimensional) Brownian motion starting at the origin. What are the drift and variance parameter?*
2. *Let $Z_t = (Z_t^1, Z_t^2)$ denote the random vector where*

   $$Z_t^1 = B_t + 2W_t, \qquad Z_t^2 = -B_t + 3W_t.$$

   Explain why Z_t is a two-dimensional Brownian motion starting at the origin with zero drift. What is the covariance matrix Γ?
3. *Find*

   $$\langle Z^1 \rangle_t, \qquad \langle Z^1, Z^2 \rangle_t.$$

Exercise 3 *Let B_t, W_t be independent standard Brownian motions. Find*

$$\mathbb{P}\{B_t \geq 2W_t - 1 \text{ for all } 0 \leq t \leq 1\}.$$

Hint: You may wish to consider $2W_t - B_t$.

Exercise 1 *Let $f(t)$ be a continuous function for $0 \le t \le 1$ and let*

$$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \right|^{3/2}.$$

What is Q

1. *If $f(t) = t^3$?*
2. *If $f(t) = 3\,B_t$ where B_t is a standard Brownian motion.*

1. $$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| \left(\frac{j}{n}\right)^3 - \left(\frac{j-1}{n}\right)^3 \right|^{\frac{3}{2}}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \left[\frac{j^3 - (j^3 - 3j^2 + 3j - 1)}{n^3} \right]^{\frac{3}{2}}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \left[\frac{3j^2 - 3j + 1}{n^3} \right]^{\frac{3}{2}}$$

$\because \frac{3j^2 - 3j + 1}{n^3} \to 0$ when $n \to \infty$

$= 0$

OR $$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| \left(\frac{j}{n}\right)^3 - \left(\frac{j-1}{n}\right)^3 \right|^{\frac{3}{2}}$$

$$\le \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{j}{n}\right)^{3\cdot\frac{3}{2}} \le \lim_{n\to\infty} n \cdot \left(\frac{j}{n}\right)^{\frac{9}{2}} = \lim_{n\to\infty} \frac{j^{\frac{9}{2}}}{n^{\frac{7}{2}}} = 0$$

2. $$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| 3B\left(\frac{j}{n}\right) - 3B\left(\frac{j-1}{n}\right) \right|^{\frac{3}{2}}$$

$$= 3^{\frac{3}{2}} \lim_{n\to\infty} \sum_{j=1}^{n} \left| B\left(\frac{j}{n}\right) - B\left(\frac{j-1}{n}\right) \right|^{\frac{3}{2}}$$

$\because \lim_{n\to\infty} \sum_{j=1}^{n} \left| B\left(\frac{j}{n}\right) - B\left(\frac{j-1}{n}\right) \right|^2 = 1$ for $\left| B\left(\frac{j}{n}\right) - B\left(\frac{j-1}{n}\right) \right|$ small value

the $\lim_{n\to\infty} \sum_{j=1}^{n} \left| B\left(\frac{j}{n}\right) - B\left(\frac{j-1}{n}\right) \right|^{\frac{3}{2}} \gg \lim_{n\to\infty} \sum_{j=1}^{n} \left| B\left(\frac{j}{n}\right) - B\left(\frac{j-1}{n}\right) \right|^2$

$\therefore$ $Q \to \infty$ when $n \to \infty$ $\quad Q = \infty$

Exercise 2 *Suppose B_t, W_t are independent standard Brownian motions.*

1. *Let $Y_t = B_t + 4W_t - 2t$. Show that Y_t is a (one-dimensional) Brownian motion starting at the origin. What are the drift and variance parameter?*

2. *Let $Z_t = (Z_t^1, Z_t^2)$ denote the random vector where*

$$Z_t^1 = B_t + 2W_t, \qquad Z_t^2 = -B_t + 3W_t.$$

Explain why Z_t is a two-dimensional Brownian motion starting at the origin with zero drift. What is the covariance matrix Γ?

3. *Find*

$$\langle Z^1 \rangle_t, \qquad \langle Z^1, Z^2 \rangle_t.$$

1. ① $Y_0 = B_0 + 4W_0 - 2 \cdot 0 = 0 + 4 \cdot 0 - 0 = 0$

② $\because$ B_t, W_t is continuous $Y_t = B_t + 4W_t - 2t$ is also continuous

③ $Y_t - Y_s = (B_t - B_s) + 4(W_t - W_s) - 2(t-s)$

$\because$ $B_t - B_s$ & $W_t - W_s$ has independent increments

$\therefore$ $Y_t - Y_s$ has independent increments

④ $Y_t - Y_s = (B_t - B_s) + 4(W_t - W_s) - 2(t-s)$

$\because$ $B_t - B_s$ & $W_t - W_s$ is normally distributed

$\therefore$ $Y_t - Y_s$ is normally distributed

drift $= -2$

$Var(Y_t) = Var(B_t) + Var(4W_t) - Var(2t)$

$= t + 16 Var(W_t) - 4Var(t)$

$= t + 16t - 0 = 17t$

2. ①② $\because$ $B_t - B_s$, $W_t - W_s$ has independent increments and is normally distributed

$\therefore\ Z_t^1 - Z_s^1 = (B_t - B_s) + 2(W_t - W_s)$

$Z_t^2 - Z_s^2 = -(B_t - B_s) + 3(W_t - W_s)$ also has independent increments and is normally distribution since they are linear combination of standard BM.

③ $Z_0^1 = B_0 + 2W_0 = 0$

$Z_0^2 = -B_0 + 3W_0 = 0 \qquad Z_0 = (Z_0^1, Z_0^2) = (0, 0)$

④ $\because B_t, W_t$ is continuous

$\therefore Z_t^1 = B_t + 2W_t,\ Z_t^2 = -B_t + 3W_t$ is also continuous

$\because$ for standard BM, B_t, W_t has drift 0

$\therefore Z_t^1, Z_t^2$ has drift 0

$\therefore Z_t$ is a two-dimensional BM starting at origin with 0 drift

$Var(Z_t^1) = Var(B_t) + Var(2W_t) = t + 4t = 5t$

$Var(Z_t^2) = Var(-B_t) + Var(3W_t) = t + 9t = 10t$

$Cov(Z_t^1, Z_t^2) = Cov(B_t + 2W_t, -B_t + 3W_t)$

$= Cov(B_t, -B_t) + \underbrace{Cov(B_t, 3W_t)}_{=0} + \underbrace{Cov(2W_t, -B_t)}_{=0} + Cov(2W_t, 3W_t)$

$\because$ independent

$= -Var(B_t) + 6Var(W_t) = -t + 6t = 5t$

$\therefore$ covariance matrix $= \begin{bmatrix} Var(Z_t^1) & Cov \\ Cov & Var(Z_t^2) \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}$

OR $\quad \Gamma = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} B_t \\ W_t \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}$

3. $\langle Z^1 \rangle_t = \langle B_t + 2W_t \rangle_t$

$$= \lim_{n\to\infty} \sum_{j\le nt} \left[B(\tfrac{j}{n}) + 2W(\tfrac{j}{n}) - B(\tfrac{j-1}{n}) - 2W(\tfrac{j-1}{n}) \right]^2$$

$$= \lim_{n\to\infty} \sum_{j\le nt} \left(B(\tfrac{j}{n})^2 - 2B(\tfrac{j}{n})B(\tfrac{j-1}{n}) + B(\tfrac{j-1}{n})^2 \right)$$

$$+ \lim_{n\to\infty} \sum_{j\le nt} \left(4W(\tfrac{j}{n})^2 - 8W(\tfrac{j}{n})W(\tfrac{j-1}{n}) + W(\tfrac{j-1}{n})^2 \right)$$

other term = 0 because W B are independent

$$= \langle B_t \rangle_t + \langle 2W_t \rangle_t$$

$$= t + 2^2 t = 5t$$

OR $\langle Z^1 \rangle_t = \langle Z^1, Z^1 \rangle_t = \Gamma_{11} t = 5t$

$\langle Z^1, Z^2 \rangle_t = \langle B_t + 2W_t, -B_t + 3W_t \rangle_t$

$$= \langle B_t, -B_t \rangle_t + \underbrace{\langle B_t, 3W_t \rangle_t}_{=0}$$

independent

$$+ \underbrace{\langle 2W_t, -B_t \rangle_t}_{=0} + \langle 2W_t, 3W_t \rangle_t$$

$$= -t + 6t = 5t$$

OR $\langle Z^1, Z^2]_t = \Gamma_{12} t = 5t$

Exercise 3 *Let B_t, W_t be independent standard Brownian motions. Find*

$$\mathbb{P}\{B_t \geq 2W_t - 1 \text{ for all } 0 \leq t \leq 1\}.$$

Hint: You may wish to consider $2W_t - B_t$.

$P\{B_t \geq 2W_t - 1\}$

$= P\{2W_t - B_t \leq 1\}$

$= P\{X_t \leq 1\}$ where $X_t = 2W_t - B_t$

$E(X_t) = 0$

$Var(X_t) = Var(2W_t) + Var(B_t) = 4t + t = 5t$

in order to change to standard BM

$\frac{X_t}{\sqrt{5}}$ is a standard BM with drift 0 and variance parameter t

$\therefore P(X_t \leq 1 \text{ for all } 0 \leq t \leq 1)$

$= 1 - P\{\max_{0 \leq t \leq 1} X_t \geq 1\}$

$= 1 - P\{\max_{0 \leq t \leq 1} \frac{X_t}{\sqrt{5}} \geq \frac{1}{\sqrt{5}}\}$

$= 1 - 2P\{Z \geq \frac{1}{\sqrt{5}}\}$

$= 1 - 2[1 - P(Z < 0.447)] = 2 \times 0.6736 - 1 = 0.3472$