FINM 345X0/STAT 390X0 Midquarter Exam, Winter 2023

You have an hour and twenty minutes (80 minutes) to complete the exam. There are 7 problems/questions, each with multiple parts, for a total of 95 points. For all multi-part problems each part is worth the same amount, 5 points. You may use the one (double sided) sheet that you prepared for the test, but you may not use any other books, notes, or electronic devices. You do not have to carry out complicated arithmetic expressions.

- On any question you may leave your answer in terms of the normal distribution function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

- Also for your reference, the moment generating function of a $N(0,1)$ random variable N is

$$\mathbb{E}[e^{sN}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} e^{st} dt = e^{s^2/2}.$$

Problem 1 (15)**.** Suppose B_t is a standard Brownian motion starting at the origin. Find the following.

1. $\mathbb{P}\{B_3 \leq 2\}$
2. The probability that by time 4 the Brownian motion has crossed the line $y = 2$.
3. $$\mathbb{P}\left\{B_6 \geq 7 \mid \max_{0 \leq s \leq 6} B_s \geq 7\right\}.$$

Problem 2 (10)**.** Let B_t be a Brownian motion with drift 2 and variance parameter 1 and filtration $\{\mathcal{F}_t\}$. Find the following conditional expectations.

1. $E[B_2^2 \mid \mathcal{F}_1]$
2. $E[e^{B_9 - 18} \mid \mathcal{F}_3]$

Problem 3 (20)**.** Suppose B_t and W_t are independent standard Brownian motions. Let

$$Y_t = 4B_t + 2W_t - t, \quad Z_t = 3B_t - W_t + 3t.$$

Y_t is a one-dimensional Brownian motion and (Y_t, Z_t) is a two-dimensional Brownian motion (you may assume that this is true — you do not have to verify this). Find the following.

1. The drift and variance parameter for Y.
2. The drift and covariance matrix for (Y, Z).
3. $$\lim_{n \to \infty} \sum_{j=1}^{2n} \left[Y\left(\frac{j}{n}\right) - Y\left(\frac{j-1}{n}\right)\right]^2.$$
4. $$\lim_{n \to \infty} \sum_{j=1}^{n} \left[Y\left(\frac{j}{n}\right) - Y\left(\frac{j-1}{n}\right)\right]\left[Z\left(\frac{j}{n}\right) - Z\left(\frac{j-1}{n}\right)\right].$$

Problem 4 (15)**.** Suppose that B_t is a standard Brownian motion and X_t, Y_t satisfy the following.

$$dX_t = 4\,X_t\,dt + dB_t, \qquad dY_t = 3\,dt + Y_t\,dB_t.$$

Find the following differentials

1. $d[e^{-t+2X_t}]$
2. $d[X_t Y_t]$
3. dS_t where
$$S_t = \exp\left\{\int_0^t (X_s + Y_s)\,ds\right\}.$$

Problem 5 (15)**.** Suppose B_t is a Brownian motion with drift 3 and variance parameter 6.

1. Give the corresponding operators L and L^*. (In the next two parts, you can write your answer in terms of L or L^*.)
2. Suppose
$$u(t,x) = \mathbb{E}\left[B_5^2 + 2\,B_5 \mid B_t = x\right], \quad 0 \le t < 5.$$
What is the PDE that $u(t,x)$ satisfies for $0 < t < 5$?
3. Suppose
$$u(t,x) = \mathbb{E}\left[B_t^2 + 2\,B_t \mid B_0 = x\right], \quad 0 \le t < 5.$$
What is the PDE that $u(t,x)$ satisfies for $0 < t < 5$?

Problem 6 (10)**.** Suppose $X_1, X_2, \ldots$ are independent random variables with distribution

$$\mathbb{P}\{X_j = 2\} = r, \quad \mathbb{P}\{X_j = 0\} = \frac{1}{2} - r, \quad \mathbb{P}\{X_j = -1\} = \frac{1}{2}$$

where $0 < r < 1/2$. Let $S_n = X_1 + \cdots + X_n$ with $S_0 = 0$ and $M_n = 2^{S_n}$.

1. For which value of r is it the case that M_n is a martingale? Use this value of r for the remainder of the problem.
2. Find
$$\mathbb{E}\left[\sum_{j=1}^{13} [M_j - M_{j-1}]^2\right].$$

Problem 7 (10)**.** Let B_t be a standard Brownian motion. Suppose $A_t = 1$ for $0 \le t < 1$ and for $t \ge 1$,

$$A_t = 0 \qquad \text{if } B_1 > 2$$

$$A_t = 1 \qquad \text{if } 0 < B_1 \le 2,$$

$$A_t = 2 \qquad \text{if } B_1 \le 0.$$

Let

$$Z_t = \int_0^t A_s\,dB_s.$$

1. Find $\mathbb{E}[A_t^2]$.
2. Find $\mathrm{Var}[Z_2]$.