STAT 525 Lecture 14

October 19, 2017

1 Topics

1. Priors for σ_θ
2. Multiple testing
3. Mixed models
4. EF

J experiments, replicates $i = 1, \cdots, n_j$

$$\begin{aligned}
\left[y_{ij} \mid \theta_j, \sigma_y^2\right] &\overset{\text{iid}}{\sim} N\left(\theta_j, \sigma_y^2\right) \text{ where } i = 1, \cdots, n_j \\
\left[\theta_j \mid \mu, \sigma_\theta\right] &\overset{\text{iid}}{\sim} N\left(\mu, \sigma_\theta^2\right) \text{ where } j = 1, \cdots, J \\
\sigma_j^2 &= \frac{\sigma_j^2}{n_j}
\end{aligned}$$

The joint likelihood is

$$\begin{aligned}
p\left(\{y_{ij}\} \mid \{\theta_j\}, \sigma_y, \mu, \sigma_\theta\right) &= \prod_{j=1}^{J} \prod_{i=1}^{n_i} p\left(y_{ij} \mid \{\theta_j\}, \sigma_y, \sigma_\theta, \mu\right) \\
&= \prod_{j=1}^{J} \prod_{i=1}^{n_i} p\left(y_{ij} \mid \theta_j, \sigma_y^2\right)
\end{aligned}$$

Priors

$$\begin{aligned}
p\left(\{\theta_j\}, \mu, \sigma_\theta\right) &= p\left(\mu, \sigma_\theta\right) p\left(\{\theta_j\} \mid \mu, \sigma_\theta\right) \\
&= p\left(\mu, \sigma_\theta\right) \prod_{j=1}^{J} N\left(\theta_j \mid \mu, \sigma_\theta^2\right) \\
p\left(\mu, \sigma_\theta\right) &= p\left(\mu\right) p\left(\sigma_\theta\right) \\
p\left(\mu\right) &\propto 1
\end{aligned}$$

Priors on σ_θ summarized previously in Lecture 13

1. $\sigma_\theta \sim \text{Uniform}\left(0, A\right)$ possibility $A \to \infty$.
2. $\sigma_\theta \sim C^{+}\left(0, A\right)$
3. $\sigma_\theta^{-2} \sim \text{Gamma}\left(\epsilon, \epsilon\right)$ $\epsilon \to 0$ not appropriate. Improper prior then posterior would be improper.

2 Half Cauchy $\sigma_\theta \sim C^+(0, A)$

$$\sigma_\theta \sim C^+(0, A)$$

the above is equivalent to the following representation

$$\begin{aligned}[\eta_\theta \mid \gamma_\theta] &\sim \text{Gamma}\left(\frac{1}{2}, \gamma_\theta\right) \\ [\gamma_\theta] &\sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{A^2}\right)\end{aligned}$$

for $\eta_\theta^{-\frac{1}{2}} = \sigma_\theta$.

$$\begin{aligned}[\eta_\theta, \gamma_\theta] &= [\eta_\theta \mid \gamma_\theta] [\gamma_\theta] \\ &= \left\{\frac{\gamma_\theta^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \eta_\theta^{\frac{1}{2}-1} \exp\left(-\gamma_\theta \eta_\theta\right)\right\} \left\{\frac{\left(\frac{1}{A^2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \gamma_\theta^{\frac{1}{2}-1} \exp\left(-\gamma_\theta / A^2\right)\right\} \\ &\propto \left\{\eta_\theta^{\frac{1}{2}-1} \exp\left(-\gamma_\theta \eta_\theta\right)\right\} \left\{\gamma_\theta^{\frac{1}{2}} \gamma_\theta^{\frac{1}{2}-1} \exp\left(-\gamma_\theta / A^2\right)\right\}\end{aligned}$$

Conditional Likelihood

$$[\theta_j \mid \mu, \sigma_\theta] \overset{\text{ind}}{\sim} N\left(\mu, \eta_\theta^{-1}\right)$$

$$\begin{aligned}p\left(\{\theta_j\} \mid \mu, \sigma_\theta\right) &= \prod_{j=1}^{J} \frac{1}{\sqrt{2\pi\eta_\theta^{-1}}} \exp\left(-\frac{\eta_\theta}{2}\left(\theta_j - \mu\right)^2\right) \\ &\propto \eta_\theta^{J/2} \exp\left(-\eta_\theta \sum_{j=1}^{J} \left(\theta_j - \mu\right)^2 / 2\right)\end{aligned}$$

Full conditionals

$$\begin{aligned}[\eta_\theta \mid \boldsymbol{y}, \cdots] &= [\eta_\theta \mid \{\theta_j\}, \mu, \gamma_\theta] \\ &\sim \text{Gamma}\left(\frac{J}{2} + \frac{1}{2}, \sum_{j=1}^{J} \left(\theta_j - \mu\right)^2 / 2 + \gamma_\theta\right) \\ [\gamma_\theta \mid \boldsymbol{y}, \cdots] &= [\gamma_\theta \mid \eta_\theta] \\ &\sim \text{Gamma}\left(1, \eta_\theta + \frac{1}{A^2}\right)\end{aligned}$$

School level mean and sd

$$\begin{aligned}\sigma_j &= \sigma_y / \sqrt{n_j} = SE(j) \\ y_j &= \bar{y}_{\cdot j} = \frac{1}{n_j} \sum_{i=1}^{n_i} y_{ij}\end{aligned}$$

3 Uniform Prior $p\left(\sigma_\theta\right) \propto 1$

The above is equivalent to $p\left(\eta_\theta\right) \propto \eta_\theta^{-3/2}$

$$
\begin{aligned}
p\left(\sigma_\theta\right) &\propto 1 \cdot \left|\frac{d}{d\left(\sigma_\theta^2\right)} \sqrt{\sigma_\theta^2}\right| \\
&= \frac{1}{\sqrt{\sigma_\theta^2}}
\end{aligned}
$$

Let $\eta_\theta = 1/\sigma_\theta^2$

$$
\begin{aligned}
p\left(\eta_\theta\right) &\propto \frac{1}{\sqrt{\eta_\theta^{-1}}} \left|\frac{d}{d\eta_\theta}\eta_\theta^{-1}\right| \\
&\propto \eta_\theta^{-\frac{3}{2}} \\
&\sim \text{"Gamma"}\left(-\frac{1}{2}, 0\right)
\end{aligned}
$$

Full conditionals

$$
\begin{aligned}
\left[\eta_\theta \mid \boldsymbol{y}, \cdots\right] &= \left[\eta_\theta \mid \left\{\theta_j\right\}, \mu\right] \\
&\sim \text{Gamma}_{\left[\frac{1}{A^2}, \infty\right)} \left(\frac{J}{2} - \frac{1}{2}, \sum_{j=1}^{J} \left(\theta_j - \mu\right)^2\right)
\end{aligned}
$$

If $\sigma_\theta \sim \text{Uniform}\left(0, A\right)$

$$
\begin{aligned}
\sigma_\theta < A &\Leftrightarrow \sigma_\theta^2 < A^2 \\
&\Leftrightarrow \eta_\theta > \frac{1}{A^2}
\end{aligned}
$$

4 Multiple Comparison

Suppose we want to test

$$
H_{0,j} : \theta_j = 0
$$

where $j = 1, \cdots, J$. (Maybe also interested in the contracts)

$$
H_{0,jk} : \theta_j - \theta_k = 0 \quad \forall j, k
$$

Suppose we have $j = 1, \cdots, m$ hypothesis to test at α level

$$
p\left(\text{reject } H_{0j} \mid H_{0j}\right) = \alpha
$$

If tests are independent, family-wise error rate

$$
\begin{aligned}
\text{FWER} &= P\left(\text{at least one false postive}\right) \\
&= 1 - P\left(\text{no false pos}\right) \\
&= 1 - \prod_{j=1}^{m} \left[1 - P\left(\text{reject } H_{0j} \mid H_{0j}\right)\right] \\
&= 1 - \left(1 - \alpha\right)^m
\end{aligned}
$$

Ex: $m = J = 8$, $\alpha = 0.05$, then FWER $= 0.34$.

Confidence intervals

$$\bar{y}_{\cdot j} \pm z_{1-\alpha/2} \left(\frac{\sigma_y}{\sqrt{J}} \right)$$

Bonferroni Correction: we use $\alpha^* = \alpha/m$ or 0.00625

Even under this correction for dependent tests

$$\text{FWER} \leq m \times \alpha$$

Consequences: 1. Lose power. 2. Widen the confidence intervals. 3. Scaling with J, if we have more groups we are more conservative.

Key Points:

1. Multiple comparison adjustment widen the intervals but they don't adjust the mean at all.
2. The partial pooling or mixed hierarchical model shifts the means towards each other and end up shrink credible intervals.

5 Mixed models

$$\begin{aligned} y_{ij} &= \theta_j + \epsilon_{ij}, \quad \epsilon_{ij} \overset{\text{iid}}{\sim} N\left(0, \sigma_y^2\right), \ \theta_j \overset{\text{iid}}{\sim} N\left(\mu, \sigma_\theta^2\right) \\ &= \underbrace{\mu}_{\text{fix effect}} + \underbrace{\gamma_j}_{\text{random effect}} + \epsilon \end{aligned}$$

where $\gamma_j \overset{\text{iid}}{\sim} N\left(0, \sigma_\theta^2\right)$ and $\gamma_j \equiv \theta_j - \mu$.

Centered

$$\begin{aligned} \left[y_{ij} \mid \theta_j, \sigma_y\right] &\sim N\left(\theta_j, \sigma_y^2\right) \\ \theta_j &\sim N\left(\mu, \sigma_\theta^2\right) \end{aligned}$$

Non-centered

$$\begin{aligned} \left[y_{ij} \mid \mu, \gamma_j, \sigma_y\right] &\sim N\left(\mu + \gamma_j, \sigma_y^2\right) \\ \gamma_j &\sim N\left(0, \sigma_\theta^2\right) \end{aligned}$$

CovarianceL conditioned on μ, σ_y and σ_θ

$$\begin{aligned} \operatorname{Cov}\left(y_{ij}, y_{i'y'}\right) &= \operatorname{Cov}\left(\mu + \gamma_j + \epsilon_{ij}, \mu + \gamma_{j'} + \epsilon_{i'j'}\right) \\ &= \begin{cases} \sigma_y^2 + \sigma_\theta & i = i', j = j^{,} \text{ (Variance)} \\ \sigma_\theta^2 & i \neq i', j = j^{,} \text{ (Within group Variance)} \\ 0 & j \neq j' \text{ (Between group Variance)} \end{cases} \end{aligned}$$