**FINM 34500/STAT 39000**
**Problem Set 5** (due February 5 with extension to February 7)
**Reading:** Notes through Chapter 3 (remember: indented parts are optional reading)

**EXAM** in-class on Monday, February 5. Includes material in lectures through February 1 and Problem Sets 1-5. Even though you have until Wednesday, February 7 to hand in Problem 5, the material will be included on the Monday exam. Closed book and electronic equipment, but you may bring one standard sheet (two-sided) page of notes that you have created yourself (no copied material).

**Exercise 1** *Use Itô's formula to find the stochastic differential* $df(t, B_t)$ *where* $B_t$ *is a standard Brownian motion and*

1. $f(t, x) = x^2 \, e^{-x}$
2. $f(t, x) = x^2 \, t \, e^{-tx}$
3. $f(x) = \cos x + x^3$
4. *Repeat these exercises for* $f(t, X_t)$ *where*

$$dX_t = X_t \, [dt - 3 \, dB_t].$$

**Exercise 2** *Suppose an asset follows the following geometric SDE,*

$$dX_t = 3 \, X_t \, dt + X_t \, dB_t.$$

1. *Write the exact solution of this equation. In other words, find* $X_t$ *as a function of* $B_t$.
2. *Suppose* $X_0 = 1$. *What is the probability that* $X_1 > 2$*?*
3. *Suppose* $X_0 = 1/2$. *What is the probability that* $X_2 < 6$*?*
4. *Let* $Y_t = \log X_t$ *where* $\log$ *denotes the natural logarithm. Find the equation that* $Y_t$ *satisfies (your answer should be in terms of* $Y_t$ *and* $B_t$ *and should not include* $X_t$*).*

**Exercise 3** *Suppose that two assets* $X_t, Y_t$ *follow the SDEs*

$$dX_t = X_t \, [, dt + dB_t],$$

$$dY_t = Y_t \, [3 \, dt - 2 \, dB_t],$$

*where* $B_t$ *is a standard Brownian motion. Suppose also that* $X_0 = Y_0 = 1$.

1. *Let* $Z_t = X_t Y_t$. *Give the SDE satisfied by* $Z_t$.
2. *Let* $Z_t = X_t / Y_t$. *Give the SDE satisfied by* $Z_t$.

**Exercise 4** *Suppose $B_t$ is a standard Brownian motion and $X_t$ satisfies*

$$dX_t = X_t\,[3X_t\,dt - 2\,dB_t]\,, \quad X_0 = 1.$$

*For each of the following find $A_t, C_t$ such that*

$$d\langle Y\rangle_t = A_t\,dt, \quad d\langle Y, X\rangle_t = C_t\,dt.$$

1. $Y_t = B_t^2 + 2t$.
2. $Y_t = X_t + \int_0^t X_s^2\,ds$.
3. $$Y_t = X_t^2 + \exp\left\{2\int_0^t X_s^2\,ds\right\}.$$

**Exercise 1** *Use Itô's formula to find the stochastic differential $df(t, B_t)$ where $B_t$ is a standard Brownian motion and*

1. $f(t, x) = x^2\, e^{-x}$
2. $f(t, x) = x^2\, t\, e^{-tx}$
3. $f(x) = \cos x + x^3$
4. *Repeat these exercises for $f(t, X_t)$ where*

$$dX_t = X_t\,[dt - 3\, dB_t].$$

1. $f(t,x) = x^2 e^{-x}$

$\dot{f}(t,x) = 0$

$f'(t,x) = 2xe^{-x} - x^2e^{-x}$

$f''(t,x) = 2e^{-x} - 2xe^{-x} - (2xe^{-x} - x^2e^{-x})$

$= 2e^{-x} - 4xe^{-x} + x^2e^{-x}$ According to Itô's Formula 2

$f(t, B_t) = f(B_0) + \int_0^t 0\, ds + \int_0^t (2B_se^{-B_s} - B_s^2e^{-B_s})\, dB_s$

$+ \frac{1}{2}\int_0^t 2e^{-B_s} - 4B_se^{-B_s} + B_s^2e^{-B_s}\, ds$

$\therefore df(t, B_t) = [2B_te^{-B_t} - B_t^2e^{-B_t}]\,dB_t + [e^{-B_t} - 2B_te^{-B_t} + \frac{1}{2}B_t^2e^{-B_t}]\,dt$

2. $f(t,x) = x^2te^{-tx}$

$\dot{f}(t,x) = x^2e^{-tx} - x^3te^{-tx}$

$f'(t,x) = 2xte^{-tx} - x^2t^2e^{-tx}$

$f''(t,x) = 2te^{-tx} - 2xt^2e^{-tx} - [2xt^2e^{-tx} - x^2t^3e^{-tx}]$

$= 2te^{-tx} - 2xt^2e^{-tx} - 2xt^2e^{-tx} + x^2t^3e^{-tx}$

According to Itô's Formula 2

$$f(t,B_t) = f(0,B_0) + \int_0^t B_s^2 e^{-sB_s} - sB_s^3 e^{-sB_s}\, ds +$$

$$\int_0^t 2sB_s e^{-sB_s} - B_s^2 s^2 e^{-sB_s}\, dB_s + \int_0^t 2se^{-sB_s} - 2B_s s^2 e^{-sB_s} -$$

$$2B_s s^2 e^{-sB_s} + B_s^2 s^3 e^{-sB_s}\, ds$$

$$df(t,B_t) = \left[B_t^2 e^{-tB_t} - tB_t^3 e^{-tB_t} + te^{-tB_t} - 2t^2 B_t e^{-tB_t} + \frac{1}{2} t^3 B_t^2 e^{-tB_t}\right] dt$$

$$+ \left[2tB_t e^{-tB_t} - t^2 B_t^2 e^{-tB_t}\right] dB_t$$

3. $f(x) = \cos x + x^3$

$f'(x) = -\sin x + 3x^2$

$f''(x) = -\cos x + 6x$

According to Itô's Formula 1

$$f(B_t) = f(B_0) + \int_0^t -\sin B_s + 3B_s^2\, dB_s + \frac{1}{2}\int_0^t -\cos B_s + 6B_s\, ds$$

$$df(B_t) = \left[-\sin B_t + 3B_t^2\right] dB_s + \left[3B_t - \frac{1}{2}\cos B_t\right] dt$$

4. $dX_t = X_t[dt - 3dB_t]$

$= X_t\, dt - 3X_t\, dB_t$

$R_t = X_t, \quad A_t = -3X_t$

① according to Itô's formula 3.

$$df(t,X_t) = [\dot{f}(t,X_t) + R_t f'(t,X_t) + \frac{1}{2} f''(t,X_t) A_t^2] dt + f'(t,X_t) A_t dB_t$$

$$= \left[X_t (2X_t e^{-X_t} - X_t^2 e^{-X_t}) + \frac{9X_t^2}{2}[2e^{-X_t} - 4X_t e^{-X_t} + X_t^2 e^{-X_t}]\right] dt$$

$$- 3X_t ( 2X_t e^{-X_t} - X_t^2 e^{-X_t} ) dB_t$$

② according to Itô's formula 3.

$$df(t,X_t) = [\dot{f}(t,X_t) + R_t f'(t,X_t) + \frac{1}{2} f''(t,X_t) A_t^2 ] dt + f'(t,X_t) A_t dB_t$$

$$= \Big[ X_t^2 e^{-tX_t} - X_t^3 t e^{-tX_t} + X_t ( 2X_t t e^{-tX_t} - X_t^2 t^2 e^{-tX_t} ) + \frac{9X_t^2}{2} ( 2t e^{-tX_t} - 4X_t t^2 e^{-tX_t} + X_t^2 t^3 e^{-tX_t} ) \Big] dt$$

$$- 3X_t ( 2X_t t e^{-tX_t} - X_t^2 t^2 e^{-tX_t} ) dB_t$$

③ according to Itô's formula 3.

$$df(t,X_t) = [\dot{f}(t,X_t) + R_t f'(t,X_t) + \frac{1}{2} f''(t,X_t) A_t^2 ] dt + f'(t,X_t) A_t dB_t$$

$$= \Big[ X_t ( -\sin X_t + 3X_t^2 ) + \frac{9X_t^2}{2} ( -\cos X_t + 6X_t ) \Big] dt$$

$$- 3X_t ( -\sin X_t + 3X_t^2 ) dB_t$$

**Exercise 2** *Suppose an asset follows the following geometric SDE,*

$$dX_t = 3\,X_t\,dt + X_t\,dB_t.$$

1. *Write the exact solution of this equation. In other words, find $X_t$ as a function of $B_t$.*
2. *Suppose $X_0 = 1$. What is the probability that $X_1 > 2$?*
3. *Suppose $X_0 = 1/2$. What is the probability that $X_2 < 6$?*
4. *Let $Y_t = \log X_t$ where log denotes the natural logarithm. Find the equation that $Y_t$ satisfies (your answer should be in terms of $Y_t$ and $B_t$ and should not include $X_t$).*

1. $X_t = X_0\, e^{(m-\frac{\sigma^2}{2})t + \sigma B_t}$

$dX_t = 3X_t\,dt + X_t\,dB_t$ in this case $m=3$, $\sigma=1$

$\therefore\ X_t = X_0\, e^{(3-\frac{1}{2})t + B_t} = X_0\, e^{\frac{5}{2}t + B_t}$

2. $X_0 = 1$. $\quad X_t = e^{\frac{5}{2}t + B_t}$

$P(X_1 > 2) = P(e^{\frac{5}{2}+B_1} > 2)$

$= P(e^{B_1} > 2e^{-\frac{5}{2}})$

$= P(B_1 > \ln(2e^{-\frac{5}{2}}))$

$= P(B_1 > \ln 2 - \frac{5}{2})$

$= 1 - \Phi(\ln 2 - \frac{5}{2})$

$= 1 - 0.0354 = 0.9646$

3. $X_0 = \frac{1}{2}$ $\quad X_t = \frac{1}{2} e^{\frac{5}{2}t + B_t}$

$P(X_2 < 6) = P(\frac{1}{2} e^{5+B_2} < 6)$

$= P(e^{B_2} < 12e^{-5})$

$= P(B_2 < \ln 12 - 5)$

$= P(B_1 < \frac{\ln 12 - 5}{\sqrt{2}}) = \Phi(\frac{\ln 12 - 5}{\sqrt{2}}) = 0.0377$

4. $dX_t = 3X_t\,dt + X_t\,dB_t$ $\qquad R_t = 3X_t\ ,\ A_t = X_t$

let $f(X_t) = Y_t = \log X_t$ $\qquad f'(X_t) = \frac{1}{X_t}$ $\quad f''(X_t) = \frac{1}{-X_t^2}$

Accordiny Itô's formula 3

$$df(t,X_t) = [\dot{f}(t,X_t) + R_t f'(t,X_t) + \frac{1}{2} f''(t,X_t) A_t^2]\,dt + f'(t,X_t) A_t\,dB_t$$

$$= [3X_t \cdot \frac{1}{X_t} + \frac{X_t^2}{2} \cdot (-\frac{1}{X_t^2})]\,dt + \frac{1}{X_t} \cdot X_t\,dB_t$$

$$= (3 - \frac{1}{2})\,dt + dB_t = \frac{5}{2}\,dt + dB_t$$

**Exercise 3** *Suppose that two assets $X_t, Y_t$ follow the SDEs*

$$dX_t = X_t\,[,dt + dB_t],$$

$$dY_t = Y_t\,[3\,dt - 2\,dB_t],$$

*where $B_t$ is a standard Brownian motion. Suppose also that $X_0 = Y_0 = 1$.*

1. *Let $Z_t = X_t Y_t$. Give the SDE satisfied by $Z_t$.*
2. *Let $Z_t = X_t / Y_t$. Give the SDE satisfied by $Z_t$.*

1. $dZ_t = d(X_t Y_t) = X_t\,dY_t + Y_t\,dX_t + d\langle X, Y\rangle_t$

$$= X_t Y_t [3dt - 2dB_t] + Y_t X_t [dt + dB_t] + (X_t \cdot (-2Y_t))\,dt$$

$$= (4X_tY_t - 2X_tY_t)\,dt - X_tY_t\,dB_t$$

$= 2Z_t dt - Z_t dB_t$

2. $Z_t = X_t / Y_t$ Let $K_t = \frac{1}{Y_t}$

$Z_t = X_t K_t$

$K_t = \frac{1}{Y_t}$ $K'_t = -\frac{1}{Y_t^2}$ $K''_t = \frac{2}{Y_t^3}$

$dY_t = Y_t(3dt - 2dB_t)$

$R_t = 3Y_t$ , $A_t = -2Y_t$

According to Itô's formula 3

$df(t,X_t) = [\dot{f}(t,X_t) + R_t f'(t,X_t) + \frac{1}{2} f''(t,X_t) A_t^2] dt + f'(t,X_t) A_t dB_t$

$= [3Y_t \cdot (-\frac{1}{Y_t^2}) + \frac{1}{2} \frac{2}{Y_t^3} \cdot 4Y_t^2] dt + (-\frac{1}{Y_t^2})(-2Y_t) dB_t$

$= K_t dt + 2K_t dB_t$

$\therefore dZ_t = d(X_t K_t) = X_t dK_t + K_t dX_t + d\langle X, K\rangle_t$

$= X_t (K_t dt + 2K_t dB_t) + K_t X_t (dt + dB_t)$

$+ (2K_t \cdot X_t) dt$

$= Z_t(1+1+2) dt + Z_t(2+1) dB_t$

$= 4Z_t dt + 3Z_t dB_t$

**Exercise 4** *Suppose $B_t$ is a standard Brownian motion and $X_t$ satisfies*

$$dX_t = X_t\left[3X_t\,dt - 2\,dB_t\right], \quad X_0 = 1.$$

*For each of the following find $A_t, C_t$ such that*

$$d\langle Y\rangle_t = A_t\,dt, \quad d\langle Y, X\rangle_t = C_t\,dt.$$

1. $Y_t = B_t^2 + 2t$.
2. $Y_t = X_t + \int_0^t X_s^2\,ds$.
3. $$Y_t = X_t^2 + \exp\left\{2\int_0^t X_s^2\,ds\right\}.$$

$dX_t = 3X_t^2\,dt - 2X_t\,dB_t$

$R_t = 3X_t^2 \quad A_t = -2X_t$

1. $Y_t = B_t^2 + 2t$

$\dot{Y}(t,B_t) = 2$, $Y'(t,B_t) = 2B_t$, $Y''(t,B_t) = 2$

According to Itô's formula 2.

$dY(t,B_t) = (2 + \frac{1}{2}\times 2)\,dt + 2B_t\,dB_t$

$= 3\,dt + 2B_t\,dB_t$

$R_t = 3, \quad A_t = 2B_t$

$d\langle Y\rangle_t = A_t^2\,dt = 4B_t^2\,dt$

$d\langle X, Y\rangle_t = (2B_t)(-2X_t)\,dt = -4X_tB_t\,dt$

2. $Y_t = X_t + \int_0^t X_s^2\,ds$

$dY_t = dX_t + X_t^2\,dt$

$= [3X_t^2\,dt - 2X_t\,dB_t] + X_t^2\,dt$

$= 4X_t^2\,dt - 2X_t\,dB_t$

$R_t = 4X_t^2 \qquad A_t = -2X_t$

$d\langle Y\rangle_t = A_t^2\,dt = 4X_t^2\,dt$

$d\langle X, Y\rangle_t = (-2X_t)(-2X_t)\,dt = 4X_t^2\,dt$

3. $Y_t = X_t^2 + e^{2\int_0^t X_s^2\,ds}$

$dY_t = 2X_t\,dX_t + e^{2\int_0^t X_s^2\,ds} \cdot 2X_t^2\,dt$

$= 2X_t[3X_t^2\,dt - 2X_t\,dB_t] + e^{2\int_0^t X_s^2\,ds} \cdot 2X_t^2\,dt$

$= [6X_t^3 + e^{2\int_0^t X_s^2\,ds} \cdot 2X_t^2]\,dt - 4X_t^2\,dB_t$

$d\langle Y\rangle_t = A_t^2\,dt = 16X_t^4\,dt$

$d\langle X, Y\rangle_t = (-4X_t^2)(-2X_t) = 8X_t^3\,dt$