FINM 34500/STAT 39000
Winter 2024
Problem Set 4 (due January 29)

Reading: Sections 2.10 – 3.2.

Exercise 1 *Let B_t be a standard (one-dimensional) Brownian motion (not necessarily starting at the origin). For the following functions $\phi(t,x), 0 < t < 3$, state the PDE that it satisfies. If you use the L or L^* notation, you must say what L or L^* is in these cases.*

1. *$\phi(t,x)$ is the density of B_t (as a function of x) given that $B_0 = 0$.*
2. *$\phi(t,x) = \mathbb{P}\{B_t < 4 \mid B_0 = x\}$*
3. *$\phi(t,x) = \mathbb{E}[B_t^3 \mid B_0 = x\}$*
4. *$\phi(t,x) = \mathbb{E}[B_3 - B_3^2 \mid B_t = x]$*
5. *Repeat the examples above where B has drift 2 and variance parameter 9.*

Exercise 2 *Suppose B_t, W_t are independent standard Brwonian motions starting at the origin, and the random vector $Z_t = (Z_t^1, Z_t^2)$ is defined by*

$$Z_t^1 = B_t + W_t - t, \qquad Z_t^2 = 2\,B_t - 6\,W_t.$$

Note that Z_t is a two-dimensional Brownian motion.

1. *What are the drift and covariance matrix for Z?*
2. *What are the operators L, L^* associated to Z?*
3. *Let $\phi(t,x), t > 0, x \in \mathbb{R}^2$ be the density of Z_t at time t. Find the PDE satisfied by $\phi(t,x)$.*

Exercise 3 *For each of these problems $\{A_t\}$ will be a simple process that only changes at time $1/3$, that is*

$$A_t = 1, \qquad 0 \leq t < 1/3.$$

$$A_t = Y, \qquad 1/3 \leq t \leq 1,$$

where Y is a random variable measurable with respect to $\mathcal{F}_{1/3}$. Let

$$Z_t = \int_0^t A_s\, dB_s.$$

For each of these examples of Y, find $\mathbb{P}\{Z_{2/3} \geq 0\}$.

1.
$$Y = \begin{cases} 0 & B_{1/3} \geq 0 \\ 1 & B_{1/3} < 0 \end{cases}$$

2.
$$Y = \begin{cases} 0 & B_{1/3} \geq 0 \\ 2 & B_{1/3} < 0 \end{cases}$$

3.
$$Y = \begin{cases} -1 & B_{1/3} \geq 0 \\ 1 & B_{1/3} < 0 \end{cases}$$

Exercise 4 *For the three cases in the last exercise, give*

$$\langle Z \rangle_t = \int_0^t A_s^2 \, ds.$$

In each case, the answer should look like two functions of t — one for the event $B_{1/3} \geq 0$ and one on the event $B_{1/3} < 0$.

Exercise 5 *For Case #2 in the last two exercises, verify directly the statement*

$$\mathrm{Var}[Z_1] = \int_0^1 \mathbb{E}[A_t^2] \, dt.$$

Exercise 1 *Let B_t be a standard (one-dimensional) Brownian motion (not necessarily starting at the origin). For the following functions $\phi(t,x), 0 < t < 3$, state the PDE that it satisfies. If you use the L or L^* notation, you must say what L or L^* is in these cases.*

1. *$\phi(t,x)$ is the density of B_t (as a function of x) given that $B_0 = 0$.*
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4. *$\phi(t,x) = \mathbb{E}[B_3 - B_3^2 \mid B_t = x]$*
5. *Repeat the examples above where B has drift 2 and variance parameter 9.*

1. $\phi(t,x)$ is density of B_t with drift 0 and variance parameter 1.

∴ according to the fact: $\frac{\partial}{\partial t}\phi(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(t,x)$

$\dot{\phi}(t,x) = L^*\phi(t,x)$ where $L^* f(x) = \frac{1}{2}f''(x)$

2. Let density of B_t be $P_t(t,x)$

$$\phi(t,x) = P\{B_t < 4 \mid B_0 = x\}$$
$$= \int_{-\infty}^{4} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$

$$\dot{\phi}(t,x) = \frac{\partial}{\partial t}\int_{-\infty}^{4} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$
$$= \int_{-\infty}^{4} \dot{P_t}(t,x)\, dy$$

$\because \dot{P_t}(t,x) = \frac{1}{2}\frac{\partial}{\partial x^2}P_t(t,x)$

$= L_x P_t(t,x)$

where $L_x = \frac{1}{2}f''(x)$

$$= \int_{-\infty}^{4} L_x P_t(t,x)\, dy$$
$$= L_x \int_{-\infty}^{4} P_t(t,x)\, dy$$
$$= L_x\, \phi(t,x)$$ where $L_x f(x) = \frac{1}{2}f''(x)$

3. $\phi(t,x) = E[B_t^3 \mid B_0 = x]$

$$= \int_{-\infty}^{\infty} B_t^3\, P_t(t,x)\, dy$$

$$\phi(\hat{t}, x) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} B_t^3 \, P_t(t, x)\, dy$$

$$= \int_{-\infty}^{\infty} B_t^3 \, \dot{P}_t(t, x)\, dy$$

$$= \int_{-\infty}^{\infty} B_t^3 \cdot L_x P_t(t, x)\, dy$$

$$= L_x \int_{-\infty}^{\infty} B_t^3 P_t(t, x)\, dy = L_x\, \phi(t, x) \quad \text{where } L_x f(x) = \frac{1}{2} f''(x)$$

4. $\phi(t, x) = E[B_3 - B_3^2 \mid B_t = x]$

Let $u(t, x) = E[B_t - B_t^2 \mid B_0 = x]$

$\therefore \quad \phi(t, x) = u(3-t, x)$

$\therefore \quad \dot{\phi}(t, x) = -\dot{u}(3-t, x)$

$= -L_x\, u(3-t, x) = -L_x\, \phi(t, x)$

where $L_x f(x) = \frac{1}{2} f''(x)$

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when B has drift 2 and variance parameter 9.

$L^* f(x) = -m f'(x) + \frac{\sigma^2}{2} f''(x) = -2f'(x) + \frac{9}{2} f''(x)$

$L f(x) = m f'(x) + \frac{\sigma^2}{2} f''(x) = 2f'(x) + \frac{9}{2} f''(x)$

1. $\phi(t, x)$ is density of B_t with drift 0 and variance parameter 1.

$\therefore$ according to the fact: $\quad \frac{\partial}{\partial t}\phi(t, x) = \frac{9}{2}\, \partial_{xx}\phi(t, x) - 2\, \partial_x \phi(t, x)$

$= L^* \phi(t, x)$

where $L^* f(x) = \frac{9}{2} f''(x) - 2f'(x)$

2. Let density of B_t be $P_t(t, x)$

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$$\phi(t,x) = P\{B_t < 4 \mid B_0 = x\}$$

$$= \int_{-\infty}^{4} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x-2t)^2}{2t}} dy$$

$$\dot{\phi}(t,x) = \frac{\partial}{\partial t} \int_{-\infty}^{4} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x-2t)^2}{2t}} dy$$

$$= \int_{-\infty}^{4} \dot{P_t}(t,x)\, dy \qquad \because \dot{P_t}(t,x) = \frac{9}{2}\frac{\partial}{\partial x^2} P_t(t,x) - 2\partial_x P_t(t,x)$$

$$= L_x P_t(t,x)$$

$$= \int_{-\infty}^{4} L_x P_t(t,x)\, dy \qquad \text{where } L_x f(x) = \frac{9}{2} f''(x) + 2f'(x)$$

$$= L_x \int_{-\infty}^{4} P_t(t,x)\, dy$$

$$= L_x \phi(t,x) \qquad \text{where } L_x f(x) = \frac{9}{2} f''(x) + 2f'(x)$$

3. $\phi(t,x) = E[B_t^3 \mid B_0 = x]$

$$= \int_{-\infty}^{\infty} B_t^3 P_t(t,x)\, dy$$

$$\dot{\phi}(t,x) = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} B_t^3 P_t(t,x)\, dy$$

$$= \int_{-\infty}^{\infty} B_t^3 \dot{P_t}(t,x)\, dy$$

$$= \int_{-\infty}^{\infty} B_t^3 \cdot L_x^* P_t(t,x)\, dy$$

$$= L_x^* \int_{-\infty}^{\infty} B_t^3 P_t(t,x)\, dy = L_x \phi(t,x)$$

$$\text{where } L_x f(x) = \frac{9}{2} f''(x) + 2f'(x)$$

4. $\phi(t,x) = E[B_3 - B_3^2 \mid B_t = x]$

Let $u(t,x) = E[B_t - B_t^2 \mid B_0 = x]$

$\therefore \quad \phi(t,x) = u(3-t, x)$

$\therefore \quad \dot{\phi}(t,x) = -\dot{u}(3-t,x)$

$= -L_x\, u(3-t,x) = -L_x\, \phi(t,x)$

where $L_x f(x) = \frac{9}{2} f''(x) + 2f'(x)$

Exercise 2 *Suppose B_t, W_t are independent standard Brwonian motions starting at the origin, and the random vector $Z_t = (Z_t^1, Z_t^2)$ is defined by*

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Note that Z_t is a two-dimensional Brownian motion.

1. *What are the drift and covariance matrix for Z?*
2. *What are the operators L, L^* associated to Z?*
3. *Let $\phi(t,x), t > 0, x \in \mathbb{R}^2$ be the density of Z_t at time t. Find the PDE satisfied by $\phi(t,x)$.*

1. $\because$ B_t, W_t are standard BM with drift 0.

$\therefore$ drift for $Z_t^1 = 0+0-1 = -1$, $Z_t^2 = 0-0 = 0$

$\therefore$ drift for $Z = (-1, 0)$

$var(Z_t^1) = var(B_t) + var(W_t) + Var(t)$

$= t + t + 0 = 2t$

$Var(Z_t^2) = var(2B_t) + Var(6W_t)$

$= 4t + 36t = 40t$

$cov(Z_t^1, Z_t^2) = (B_t + W_t - t,\ 2B_t - 6W_t)$

$= Cov(B_t, 2B_t) + Cov(W_t, -6W_t)$

$= 2t - 6t = -4$

$\therefore$ covariance matrix $= \begin{bmatrix} 2 & -4 \\ -4 & 40 \end{bmatrix}$

OR $\Pi = \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} B_t \\ W_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 40 \end{bmatrix}$

2. $\because$ Z is 2-dimensional BM with drift 0 and covariance matrix $\begin{bmatrix} 2 & -4 \\ -4 & 40 \end{bmatrix}$, then

$$Lf(x) = \sum_{j=1}^{J} m_j \partial_{x_j} f(x) + \frac{1}{2} \sum_{j \le j,k \le d} \Pi_{jk} \partial^2_{x_j x_k} f(x)$$

$$Lf(x) = -\partial_{x_1} f(x) + \frac{1}{2}\left[2\partial^2_{x_1 x_1} f(x) - 8\partial^2_{x_1 x_2} f(x) + 40\partial^2_{x_2 x_2} f(x) \right]$$
$$= -\partial_{x_1} f(x) + \partial^2_{x_1 x_1} f(x) - 4\partial^2_{x_1 x_2} f(x) + 20\partial^2_{x_2 x_2} f(x)$$

$$L^* f(x) = -\sum_{j=1}^{J} m_j \partial_{x_j} f(x) + \frac{1}{2} \sum_{j \le j,k \le d} \Pi_{jk} \partial^2_{x_j x_k} f(x)$$
$$= \partial_{x_1} f(x) + \frac{1}{2}\left[2\partial^2_{x_1 x_1} f(x) - 8\partial^2_{x_1 x_2} f(x) + 40\partial^2_{x_2 x_2} f(x) \right]$$
$$= \partial_{x_1} f(x) + \partial^2_{x_1 x_1} f(x) - 4\partial^2_{x_1 x_2} f(x) + 20\partial^2_{x_2 x_2} f(x)$$

3. $\because$ $Z_0 = [B_0 + W_0 - 0,\ 2B_0 - 6W_0] = [0, 0]$

$$\dot{\phi}(t, x) = L_x^* \phi(t, x)$$

where $L_x^* f(x) = \partial^2_{x_1} f(x) - 4\partial^2_{x_1 x_2} f(x) + 20\partial^2_{x_2^2} f(x)$

Exercise 3 *For each of these problems $\{A_t\}$ will be a simple process that only changes at time $1/3$, that is*

$$A_t = 1, \qquad 0 \le t < 1/3.$$

$$A_t = Y, \qquad 1/3 \le t \le 1,$$

where Y is a random variable measurable with respect to $\mathcal{F}_{1/3}$. Let

$$Z_t = \int_0^t A_s \, dB_s.$$

For each of these examples of Y, find $\mathbb{P}\{Z_{2/3} \ge 0\}$.

1.
$$Y = \begin{cases} 0 & B_{1/3} \ge 0 \\ 1 & B_{1/3} < 0 \end{cases}$$

2.
$$Y = \begin{cases} 0 & B_{1/3} \ge 0 \\ 2 & B_{1/3} < 0 \end{cases}$$

3.
$$Y = \begin{cases} -1 & B_{1/3} \ge 0 \\ 1 & B_{1/3} < 0 \end{cases}$$

1. $Z_t = \int_0^t A_s \, dB_s$

$= \int_0^{\frac{1}{3}} 1 \, dB_s + \int_{\frac{1}{3}}^{t} Y \, dB_s$

if $B_{\frac{1}{3}} \geqslant 0$ $\quad Z_t = B_{\frac{1}{3}} - 0 + 0 = B_{\frac{1}{3}}$

$\therefore Z_{\frac{2}{3}} = B_{\frac{1}{3}} \geqslant 0$

if $B_{\frac{1}{3}} < 0$ $\quad Z_t = B_{\frac{1}{3}} + \int_{\frac{1}{3}}^{t} 1 \, dB_s$

$\therefore Z_{\frac{2}{3}} = B_{\frac{1}{3}} + \int_{\frac{1}{3}}^{\frac{2}{3}} 1 \, dB_s$

$= B_{\frac{1}{3}} + (B_{\frac{2}{3}} - B_{\frac{1}{3}}) = B_{\frac{2}{3}}$

$\therefore P(Z_{\frac{2}{3}} \geqslant 0) = P(B_{\frac{1}{3}} \geqslant 0) \cdot 1 + P(B_{\frac{2}{3}} > 0, B_{\frac{1}{3}} < 0)$

$= \frac{1}{2} + P(B_1 < 0 \mid B_2 > 0) \cdot P(B_2 > 0)$

$= \frac{1}{2} + (1 - P(B_1 > 0 \mid B_2 > 0)) \cdot \frac{1}{2}$

$= \frac{1}{2} + (1 - \frac{3}{4}) \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$

2. if $B_{\frac{1}{3}} \geq 0$ $\quad Z_t = B_{\frac{1}{3}} + 0 = B_{\frac{1}{3}}$

$$Z_{\frac{2}{3}} = B_{\frac{1}{3}} \geq 0$$

if $B_{\frac{1}{3}} < 0$, $\quad Z_t = B_{\frac{1}{3}} + \int_{\frac{1}{3}}^{t} 2 \, dB_s$

$$Z_{\frac{2}{3}} = B_{\frac{1}{3}} + 2(B_{\frac{2}{3}} - B_{\frac{1}{3}}) = 2B_{\frac{2}{3}} - B_{\frac{1}{3}}$$

$$P(Z_{\frac{2}{3}} > 0) = P(B_{\frac{1}{3}} \geq 0) \cdot 1 + P(B_{\frac{1}{3}} + 2(B_{\frac{2}{3}} - B_{\frac{1}{3}}), B_{\frac{1}{3}} < 0)$$

$$= \frac{1}{2} + \frac{\frac{\pi}{2} - \arctan(\frac{1}{2})}{2\pi}$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2\pi} \arctan(\frac{1}{2})$$

$$= \frac{3}{4} - \frac{1}{2\pi} \arctan(\frac{1}{2})$$

3. if $B_{\frac{1}{3}} \geq 0$ $\quad Z_t = B_{\frac{1}{3}} + \int_{\frac{1}{3}}^{t} -1 \, dB_s$

$$Z_{\frac{2}{3}} = B_{\frac{1}{3}} + (-B_{\frac{2}{3}} + B_{\frac{1}{3}}) = 2B_{\frac{1}{3}} - B_{\frac{2}{3}}$$

if $B_{\frac{1}{3}} < 0$ $\quad Z_t = B_{\frac{1}{3}} + \int_{\frac{1}{3}}^{t} 1 \, dB_s$

$$Z_{\frac{2}{3}} = B_{\frac{1}{3}} + (B_{\frac{2}{3}} - B_{\frac{1}{3}}) = B_{\frac{2}{3}}$$

$$\therefore P(Z_{\frac{2}{3}} > 0) = P(B_{\frac{1}{3}} \geq 0, B_{\frac{1}{3}} + (B_{\frac{1}{3}} - B_{\frac{2}{3}}) > 0) + P(B_{\frac{1}{3}} < 0, B_{\frac{2}{3}} > 0)$$

$$= P(B_{\frac{1}{3}} \geq 0, B_{\frac{1}{3}} - (B_{\frac{2}{3}} - B_{\frac{1}{3}}) > 0) + (1 - P(B_1 > 0 \mid B_2 > 0)) \cdot P(B_2 > 0)$$

$$= \frac{3}{8} + (1 - \frac{3}{4}) \cdot \frac{1}{2} = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

Exercise 4 *For the three cases in the last exercise, give*

$$\langle Z\rangle_t = \int_0^t A_s^2\, ds.$$

In each case, the answer should look like two functions of t — one for the event $B_{1/3} \geq 0$ and one on the event $B_{1/3} < 0$.

① $\int_0^t A_s^2\, ds = \int_0^{\frac{1}{3}} 1\, ds + \int_{\frac{1}{3}}^t Y\, ds$

when $B_{\frac{1}{3}} \geqslant 0$ $\quad Y=0.$

$$\langle Z\rangle_t = \int_0^{\frac{1}{3}} 1\, ds + 0 = \frac{1}{3}$$

when $B_{\frac{1}{3}} < 0$ $\quad Y=1$

$$\langle Z\rangle_t = \frac{1}{3} + \int_{\frac{1}{3}}^t 1\, ds = \frac{1}{3} + (t-\frac{1}{3}) = t$$

② when $B_{\frac{1}{3}} \geqslant 0$ $\quad Y=0$

$$\langle Z\rangle_t = \frac{1}{3}$$

when $B_{\frac{1}{3}} < 0$ $\quad Y=2$

$$\langle Z\rangle_t = \frac{1}{3} + \int_{\frac{1}{3}}^t 4\, ds = \frac{1}{3} + 4(t-\frac{1}{3}) = 4t-1$$

③ when $B_{\frac{1}{3}} \geqslant 0$ $\quad Y=-1$

$$\langle Z\rangle_t = \frac{1}{3} + \int_{\frac{1}{3}}^t 1\, ds = \frac{1}{3} + (t-\frac{1}{3}) = t$$

when $B_{\frac{1}{3}} < 0$ $Y = -1$

$$\langle Z \rangle_t = \frac{1}{3} + \int_{\frac{1}{3}}^{t} 1 \, ds = t$$

Exercise 5 *For Case #2 in the last two exercises, verify directly the statement*

$$\mathrm{Var}[Z_1] = \int_0^1 \mathbb{E}[A_t^2] \, dt.$$

case #2 :

$$Y = \begin{cases} 0 & , B_{\frac{1}{3}} \geq 0 \\ 2 & , B_{\frac{1}{3}} < 0 \end{cases}$$

$\because$ $Z_0 = E(Z_0) = E(Z_t) = 0$

$\therefore$ $\mathrm{Var}(Z_t) = E(Z_t^2) - (E(Z))^2 = E(Z_t^2)$

$$\mathrm{Var}(Z_1) = E(Z_1^2) = \sum_{j=1}^{K} E(Y_j^2 (B_{t_j} - B_{t_{j-1}})^2)$$

$$+ \sum_{j \neq k} \underline{E[Y_j (B_{t_j} - B_{t_{j-1}}) Y_k (B_{t_k} - B_{t_{k-1}})]}$$

since it is equal to

$$= E(E[Y_j (B_{t_j} - B_{t_{j-1}}) Y_k (B_{t_k} - B_{t_{k-1}}) | F_{t_{k-1}}])$$

$$= Y_j (B_{t_j} - B_{t_{j-1}}) Y_k \underbrace{E(B_{t_k} - B_{t_{k-1}})}_{=0} = 0$$

$$= \sum_{j=1}^{K} E(Y_j^2 (B_{t_j} - B_{t_{j-1}})^2)$$

$$= E(1\cdot(B_{\frac{1}{3}}-B_0)^2) + E(Y^2(B_1-B_{\frac{1}{3}})^2)$$

$$= E(B_{\frac{1}{3}}^2) + \underline{E(Y^2 B_{\frac{2}{3}}^2)}$$

when $B_t \geq 0$ $Y=0$

$E(Y^2 B_{\frac{2}{3}}^2) = E(0) = 0$

when $B_t < 0$ $Y=2$

$E(4B_{\frac{2}{3}}^2) = Var(2B_{\frac{2}{3}}) + (E(2B_{\frac{2}{3}}))^2$

$= Var(2B_{\frac{2}{3}}) = 4\cdot\frac{2}{3} = \frac{8}{3}$

$E[Y^2 B_{\frac{2}{3}}^2] = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}$

$$= Var(B_{\frac{1}{3}}) + (E(B_{\frac{1}{3}}))^2 + \frac{4}{3}$$

$$= \frac{1}{3} + 0 + \frac{4}{3} = \frac{5}{3}$$

$$\int_0^1 E(A_t^2)\,dt = \int_0^{\frac{1}{3}} E(1)\,dt + \int_{\frac{1}{3}}^1 E(Y^2)\,dt$$

$$= \int_0^{\frac{1}{3}} 1\,dt + \int_{\frac{1}{3}}^1 \left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2^2\right)dt$$

$$= \frac{1}{3} + \int_{\frac{1}{3}}^1 2\,dt = \frac{1}{3} + 2\left(1-\frac{1}{3}\right) = \frac{5}{3}$$

$$\therefore \quad Var(Z_1) = \int_0^1 E(A_t^2)\,dt$$