# Lecture 6:
# Time Diversification

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Autumn 2023

FINM 36700: Portfolio Management

# Notation

| notation | description | |
|---|---|---|
| $r_t$ | | return rate, $(t-1 \text{ to } t)$ |
| $r_{t,t+h}$ | $\left(\prod_{\tau=1}^{h} R_{t+\tau}\right) - 1$ | cumulative return rate, from $t$ to $t+h$ |
| $\mathtt{r}_t$ | $\log(1+r_t)$ | log return from $t-1$ to $t$ |
| $\mathtt{r}_{t,t+h}$ | $\log(1+r_{t,t+h})$ | log cumulative return from $t$ to $t+h$. |

# Cumulative return risk and autocorrelation

Log cumulative returns are simply a portfolio of single-period returns.

- Define $\mu$ and $\sigma^2$ as

$$\mathbb{E}\left[\mathbf{r}_{t,t+1}\right] = \mu, \qquad \text{var}\left[\mathbf{r}_{t,t+1}\right] = \sigma^2$$

- For the $h$-period log return,

$$\mathbb{E}\left[\mathbf{r}_{t,t+h}\right] = h\mu$$

$$\text{var}\left[\mathbf{r}_{t,t+h}\right] = \sum_{j=1}^{h}\sum_{i=1}^{h} \text{cov}\left[\mathbf{r}_{t+i}, \mathbf{r}_{t+j}\right]$$

very small 因为如果period有大cov

easy to make money

across time return

(not different asset)

# Autocorrelation models

As seen in the previous slide,

- The variance of cumulative returns, $\mathtt{r}_{t,t+h}$, depends critically on the auto-covariance of the return series, denoted as

$$\text{cov}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right], \quad \text{or} \quad \sigma_{t,t+i}$$

- Specifying a form for the autocorrelations, $\text{corr}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right]$, is equivalent.

- A model of autocorrelations uniquely specifies a linear time series model.

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# AR(1) models

Autoregressive (AR) models are among the most popular in time-series statistics.

Consider the AR(1) model,

$$\text{cov}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right] = \rho^i \sigma^2$$

$$\text{corr}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right] = \rho^i$$

With AR models, covariances are easy to scale over time.

# Diversification and cumulative returns

Mean returns scale linearly in horizon $h$,

$$\mathbb{E}\left[\mathbf{r}_{t,t+h}\right] = h\mu$$

But scaling of variance depends on correlation,

var = cov = $h^2 \sigma^2$

- $\rho = 1.$ Std.Dev. is linear in cumulation: $\text{std}\left[\mathbf{r}_{t,t+h}\right] = h\sigma$

h period $\cdot \sigma^2$

- $\rho = 0.$ Variance is linear in cumulation: $\text{var}\left[\mathbf{r}_{t,t+h}\right] = h\sigma^2$

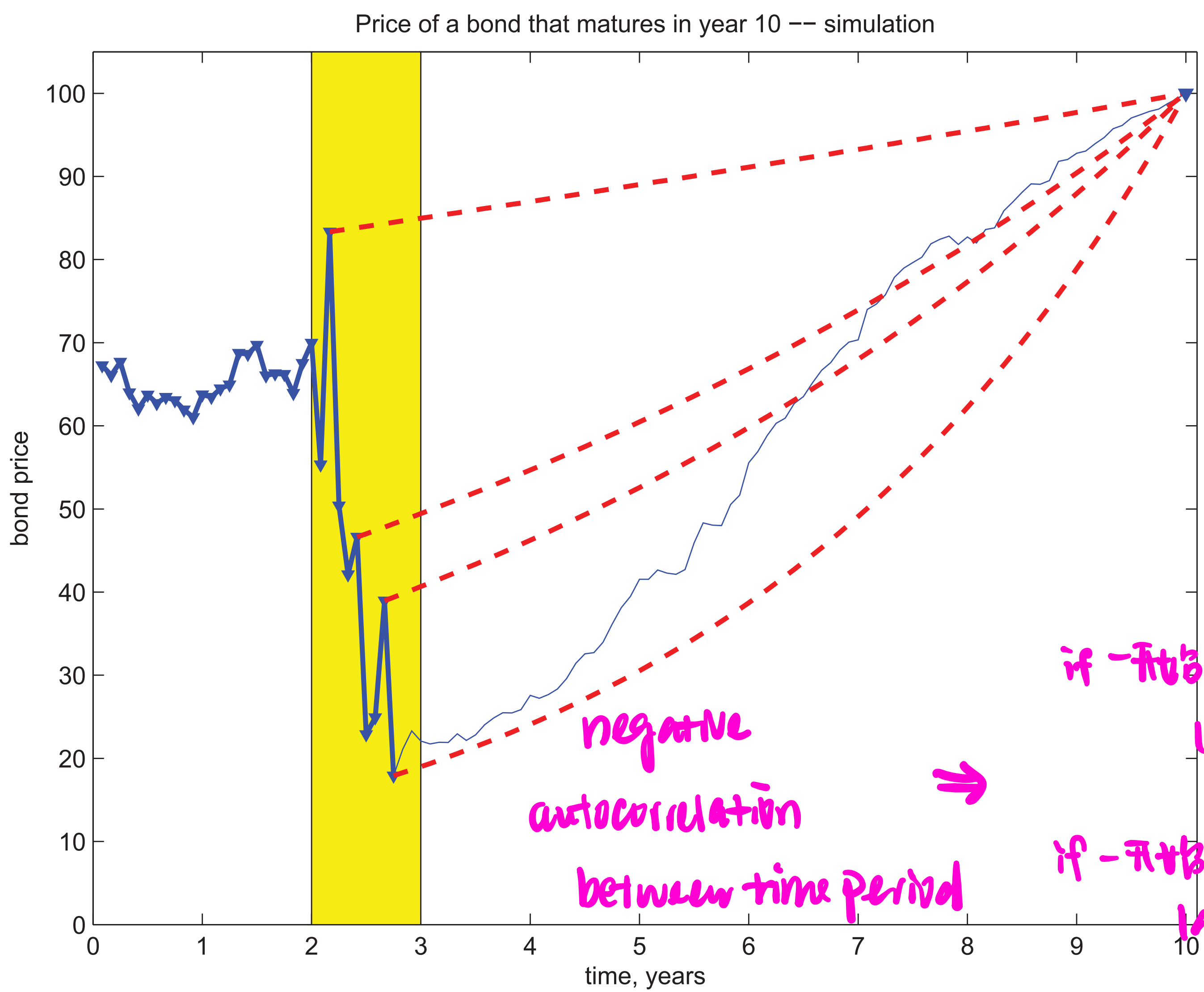
std = $\sqrt{h}\sigma$

- $\rho = -1.$ The return is riskless: $\text{var}\left[\mathbf{r}_{t,t+h}\right] = 0$

# Example: Riskless bond

At time $t = 0$, consider a bond with riskless payout at $t = 10$. The yield of the bond at $t = 0$ is 5%.

- The 10-year cumulative return, $\mathtt{r}_{0,10}$ is riskless, and equals $5\% \times 10$.
- At any intermediate time, $(t$, such that $0 < t < 10,)$ the bond price is uncertain.
- Thus, the intermediate returns, $\mathtt{r}_{0,t}$ and $\mathtt{r}_{t,10}$ are uncertain.
- However, $\mathtt{r}_{0,10} = \mathtt{r}_{0,t} + \mathtt{r}_{t,10}$.
- So if $\mathtt{r}_{0,t}$ is unexpectedly high, then $\mathtt{r}_{t,10}$ must be relatively low, such that the riskless return $\mathtt{r}_{0,10}$ ends up at $5\% \times 10$.




negative autocorrelation between time period ⇒

if 一开始 return high, then later return low

if 一开始 return low, the later return high

Figure: A ten-year risk-free bond has a 10-year return with perfect mean reversion. Source: Cochrane (2011).

# Negative serial correlation in bonds

Thus, riskless bonds should have negatively autocorrelated returns:

0-3 year to 3-10 year have perfect negative return

$$\text{corr}\left(\mathrm{r}_{t}, \mathrm{r}_{t+1}\right) < 0$$

And if the bond matures at $T$, then for any $0 < h < T - t$,

$$\text{corr}\left(\mathrm{r}_{t,t+h}, \mathrm{r}_{t+h,T}\right) = -1$$

Default-free bonds are safer in the long-run, with $\text{var}\left[\mathrm{r}_{t,T}\right] = 0$.

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# Cumulative Sharpe ratios in AR(1) model

Consider again the cumulative return, $\mathtt{r}_{t,t+h}$.

- For $\rho = 1$,

$$\text{SR}(\mathtt{r}_{t,t+h}) = \text{SR}(\mathtt{r}_t)$$

- For $|\rho| < 1$,

$$\text{SR}(\mathtt{r}_{t,t+h}) > \text{SR}(\mathtt{r}_t)$$

- For $\rho = 0$, for most of stocks

var $= \rho\sigma^2$

$$\text{SR}(\mathtt{r}_{t,t+h}) = \sqrt{h}\,\text{SR}(\mathtt{r}_t)$$

longer term ⇒ higher SR

和 different asset 一样，period 之间有 low autocorrelation → reduce risk when period is long

# Mean annualized returns

The annualized mean (log) return on the $h$-period investment, $\mathtt{r}_{t,t+h}$ is

$$\frac{\mathtt{r}_{t,t+h}}{h} = \frac{\sum_{i=1}^{h} \mathtt{r}_{t+i}}{h}$$

This is just the usual sample estimate of the mean of one-period returns, $\bar{r}$, based on a sample-size of $h$!

| For any $\rho$, | For $\rho = 0$, |
|---|---|
| $\mathbb{E}[\bar{r}] = \mu$ | $\text{var}[\bar{r}] = \frac{\sigma^2}{h}$ |

average annual var

- So as the investment horizons gets large, the mean annualized return, $\bar{r}$, converges to the true annual mean return, $\mu$.
- Even if $\rho \neq 0$ we can still conclude that $\bar{r} \to \mu$. (Law of large numbers still holds.)

× cumulative return
but average annual ret

E[r̄]=μ 永远 hold.

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# Time diversification

Time-diversification refers to this idea that mean annualized return becomes riskless for large investment horizons.

- True, as horizon increases the variance of the annualized return goes to zero.
- However, the variance of the cumulative return, $\mathtt{r}_{t,t+h}$ is still growing.

So-called time diversification depends on the risk one is measuring.

average : safer

cumulative : riskier

# Evidence: Are equity returns serially correlated?

Table: Auto-regression estimates for market returns, risk-free rate, and excess market returns. Regression estimates of $y_{t+1} = a + \rho y_t + \epsilon_{t+1}$

| | Monthly | | | Annual | | |
|---|---|---|---|---|---|---|
| $y = \ldots$ | $r^m$ | $r^f$ | $\tilde{r}^m$ | $r^m$ | $r^f$ | $\tilde{r}^m$ |
| $\hat{\rho}$ | 0.11 | 0.89 | 0.12 | 0.01 | 0.92 | 0.02 |
| $t(\hat{\rho})$ | 2.02 | 30.38 | 2.05 | 0.09 | 13.31 | 0.15 |
| $R^2$ | 0.01 | $\boxed{0.80}$ | 0.01 | 0.00 | 0.83 | 0.00 |

risk-free has high serial corr

negligible

Source: CRSP value-weighted equity markets, 1927-2010. CRSP 3-month U.S. treasury bill. GMM standard errors.

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# Serial correlation of equities

- The table shows that empirically the serial correlation of excess equity returns is small.
- Not surprisingly, the serial correlation of the risk-free rate is very high.
- The serial correlation is much smaller for annual returns.

Does estimating the autocorrelation of longer-horizon returns tell a different story?

# Evidence: Cumulative equity returns

Table: Std.dev., Sharpe-ratios, and serial correlation, of excess cumulative returns.

SR grow at $\sqrt{h}$ SR when $\rho = 0$

| | Horizon h (years) | | | | |
|---|---|---|---|---|---|
| | 1 | 3 | 5 | 7 | 10 |
| $\sigma\left(\tilde{r}^m_{t,t+h}\right)/\sqrt{h}$ | 0.21 | 0.24 | 0.28 | 0.26 | 0.32 |
| $SR/\sqrt{h}$ 趋于相同 | 0.36 | 0.36 | 0.32 | 0.34 | 0.30 |
| $\hat{\rho}$ | 0.01 | -0.29 | -0.30 | 0.40 | 0.06 |

逐渐变小

$\rho$ little bit > 0

autocorrelation not zero

- $\tilde{r}^m_{t,t+h}$ denotes the $h$-year excess return of the CRSP U.S. equity index over the 90-day t-bill.
- $\hat{\rho}$ is the serial correlation of long-horizon excess returns estimated by regressing $\tilde{r}^m_{t,t+h}$ on $\tilde{r}^m_{t-h,t}$.

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# Long-run uncertainty

riskier when period is longer

- The results in the table above show that normalized by $\sqrt{h}$, the Sharpe ratios remain almost constant across return horizon, or slightly decrease.
- This would be consistent with market equity excess returns having small serial correlation.
- Unfortunately, the basic estimates in the previous table are not statistically conclusive.
- Not surprising, since there are not a lot of long-horizon data points available.