The problem of Turán numbers of two disjoint stars

Niping Yi

Abstract

In this paper, we investigate the Turán numbers of two disjoint star graphs, denoted as S_k and S_l, where $l \leq k$. We aim to determine the maximum number of edges in a graph of order n that avoids containing the graph $S_k \cup S_l$ as a subgraph. Our main result provides a formula for the Turán number $ex(n, S_k \cup S_l)$ based on the values of n, k, and l.

Keywords: Turán number, Disjoint Stars, Extremal Graph

1. Introduction

Our notation in this paper is standard. Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ is the vertex set with size $v(G)$ and $E(G)$ is the edge set with size $e(G)$. The degree of $v \in V(G)$, the number of edges incident to v, is denoted by $d_G(v)$ and the set of neighbors of v is denoted by $N(v)$. Let G and H be two disjoint graphs, denote by $G \cup H$ the disjoint union of G and H and by $k \cdot G$ the disjoint union of k copies of a graph G. Let S_k denote the star on $k + 1$ vertices.

The Turán number of a graph H, $\mathrm{ex}(n, H)$, is the maximum number of edges in a graph of order n which does not contain H as a subgraph. We say that a graph is H-free if it does not contain H as a subgraph.

In this article, we determine $ex(n, S_k \cup S_l)$ for all values of n, k, l, where $l \leq k$.

We will prove the following theorem.

Theorem 1. *If $l \leq k$, then*

$$ex(n, S_k \cup S_l) = \begin{cases} \binom{n}{2}, & \text{if } n \leq l + k + 1, \\ \max\left\{ \left\lfloor \frac{(l-1)n+(k+1)(l+k+1)}{2} \right\rfloor, \left\lfloor \frac{(l+1)n+(k+1)(l+1)}{2} \right\rfloor, \left\lfloor \frac{(k-1)n}{2} \right\rfloor \right\}, & \text{if } n \geq l + k + 2. \end{cases}$$

Email address: `yiniping123@gmail.com` (Niping Yi)

It is easy to check that

$$ex(n, 2 \cdot S_l) \leq ex(n, S_l \cup S_k) \leq ex(n, 2 \cdot S_k)$$

where

$$ex(n, 2 \cdot S_l) = \begin{cases} \binom{n}{2}, & \text{if } n \leq 2(l+1) \\ \left\lfloor \dfrac{(l-1)n + (l+1)(2l+1)}{2} \right\rfloor, & \text{if } 2(l+1) \leq n \leq (l+1)^2 \\ \left\lfloor \dfrac{(l+1)n - (l+1)}{2} \right\rfloor, & \text{if } n \geq (l+1)^2 \end{cases} \tag{1}$$

As shown in Equation 1, [1]

2. Proof of Theorem 1

Proof. Assume that $l < k$. Denote

$$f(n,k,l) = \max\left\{ \left\lfloor \frac{(l-1)n + (k+1)(l+k+1)}{2} \right\rfloor, \left\lfloor \frac{(l+1)n + (k+1)(l+1)}{2} \right\rfloor, \left\lfloor \frac{(k-1)n}{2} \right\rfloor \right\}$$

Let G be an $S_k \cup S_l$-free graph with $e(G) \geq f(n,k,l)$, and let $u \in V(G)$ be a vertex of maximum degree in G. We have the following claims.

Claim 1: $\Delta(G) \geq k$.

Proof. Suppose $\Delta(G) \leq k-1$. Then $e(G) \leq \left\lfloor \frac{(k-1)n}{2} \right\rfloor$, which implies $e(G) \leq f(n,k,l)$, leading to a contradiction. Therefore, $\Delta(G) \geq k$.

Claim 2: $\Delta(G) \leq k + l$.

Proof. Suppose $\Delta(G) \geq k + l + 1$. For any $v \in V(G) \setminus \{u\}$, we have $d(v) \leq l$. Otherwise, if there exists a $v \in V(G)$ such that $d(v) \geq l+1$, then G contains $S_k \cup S_l$, which is a contradiction. Then $e(G) \leq \left\lfloor \frac{n-1+l(n-1)}{2} \right\rfloor$, which implies $e(G) \leq f(n,k,l)$, a contradiction.

Claim 3: $\delta(G) \in [l-1, k-2]$.

Proof. By the extremality of $ex(n, S_k \cup S_l)$, it is easy to see that $\delta(G) \geq l-1$. On the other hand, if $\delta(G) \geq k-1$, and since $\Delta(G) \geq k$, then G contains $S_k \cup S_l$, which is a contradiction.

Claim 4: There are at least two vertices in G with degree greater than l.

Proof. By contradiction. Suppose there is only one vertex with degree greater than l, and all other vertices have degree at most l. Then $e(G) \leq \left\lfloor \frac{(n-1)+l(n-1)}{2} \right\rfloor$, which implies $e(G) \leq f(n,k,l)$, leading to a contradiction.

Based on the above claims, let us assume another vertex with degree greater than l is y.

Claim 5: $\Delta(G) \leq k+l-1$.

Proof. Suppose $\Delta(G) = k+l$. Then we have $N(y) \subseteq N(u)$, otherwise G contains $S_k \cup S_l$, which is a contradiction.

Now we consider $k \leq \Delta(G) \leq k+l-i$, where $i \geq 1$. We define the following notations. Let $Z = \{x \mid x \in N(u), d(x) \geq l+1\}$, and let $z = |Z|$. Let $V_1 = \{x \mid x \in V(G) \setminus N(u), d(x) \geq l\}$, and $V_2 = V(G) \setminus (N(u) \cup V_1)$. Let m_{l+t} be the number of vertices in V_1 with degree $l+t$, where $0 \leq t \leq k-i$. Let m_q be the number of vertices in V_2 with degree q, where $0 \leq q \leq l-1$.

(a) We claim that for any $x \in Z$, $|N(x) \setminus N(u)| \leq i$. Otherwise, if there exists an x such that $|N(x) \setminus N(u)| \geq i+1$, then G contains $S_k \cup S_l$, which is a contradiction.

(b) We claim that for any $y \in V_1$, $|N(y) \cap N(u)| \geq l+t-(i-1)$. If $|N(y) \cap N(u)| \leq l+t-i$, then the degree of vertices in $N(y)$ intersecting with $V(G) \setminus N(u)$ is at least l, which implies that G contains $S_k \cup S_l$, leading to a contradiction.

(c)
$$\begin{aligned} e(N(u), V_1 \cup V_2) &= \sum_{x \in Z} |N(x) \cap N(u)| + \sum_{x \in N(u) \setminus Z} |N(x) \cap N(u)| \\ &\leq iz + (l-1)|N(u) \setminus Z| = iz + (l-1)(l+k-i-z) \end{aligned}$$

$$
\begin{aligned}
e(N(u), V_1 \cup V_2) &= \sum_{x \in V_1} |N(x) \cap N(u)| + \sum_{x \in V_2} |N(x) \cap N(u)| \\
&\geq \sum_{t=0}^{k-i} (l + t - (i-1)) m_{l+t}
\end{aligned}
$$

$$
\begin{aligned}
2e(G) &= d(u) + \sum_{x \in Z} d(x) + \sum_{x \in N(u) \setminus Z} d(x) + \sum_{x \in V_1} d(x) + \sum_{x \in V_2} d(x) \\
&\leq k + l - i + (k + l - i)z + (k + l - i - z)l + \sum_{t=0}^{k-i} (l+t) m_{l+t} + \sum_{q=0}^{l-1} q m_q
\end{aligned}
$$

$$
\begin{aligned}
2e(G) &\geq 2f(n, k, l) + 2 \\
&\geq 2 \left\lfloor \frac{(l-1)n + (k+1)(l+k+1)}{2} \right\rfloor + 2 \\
&\geq (l-1)n + (k+1)(l+k+1) \\
&= (l-1) \left(\sum_{t=0}^{k-i} m_{l+t} + \sum_{q=0}^{l-1} m_q + k + l - i + 1 \right) + (k+1)(l+k+1)
\end{aligned}
$$

From (c), we have

$$
\begin{aligned}
& iz + (l-1)(k+l-i-z) + (k+l-i) + (k+l-i)z + (k+l-i-z)l + \left(\sum_{t=0}^{k-i} (l+t) m_{l+t} + \sum_{q=0}^{l-1} q m_q \right) \\
& \geq \sum_{t=0}^{k-i} (l + t - (i-1)) m_{l+t} + (l-1) \left(\sum_{t=0}^{k-i} m_{l+t} + \sum_{q=0}^{l-1} m_q + k + l - i + 1 \right) + (k+1)(l+k+1)
\end{aligned}
$$

Thus, we obtain

$$
z \geq \frac{\left(\sum_{t=0}^{k-i} (l-i) m_{l+t} + \sum_{q=0}^{l-1-} (l-1-q) m_q \right) + (k^2 - l^2) + li + l + k + i}{k - l + 1}
$$

Given that

$$
\frac{(k^2 - l^2) + li + l + k + i}{k - l + 1} > k + l - i,
$$

it follows that

$$z > k + l - i$$

.

This leads to a contradiction. ■

References

[1] Sha-Sha Li and Jian-Hua Yin, "Two results about the Turán number of star forests," Discrete Mathematics, vol. 343, no. 11, pp. 111702, 2020.