Intro to Signal Processing – HW2

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Part I
Theory

1 Question 1

1.1 a:

1.1.1 (a)

Let us denote $S := vec(\beta_{i_1}, ..., \beta_{i_k})$ let us denote $\phi_A(x)$ of the projection of x on A and so for $\hat{f} \in S$ we need to get SE to a minimum in the following way:

$$\|f - \hat{f}\|_2^2 = \|f - \phi_F(f) + \phi_F(f) - \hat{f}\|_2^2 = \|f - \phi_F(f)\|_2^2 + \|\phi_F(f) - \hat{f}\|_2^2$$

$$= \|f - \phi_F(f)\|_2^2 + \|\phi_F(f) - \phi_S(f) + \phi_S(f) - \hat{f}\|_2^2$$

$$= \|f - \phi_F(f)\|_2^2 + \|\phi_F(f) + \phi_S(f)\|_2^2 + \|\phi_S(f) - \hat{f}\|_2^2 \tag{1.1}$$

We can notice that the first two terms in this equation will be constant so for the best representation we can choose $\hat{f} = \phi_S(f) = \sum_{j=1}^{k} < \beta_{i_j}, f > \beta_{i_j}$ and also that $\phi_F(f) = \sum_{i=1}^{n} < \beta_i, f > \beta_i$ and so let us calculate SE for $\hat{F}$:

$$\|f - \hat{f}\|_2^2 = \|f - \sum_{i=1}^{n} < \beta_i, f > \beta_i\|_2^2 + \|\sum_{i=1}^{n} < \beta_i, f > \beta_i - \sum_{j=1}^{k} < \beta_{i_j}, f > \beta_{i_j}\|_2^2$$

$$= \|f\|_2^2 - \sum_{i=1}^{n} < \beta_i, f >^2 + \sum_{\beta_j \in B/\{\beta_{i_1}, ..., \beta_{i_k}\}} < \beta_j, f >^2 = \|f\|_2^2 - \sum_{\beta_j \in \{\beta_{i_1}, ..., \beta_{i_k}\}} < \beta_j, f >^2 \tag{1.2}$$

1.1.2 (b)

From what we got in the previous sub-question that the base for k-approximation will be

$$B*_k = argmin_{\{\beta_{i_1}, ..., \beta_{i_k}\}} \|f\|_2^2 - \sum_{\beta_j \in \{\beta_{i_1}, ..., \beta_{i_k}\}} < \beta_j, f >^2 = argmax_{\{\beta_{i_1}, ..., \beta_{i_k}\}} \sum_{\beta_j \in \{\beta_{i_1}, ..., \beta_{i_k}\}} < \beta_j, f >^2 . \tag{1.3}$$

And so we can deduce that because each part of the sum is independent our answer will be the k functions in which $< \beta_i, f >$ will be the largest. Notice that the inner product can be the same for all choices and therefore it does not have to be unique.

1.2 b:

1.2.1 (a)

Let us denote the first family as B and the second as $\tilde{B}$ family, they are both basses of F. and so we get that $\sum_{i=1}^{n} < \tilde{\beta}_i, f > \tilde{\beta}_i = \hat{f} = \sum_{i=1}^{n} < \beta_i, f > \beta_i$, and from here we can deduce that the SE will be the same for both cases.

1.2.2 (b)

As discussed in the previous section we will get that for $\{\beta_{i_1}, ..., \beta_{i_k}\} \subset B$ the SE will be $\|f\|_2^2 - \sum_{\beta_j \in \{\beta_{i_1}, ..., \beta_{i_k}\}} < \beta_j, f >^2$, and for $\{\tilde{\beta}_{i_1}, ..., \tilde{\beta}_{i_k}\} \subset \tilde{B}$ the SE will be $\|f\|_2^2 - \sum_{\tilde{\beta}_j \in \{\tilde{\beta}_{i_1}, ..., \tilde{\beta}_{i_k}\}} < \tilde{\beta}_j, f >^2$. We don't have any additional information the tasses and so we don't have any information regarding the relation between these two SE.

2 Question 2

Given $t \in [0,1]$, we'll look at the signal:

$$\phi(t) = a + b\cos(2\pi t) + c\cos^2(\pi t)$$

where $a, b, c \in \mathbb{R}$ are constants.

2.1 a

We have the Haar matrix of 4×4:

$$H_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}$$

2.1.1 i

We'll show that H_4 is unitary:

$$H_4^* = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

and:

$$\begin{aligned} H_4 \cdot H_4^* = \tfrac{1}{4}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ = \tfrac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = I_4 \end{aligned}$$

We got that $H_4^* \cdot H_4 = H_4^{-1} \cdot H_4 = I^4$, and therefore H_4 is unitary.

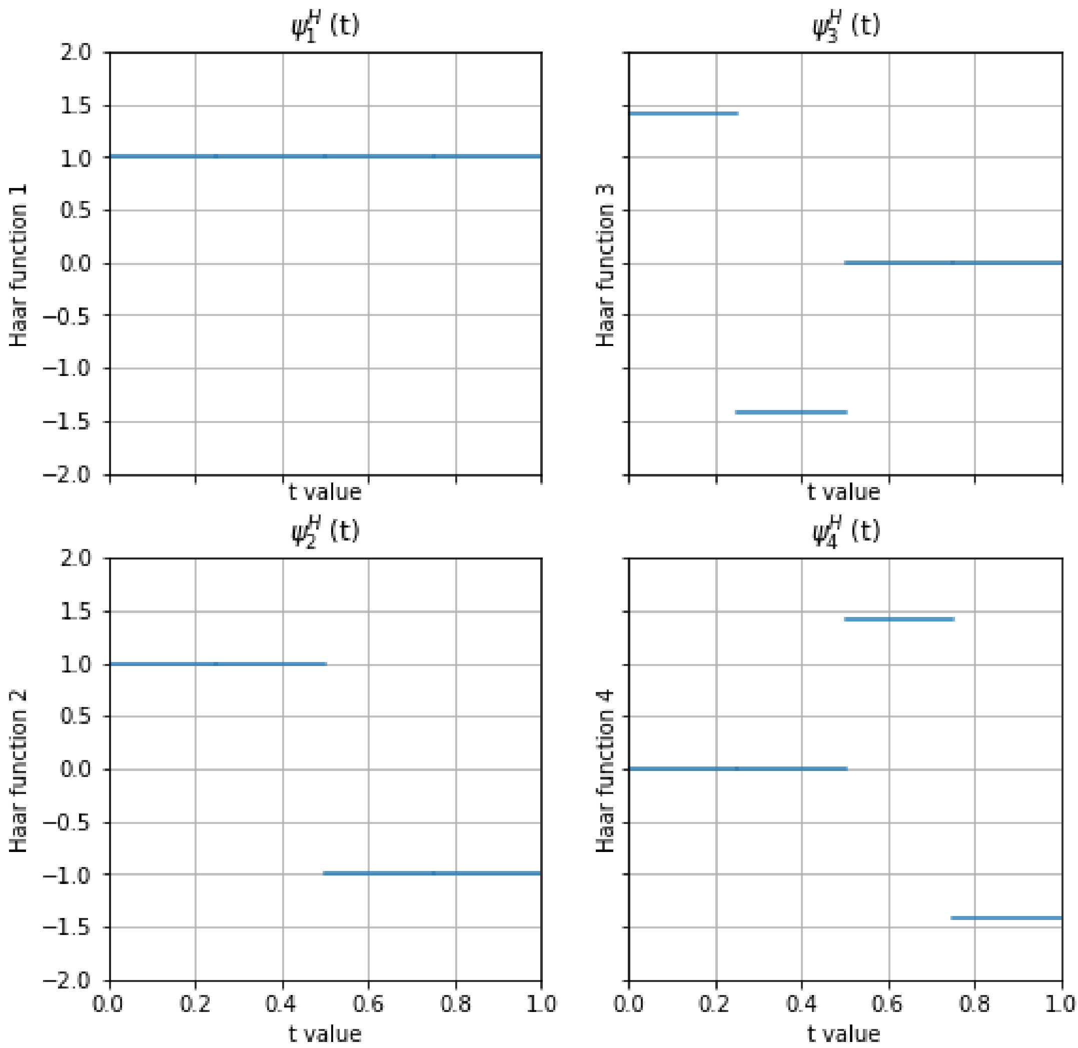
2.1.2 ii

The set of the othonormal Haar functions $\{\psi_i^H(t)\}_{i=1}^4$ can be calculated as seen in the lecture, by:

$$\psi^H(t) = H_4^* \cdot \begin{pmatrix} \sqrt{4}\cdot 1_{\Delta_1}(t) \\ \sqrt{4}\cdot 1_{\Delta_2}(t) \\ \sqrt{4}\cdot 1_{\Delta_3}(t) \\ \sqrt{4}\cdot 1_{\Delta_4}(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1_{\Delta_1}(t) \\ 1_{\Delta_2}(t) \\ 1_{\Delta_3}(t) \\ 1_{\Delta_4}(t) \end{pmatrix}$$

$$\psi^H(t) = \begin{pmatrix} \psi_1^H(t) \\ \psi_2^H(t) \\ \psi_3^H(t) \\ \psi_4^H(t) \end{pmatrix} = \begin{pmatrix} 1_{\Delta_1}(t) + 1_{\Delta_2}(t) + 1_{\Delta_3}(t) + 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) + 1_{\Delta_2}(t) - 1_{\Delta_3}(t) - 1_{\Delta_4}(t) \\ \sqrt{2}\cdot 1_{\Delta_1}(t) - \sqrt{2}\cdot 1_{\Delta_2}(t) \\ \sqrt{2}\cdot 1_{\Delta_3}(t) - \sqrt{2}\cdot 1_{\Delta_4}(t) \end{pmatrix}$$

And visualy we get:



2.1.3 iii

We saw that the best approximation of ϕ using this Haar basis is achieved by taking the projection onto $F = span\{\psi_1^H(t), \psi_2^H(t), \psi_3^H(t), \psi_4^H(t)\}$, and getting:

$$\hat{\phi}(t) = \sum_{i=1}^{4} \langle \psi_i^H(t), \phi(t) \rangle \cdot \psi_i^H(t)$$

To get the optimal Haar wavelet coefficients, we'll calculate: $\psi_i^* = \langle \psi_i^H(t), \phi(t) \rangle$, like so:

$$\begin{aligned}
\langle \psi_i^H(t), \phi(t) \rangle = & \int_0^1 \phi(t) \cdot \psi^H(t) dt \\
= & \int_0^1 \phi(t) \cdot H_4^* \cdot \begin{pmatrix} \sqrt{4} \cdot 1_{\Delta_1}(t) \\ \sqrt{4} \cdot 1_{\Delta_2}(t) \\ \sqrt{4} \cdot 1_{\Delta_3}(t) \\ \sqrt{4} \cdot 1_{\Delta_4}(t) \end{pmatrix} dt \\
= & H_4^* \cdot \int_0^1 \phi(t) \cdot \begin{pmatrix} \sqrt{4} \cdot 1_{\Delta_1}(t) \\ \sqrt{4} \cdot 1_{\Delta_2}(t) \\ \sqrt{4} \cdot 1_{\Delta_3}(t) \\ \sqrt{4} \cdot 1_{\Delta_4}(t) \end{pmatrix} dt \\
= & H_4^* \cdot \begin{pmatrix} \psi_1^{E*} \\ \psi_2^{E*} \\ \psi_3^{E*} \\ \psi_4^{E*} \end{pmatrix}
\end{aligned}$$

We'll write the standard basis coefficients of the optimal solution as ψ_i^{E*}.
When writing the Haar basis coefficients as ψ_i^{H*}, we get:

$$\begin{aligned}
\hat{\phi}(t) &= \sum_{i=1}^{4} \langle \psi_i^H(t), \phi(t) \rangle \cdot \psi_i^H(t) \\
&= \begin{pmatrix} \psi_1^{H*} & \psi_2^{H*} & \psi_3^{H*} & \psi_4^{H*} \end{pmatrix} \begin{pmatrix} \psi_1^H(t) \\ \psi_2^H(t) \\ \psi_3^H(t) \\ \psi_4^H(t) \end{pmatrix} \\
&= \begin{pmatrix} \psi_1^{E*} & \psi_2^{E*} & \psi_3^{E*} & \psi_4^{E*} \end{pmatrix} \cdot H_4 \cdot H_4^* \cdot \begin{pmatrix} \sqrt{4} \cdot 1_{\Delta_1}(t) \\ \sqrt{4} \cdot 1_{\Delta_2}(t) \\ \sqrt{4} \cdot 1_{\Delta_3}(t) \\ \sqrt{4} \cdot 1_{\Delta_4}(t) \end{pmatrix} \\
&= \begin{pmatrix} \psi_1^{E*} & \psi_2^{E*} & \psi_3^{E*} & \psi_4^{E*} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{4} \cdot 1_{\Delta_1}(t) \\ \sqrt{4} \cdot 1_{\Delta_2}(t) \\ \sqrt{4} \cdot 1_{\Delta_3}(t) \\ \sqrt{4} \cdot 1_{\Delta_4}(t) \end{pmatrix} \\
&= \sum_{i=1}^{4} \langle 2 \cdot 1_{\Delta_i}(t), \phi(t) \rangle \cdot 2 \cdot 1_{\Delta_i}(t)
\end{aligned}$$

We'll calculate the standard basis coefficients:

$$\begin{aligned}
\psi_1^{E*} &= 2 \cdot \langle 1_{\Delta_i}(t), \phi(t) \rangle \\
&= 2 \cdot \int_0^1 1_{\Delta_1}(t) \cdot (a + b\cos(2\pi t) + c\cos^2(\pi t)) dt \\
&= 2 \cdot \int_0^{0.25} a + b\cos(2\pi t) + c\cos^2(\pi t) dt \\
&= \tfrac{1}{2}a + 2b \cdot \left(\tfrac{1}{2\pi}\sin 2\pi t\right) |_0^{0.25} + c \cdot \left(\tfrac{1}{2\pi}\sin 2\pi t + t\right) |_0^{0.25} \quad \left\{ \int \cos^2 \pi t = \frac{1}{2\pi}\sin 2\pi t + t \right\} \\
&= \tfrac{a}{2} + \tfrac{b}{\pi} + \tfrac{c}{2\pi} + \tfrac{c}{4} \\
\psi_2^{E*} &= \tfrac{1}{2}a + b \cdot \left(\tfrac{1}{\pi}\sin 2\pi t\right) |_{0.25}^{0.5} + c \cdot \left(\tfrac{1}{2\pi}\sin 2\pi t + t\right) |_{0.25}^{0.5} \\
&= \tfrac{a}{2} - \tfrac{b}{\pi} - \tfrac{c}{2\pi} + \tfrac{c}{4} \\
\psi_3^{E*} &= \tfrac{1}{2}a + b \cdot \left(\tfrac{1}{\pi}\sin 2\pi t\right) |_{0.5}^{0.75} + c \cdot \left(\tfrac{1}{2\pi}\sin 2\pi t + t\right) |_{0.5}^{0.75} \\
&= \tfrac{a}{2} - \tfrac{b}{\pi} - \tfrac{c}{2\pi} + \tfrac{c}{4} \\
\psi_4^{E*} &= \tfrac{1}{2}a + b \cdot \left(\tfrac{1}{\pi}\sin 2\pi t\right) |_{0.75}^{1} + c \cdot \left(\tfrac{1}{2\pi}\sin 2\pi t + t\right) |_{0.75}^{1} \\
&= \tfrac{a}{2} + \tfrac{b}{\pi} + \tfrac{c}{2\pi} + \tfrac{c}{4}
\end{aligned}$$

And so we can now calculate the approximation $\hat{\phi}(t)$
We can calculate the MSE of our approximation:

$$\begin{aligned}
\Psi_{MSE}(\phi - \hat{\phi}) &= \int_0^1 \phi^2(t) dt - \sum_{i=1}^4 (\Psi_i^{E*})^2 \\
&= \int_0^1 (a + b\cos(2\pi t) + c\cos^2(\pi t))^2 dt - \sum_{i=1}^4 (\Psi_i^{E*})^2 \\
&= \int_0^1 \left(a^2 + ab\cos 2\pi t + ac\cos^2 \pi t + b^2 \cos^2 2\pi t + bc \cos 2\pi t \cos^2 \pi t + c^2 \cos^4 \pi t\right) dt - \sum_{i=1}^4 (\Psi_i^{E*})^2 \\
&= \left(\tfrac{1}{32\pi}\left(4\pi t(8a^2 + 4ac + 4b^2 + 2bc + 3c^2) + 8\sin 2\pi t \cdot (2ab + ac + bc + c^2) + (4b^2 + 2bc + c^2) \cdot \sin 4\pi t\right)\right) |_0^1 - \sum_{i=1}^4 (\Psi_i^{E*})^2 \\
& a^2 + ac + \tfrac{3c}{8} + \tfrac{b^2}{2} - \tfrac{bc}{2} - 2\left(\tfrac{a}{2} - \tfrac{b}{\pi} - \tfrac{c}{2\pi} + \tfrac{c}{4}\right)^2 - 2\left(\tfrac{a}{2} + \tfrac{b}{\pi} + \tfrac{c}{2\pi} + \tfrac{c}{4}\right)^2
\end{aligned}$$

2.1.4 iv

We assume now $a \geq b \geq 0, c \geq 0$.

From the last part, we now know that the Haar approximation coefficients are:

$$\begin{pmatrix}\psi_1^{H*}(t)\\ \psi_2^{H*}(t)\\ \psi_3^{H*}(t)\\ \psi_4^{H*}(t)\end{pmatrix} = H_4^* \cdot \begin{pmatrix}\psi_1^{E*}\\ \psi_2^{E*}\\ \psi_3^{E*}\\ \psi_4^{E*}\end{pmatrix}$$
$$= \tfrac{1}{2}\begin{pmatrix}\psi_1^{E*}+\psi_2^{E*}+\psi_3^{E*}+\psi_4^{E*}\\ \psi_1^{E*}+\psi_2^{E*}-\psi_3^{E*}-\psi_4^{E*}\\ \sqrt{2}\psi_1^{E*}-\sqrt{2}\psi_2^{E*}\\ \sqrt{2}\psi_3^{E*}-\sqrt{2}\psi_4^{E*}\end{pmatrix}$$
$$= \tfrac{1}{2}\begin{pmatrix}4\cdot(\frac{a}{2}+\frac{c}{4})\\ 0\\ \sqrt{2}(\frac{a}{2}+\frac{b}{\pi}+\frac{c}{2\pi}+\frac{c}{4})-\sqrt{2}(\frac{a}{2}-\frac{b}{\pi}-\frac{c}{2\pi}+\frac{c}{4})\\ \sqrt{2}(\frac{a}{2}-\frac{b}{\pi}-\frac{c}{2\pi}+\frac{c}{4})-\sqrt{2}(\frac{a}{2}+\frac{b}{\pi}+\frac{c}{2\pi}+\frac{c}{4})\end{pmatrix}$$
$$= \begin{pmatrix}a+\frac{c}{2}\\ 0\\ \sqrt{2}(\frac{b}{\pi}+\frac{c}{2\pi})\\ -\sqrt{2}(\frac{b}{\pi}+\frac{c}{2\pi})\end{pmatrix}$$

In order to get the k-term approximation, we need to take the k largest coefficients. Given $a \geq b$, we get that $a+\frac{c}{2} \geq \sqrt{2}(\frac{b}{\pi}+\frac{c}{2\pi})$. So the k term approximations will be:

$$\begin{aligned}
k=1 :& \hat{\phi}_1(t) = \big(a+\frac{c}{2}\big)\psi_1^H(t)\\
k=2 :& \hat{\phi}_2(t) = \big(a+\frac{c}{2}\big)\psi_1^H(t) + \Big\{\sqrt{2}(\frac{b}{\pi}+\frac{c}{2\pi})\psi_3^H(t) \text{ or } -\sqrt{2}(\frac{b}{\pi}+\frac{c}{2\pi})\psi_4^H(t)\Big\}\\
k=3 :& \hat{\phi}_3(t) = \big(a+\frac{c}{2}\big)\psi_1^H(t) + \sqrt{2}(\frac{b}{\pi}+\frac{c}{2\pi})\psi_3^H(t) - \sqrt{2}(\frac{b}{\pi}+\frac{c}{2\pi})\psi_4^H(t)\\
k=4 :& \hat{\phi}_4(t) = \hat{\phi}_3(t)
\end{aligned}$$

2.1.5 v

Given $a=\frac{1}{\pi}, b=1, c=\frac{3}{2}$, we can calculate:

$$\begin{pmatrix}\psi_1^{H*}(t)\\ \psi_2^{H*}(t)\\ \psi_3^{H*}(t)\\ \psi_4^{H*}(t)\end{pmatrix} = \begin{pmatrix}a+\frac{c}{2}\\ 0\\ \sqrt{2}(\frac{b}{\pi}+\frac{c}{2\pi})\\ -\sqrt{2}(\frac{b}{\pi}+\frac{c}{2\pi})\end{pmatrix} = \begin{pmatrix}\frac{1}{\pi}+\frac{3}{4}\\ 0\\ \frac{\sqrt{2}}{\pi}+\frac{3\sqrt{2}}{4\pi}\\ -\frac{\sqrt{2}}{\pi}-\frac{3\sqrt{2}}{4\pi}\end{pmatrix}$$

And the k-term approximations will be:

$$\begin{aligned}
k=1 :& \hat{\phi}_1(t) = (\frac{1}{\pi}+\frac{3}{4})\psi_1^H(t)\\
k=2 :& \hat{\phi}_2(t) = (\frac{1}{\pi}+\frac{3}{4})\psi_1^H(t) + \Big\{(\frac{\sqrt{2}}{\pi}+\frac{3\sqrt{2}}{4\pi})\psi_3^H(t) \text{ or } -(\frac{\sqrt{2}}{\pi}+\frac{3\sqrt{2}}{4\pi})\psi_4^H(t)\Big\}\\
k=3 :& \hat{\phi}_3(t) = (\frac{1}{\pi}+\frac{3}{4})\psi_1^H(t) + (\frac{\sqrt{2}}{\pi}+\frac{3\sqrt{2}}{4\pi})\psi_3^H(t) - (\frac{\sqrt{2}}{\pi}+\frac{3\sqrt{2}}{4\pi})\psi_4^H(t)\\
k=4 :& \hat{\phi}_4(t) = \hat{\phi}_3(t)
\end{aligned}$$

2.2 b

The 4×4 Walsh-Hadamard matrix is given by:

$$W_4 = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix}$$

and its columns are used to form a set of 4 orthonormal functions $\{\chi_i^W(t)\}_{i=1}^4$, defined for $t \in [0,1]$

2.2.1 i

We'll prove that W_4 is unitary, meaning $W_4 \cdot W_4^* = I$.

Since W_4 is symetrical and real, we get $W_4 = W_4^*$.

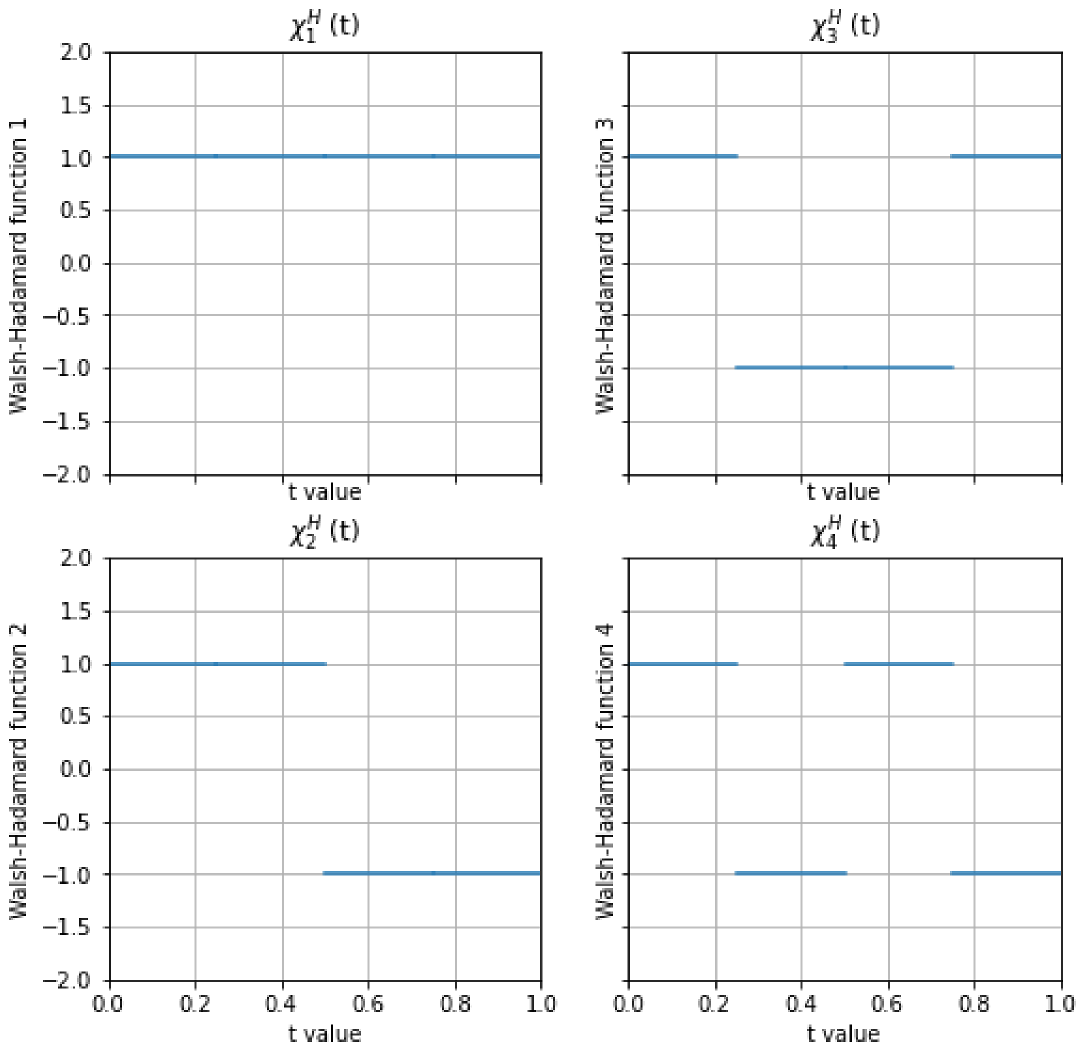
$$\begin{aligned} W_4 W_4^* &= W_4^2 \\ = \quad & \tfrac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^2 \\ = \quad & \tfrac{1}{4} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = I_4 \end{aligned}$$

And we got that W_4 is unitary.

2.2.2 ii

Same as clause a, we'll find the set of the orthonormal Walsh-Hadamard functions:

$$\begin{aligned} \begin{pmatrix} \chi_1^W(t) \\ \chi_2^W(t) \\ \chi_3^W(t) \\ \chi_4^W(t) \end{pmatrix} &= W_4^* \begin{pmatrix} \sqrt{4} 1_{\Delta_1}(t) \\ \sqrt{4} 1_{\Delta_2}(t) \\ \sqrt{4} 1_{\Delta_3}(t) \\ \sqrt{4} 1_{\Delta_4}(t) \end{pmatrix} = 2 \cdot W_4 \cdot \begin{pmatrix} 1_{\Delta_1}(t) \\ 1_{\Delta_2}(t) \\ 1_{\Delta_3}(t) \\ 1_{\Delta_4}(t) \end{pmatrix} \\ &= \begin{pmatrix} 1_{\Delta_1}(t) + 1_{\Delta_2}(t) + 1_{\Delta_3}(t) + 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) + 1_{\Delta_2}(t) - 1_{\Delta_3}(t) - 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) - 1_{\Delta_2}(t) - 1_{\Delta_3}(t) + 1_{\Delta_4}(t) \\ 1_{\Delta_1}(t) - 1_{\Delta_2}(t) + 1_{\Delta_3}(t) - 1_{\Delta_4}(t) \end{pmatrix} \end{aligned}$$



2.2.3 iii

We saw that the best approximation of ϕ using this Walsh-Hadamard basis is achieved by taking the projection onto $F = span\{\chi_1^W(t), \chi_2^W(t), \chi_3^W(t), \chi_4^W(t)\}$, and getting:

$$\hat{\phi}(t) = \sum_{i=1}^{4} \langle \chi_i^W(t), \phi(t) \rangle \cdot \chi_i^W(t)$$

To get the optimal Walsh-Hadamard wavelet coefficients, we'll calculate: $\chi_i^* = \langle \chi_i^W(t), \phi(t) \rangle$, like so:

$$\begin{aligned}
\langle \chi_i^W(t), \phi(t) \rangle &= \int_0^1 \phi(t) \cdot \chi^W(t) dt \\
&= \int_0^1 \phi(t) \cdot W_4^* \cdot \begin{pmatrix} \sqrt{4} \cdot 1_{\Delta_1}(t) \\ \sqrt{4} \cdot 1_{\Delta_2}(t) \\ \sqrt{4} \cdot 1_{\Delta_3}(t) \\ \sqrt{4} \cdot 1_{\Delta_4}(t) \end{pmatrix} dt \\
&= W_4 \cdot \int_0^1 \phi(t) \cdot \begin{pmatrix} \sqrt{4} \cdot 1_{\Delta_1}(t) \\ \sqrt{4} \cdot 1_{\Delta_2}(t) \\ \sqrt{4} \cdot 1_{\Delta_3}(t) \\ \sqrt{4} \cdot 1_{\Delta_4}(t) \end{pmatrix} dt \\
&= W_4 \cdot \begin{pmatrix} \chi_1^{E*} \\ \chi_2^{E*} \\ \chi_3^{E*} \\ \chi_4^{E*} \end{pmatrix}
\end{aligned}$$

We'll write the standard basis coefficients of the optimal solution as χ_i^{E*}.
When writing the Walsh-Hadamard basis coefficients as χ_i^{W*}, we get:

$$\begin{aligned}
\hat{\phi}(t) = & \quad \textstyle\sum_{i=1}^{4} \langle \chi_i^W(t), \phi(t) \rangle \cdot \chi_i^W(t) \\
= & \quad \begin{pmatrix} \chi_1^{W*} & \chi_2^{W*} & \chi_3^{W*} & \chi_4^{W*} \end{pmatrix} \begin{pmatrix} \chi_1^W(t) \\ \chi_2^W(t) \\ \chi_3^W(t) \\ \chi_4^W(t) \end{pmatrix} \\
= & \quad \begin{pmatrix} \chi_1^{E*} & \chi_2^{E*} & \chi_3^{E*} & \chi_4^{E*} \end{pmatrix} \cdot W_4 \cdot W_4^* \cdot \begin{pmatrix} \sqrt{4} \cdot 1_{\Delta_1}(t) \\ \sqrt{4} \cdot 1_{\Delta_2}(t) \\ \sqrt{4} \cdot 1_{\Delta_3}(t) \\ \sqrt{4} \cdot 1_{\Delta_4}(t) \end{pmatrix} \\
= & \quad \begin{pmatrix} \chi_1^{E*} & \chi_2^{E*} & \chi_3^{E*} & \chi_4^{E*} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{4} \cdot 1_{\Delta_1}(t) \\ \sqrt{4} \cdot 1_{\Delta_2}(t) \\ \sqrt{4} \cdot 1_{\Delta_3}(t) \\ \sqrt{4} \cdot 1_{\Delta_4}(t) \end{pmatrix} \\
= & \quad \textstyle\sum_{i=1}^{4} \langle 2 \cdot 1_{\Delta_i}(t), \phi(t) \rangle \cdot 2 \cdot 1_{\Delta_i}(t)
\end{aligned}$$

The standard basis coefficients are the same as the Haar approximation, so:

$$\begin{aligned}
\chi_1^{E*} &= \tfrac{a}{2} + \tfrac{b}{\pi} + \tfrac{c}{2\pi} + \tfrac{c}{4} \\
\chi_2^{E*} &= \tfrac{a}{2} - \tfrac{b}{\pi} - \tfrac{c}{2\pi} + \tfrac{c}{4} \\
\chi_3^{E*} &= \tfrac{a}{2} - \tfrac{b}{\pi} - \tfrac{c}{2\pi} + \tfrac{c}{4} = \chi_2^{E*} \\
\chi_4^{E*} &= \tfrac{a}{2} + \tfrac{b}{\pi} + \tfrac{c}{2\pi} + \tfrac{c}{4} = \chi_1^{E*}
\end{aligned}$$

And so we can now calculate the approximation $\hat{\phi}(t)$
The MSE of our approximation is the same as the Haar approximation:

$$\Psi_{MSE}(\phi - \hat{\phi}) = a^2 + ac + \frac{3c}{8} + \frac{b^2}{2} - \frac{bc}{2} - 2\Big(\frac{a}{2} - \frac{b}{\pi} - \frac{c}{2\pi} + \frac{c}{4}\Big)^2 - 2\Big(\frac{a}{2} + \frac{b}{\pi} + \frac{c}{2\pi} + \frac{c}{4}\Big)^2$$

2.2.4 iv

As we found the basis coefficients of WH are the same as Haar, we can simply calculate:

$$\begin{pmatrix} \chi_1^{W*}(t) \\ \chi_2^{W*}(t) \\ \chi_3^{W*}(t) \\ \chi_4^{W*}(t) \end{pmatrix} = \tfrac{1}{2} \cdot \begin{pmatrix} \chi_1^{E*} + \chi_2^{E*} + \chi_3^{E*} + \chi_4^{E*} \\ \chi_1^{E*} + \chi_2^{E*} - \chi_3^{E*} - \chi_4^{E*} \\ \chi_1^{E*} - \chi_2^{E*} - \chi_3^{E*} + \chi_4^{E*} \\ \chi_1^{E*} - \chi_2^{E*} + \chi_3^{E*} - \chi_4^{E*} \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 4 \cdot (\frac{a}{2} + \frac{c}{4}) \\ 0 \\ 4 \cdot (\frac{b}{\pi} + \frac{c}{2\pi}) \\ 0 \end{pmatrix} = \begin{pmatrix} a + \frac{c}{2} \\ 0 \\ \frac{2b}{\pi} + \frac{2c}{\pi} \\ 0 \end{pmatrix}$$

In order to get the k-term approximation, we need to take the k largest coefficients.
Given $a \geq b$, we get that $a + \frac{c}{2} \geq \frac{2b}{\pi} + \frac{2c}{\pi}$). So the k term approximations will be:

$$\begin{aligned}
k = 1 : & \hat{\phi}_1(t) = \big(a + \frac{c}{2}\big)\chi_1^W(t) \\
k = 2 : & \hat{\phi}_2(t) = \big(a + \frac{c}{2}\big)\chi_1^W(t) + \big(\frac{2b}{\pi} + \frac{2c}{\pi}\big)\chi_2^W(t) \\
k = 3 : & \hat{\phi}_3(t) = \hat{\phi}_2(t) \\
k = 4 : & \hat{\phi}_4(t) = \hat{\phi}_2(t)
\end{aligned}$$

2.2.5 v

Given $a = \frac{1}{\pi}, b = 1, c = \frac{3}{2}$, we can calculate:
$\big(a + \frac{c}{2}\big) = \frac{1}{\pi} + \frac{3}{4}$, and $\big(\frac{2b}{\pi} + \frac{2c}{\pi}\big) = \frac{2}{\pi} + \frac{3}{2\pi} = \frac{7}{2\pi}$

$$
\begin{aligned}
k = 1 :&\hat{\phi}_1(t) = (\frac{1}{\pi} + \frac{3}{4})\chi_1^W(t) \\
k = 2 :&\hat{\phi}_2(t) = (\frac{1}{\pi} + \frac{3}{4})\chi_1^W(t) + \frac{7}{2\pi}\chi_2^W(t) \\
k = 3 :&\hat{\phi}_3(t) = \hat{\phi}_2(t) \\
k = 4 :&\hat{\phi}_4(t) = \hat{\phi}_2(t)
\end{aligned}
$$

3 Question 3

3.1 a:

We will show the claim in induction on n. For the base H_{2^1} we can write as the following equation $H_{2^1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. It is clear that H_2^1 is symmetric and real. Let us show that it is unitary

$$H_{2^1}H_{2^1}^T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \frac{2}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I \tag{3.1}$$

and so by definition H_{2^1} is unitary.

We also get that we can write $\lambda_{2^1} = \frac{1}{\sqrt{2}}$ and that $A = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ such that $H_{2^1} = \lambda_{2^1}A$.

Let us assume that the claim is true until n-1, let us show that it is true for n. By definition we that $H_{2^n} = \frac{1}{\sqrt{2}}\begin{pmatrix}H_{2^{n-1}} & H_{2^{n-1}}\\ H_{2^{n-1}} & -H_{2^{n-1}}\end{pmatrix}$. Let us show by definition that H_{2^n} is symmetric:

$$\begin{aligned} H_{2^n}^T &= (\frac{1}{\sqrt{2}}\begin{pmatrix}H_{2^{n-1}} & H_{2^{n-1}}\\ H_{2^{n-1}} & -H_{2^{n-1}}\end{pmatrix})^T = \frac{1}{\sqrt{2}}\begin{pmatrix}H_{2^{n-1}}^T & H_{2^{n-1}}^T\\ H_{2^{n-1}}^T & -H_{2^{n-1}}^T\end{pmatrix} \\ &=^* \frac{1}{\sqrt{2}}\begin{pmatrix}H_{2^{n-1}} & H_{2^{n-1}}\\ H_{2^{n-1}} & -H_{2^{n-1}}\end{pmatrix} = H_{2^n} \end{aligned} \tag{3.2}$$

* Where we used the fact the $H_{2^{n-1}}$ is symmetric.

Let us show that H_{2^n} is real. We know that H_{2^n} is made of the multiplication of a matrix of $H_{2^{n-1}}$ which by induction makes the matrix real and that $\frac{1}{\sqrt{2}}$ is also real and therefore we get that H_{2^n} is real.

Let us show by definition that H_{2^n} is unitary:

$$\begin{aligned} H_{2^n}H_{2^n}^T &= \frac{1}{\sqrt{2}}\begin{pmatrix}H_{2^{n-1}} & H_{2^{n-1}}\\ H_{2^{n-1}} & -H_{2^{n-1}}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}H_{2^{n-1}}^T & H_{2^{n-1}}^T\\ H_{2^{n-1}}^T & -H_{2^{n-1}}^T\end{pmatrix} = \frac{1}{2}\begin{pmatrix}H_{2^{n-1}} & H_{2^{n-1}}\\ H_{2^{n-1}} & -H_{2^{n-1}}\end{pmatrix}\begin{pmatrix}H_{2^{n-1}} & H_{2^{n-1}}\\ H_{2^{n-1}} & -H_{2^{n-1}}\end{pmatrix} \\ &=^* \frac{1}{2}\begin{pmatrix}2H_{2^{n-1}}^2 & 0\\ 0 & -2H_{2^{n-1}}^2\end{pmatrix} = \frac{2}{2}\begin{pmatrix}H_{2^{n-1}}^2 & 0\\ 0 & -H_{2^{n-1}}^2\end{pmatrix} =^* \begin{pmatrix}I & 0\\ 0 & I\end{pmatrix} = I \end{aligned} \tag{3.3}$$

* Where we used the fact the $H_{2^{n-1}}$ is unitary and symmetric. and so by definition H_{2^n} is unitary.

From the induction we know that there are A_{n-1} and $\lambda_{2^{n-1}}$ such that $H_{2^{n-1}} = \lambda_{2^{n-1}}A_{n-1}$ and so we can write H_{2^n} as

$$\begin{aligned} H_{2^n} &= \frac{1}{\sqrt{2}}\begin{pmatrix}H_{2^{n-1}} & H_{2^{n-1}}\\ H_{2^{n-1}} & -H_{2^{n-1}}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\lambda_{2^{n-1}}A_{n-1} & \lambda_{2^{n-1}}A_{n-1}\\ \lambda_{2^{n-1}}A_{n-1} & -\lambda_{2^{n-1}}A_{n-1}\end{pmatrix} \\ &=^* \frac{\lambda_{2^{n-1}}}{\sqrt{2}}\begin{pmatrix}A_{n-1} & A_{n-1}\\ A_{n-1} & -A_{n-1}\end{pmatrix} \end{aligned} \tag{3.4}$$

and so let us denote $\lambda_{2^n} = \frac{\lambda_{2n-1}}{\sqrt{2}}$ and let us denote $A_n = \begin{pmatrix}A_{n-1} & A_{n-1}\\ A_{n-1} & -A_{n-1}\end{pmatrix}$, and so the condition works on H_n. We get in general that the induction holds and so we get that the claim holds for all $n \in \mathbb{N}$ as required.

3.2 b:

3.2.1 (i)

As the hint suggests let us consider two cases. In the case that both the last number of s_1 and the first number of s_2 have the same sign then we get that $S(s_1s_2) = S(s_1) + S(s_2)$. In the case that they are different signs we will get that $S(s_1s_2) = S(s_1) + S(s_2) + 1$.

3.2.2 (ii)

Let us prove this claim using induction on n. For the base H_{2^1} from straight calculation we get that $S(r_1) = 0$ and that $S(r_2) = 1$, and so the condition holds. Let us assume that the claim is true until n-1, let us show that it is true for n. We know that $H_{2^n} = \frac{1}{\sqrt{2}}\begin{pmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{pmatrix}$, and that from the induction that the number of sign changes for all the rows of $H_{2^{n-1}}$ is $\{0, 1, ..., 2^{n-1} - 1\}$. for $i \in \{0, 1, ..., 2^n - 1\}$ Let us denote $k = \lfloor \frac{i}{2} \rfloor$ we can get from i's range that k's range will be $k \in \{0, 1, ..., 2^{n-1} - 1\}$. We get that from the induction $H_{2^{n-1}}$ that it has a row the k changes let us denote it to be s_k. Let us note that from how H_{2^n} is defined there will be two different rows that will start with s_k one which it will be $(s_k s_k)$ and one which will be $(s_k(-s_k))$, and so from the question 3.b.i we get that one of them will be $2S(r_i) = 2k$ let us mark that row as s'_{2k} and the other one will be $2S(r_i) + 1 = 2k + 1$ let us mark that row as s'_{2k+1} we get both cases because there is one sign change between those 2 rows that leads to the difference. And so let us look at $m\ in \{0, 1, ..., 2^n - 1\}$ for all m's that are even we get that there is k such that $m = 2k$ and so we will get the the row s'_{2k} will have m sign changes, and for all m's that are even we get that there is k such that $m = 2k + 1$ and so the row s'_{2k+1} will have m sign changes. We got in general that $\{0, 1, ..., 2^n - 1\} \subseteq \{S(r_1), ..., S(r_N)\}$ and due to the size of H_{2^n} that $\{S(r_1), ..., S(r_N)\} \subseteq \{0, 1, ..., 2^n - 1\}$ and so we get in general that $\{S(r_1), ..., S(r_N)\} = \{0, 1, ..., 2^n - 1\}$ as required.

4 On Haar Matrices

a Is H_N symetrical if $N \geq 1$:
No, as we can see for $N = 4$:

$$H_4 = \begin{pmatrix} H_2 \otimes (1,1) \\ I_N \otimes (1,-1) \end{pmatrix}$$

When H_2 is the Haar matrix: $H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
We get:

$$H_4 = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes (1,1) \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1,-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Which we can see isn't symetrical.

b We can check if H_4 is unitary:

$$H_4^* \cdot H_4 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

We got $H_4^* H_4 \neq I_4$, so H_4 isn't unitary.

c We'll create the algorithm to create the normalised $\tilde{H}_N$:

$$\tilde{H}_2 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\tilde{H}_{2N} = \begin{pmatrix} \tilde{H}_N \otimes \begin{pmatrix} 1 & 1 \end{pmatrix} \\ I_N \otimes \begin{pmatrix} 1 & -1 \end{pmatrix} \end{pmatrix}$$

d We need to prove: $(A \otimes B)^T = A^T \otimes B^T$

$$(A \otimes B)^T = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2.1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,n}B \end{pmatrix}^T = \begin{pmatrix} a_{1,1}B^T & a_{2,1}B^T & \cdots & a_{n,1}B^T \\ a_{1,2}B^T & a_{2,2}B^T & \cdots & a_{n.2}B^T \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n}B^T & a_{2,n}B^T & \cdots & a_{n,n}B^T \end{pmatrix} = A^T \otimes B^T$$

e We start by writing the $\hat{H}_2$ matrix:

$$\hat{H}_2 = \tilde{H}_2^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

With the equation we found in clause d, we can now calculate:

$$\begin{aligned} \hat{H}_{2N} = \quad & \tilde{H}_{2N}^T = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \tilde{H}_N \otimes \begin{pmatrix} 1 & 1 \end{pmatrix} \\ I_N \otimes \begin{pmatrix} 1 & -1 \end{pmatrix} \end{pmatrix}^T = \tfrac{1}{\sqrt{2}} \left(\left(\tilde{H}_N \otimes \begin{pmatrix} 1 & 1 \end{pmatrix}\right)^T \quad \left(I_N \otimes \begin{pmatrix} 1 & -1 \end{pmatrix}\right)^T \right) \\ = \quad & \tfrac{1}{\sqrt{2}} \left(\tilde{H}_N^T \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad I_N^T \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \\ = \quad & \tfrac{1}{\sqrt{2}} \left(\hat{H}_N \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad I_N \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \end{aligned}$$

Part II
Implementation

5 Numerical and Practical Bit Allocation for Two- Dimensional Signals

5.1 a.

Let us calculate the value range, let as use the fact that $sin(x), cos(x) \in [-1, 1]$

$$\phi(x,y) = Acos(2\pi\omega_x x)sin(2\pi\omega_y y) \leq Acos(2\pi\omega_x x) \leq A \tag{5.1}$$

$$\phi(x,y) = Acos(2\pi\omega_x x)sin(2\pi\omega_y y) \geq -Acos(2\pi\omega_x x) \geq A \tag{5.2}$$

And so we get that the range $[-A, A]$.

Let us calculate the derivative by x and y:

$$\frac{\partial\phi(x,y)}{\partial x} = -2\pi w_x A\sin(2\pi w_x x)\sin(2\pi w_y y) \tag{5.3}$$

$$\frac{\partial\phi(x,y)}{\partial y} = 2\pi w_y A\cos(2\pi w_x x)\cos(2\pi w_y y) \tag{5.4}$$

From what we learned in calculus we get that $\int_a^b \cos^2(\alpha t)dt = \frac{1}{2}(\frac{\sin(2\alpha t)}{2\alpha} + t)\Big|_a^b$, and also that $\int_a^b \sin^2(\alpha t)dt = \frac{1}{2}(t - \frac{\sin(2\alpha t)}{2\alpha})\Big|_a^b$. Let us calculate the horizontal and vertical derivative energies. Horizontal-derivative energy:

$$\int_0^1\int_0^1 (\frac{\partial\phi(x,y)}{\partial x})^2 dxdy = \int_0^1\int_0^1 (-2\pi w_x A\sin(2\pi w_x x)\sin(2\pi w_y y)^2 dxdy$$

$$= (2\pi w_x A)^2 \int_0^1\int_0^1 (\sin(2\pi w_x x)\sin(2\pi w_y y))^2 dxdy = (2\pi w_x A)^2 \int_0^1 (\sin(2\pi w_x x))^2 dx \int_0^1 (\sin(2\pi w_y y))^2 dy$$

$$= (2\pi w_x A)^2 (\frac{1}{2}(x - \frac{\sin(4\pi w_x x)}{4\pi w_x x})\Big|_0^1)(\frac{1}{2}(y - \frac{\sin(4\pi w_y y)}{4\pi w_y y})\Big|_0^1) = \frac{(2\pi w_x A)^2}{4} = 4\pi^2 A^2 = 246740110 \tag{5.5}$$

Vertical-derivative energy:

$$\int_0^1\int_0^1 (\frac{\partial\phi(x,y)}{\partial y})^2 dxdy = \int_0^1\int_0^1 (2\pi w_y A\cos(2\pi w_x x)\cos(2\pi w_y y))^2 dxdy$$

$$= (2\pi w_y A)^2 \int_0^1\int_0^1 (\cos(2\pi w_x x)\cos(2\pi w_y y))^2 dxdy = (2\pi w_y A)^2 \int_0^1 (\cos(2\pi w_x x))^2 dx \int_0^1 (\cos(2\pi w_y y))^2 dy$$

$$= (2\pi w_y A)^2 (\frac{1}{2}(\frac{\sin(4\pi w_x x)}{4\pi w_x} + x)\Big|_0^1)(\frac{1}{2}(\frac{\sin(4\pi w_y y)}{4\pi w_y} + y)\Big|_0^1) = \frac{(2\pi w_y A)^2}{4} = 49\pi^2 A^2 = 3022566348 \tag{5.6}$$

5.2 b.

The signal we got was:

5.3 c.

The numerical results we got are:

```
Value Range: 4999
Horizontal-Derivative energy: 246695987.52134135
Vertical-Derivative energy: 3021599202.9528975
```

When comparing we can see that for x we got 246740110 and 246695987 and for y we got that 3022566348 and 3021599202 and for range we got 5000 and 4999 which as we can see the numerical calculations we got are close to the what we calculated in 1.a, if we improve the resolution even more we will get even a better approximation.

5.4 d.

We want to minimize the MSE function.Let us denote Δ as the range of the function. We have the constraint that we want $N_x N_y b = B$. The MSE function we get is

$$MSE = \frac{E_x}{12 * N_x^2} + \frac{E_y}{12 * N_y^2} + \frac{\Delta^2}{12 * 2^{2b}} = \frac{b\sqrt{E_x E_y}}{6B} + \frac{\Delta^2}{12 * 2^{2b}} \tag{5.7}$$

If we derivative compare to 0 and place B instead we will get that following equation

$$b = \frac{1}{2}\log_2\left(\frac{\Delta B \ln(2)}{\sqrt{E_x E_y}}\right) \tag{5.8}$$

When we derivative by N_x and compare to zero we will get the equation

$$0 = \frac{2N_y E_x}{12N_x^2} - \frac{2E_y}{12N_y^3} \tag{5.9}$$

$$N_y^2 = \frac{E_y}{E_x} N_x^2 \tag{5.10}$$

and so using this conclusion and the fact that $N_x N_y b = B$ we get that:

$$N_x = \sqrt{\sqrt{\frac{E_x}{E_y}}\frac{B}{b}} \tag{5.11}$$

$$N_y = \sqrt{\sqrt{\frac{E_y}{E_x}}\frac{B}{b}} \tag{5.12}$$

5.5 e.

```
B low:
b = 3,  N_x = 21,  N_y = 79
B high:
b = 5,  N_x = 54,  N_y = 185
```

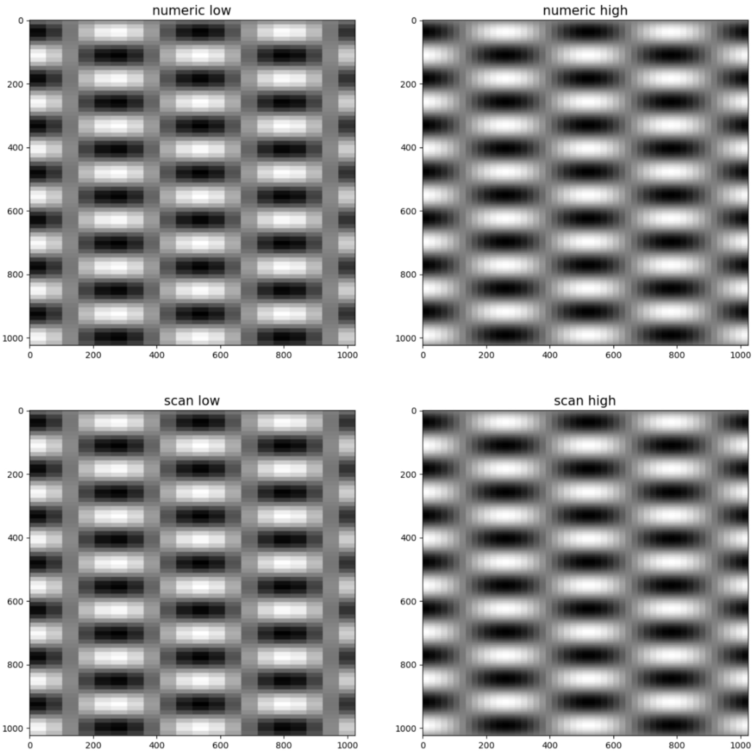
5.6 f.

Done in the corresponding notebook

5.7 g.

```
B low:
b = 3,  N_x = 21,  N_y = 79
B high:
b = 5,  N_x = 54,  N_y = 185
```

We have iterated overall all possible combinations of parameters (with jumping steps of 1) and saved the result that had the best MSE (lowest one). As we can see the results we got are similar to the previous method and so we can assume that we have found optimal representation and that is why there is no difference. Reconstructed images in the experiments:



5.8 h.

5.8.1 a

From the same way of calculation as the first part we can notice that the result will just switch as the parameters switched and therefore we get that the Horizontal-derivative energy will be 3022566348 and the Vertical-derivative energy will be 246740110, the value range will remain the same.

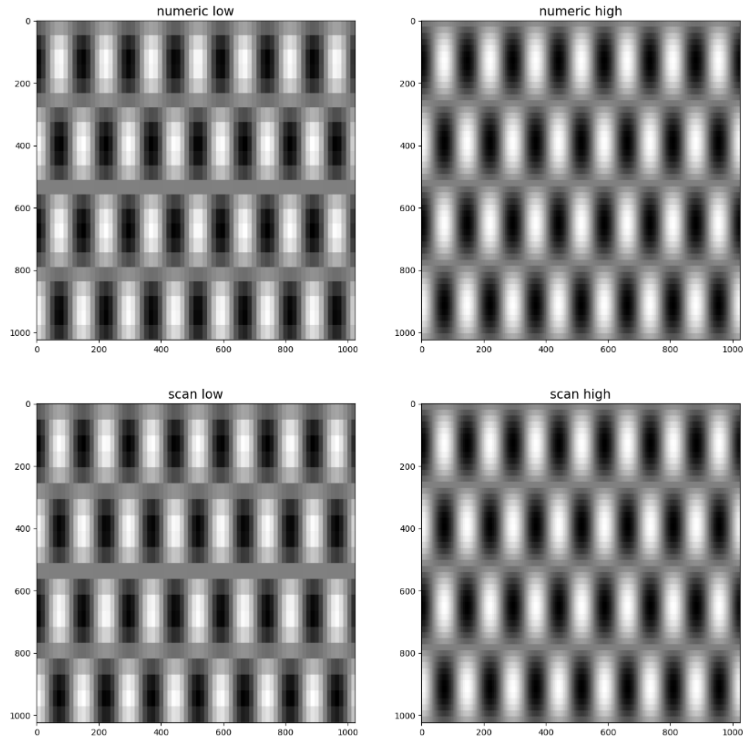
5.8.2 b

5.8.3 c-g

```
part c:
Value Range: 4999
Horizontal-Derivative energy: 3021599202
Vertical-Derivative energy: 246695987

part e:
B low:
b = 3,  N_x = 72,  N_y = 23
B high:
b = 5,  N_x = 188,  N_y = 53

part g:
B low:
b = 3,  N_x = 79,  N_y = 21
B high:
b = 5,  N_x = 185,  N_y = 54
```



We can see that the energy has switched has caused N_x and N_y to switch but b has remained the same.

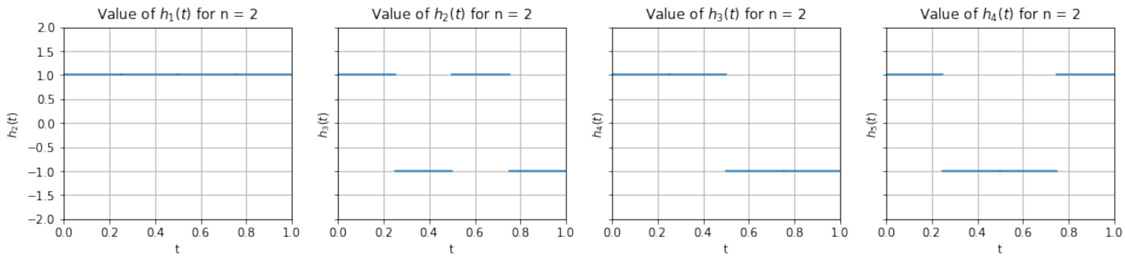
6 Hadamard, Hadamard-Walsh, and Haar matrices

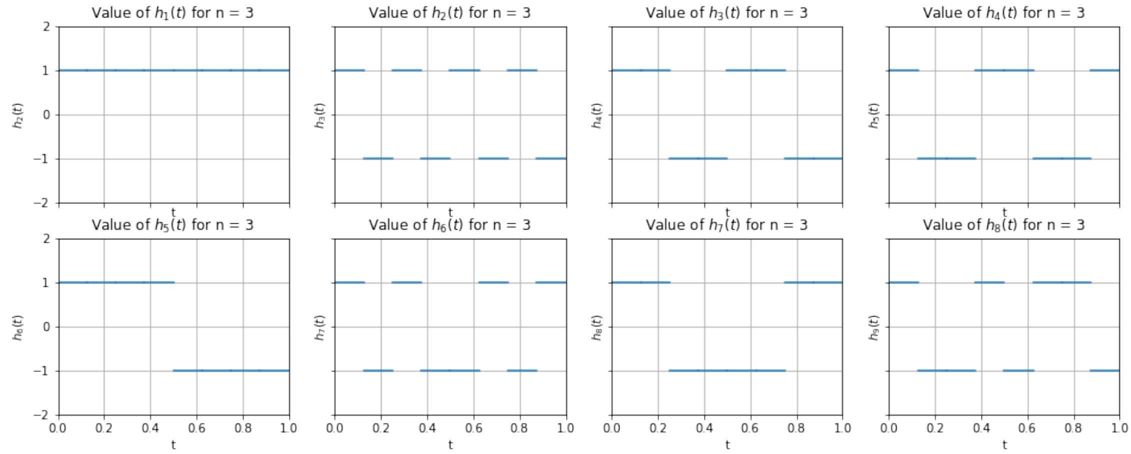
6.1 a

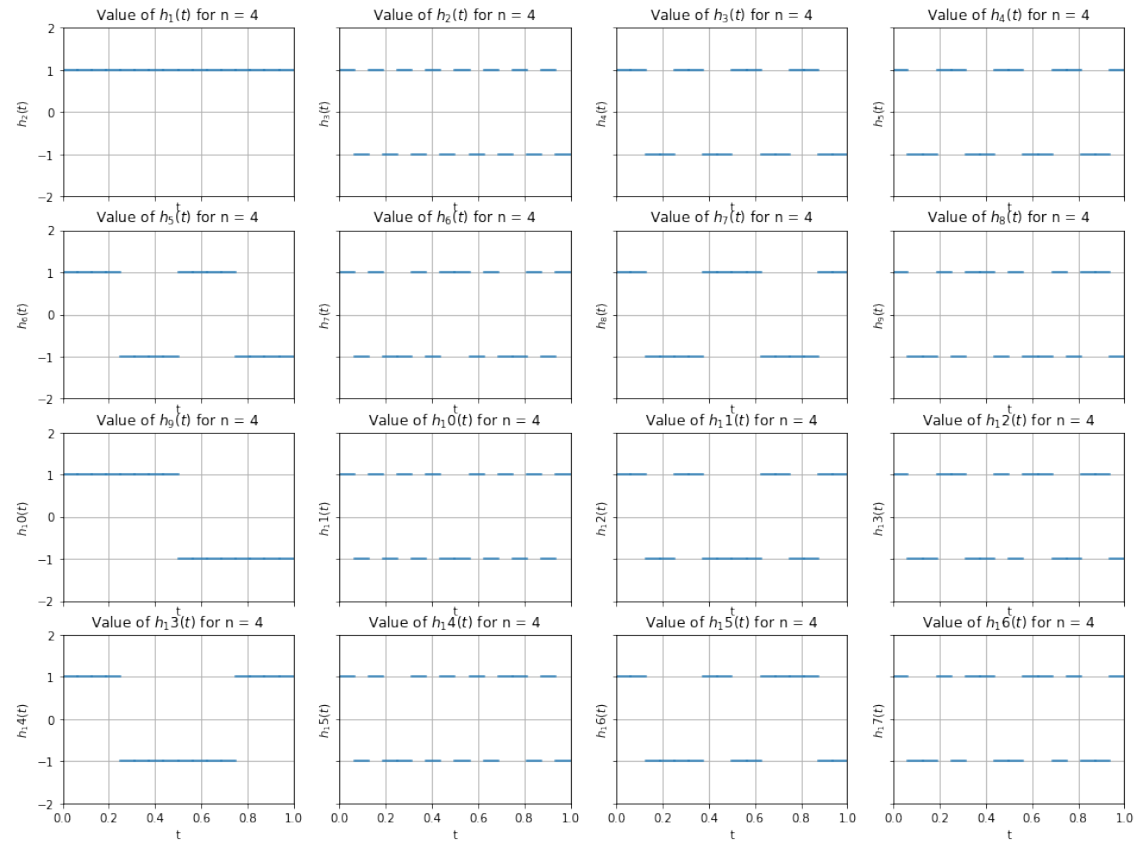
Hadamard code is implemented in the code in wetq2.ipynb

6.2 b

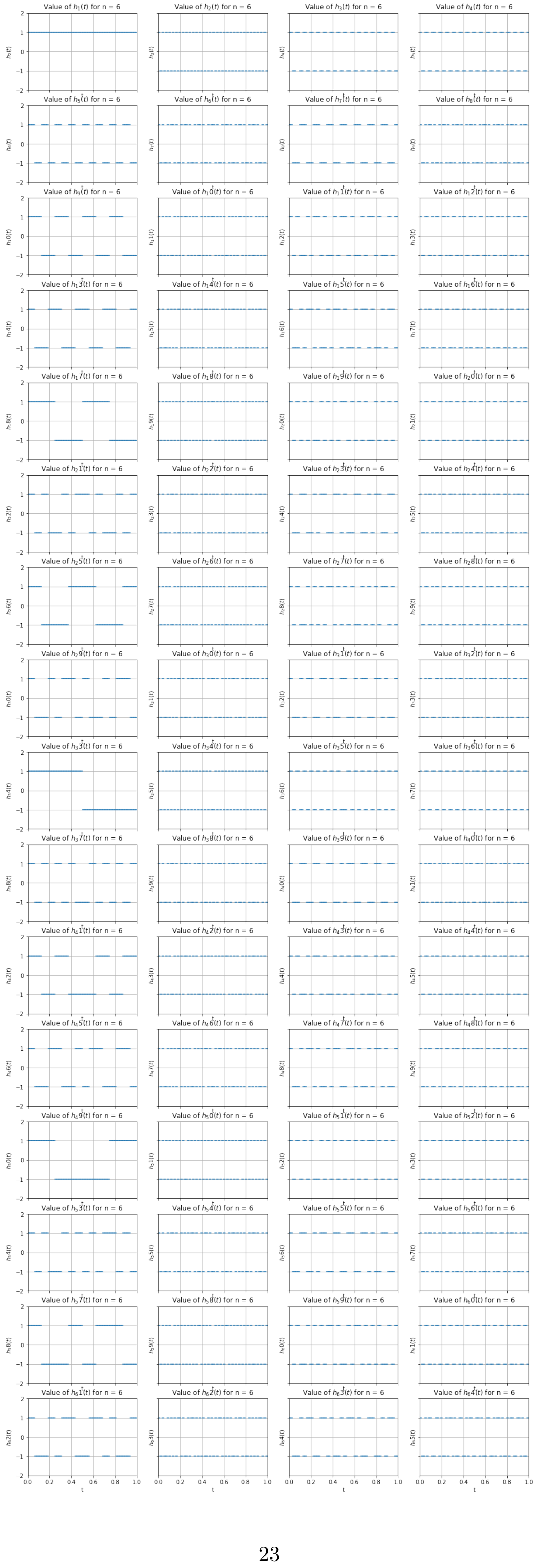
Plots of the Hadamard functions $\{h_i\}_{i=1}^{2^n}$:






Value of $h_1(t)$ for n = 4
Value of $h_2(t)$ for n = 4
Value of $h_3(t)$ for n = 4
Value of $h_4(t)$ for n = 4
Value of $h_5(t)$ for n = 4
Value of $h_6(t)$ for n = 4
Value of $h_7(t)$ for n = 4
Value of $h_8(t)$ for n = 4
Value of $h_9(t)$ for n = 4
Value of $h_10(t)$ for n = 4
Value of $h_11(t)$ for n = 4
Value of $h_12(t)$ for n = 4
Value of $h_13(t)$ for n = 4
Value of $h_14(t)$ for n = 4
Value of $h_15(t)$ for n = 4
Value of $h_16(t)$ for n = 4
t


Value of $h_1(t)$ for n = 5
Value of $h_2(t)$ for n = 5
Value of $h_3(t)$ for n = 5
Value of $h_4(t)$ for n = 5
Value of $h_5(t)$ for n = 5
Value of $h_6(t)$ for n = 5
Value of $h_7(t)$ for n = 5
Value of $h_8(t)$ for n = 5
Value of $h_9(t)$ for n = 5
Value of $h_10(t)$ for n = 5
Value of $h_11(t)$ for n = 5
Value of $h_12(t)$ for n = 5
Value of $h_13(t)$ for n = 5
Value of $h_14(t)$ for n = 5
Value of $h_15(t)$ for n = 5
Value of $h_16(t)$ for n = 5
Value of $h_17(t)$ for n = 5
Value of $h_18(t)$ for n = 5
Value of $h_19(t)$ for n = 5
Value of $h_20(t)$ for n = 5
Value of $h_21(t)$ for n = 5
Value of $h_22(t)$ for n = 5
Value of $h_23(t)$ for n = 5
Value of $h_24(t)$ for n = 5
Value of $h_25(t)$ for n = 5
Value of $h_26(t)$ for n = 5
Value of $h_27(t)$ for n = 5
Value of $h_28(t)$ for n = 5
Value of $h_29(t)$ for n = 5
Value of $h_30(t)$ for n = 5
Value of $h_31(t)$ for n = 5
Value of $h_32(t)$ for n = 5
t

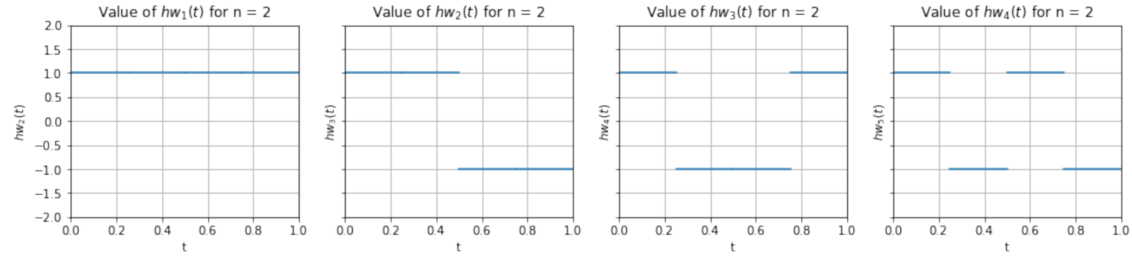


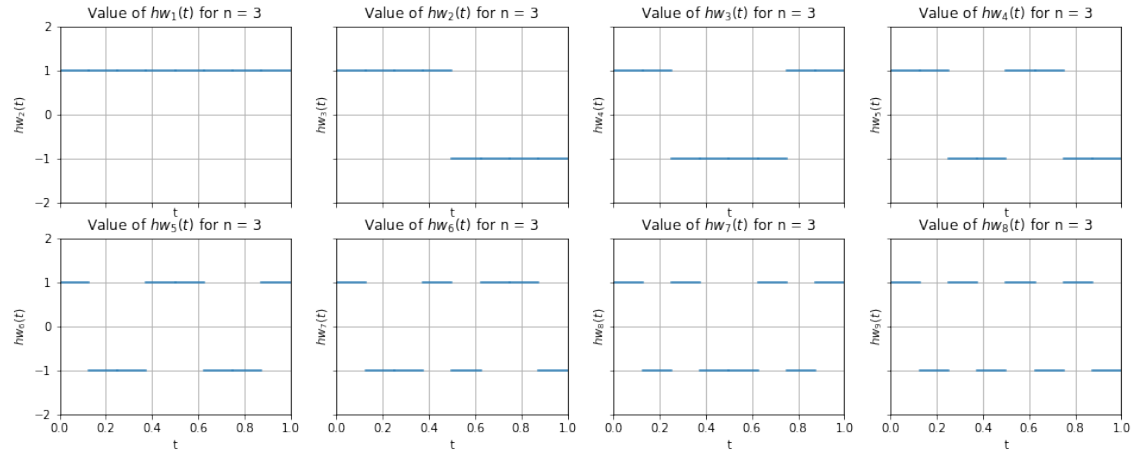
6.3 c

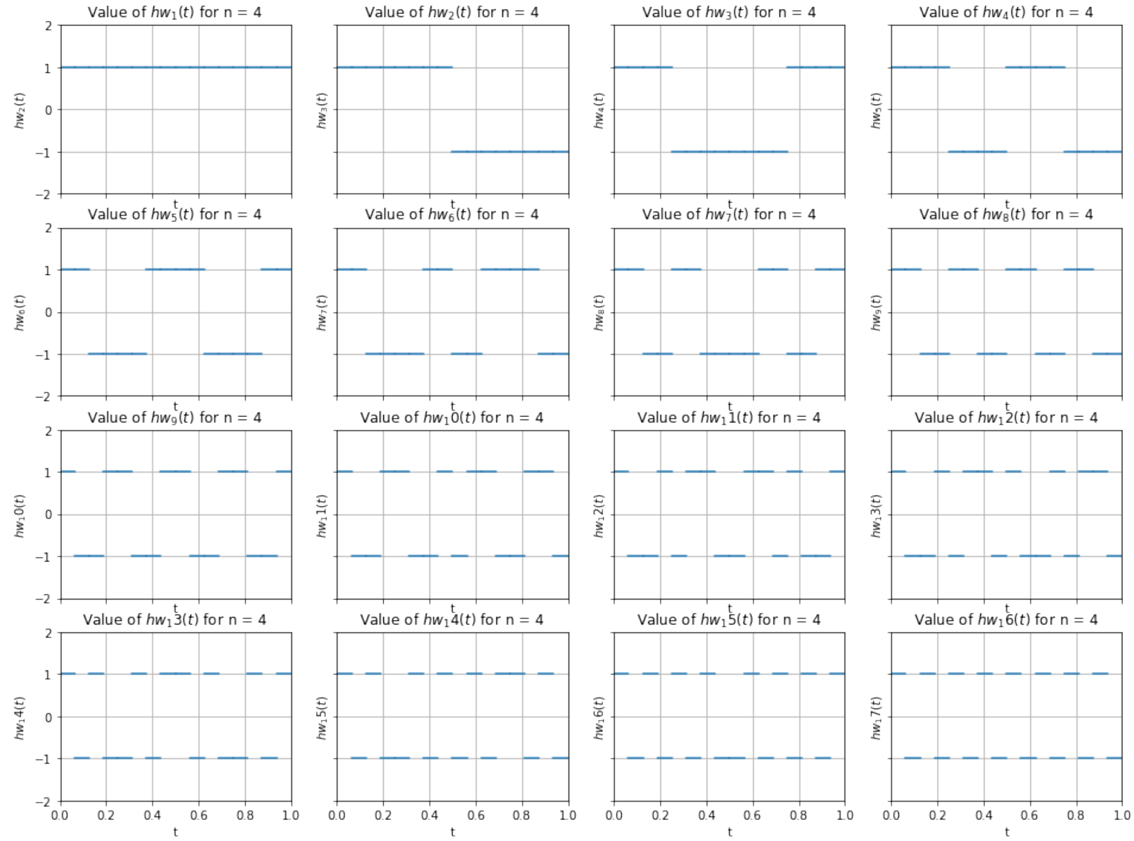
Walsh-Hadamard implementation is in the code in wetq2.ipynb

6.4 d

Plots of the Walsh-Hadamard functions $\{wh_i\}_{i=1}^{2^n}$:






Value of $hw_1(t)$ for n = 4
Value of $hw_2(t)$ for n = 4
Value of $hw_3(t)$ for n = 4
Value of $hw_4(t)$ for n = 4
Value of $hw_5(t)$ for n = 4
Value of $hw_6(t)$ for n = 4
Value of $hw_7(t)$ for n = 4
Value of $hw_8(t)$ for n = 4
Value of $hw_9(t)$ for n = 4
Value of $hw_10(t)$ for n = 4
Value of $hw_11(t)$ for n = 4
Value of $hw_12(t)$ for n = 4
Value of $hw_13(t)$ for n = 4
Value of $hw_14(t)$ for n = 4
Value of $hw_15(t)$ for n = 4
Value of $hw_16(t)$ for n = 4
t

Value of $hw_1(t)$ for n = 5
Value of $hw_2(t)$ for n = 5
Value of $hw_3(t)$ for n = 5
Value of $hw_4(t)$ for n = 5
Value of $hw_5(t)$ for n = 5
Value of $hw_6(t)$ for n = 5
Value of $hw_7(t)$ for n = 5
Value of $hw_8(t)$ for n = 5
Value of $hw_9(t)$ for n = 5
Value of $hw_10(t)$ for n = 5
Value of $hw_11(t)$ for n = 5
Value of $hw_12(t)$ for n = 5
Value of $hw_13(t)$ for n = 5
Value of $hw_14(t)$ for n = 5
Value of $hw_15(t)$ for n = 5
Value of $hw_16(t)$ for n = 5
Value of $hw_17(t)$ for n = 5
Value of $hw_18(t)$ for n = 5
Value of $hw_19(t)$ for n = 5
Value of $hw_20(t)$ for n = 5
Value of $hw_21(t)$ for n = 5
Value of $hw_22(t)$ for n = 5
Value of $hw_23(t)$ for n = 5
Value of $hw_24(t)$ for n = 5
Value of $hw_25(t)$ for n = 5
Value of $hw_26(t)$ for n = 5
Value of $hw_27(t)$ for n = 5
Value of $hw_28(t)$ for n = 5
Value of $hw_29(t)$ for n = 5
Value of $hw_30(t)$ for n = 5
Value of $hw_31(t)$ for n = 5
Value of $hw_32(t)$ for n = 5
t

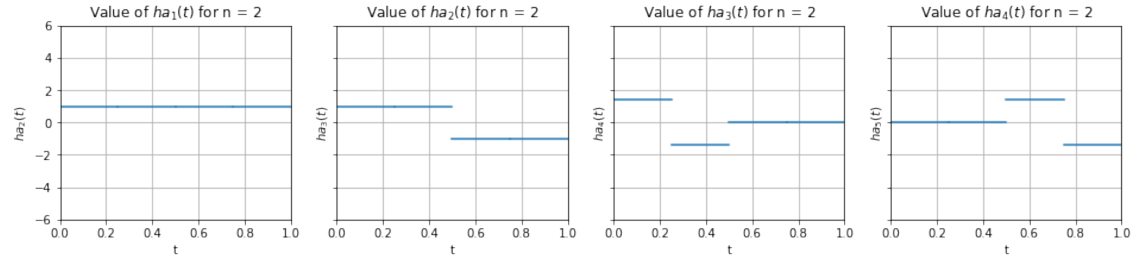
Value of $hw_1(t)$ for n = 6
Value of $hw_2(t)$ for n = 6
Value of $hw_3(t)$ for n = 6
Value of $hw_4(t)$ for n = 6

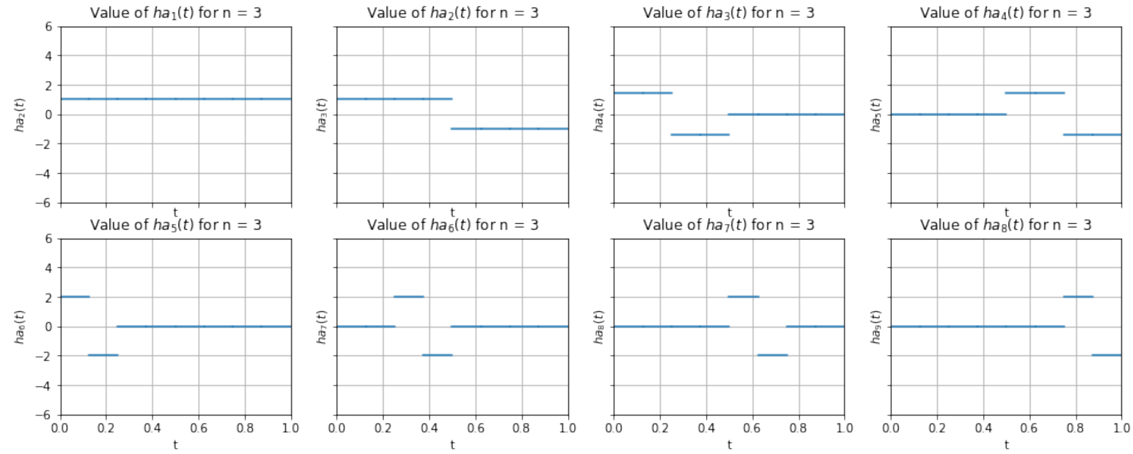
6.5 e

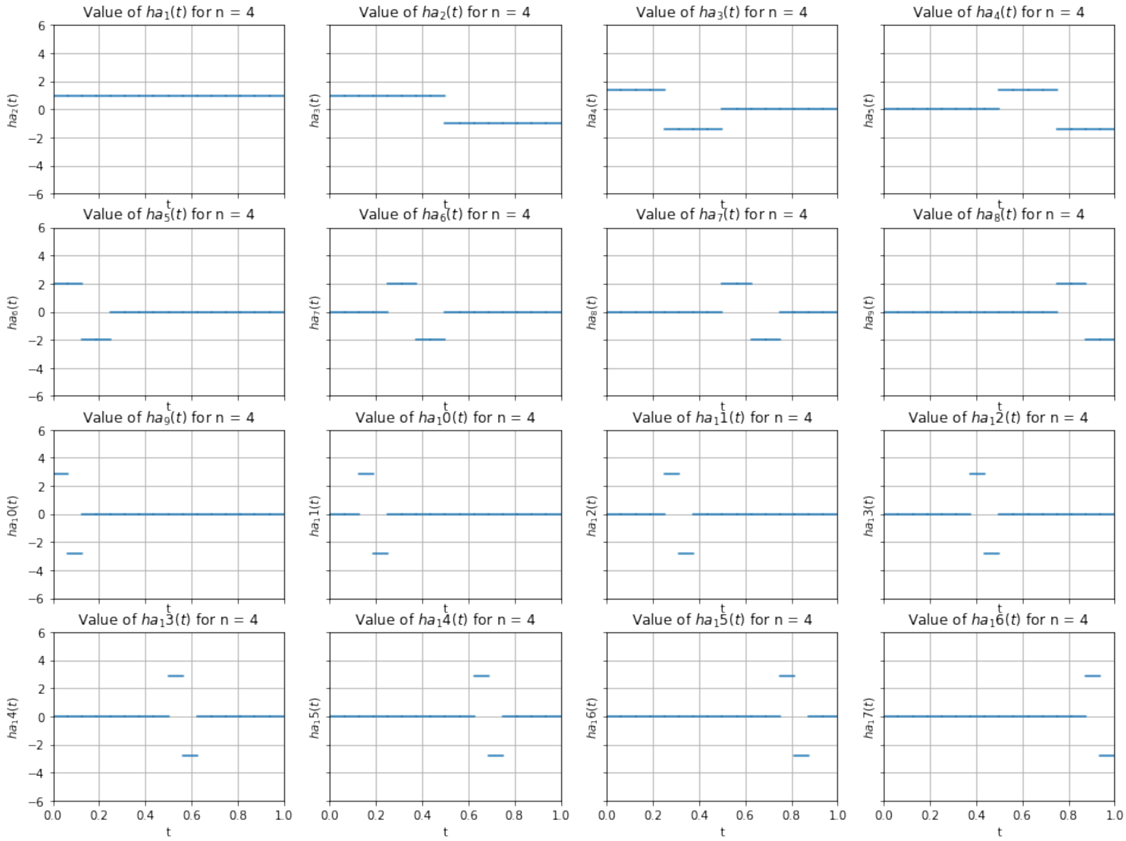
Haar implementation is in the code in wetq2.ipynb

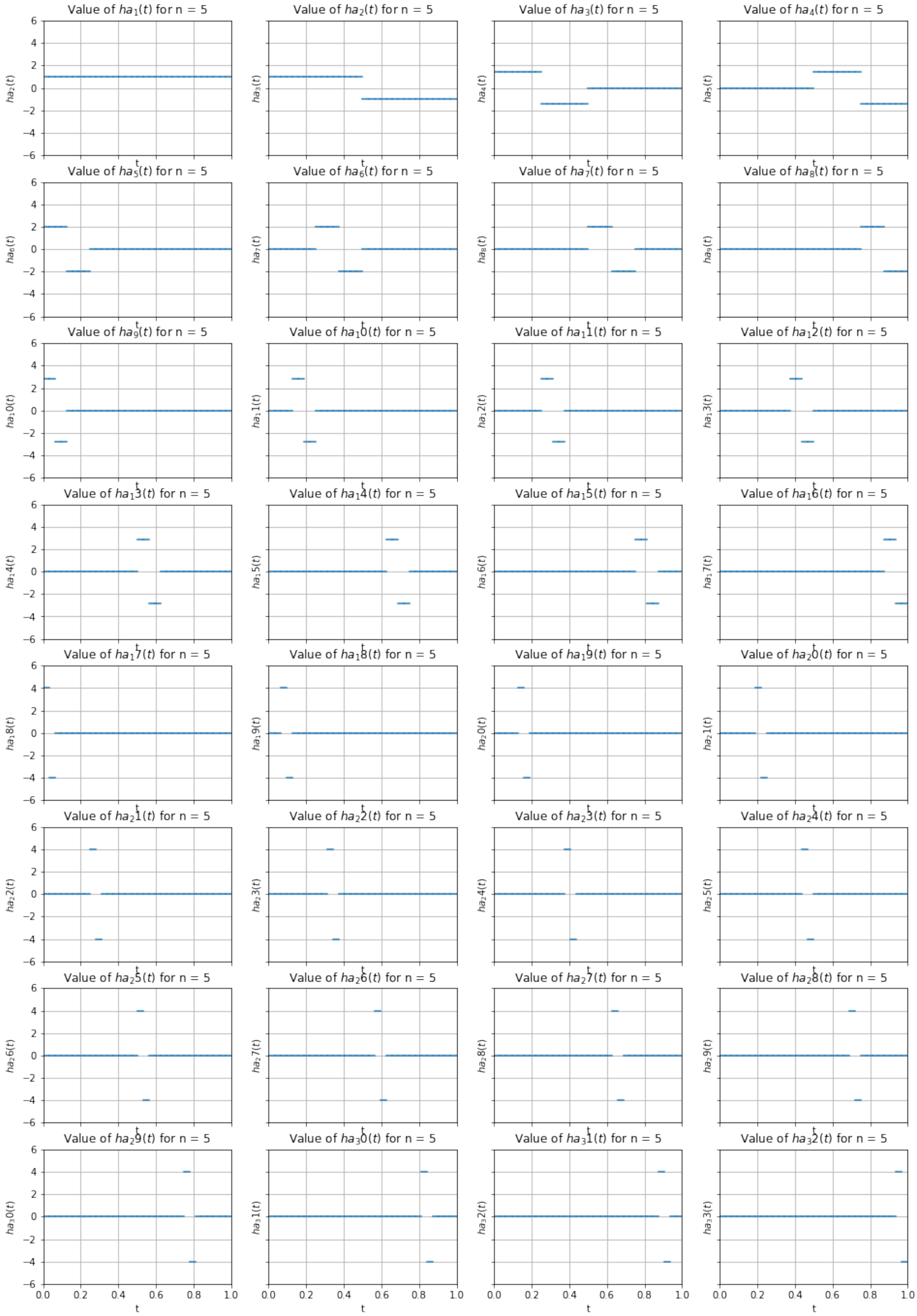
6.6 f

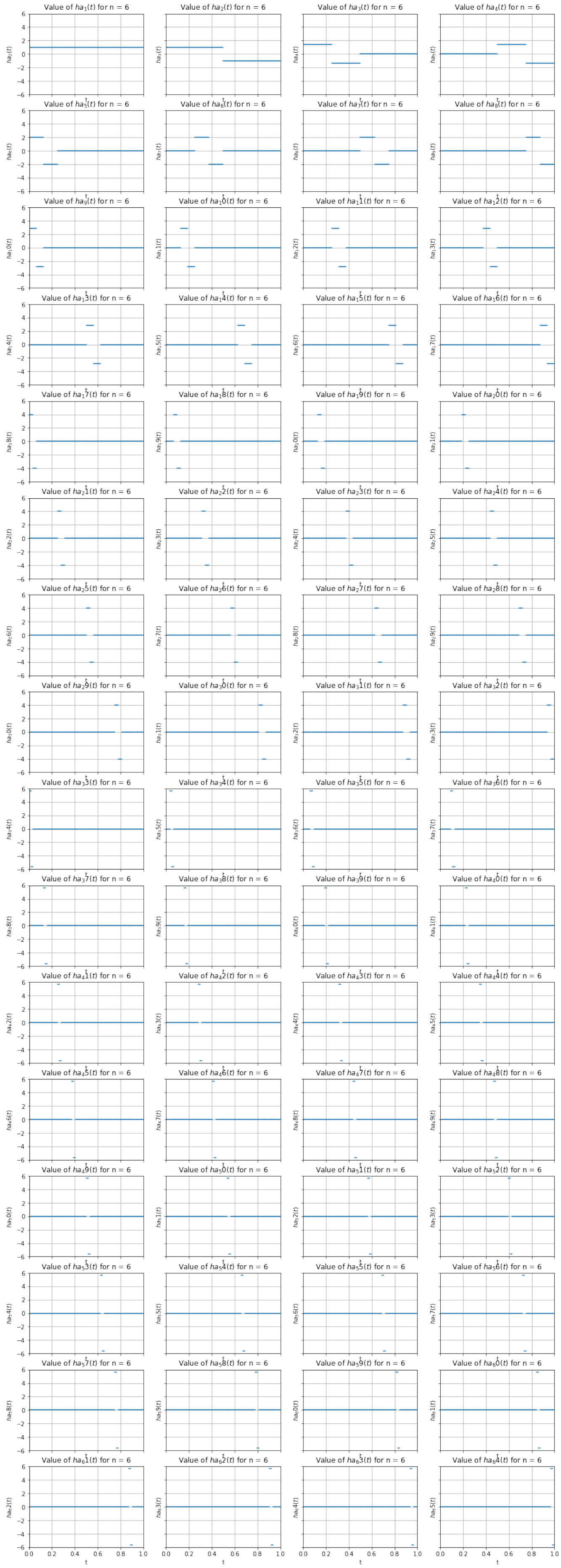
Plots of the Haar functions $\{ha_i\}_{i=1}^{2^n}$:





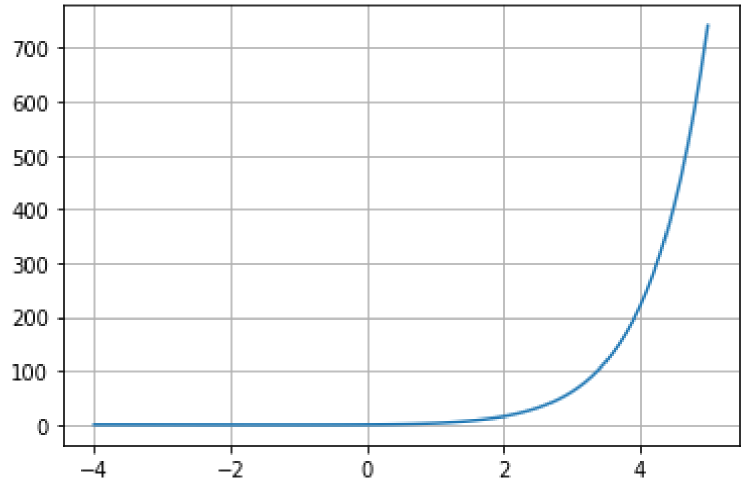

Value of $ha_1(t)$ for n = 4
Value of $ha_2(t)$ for n = 4
Value of $ha_3(t)$ for n = 4
Value of $ha_4(t)$ for n = 4
Value of $ha_5(t)$ for n = 4
Value of $ha_6(t)$ for n = 4
Value of $ha_7(t)$ for n = 4
Value of $ha_8(t)$ for n = 4
Value of $ha_9(t)$ for n = 4
Value of $ha_10(t)$ for n = 4
Value of $ha_11(t)$ for n = 4
Value of $ha_12(t)$ for n = 4
Value of $ha_13(t)$ for n = 4
Value of $ha_14(t)$ for n = 4
Value of $ha_15(t)$ for n = 4
Value of $ha_16(t)$ for n = 4
t


Value of $ha_1(t)$ for n = 5
Value of $ha_2(t)$ for n = 5
Value of $ha_3(t)$ for n = 5
Value of $ha_4(t)$ for n = 5
Value of $ha_5(t)$ for n = 5
Value of $ha_6(t)$ for n = 5
Value of $ha_7(t)$ for n = 5
Value of $ha_8(t)$ for n = 5
Value of $ha_9(t)$ for n = 5
Value of $ha_1 0(t)$ for n = 5
Value of $ha_1 1(t)$ for n = 5
Value of $ha_1 2(t)$ for n = 5
Value of $ha_1 3(t)$ for n = 5
Value of $ha_1 4(t)$ for n = 5
Value of $ha_1 5(t)$ for n = 5
Value of $ha_1 6(t)$ for n = 5
Value of $ha_1 7(t)$ for n = 5
Value of $ha_1 8(t)$ for n = 5
Value of $ha_1 9(t)$ for n = 5
Value of $ha_2 0(t)$ for n = 5
Value of $ha_2 1(t)$ for n = 5
Value of $ha_2 2(t)$ for n = 5
Value of $ha_2 3(t)$ for n = 5
Value of $ha_2 4(t)$ for n = 5
Value of $ha_2 5(t)$ for n = 5
Value of $ha_2 6(t)$ for n = 5
Value of $ha_2 7(t)$ for n = 5
Value of $ha_2 8(t)$ for n = 5
Value of $ha_2 9(t)$ for n = 5
Value of $ha_3 0(t)$ for n = 5
Value of $ha_3 1(t)$ for n = 5
Value of $ha_3 2(t)$ for n = 5
t



7 g

Our original function figure:



We found each base needed (taken from the theory part), and normalized it to get an orthonormal basis to work with in our range $[-4, 5]$.

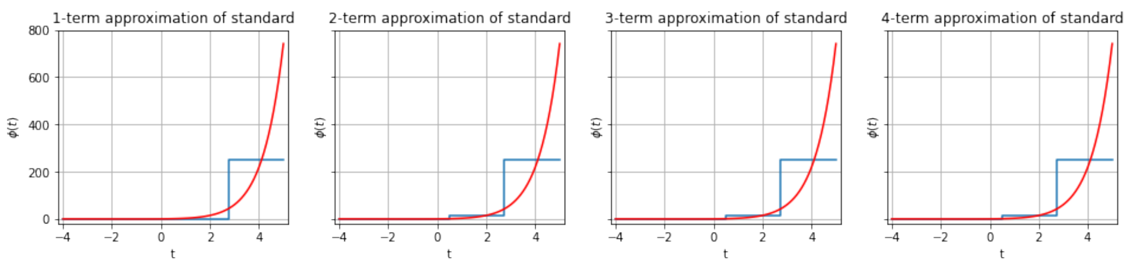
We calculated our needed MSE with:

$$MSE = \int_{-4}^{5} t \cdot \exp(t)dt - \sum_{i=1}^{k} \phi_k^2 \tag{7.1}$$

Where ϕ_k are the base coefficients calculated.

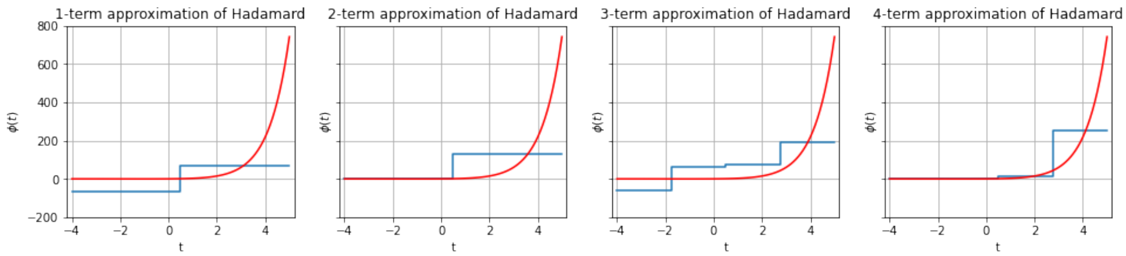
k-term approximation with the standard base

1. $k = 1$: $MSE = 83250.91186720502$
2. $k = 2$: $MSE = 82897.49777481775$
3. $k = 3$: $MSE = 82897.43145134844$
4. $k = 4$: $MSE = 82897.3780958027$



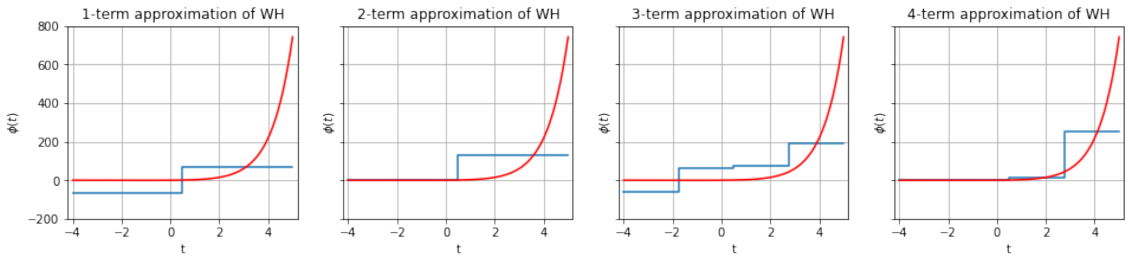
k-term approximation with the Hadamard base

1. $k = 1$: $MSE = 186407.41744756108$
2. $k = 2$: $MSE = 147237.1738501598$
3. $k = 3$: $MSE = 115062.51482711299$
4. $k = 4$: $MSE = 82897.37809580273$



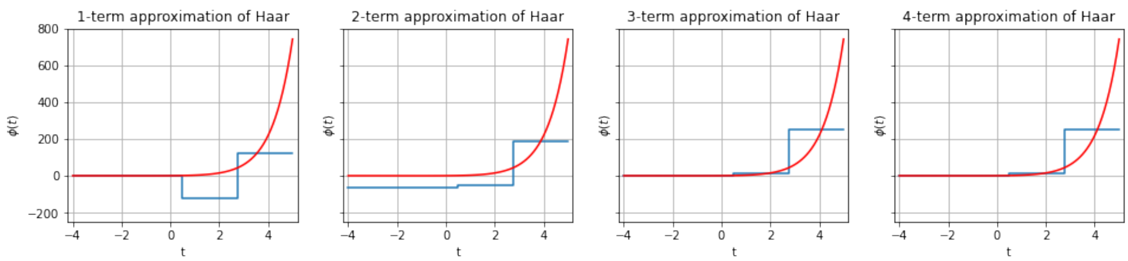
k-term approximation with the Walsh-Hadamard base

1. $k = 1$: $MSE = 186407.41744756108$
2. $k = 2$: $MSE = 147237.1738501598$
3. $k = 3$: $MSE = 115062.51482711299$
4. $k = 4$: $MSE = 82897.37809580273$



k-term approximation with the Haar base

1. $k = 1$: $MSE = 161431.47555624472$
2. $k = 2$: $MSE = 122067.62204552887$
3. $k = 3$: $MSE = 82897.3784481276$
4. $k = 4$: $MSE = 82897.37809580273$



Conclusion As expected, we got that for $k = 4$, all bases give the same MSE.