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TECHNION
ISRAEL INSTITUTE OF TECHNOLOGY

Exam Booklet

2023

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Subject Name _____

Subject Number 236201

Exam Room Ullman 801

Faculty CS

Semester Winter 22-23

Date 20/3/23

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6) $H_{CPR}(\varphi) = \begin{pmatrix} \varphi_n \cdot \ldots \cdot \varphi_1 \\ \vdots \\ \varphi_n \cdot \ldots \cdot \varphi_k \\ \vdots \\ \varphi_n \end{pmatrix}$

What about other d?

X – 1

X – 1

$\lambda \in \mathbb{R}$, $H_{+,\lambda} = H_{CP} + \lambda H_{CPR}$, $H_{\times,\lambda} = H_{CP} \odot H_{CPR}$

a) is $H_{+,\lambda}$ linear? NO! for $\lambda = 0$ it's H_{CP} which isn't linear ✓

~~b) is $H_{+,\lambda}$ SI? NO! same reason, $\lambda = 0$: $H_{+,\lambda} = H_{CP}$ which is not SI~~

c) is $H_{\times,\lambda}$ linear?

$$H_{\times,\lambda}(\varphi) = \begin{pmatrix} \varphi_1 \cdot \varphi_1 \cdot \ldots \cdot \varphi_n \\ \vdots \\ \varphi_k \cdot \varphi_n \cdot \ldots \cdot \varphi_k \\ \vdots \\ \varphi_n \cdot \varphi_n \end{pmatrix} \lambda \begin{pmatrix} \varphi_1 \cdot \varphi_1 \cdot \ldots \cdot \varphi_n \\ \vdots \\ \varphi_1 \cdot \ldots \cdot \varphi_k \cdot \varphi_n \cdot \ldots \cdot \varphi_k \\ \vdots \\ \varphi_1 \cdot \ldots \cdot \varphi_n \cdot \varphi_n \end{pmatrix} =$$

$$= \lambda \begin{pmatrix} \varphi_1 \cdot \ldots \cdot \varphi_n \\ \vdots \\ \varphi_1 \cdot \ldots \cdot \varphi_n \end{pmatrix} \odot \varphi$$

X – 1

$\varphi' = 2\varphi$

$$\Rightarrow H_{\times,\lambda}(\varphi') = \lambda \begin{pmatrix} 2^{n+1} \varphi_1 \cdot \ldots \cdot \varphi_n \times \varphi_1 \\ \vdots \\ 2^{n+1} \varphi_1 \cdot \ldots \cdot \varphi_n \times \varphi_n \end{pmatrix} \odot 2\varphi \neq 2 H_{\times,\lambda}(\varphi) \Rightarrow H_{\times,\lambda} \text{ is } \underline{\text{not linear}}$$ ✓

d) is $H_{\times,\lambda}$ SI? <u>YES!</u> denote $\psi_\varphi = \lambda \begin{pmatrix} \varphi_1 \cdot \ldots \cdot \varphi_n \times \varphi_1 \\ \vdots \\ \varphi_1 \cdot \ldots \cdot \varphi_n \times \varphi_n \end{pmatrix}$ ~~a constant~~

$$T(H(\varphi)) = T(\psi \odot \varphi) \overset{\psi \text{ unaffected}}{=} \psi \odot T(\varphi) = H(T(\varphi))$$ ✓

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4) $H_{max}(\varphi) = \begin{pmatrix} \max(\varphi_n, \varphi_1, \varphi_2) \\ \vdots \\ \max(\varphi_{k-1}, \varphi_k, \varphi_{k+1}) \\ \vdots \\ \max(\varphi_{n-1}, \varphi_n, \varphi_1) \end{pmatrix}$

$H_{max,l}(\varphi) = H_{max} \circ \ldots \circ H_{max}(\varphi)$

a) is $H_{max,l}$ linear? NO!

$\varphi_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\varphi_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ for $l = 1$:

$H_{max,l}(\varphi_1 + \varphi_2) = H(0) = 0$

$H(\varphi_1) + H(\varphi_2) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \neq 0 \Rightarrow$ not linear

Generalise to $\mathbb{R}^{m \geq 4}$? X −1

b) is $H_{max,l}$ SI?

we can see that H_{max} is SI, as $H_{max}(T(\varphi)) =$

$= T(H_{max}(\varphi))$ (some index play needed)

also, each activation of H_{max} is not a function of index, but elements in vicinity of index. Therefore, $H_{max,l}$ is invariant to shifts – and therefore SI

5) $H_{cp}(\varphi) = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_1 \ldots \varphi_k \\ \vdots \\ \varphi_1 \ldots \varphi_n \end{pmatrix}$

a) linear? NO: $\varphi_2 = 2\varphi_1 \Rightarrow H_{cp}(\varphi_2) = \begin{pmatrix} 2\varphi_1 \\ \vdots \\ 2^k(\varphi_1 \ldots \varphi_k) \\ \vdots \\ 2^n(\varphi_1 \ldots \varphi_k) \end{pmatrix} \neq 2H_{cp}(\varphi_1)$

b) SI? No: $\varphi = (1, 0)^T$

$H_{cp}(\varphi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $T(\varphi) = (0\ 1)^T$, $H(T(\varphi)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$T(H_{cp}(\varphi)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq H(T(\varphi)) \Rightarrow$ not SI

Generalise to $\mathbb{R}^{n \geq 3}$? X −1

Ex2

a) + b) 3)

$\varphi \in \mathbb{R}^n$

$\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3, \quad x = (x_1, x_2, x_3)$

$h_\alpha(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$

$$H_\alpha(\varphi) = \begin{pmatrix} h_\alpha(\varphi_n, \varphi_1, \varphi_2) \\ \vdots \\ h_\alpha(\varphi_{k-1}, \varphi_k, \varphi_{k+1}) \\ \vdots \\ h_\alpha(\varphi_{n-1}, \varphi_n, \varphi_1) \end{pmatrix}$$

let $l \geq 1$ integer. $H_{\alpha,l}$ is applying H_α l times.

$H_{\alpha,l}(\varphi) = H_\alpha \circ \dots \circ H_\alpha(\varphi)$

First, we'll see that

$$H_\alpha(\varphi) = \begin{pmatrix} \alpha_2 & \alpha_3 & 0 & \cdots & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_3 & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \alpha_3 & 0 & \cdots & 0 & \alpha_1 \; \alpha_2 \end{pmatrix} \cdot \varphi$$

we can see that H_α is represented by matrix $\Rightarrow H_\alpha$ is linear ✓

also, we can see that H_α is circulant $\Rightarrow H_\alpha$ is SI ✓

$\Rightarrow H_{\alpha,l}(\varphi) = H_\alpha \cdot H_\alpha \cdot \dots \cdot H_\alpha \cdot \varphi$ and we know that

H_α^l is $H_{\alpha,l}$ represented matrix $\Rightarrow H_{\alpha,l}$ is linear

and multiplying circulant matrices $= \underbrace{\text{circulant}}_{H_\alpha^l \text{ circ}} \Rightarrow H_{\alpha,l}$ is SI

$\Rightarrow H_{\alpha,l}$ is LSI ✓

Ex2

2) $g_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\quad H_{\mu,\sigma} : \mathbb{R}^n \to \mathbb{R}^n$:

$$H_{\mu,\sigma}(\varphi) = \begin{pmatrix} \sum_{l=0}^{n-1} g_{\mu,\sigma}(l)\, \varphi_{1-l} \\ \vdots \\ \sum_{l=0}^{n-1} g_{\mu,\sigma}(l)\, \varphi_{k-l} \\ \vdots \\ \sum_{l=0}^{n-1} g_{\mu,\sigma}(l)\, \varphi_{n-l} \end{pmatrix}$$

we can easily see that $H_{\mu,\sigma}$ is the convolution between $\bar{g} = [g(0), g(1) \dots g(n-1)]^T$ and signal φ.

2.a) is $H_{\mu,\sigma}$ linear?

we know that convolution is linear, but we'll show:

$$H_{\mu,\sigma}(a\varphi_1 + b\varphi_2)_k = [g \circledast (a\varphi_1 + b\varphi_2)]_k = \sum_{l=0}^{n-1} g_{\mu,\sigma}(l)(a\varphi^1_{k-l} + b\varphi^2_{k-l}) =$$

$$= a\sum g(l)\varphi^1_{k-l} + b\sum g(l)\varphi^2_{k-l} = aH(\varphi_1) + bH(\varphi_2)$$

2.b) is H SI? $\quad \varphi' = T(\varphi) = (\varphi_n\, \varphi_1 \dots \varphi_{n-1})^T$

$$H(\varphi')_k = \begin{pmatrix} \sum g(l)\, \varphi_{n-l} \\ \vdots \\ \sum g(l)\, \varphi_{k-1-l} \\ \vdots \\ \sum g(l)\, \varphi_{n-1-l} \end{pmatrix} = T(H(\varphi)) \text{ as we can see}$$

$\Rightarrow H_{\mu,\sigma}$ is shift invariant

2.1.b) Where $\Lambda_{A'} = \begin{pmatrix} a_1' & & \\ & \ddots & \\ & & a_{n-1}' \end{pmatrix}$

$\bar{a}' = DFT^* \cdot \left(\text{col 1 of } A_\varphi' \right) = DFT^* \begin{pmatrix} \varphi_1 \\ \varphi_0 \\ \varphi_{n-1} \\ \vdots \\ \varphi_2 \end{pmatrix}$

$\Rightarrow$ $\bar{a}'$ has the same values as $\bar{a}$ (eigenvalues of A_φ)

but with changed order and different eigenvectors for different eigenvalues (with a shift)

$a_k = \sqrt{N} \cdot \langle (\varphi_0 \cdots \varphi_1)^T, k \text{ row of } DFT^* \rangle$

$a_k' = \sqrt{N} \cdot \langle (\varphi_1 \cdots \varphi_2)^T, k \text{ row of } DFT^* \rangle$

$\Rightarrow A_\varphi' = DFT^* \Lambda_{A'} DFT$; $\Lambda_{A'} = \begin{pmatrix} 0 & & 1 \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix} \Lambda_A = \begin{pmatrix} a_{n-1} & & & 0 \\ & a_0 & & \\ & & \ddots & \\ 0 & & & a_1 \end{pmatrix}$

$\Rightarrow B_\varphi' = A_\varphi'^{\dagger} = DFT^* \Lambda_{A'}^{\dagger} DFT$

2.1.b) is H_{P1} SI? $\Leftrightarrow H_{P1}(T(\varphi)) = T(H_{P1}(\varphi)) = T(\psi_\varphi)$

$T(\varphi) = P = \begin{pmatrix} 0 & & 1 \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix}$; $T(\varphi) = P\varphi$; P - circulant

$\psi_\varphi = B_\varphi^T \text{ col 1} = B_{\varphi\,\text{col 1}}^T = (A_\varphi^{\dagger})^T \text{ col 1}$

circ: $A_\varphi = DFT^* \Lambda_A DFT$; $\Lambda_A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$

$B_\varphi = A_\varphi^{\dagger} = DFT^* \Lambda_B DFT$; $\Lambda_B = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$ $\begin{cases} b_i = \frac{1}{a_i} & a_i \neq 0 \\ b_i = 0 & a_i = 0 \end{cases}$

P is full rank $\Rightarrow$ all $\lambda_i \neq 0$

(A_φ' with first row $P\varphi$) $= PA_\varphi$ {same shift down happens to all A_φ rows}

$PA_\varphi = DFT^* \Lambda_P DFT DFT^* \Lambda_A DFT = DFT^* \Lambda_P \Lambda_A DFT$

$B_\varphi' = (PA_\varphi)^{\dagger} = DFT^* (\Lambda_P \Lambda_A)^{\dagger} DFT$; $(\Lambda_P \Lambda_A)^{\dagger} = \text{diag}\left\{ \frac{1}{P_i \cdot a_i}, 0 \right\}$ $a_i \neq 0$ $a_i = 0$

we see that first, B_φ' is circulant (diagonalized by DFT)

$B_\varphi' = (PA_\varphi)^{\dagger} = A_\varphi^{\dagger} P^{\dagger} = \underbrace{A_\varphi^{\dagger}}_{\text{circulant}} P^T \cancel{= PA_\varphi^{\dagger}} = \cancel{PB_\varphi}$

X - 3 NO

first row of B_φ' is the last row of B_φ which is $P\psi_\varphi \Rightarrow H_{P1}$ is shift invariant

Ex 2 $\quad \varphi \in \mathbb{R}^n,\ n>3$

1) A_φ be the circ matrix with first row φ^T

$B_\varphi = A_\varphi^{\#}$ be the pseudo-inverse of A_φ : $B_\varphi = A_\varphi^{\dagger}$

ψ_φ^T is the first row of B_φ.

Let $H_{PI} : \mathbb{R}^n \to \mathbb{R}^n$ be the system: $H_{PI}(\varphi) = \psi_\varphi$

1.a) is H_{PI} linear?

first; $A_\varphi = DFT^* \Lambda_A DFT^*$ $\quad \Lambda_A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$

$B_\varphi = DFT^* \Lambda_B DFT$; $\Lambda_B = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$; $b_i = \begin{cases} \frac{1}{a_i} & a_i \neq 0 \\ 0 & a_i = 0 \end{cases}$

– let's look at 2φ. $A_{2\varphi} = 2A_\varphi$ (just multipy)

$\Rightarrow \Lambda_{2A} = 2\Lambda_A$

$\Rightarrow B_{2\varphi} = A_{2\varphi}^{\dagger}$ has $\Lambda_{2B} = \begin{pmatrix} b_1' & & 0 \\ & \ddots & \\ 0 & & b_n' \end{pmatrix}$ where $b_i' = \begin{cases} \frac{1}{a_i'} = \frac{1}{2a_i} & a_i \neq 0 \\ 0 & a_i = 0 \end{cases}$

$\Rightarrow$ the first row of $B_{2\varphi}$ is $\frac{1}{2}\psi_\varphi^T$:

$B_{2\varphi} = DFT^* \Lambda_{2B} DFT = \frac{1}{2} DFT^* \Lambda_B DFT = \frac{1}{2} B_\varphi$

$\Rightarrow H_{PI}(2\varphi) = \frac{1}{2}\psi_\varphi \Rightarrow H_{PI}$ is <u>not linear</u>

1.b) is H_{PI} shift invariant?

~~let us test if we use H_{PI} on the second row of A_φ → will we get the second row of B_φ?~~

~~well shift φ by one space : $\varphi' = (\varphi_1, \dots, \varphi_{n-1}, \varphi_0)^T$~~

~~A_φ' is circ = DFT diags it (DFT times) and first col of A_φ gives the eigenvalues. and $A_\varphi' = \begin{pmatrix} 0 & & 1 \\ 1 & \ddots & \\ & \ddots & 0 \end{pmatrix} A_\varphi = \begin{pmatrix} 0 & & 1 \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix} DFT^* \Lambda_A DFT$~~

~~$= DFT^* \Lambda_{A'} DFT$~~

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→ so we have $\widehat{D\varphi} = \Lambda_D \hat{\varphi}$, $\Lambda_D = \mathrm{diag}\{(d_1, \dots, d_d)\}$ =

$$d_k = e^{-\frac{2i\pi(d-1)k}{d}} - 1$$

$\varphi_0 = 0$, $\|\varphi\|_2 = 1$

$$\hat{\varphi} = \frac{1}{\sqrt{d}} DFT \cdot \varphi \Rightarrow \varphi_k^F = \frac{1}{\sqrt{d}}(k \text{ row of } DFT) \cdot \varphi = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} e^{\frac{-2i\pi kl}{d}} \cdot \varphi_l$$

$$(*) \Rightarrow \widehat{D\varphi}_k = d_k \cdot \varphi_k^F = \frac{1}{\sqrt{d}} \left(e^{\frac{-2i\pi(d-1)k}{d}} - 1\right) \sum_{l=0}^{d-1} e^{\frac{-2i\pi kl}{d}} \cdot \varphi_l =$$

$$= \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \varphi_l \cdot \left(e^{\frac{-2i\pi k(d+l-1)}{d}} - e^{\frac{-2i\pi kl}{d}}\right) =$$

$$= \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \varphi_l \cdot \underbrace{\left(e^{\frac{-2i\pi k(l-1)}{d}} - e^{\frac{-2i\pi kl}{d}}\right)}_{\text{this comes from } DFT \cdot D} = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \varphi_l \cdot e^{\frac{-2i\pi kl}{d}} \left(e^{\frac{+2i\pi k}{d}} - 1\right)$$

4) deduce: $\underbrace{\left[\sum_{l=0}^{d-1} \left(\frac{l}{d}\right)^2 |\hat{\varphi}_l|^2\right]}_{①} \cdot \underbrace{\left[\sum_{k=0}^{d-1} k^2 \varphi_k^2\right]}_{②} \geq \frac{1}{16\pi^2}$

$$② = \sum_{k=0}^{d-1} (t \odot \varphi)_k^2$$

$$① = \sum_{l=0}^{d-1} \left(\frac{l}{d}\right)^2 |\hat{\varphi}_l|^2 = \frac{1}{d^2} \sum l^2 |\varphi_l^F|^2 =$$

$$\frac{1}{d^2} \sum_{l=0}^{d-1} l^2 \cdot \left|\frac{1}{d} \sum_{k=0}^{d-1} e^{\frac{-2i\pi kl}{d}} \cdot \varphi_l\right|^2 =$$

$$= \frac{1}{d^4} \sum_{l=0}^{d-1} l^2 \cdot \sum_{k=0}^{d-1} \varphi_l \cdot e^{\frac{-2i\pi kl}{d}}$$

in ①: $\widehat{D\varphi} = \Lambda_D \hat{\varphi} \Rightarrow \hat{\varphi} = \Lambda_D^{-1} \widehat{D\varphi} \Rightarrow |\hat{\varphi}_l|^2 = (d_l^{-1} \cdot \widehat{D\varphi}_l)^2$

NO TIME! but I think that I can use ② to get to a function of $A\varphi \odot t$, and then open the sums and start deleting elements that go to 0 in the sum, as I'm summing also $e^{\frac{-i2\pi k}{d}}$ elements - sum of $e^{\frac{-i2\pi(k-l)}{d}}$ over all is $\delta_{k,l}$

You did not solve the problem
Not done

X − 24

1.2 $2\sum(t \circ D\varphi \circ D\varphi)_k = 2\sum_{k=0}^{d-1} \frac{k}{2}(\varphi_{k+1}^2 - \varphi_k^2) =$

$= \sum k(\varphi_{k+1}^2 - \varphi_k^2) = -\varphi_0^2 - \varphi_1^2 - \varphi_2^2 - \dots - \varphi_{d-1}^2 = -\sum_{k=0}^{d-1} \varphi_k^2 = -\|\varphi\|_2 = \underline{-1}$

3) Compute DFT of $D\varphi$ $\quad D\varphi[k] = \varphi_{k+1} - \varphi_k$

$\widehat{D\varphi} = DFT \cdot D\varphi$

$\widehat{D\varphi}[k] = DFT[k] \cdot D\varphi = \frac{1}{\sqrt{d}}(w^{*0\cdot k} \dots w^{*(N-1)k}) \cdot \begin{pmatrix} \varphi_1 - \varphi_0 \\ \varphi_2 - \varphi_1 \\ \vdots \\ \varphi_{d-1} - \varphi_{d-2} \\ \varphi_0 - \varphi_{d-1} \end{pmatrix} = (*)$

$= \cancel{\frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} \varphi_p \cdot w^{*k\cdot(p-1) \bmod (d)} - \varphi_p w^{*k\cdot p}} =$

$\cancel{\frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} \varphi_p \left(w^{*k} \cdot (w^{*p-1} - w^{*p}) \right) = \frac{1}{\sqrt{d}} \sum \varphi_p \cdot e^{-\frac{i2\pi}{d} kp} \cdot (e^{\frac{i2\pi}{d}} - 1)}$

$\cancel{= \frac{1}{\sqrt{d}} (e^{\frac{i2\pi}{d}} - 1) \cdot e^{\frac{-i2\pi k}{d}} \cdot \sum_{p=0}^{d-1} \varphi_p e^{\frac{-2i\pi p}{d}}}$

well see that in $(*)$ the arguments are sum of:

$\varphi_1 w^{*0} - \varphi_1 w^{*k} = \varphi_1 (1 - e^{-\frac{i2\pi k}{d}})$

$\varphi_2 w^{*k} - \varphi_2 w^{*2k} = \varphi_2 \cdot w^{*k} (1 - e^{-\frac{i2\pi k}{d}})$

$\vdots$

$\varphi_j w^{*k(j-1)} - \varphi_j w^{*kj} = \varphi_j w^{*k(j-1)} (1 - e^{-\frac{i2\pi k}{d}})$

$\vdots$

$\varphi_0 w^{*(N-1)k} - \varphi_0 w^{*0k} = \varphi_0 w^{*-k} (1 - e^{-\frac{i2\pi k}{d}})$

also, D is circulant $\Rightarrow D = DFT^* \Lambda_D DFT$

$\widehat{D\varphi} = DFT^* \cdot D\varphi = DFT^* DFT^* \Lambda_D DFT\varphi = \Lambda_D \cdot \hat{\varphi}$; $\Lambda_D = \begin{pmatrix} d_0 & & \\ & \ddots & \\ & & d_{d-1} \end{pmatrix}$

$\begin{pmatrix} d_0 \\ \vdots \\ d_{d-1} \end{pmatrix} = DFT^* \cdot \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-\frac{i2\pi(d-1)\cdot 0}{d}} - 1 \longrightarrow = 0 \\ e^{-\frac{i2\pi(d-1)\cdot k}{d}} - 1 \\ \vdots \\ e^{-\frac{i2\pi(d-1)}{d}(d-1)} - 1 \end{pmatrix}$

Ex1

$$\mathbb{R}^{d\times d} \ni D = \begin{pmatrix} -1 & 1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & 0 & \ddots & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix} \qquad A = \frac{1}{2}\begin{pmatrix} 1 & 1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & 0 & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Derivative

Moving 2-average forward

compute

1) $$\sum_{k=0}^{d-1} (a_{k+1} - a_k) b_k + \sum_{k=0}^{d-1} a_{k+1}(b_{k+1} - b_k)$$

~~given $\bar{a}_k = (a_1 \dots a_d)^T$; $\bar{b} = (b_1 \dots b_d)^T$~~

~~we can see: $D\bar{a}[i] = a_{i+1} - a_i$~~

~~$\bar{b}^T D[i] =$~~

$$\sum_{k=0}^{d-1} (a_{k+1} - a_k) b_k + \sum_{k=0}^{d-1} a_{k+1}(b_{k+1} - b_k) =$$

$$= \sum_{k=0}^{d-1} \cancel{a_{k+1} b_k} - a_k b_k + a_{k+1} b_{k+1} - \cancel{a_{k+1} b_k} =$$

$$= \sum_{k=0}^{d-1} (a_{k+1} b_{k+1} - a_k b_k) = \cancel{a_1 b_1} - \cancel{a_0 b_0} + \cancel{a_2 b_2} - \cancel{a_1 b_1} + \dots + \overset{a_0 b_0}{\overset{\shortparallel}{a_d b_d}} - \cancel{a_{d-1} b_{d-1}}$$

$$= \underline{\underline{0}}$$

2) $t = (0, 1, \dots, d-1)^T$ prove: $2\sum_{k=0}^{d-1} (t \odot (D\varphi) \odot (A\varphi))_k = -1$

where $\odot$ is Hadamard (point-wise) prod, $\varphi \in \mathbb{R}^d$, $\|\varphi\|_2 = 1$, $\varphi_0 = 0$

$D\varphi[k] = \varphi_{k+1} - \varphi_k$

$A\varphi[k] = \dfrac{\varphi_{k+1} + \varphi_k}{2}$

$$2(t \odot D\varphi \odot A\varphi)_k = t_k \cdot D\varphi[k] \cdot A\varphi[k] = \frac{k}{2} \cdot (\varphi_{k+1} - \varphi_k)(\varphi_{k+1} + \varphi_k) =$$

$$= \frac{k}{2}\left(\varphi_{k+1}^2 - \varphi_k^2\right)$$