Adjoint Method and Differential Method

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1 Introduction to the Methods

In the Maxwell FDFD solving code, The expression of electric field spreading out from a point source is directly affected by surrounding properties. And we are considering how this field be changed when a surrounding property changes which here is electric permittivity. We can compare pre- and post change by observing the different value in each domain grids one by one. There are two main methods used here to solve this problem, Adjoint method and direct differentiate method.

2 Adjoint Method and Differential Method

Local density of electromagnetic states or as known as LDOS is the quantity that we do use to consider how good the point source and the surrounding environment are, the greater the better. Therefore we only observe LDOS here instead of pure electric field when electric permittivity changes. LDOS here is represented by the product of electric source and electric field in the same direction.

$$LDOS = -Re[\int \vec{J} \cdot \vec{E} dV] \tag{1}$$

And the relationship between E and J is shown as the following .

$$[\nabla \times \nabla \times -\omega^2 \epsilon]\vec{E} = i\epsilon \vec{J} \tag{2}$$

2.1 Adjoint Method

The observed values are put into some simples variables as the below table.

Variable	Representing
g	LDOS function Eq.(1)
f	Maxwell Equation Eq.(2)
u	$\vec{E}$ electric field at each pixel
p	ϵ permittivity at each pixel

Table 1: variable table

Here we know the constraint f = 0, therefore

$$\frac{\mathrm{d}f}{\mathrm{d}p} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial f}{\partial p} = 0 \tag{3}$$

$$\frac{\partial u}{\partial p} = -(\frac{\partial f}{\partial u})^{-1}\frac{\partial f}{\partial p} \tag{4}$$

As we mentioned before, we are observing how LDOS (g) be changed when ϵ (p) changes which also contains $\frac{\partial u}{\partial p}$, so we can directly use Eq.(4) in the equation.

$$\frac{\mathrm{d}g}{\mathrm{d}p} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial g}{\partial p} \tag{5}$$

$$\frac{\mathrm{d}g}{\mathrm{d}p} = \frac{\partial g}{\partial u}[-(\frac{\partial f}{\partial u})^{-1}\frac{\partial f}{\partial p}] + \frac{\partial g}{\partial p} \tag{6}$$

Here instead of directly calculate a huge product of $(\frac{\partial f}{\partial u})^{-1}\frac{\partial f}{\partial p}$, we switch the order for $\frac{\partial g}{\partial u}$ and $(\frac{\partial f}{\partial u})^{-1}$ first which helps us a lot in term of reducing calculation amount . This method is called 'Adjoint method"

$$\frac{\mathrm{d}g}{\mathrm{d}p} = -[\frac{\partial g}{\partial u}[\frac{\partial f}{\partial u}]^{-1}]\frac{\partial f}{\partial p} + \frac{\partial g}{\partial p} \tag{7}$$

And furthermore, we need to be able to dig all variables we know as many as we can to get to our target. Here is all the numerical progress

2.1.1 $\frac{\partial g}{\partial p}$ Part

$$\frac{\partial g}{\partial p} = \frac{\partial[\int \vec{J} \cdot \vec{E} dV]}{\partial \epsilon} \tag{8}$$

We set that ϵ and E are indepedent variables or $\frac{\partial E}{\partial \epsilon} = 0$. And according to the relationship between J and E, $\int \frac{\partial J}{\partial \epsilon} E + \frac{\partial E}{\partial \epsilon} J dV = 0$, we also get $\frac{\partial J}{\partial \epsilon} = 0$.

2.1.2 $\frac{\partial g}{\partial u}$ Part

$$\frac{\partial g}{\partial u} = \frac{\partial[\int \vec{J} \cdot \vec{E} dV]}{\partial \vec{E}} \tag{9}$$

from $g = -Re[\Sigma_i (J_i E_i)]$

$$\frac{\partial g}{\partial E_i} = -Re[\Sigma_i \frac{\partial J_i}{\partial E_j} E_i + \Sigma_i \frac{\partial J_i}{\partial E_j} J_i] \tag{10}$$

we know $\frac{\partial J_i}{\partial E_j} = 0$, If i = j, $\frac{\partial J_i}{\partial E_j} = 1$ and zero in another case (δ), so

$$\frac{\partial g}{\partial u} = -Re[J] \tag{11}$$

2.1.3 $\frac{\partial f}{\partial u}$ Part

At the very beginning, the left-hand side Eq.(2) $\nabla \times \nabla \times -\omega^2 \epsilon$ from now on is represented by "M" variable. from $f(u,p) = f(\epsilon, E) = [\nabla \times \nabla \times -\omega^2 \epsilon]\vec{E} - i\epsilon\vec{J} = 0$, then we know

$$(\frac{\partial f}{\partial u})^{-1} = M^{-1} \tag{12}$$

2.1.4 $\frac{\partial f}{\partial p}$ Part

$$\frac{\partial f}{\partial p} = \frac{\partial f}{\partial \epsilon} = -\omega^2 E \tag{13}$$

2.1.5 Comparing the whole Equations

$$\frac{\mathrm{d}g}{\mathrm{d}p} = -[\frac{\partial g}{\partial u}[\frac{\partial f}{\partial u}]^{-1}]\frac{\partial f}{\partial p} + \frac{\partial g}{\partial p} \tag{14}$$

$$\frac{\mathrm{d}g}{\mathrm{d}p} = [Re(J)M^{-1}](-\omega^2 E) + 0 \tag{15}$$

In the blanket JM^{-1}, by the meaning of production between two arrays it has to be written by $J^T M^{-1} = \lambda^T$. And futhurmore, we look bank at the Eq(2) $ME = i\omega J$, we get $(J^T M^{-1})^T = (M^{-1})^T J$. Then we know one more E,J relationship.

$$J^T M^{-1} = E^T/(i\omega) \tag{16}$$

Finally, we put new Eq.(16) back into Eq.(15), then we get the most simplest form.

$$\frac{\mathrm{d}g}{\mathrm{d}p} = -i\omega E^T E \tag{17}$$

2.2 The difference between methods

As we have seen above, The adjoint method is the math progess that interacts directly with matrices. So there is the fastest and easiest way to calculate the whole brunch of output matrices when we compare with differential method calculating only one grid box in one time calculation. But ater all, We can use there mothods to recheck the precision of our programs, the answers from these two methods are supposed to be closed or exactly the same if the program is written properly.