Nontrivial dispersion relation and QCD thermodynamics in the low energy effective model

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We study the...

I. INTRODUCTION

The QCD phase structure and the search of the critical end point are the most popular research direction in both experimental and theoretical field.

II. DISPERSION RELATIONSHIP IN THE PQM MODEL

The Low-energy effective model is a effective research tool to study the QCD matter under the finite temperature and density. In the finite temperature quantum field theory, the $O(4)$-symmetry in the vacuum is broken. The $O(4)$-symmetry changes into a $\mathbb{Z}_2 \otimes O(3)$ symmetry, which leads to the split of the magnetic component and electric component. These two components are related to the transversal and longitudinal direction to the heat bath. This division makes the 4-dimension momentum into two parts, the Matsubara mode part and the space part. For example, the internal momentum under the finite temperature should be divided into $q_0 = 2\pi T(n + \frac{1}{2})$ and $\vec{q}$. Naturally, this division will divide the propagator into these two parts and the wave function renormalization coefficient is also. So we should study the QCD matter in the case of considering the difference between the two components of the dressing function Z.

However, in the past research of the Polyakov-Quark-Meson model under functional renormalization group, we always employ an approximation that assume $Z^{\perp} = Z^{\parallel}$. Under this approximation we can simplify calculations as much as possible without losing too much information. Nevertheless, the transversal component plays a major role in the physics we should also study their nontrivial influence on the Matsubara frequency and the space momentum.

In the theory of the PQM we have two kind of dressing functions, the meson Z_ϕ and the quark Z_q. In the previous research of PQM model, the approximation that $Z^{\perp} = Z^{\parallel} = Z$ is widely used. The transversal and longitudinal components are assumed to be equal. However, the magnitude of this approximation has not been studied. Thus, it is necessary to check the reliability of this approximation.

In this work, we investigate the affect of the meson dressing function and compare the numerical results which are obtained under $Z_\phi^{\perp} = Z_\phi^{\parallel} = Z_\phi$ with $Z_\phi^{\perp} \neq Z_\phi^{\parallel}$. The wave function renormalizations are scale- dependent, if we want to discuss the affect of them, the numerical calculation must be performed beyond the local potential approximation (LPA). The flow of the dressing function should meet the function written below

$$\partial_t Z_{\phi,k}^{\perp} \neq 0, \qquad \partial_t Z_{\phi,k}^{\parallel} \neq 0 \tag{1}$$

Therefore, the flow of $Z_{\phi,k}$ also working in the meson anomalous dimensions, which follows

$$\eta_{\phi,k} = -\frac{\partial_t Z_{\phi,k}}{Z_{\phi,k}} \tag{2}$$

With the broken of the $O(4)$-symmetry the meson anomalous dimension will split into

$$\eta_{\phi,k}^{\perp} = -\frac{\partial_t Z_{\phi,k}^{\perp}}{Z_{\phi,k}^{\perp}}, \qquad \eta_{\phi,k}^{\parallel} = -\frac{\partial_t Z_{\phi,k}^{\parallel}}{Z_{\phi,k}^{\parallel}} \tag{3}$$

III. POLYAKOV-QUARK-MESON MODEL UNDER THE FRG APPROXIMATION

As is shown in the previous work, the Polyakov-quark-meson model of two-flavor can be studied with the flow equation of the scale-dependent effective action $\Gamma_k[\Phi]$, Φ is the superfield, it can be written as $\Phi = (q, \bar{q}, \phi, ...)$. The flow equation within the framework of FRG can be written as

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} Tr\big(G_{\phi\phi} \partial_t R_k^{\phi}\big) - Tr\big(G_{q\bar{q}} \partial_t R_k^{q}\big) \tag{4}$$

$$G_{\phi\phi/q\bar{q}}[\Phi] = \left(\frac{1}{\frac{\delta^2 \Gamma_k[\Phi]}{\delta \Phi^2} + R_k^{\Phi}} \right)_{\phi\phi/q\bar{q}} \tag{5}$$

is the propagator which is full field-dependent. The effective action that depends on the scale of the quark-meson model can be written like this

$$\begin{aligned}\Gamma_k = \int_x \big\{ & Z_{q,k} \bar{q} [\gamma_\mu \partial_\mu - \gamma_0(\mu + igA_0)] q + \frac{1}{2} Z_{\phi,k} (\partial_\mu \phi)^2 \\ & + h_k \bar{q} (T^0 \sigma + i\gamma_5 \vec{T} \cdot \vec{\pi}) q + V_k(\rho) - c\sigma \big\} + \cdots \end{aligned} \tag{6}$$

here we omited the higher-order terms. The integral sign can be written as $\int_x = \int_0^{1/T} dx_0 \int d^3x$.The meson

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field is $\phi = (\sigma, \vec{\pi})$. $V_k(\rho)$ is field-dependent effective potential which is $O(4)$ invariant, with $\rho = \phi^2/2$.The k is the infrared cutoff scale in FRG; Λ is the UV cutoff renormalization scale, in this work Λ takes 700Mev ; μ is the chemical potential of quark. T is the generators of flavor space $\mathrm{Tr}(T^i T^j) = \frac{1}{2}\delta^{ij}$ and $T^0 = \frac{1}{\sqrt{2N_f}}\mathbb{1}_{N_f \times N_f}$. In this work, we only consider the situation of two flavors that $N_f = 2$. The linear term $-c\sigma$ breaks the chiral symmetry. At the same time, the linear breaking parameter c is related to the mass of $\vec{\pi}$; h_k is the Yukawa coupling.

In the effective action and the flow equation, we only consider the matter part that is composed of quark loop and meson loop. The glue part is working as a input background. For the purpose to encode the information of de-confinement, we adopted the Polyakov loop. The Polyakov loop plays the role of the back ground field potential which has the same effect as the gluon. It works in the model by the expectation value of the traced Polyakov loop which can be written as

$$L = \frac{1}{N_c}\langle \mathrm{Tr}\,\mathcal{P}\rangle, \qquad \bar{L} = \frac{1}{N_c}\langle \mathrm{Tr}\,\mathcal{P}^\dagger\rangle \tag{7}$$

and

$$\mathcal{P}(\vec{x}) = \mathcal{P}\exp\left(ig\int_0^\beta d\tau A_0(\vec{x},\tau)\right) \tag{8}$$

L and $\bar{L}$ can be considered as the order-parameter of the phase transition from confinement to the de-confinement.

The bosonic and the fernionic distribution satisfy the equation given below

$$n_B(\bar{m}^2_{\phi,k}, z_\phi; T) = \frac{1}{\exp\{\frac{1}{T}\frac{k}{z_\phi^{1/2}}\sqrt{1+\bar{m}^2_{\phi,k}}\} - 1} \tag{9}$$

and

$$n_F(\bar{m}^2_{q,k}, z_q; T) = \frac{1}{\exp\{\frac{1}{T}\frac{k}{z_q}\sqrt{1+\bar{m}^2_{q,k}} - \mu\} + 1} \tag{10}$$

With the addition of the Polyakov loop, the distribution function of the fermion can be modified by the following form

$$n_F(x, z_q, T, L, \bar{L}) = \frac{1 + 2\bar{L}e^{x/T} + Le^{2x/T}}{1 + 3\bar{L}e^{x/T} + 3Le^{2x/T} + e^{3x/T}} \tag{11}$$

The x stands for

$$\begin{aligned} x &= \frac{k}{z_q}\sqrt{1+\bar{m}^2_{q,k}} - \mu \\ \bar{x} &= \frac{k}{z_q}\sqrt{1+\bar{m}^2_{q,k}} + \mu \end{aligned} \tag{12}$$

A. The flow equations of the effective potential

We use the three-dimension regulators throughout our calculation of the threshold functions. Through the derivation of the effective action 4 and 5 we can get the flow equation of the effective potential under the constant mesonic fields:

$$\begin{aligned} \partial_t V_k(\rho) = &\frac{k^4}{4\pi^2}[(N_f^2 - 1)l_0^{(B,4)}(\bar{m}^2_{\pi,k}, \eta^\perp_{\phi,k}; T) \\ &+ l_0^{(B,4)}(\bar{m}^2_{\sigma,k}, \eta^\perp_{\phi,k}; T) \\ &- 4N_c N_f l_0^{(F,4)}(\bar{m}^2_{q,k}, \eta_{q,k}; T, \mu)] \end{aligned} \tag{13}$$

The $l_0^{(B/F,n)}$ stands for the threshold functions of the boson and fermion. The analytical form of the threshold functions are given in the Appendix **??**. Below are the quark mass and meson masses which are renormalized and dimensionless

$$\begin{aligned} \bar{m}^2_{\pi,k} &= \frac{V'_k(\rho)}{k^2 Z^\perp_{\phi,k}} \\ \bar{m}^2_{\sigma,k} &= \frac{V'_k(\rho) + 2\rho V''_k(\rho)}{k^2 Z^\perp_{\phi,k}} \\ \bar{m}^2_{q,k} &= \frac{h_k^2 \rho}{2k^2 Z^2_{q,k}} \end{aligned} \tag{14}$$

In the finite temperature the meson wave function renormalization $Z_{\phi,k}$ is split into $Z^\parallel$ and $Z^\perp$. In the past calculation we use a approximation that we assume $Z^\parallel = Z^\perp$. In this work, we abolished this approximation and observe if the change have any influence on the results.By considering the difference between $Z^\perp_{\phi,k}$ and $Z^\parallel_{\phi,k}$ we can calculate the anomalous dimensions respectively.The definition of the anomalous dimensions is given by

$$\eta^\perp_{\phi,k} = -\frac{\partial_t Z^\perp_{\phi,k}}{Z^\perp_{\phi,k}}, \qquad \eta^\parallel_{\phi,k} = -\frac{\partial_t Z^\parallel_{\phi,k}}{Z^\parallel_{\phi,k}} \tag{15}$$

and the definition of the quark anomalous dimension is

$$\eta_{q,k} = -\frac{\partial_t Z_{q,k}}{Z_{q,k}} \tag{16}$$

The frequency and spatial momentum are independent when we calculating the anomalous dimensions. So we can easily obtain the longitudinal component of the meson anomalous dimension with the same derivation method of the transversal component. The analytic form of the transversal component is consistent with the anomalous dimension under $Z^\perp_{\phi,k} = Z^\parallel_{\phi,k}$. The calculation of anomalous dimensions is ultimately the calculation of the flow of the wave-function renormalizations. We can use the Wetterich equation which is mentioned in Eq 5.

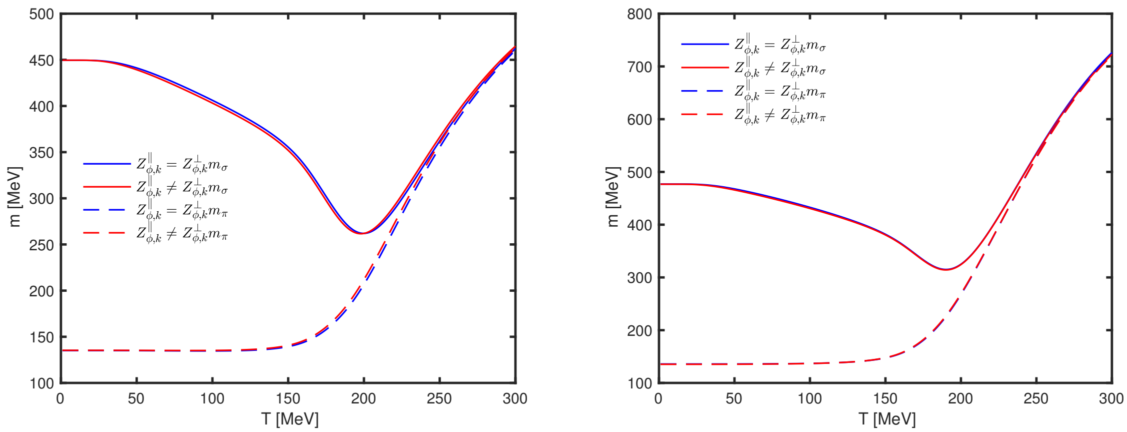

FIG. 1. blue color stands for the results with $Z^{\perp}_{\phi,k} = Z^{\parallel}_{\phi,k}$ and the red color stands for the $Z^{\perp}_{\phi,k} \neq Z^{\parallel}_{\phi,k}$. The left diagram is the results under the fix-point-expansion of the effective potential, and the right picture is the results of the physical-point-expansion.

Here we consider two expansion ways of the effective potential. On one hand, the Taylor expansion of the effective potential is about a field value κ which is un-renormalised and fixed. The effective potential can be written as

$$\bar{V}_k(\bar{\rho}) = \sum_{n=0}^{N} \frac{\bar{\lambda}_{n,k}}{n!}(\bar{\rho} - \bar{\kappa}_k)^n \tag{17}$$

with $\bar{\lambda}_{n,k} = \lambda_{n,k}/Z^{\perp n}_{\phi,k}$ and $\bar{\kappa}_k = Z^{\perp}_{\phi,k}\kappa$. The other hand, we make the expansion point κ depends on the cutoff scales k. Then the $\bar{\kappa}_k$ can be written as

$$\bar{\kappa}_k = Z^{\perp}_{\phi,k}\kappa_k \tag{18}$$

B. Baryon number fluctuation and kurtosis

The definition of the each order baryon number fluctuation can be written as

$$\chi^B_n = \frac{\partial^n}{\partial(\mu_B/T)^n}\frac{p}{T^4} \tag{19}$$

The baryon chemical potential satisfying the relationship of $\mu_B = 3\mu$. The quadratic and quartic fluctuations can be expressed using the average baryon number fluctuation $\langle N_B \rangle$

$$\begin{aligned} \chi^B_2 &= \frac{1}{VT^3}\langle \delta {N_B}^2 \rangle \\ \chi^B_4 &= \frac{1}{VT^3}\left(\langle \delta {N_B}^4 \rangle - 3\langle \delta {N_B}^2 \rangle^2\right) \end{aligned} \tag{20}$$

The definition of the kurtosis can be written as

$$\kappa\sigma^2 = \frac{\chi^B_4}{\chi^B_2} \tag{21}$$

At the same time, we also calculated the chemical freeze-out line as the function of the collision energy and the kurtosis of the baryon number fluctuation. The value of the kurtosis is determined by the temperature and the baryon chemical potential. We use the correspondence of the collision energy and temperature and chemical potential which is give in the Table I.

$\sqrt{s}$ [GeV]	200	62.4	39	27	19.6	11.5	7.7
$\mu_{B,N_f=2}$ [MeV]	25.3	78.1	121	168.7	222.7	343	459.4
$T_f(\chi^{\mathrm{B}}_3/\chi^{\mathrm{B}}_2)$ [MeV]	180	186	185	183	180	167	149
$T_f(\chi^{\mathrm{B}}_2/\chi^{\mathrm{B}}_1)$ [MeV]	178	183	188	182	178	168	154

TABLE I.

C. numerical results

In the end, we give the numerical calculation results of the physical quantity. In this section we will show the results of the comparison in the two flavor PQM model. Two comparisons have been made, one is $Z^{\parallel}_{\phi} = Z^{\perp}_{\phi}$ and $Z^{\parallel}_{\phi} \neq Z^{\perp}_{\phi}$, the other one is the fix point and physical point expansion of the effective potential. As we can see in the Fig. III A, the mass of the pion and the sigma meson have been computed. The results are clear,

As is shown in the Fig. III B, the baryon chemical potential has little influence to the numerical result of quadratic fluctuation. However, we can see clearly the influence of the change of wave function renormalization factors.

The results of the quartic fluctuations under two kinds of wave function renormalizations have larger difference at the peak of the curves, the results of $Z^{\perp}_{\phi,k} \neq Z^{\parallel}_{\phi,k}$ are lower than the results of $Z^{\perp}_{\phi,k} = Z^{\parallel}_{\phi,k}$.

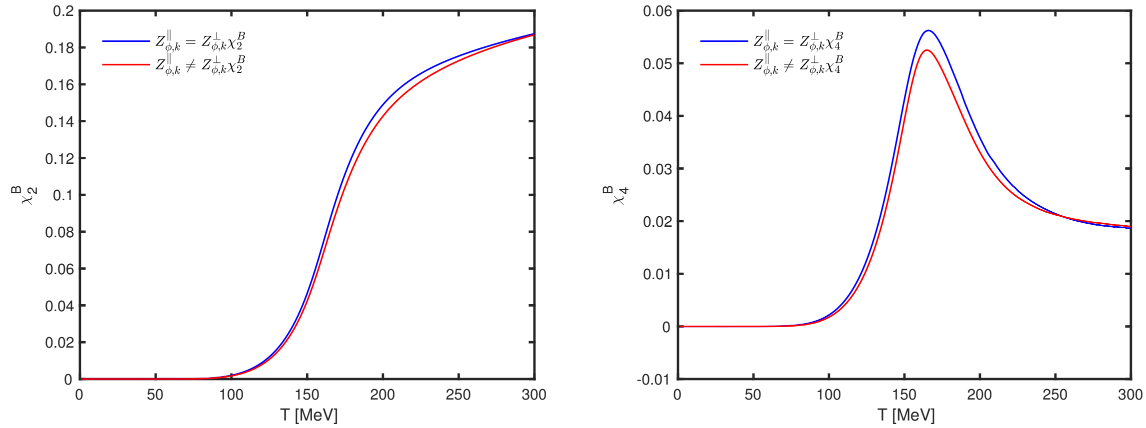

FIG. 2. blue color stands for the results with $Z^{\perp}_{\phi,k} = Z^{\|}_{\phi,k}$ and the red color stands for the $Z^{\perp}_{\phi,k} \neq Z^{\|}_{\phi,k}$. The full and dotted lines are the results under the chemical potential of 50 MeV and 100 MeV. These results are under the physical point expansion of the effective potential.

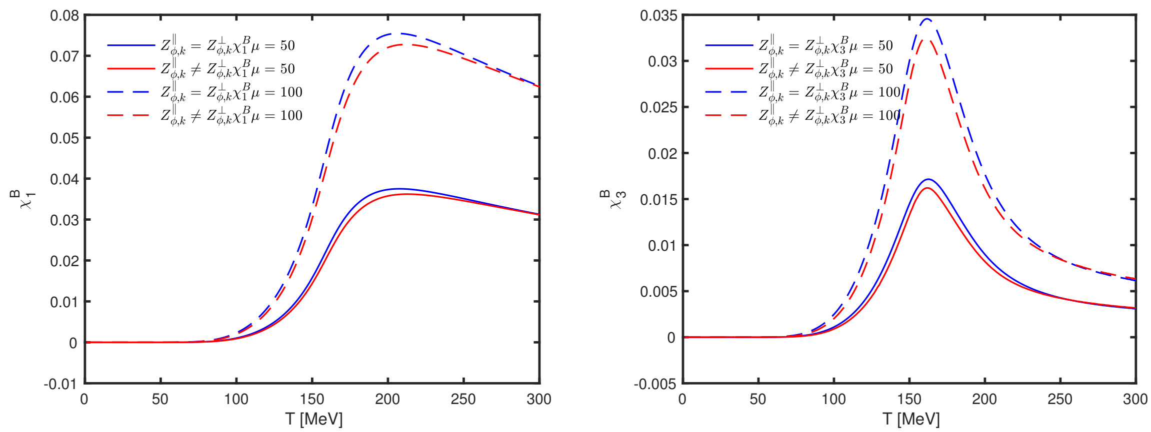

FIG. 3. blue color stands for the results with $Z^{\perp}_{\phi,k} = Z^{\|}_{\phi,k}$ and the red color stands for the $Z^{\perp}_{\phi,k} \neq Z^{\|}_{\phi,k}$. The full and dotted lines are the results under the chemical potential of 50 MeV and 100 MeV. These results are under the physical point expansion of the effective potential.

IV. CONCLUSION

In this work, we investigated the baryon number fluctuations and some thermodynamic quantities with in the functional renormalization group. We made a comparison of the results between the approximation of the mesonic wave function renormalizations and without the approximation. The calculation are accomplished under the fix point and physical point expansion of the effective potential.

The result of the comparison is obvious, the approximation of the mesonic wave function renormalizations $Z^{\|}_{\phi} = Z^{\perp}_{\phi}$ cause few difference in the numerical computation, thus this approximation is good enough to use in the future work. And the different expansion methods of the effective potential do lead to some difference results of the meson mass.

ACKNOWLEDGMENTS

Thanks

Three-d flat regulators have been used in this work. The regulator functions of the meson field and quark field can be written as

$$\begin{aligned} R^{\phi}_k(q_0,\vec{q}) &= Z^{\perp}_{\phi,k}\vec{q}^2 r_B(\vec{q}^2/k^2), \\ R^{q}_k(q_0,\vec{q}) &= Z_{q,k} i\vec{\gamma}\cdot\vec{q} r_F(\vec{q}^2/k^2) \end{aligned} \tag{22}$$

in which

$$\begin{aligned} r_B(x) &= \left(\frac{1}{x}-1\right)\Theta(1-x), \\ r_F(x) &= \left(\frac{1}{\sqrt{x}}-1\right)\Theta(1-x) \end{aligned} \tag{23}$$

Then we give a definition to the meson and quark prop-

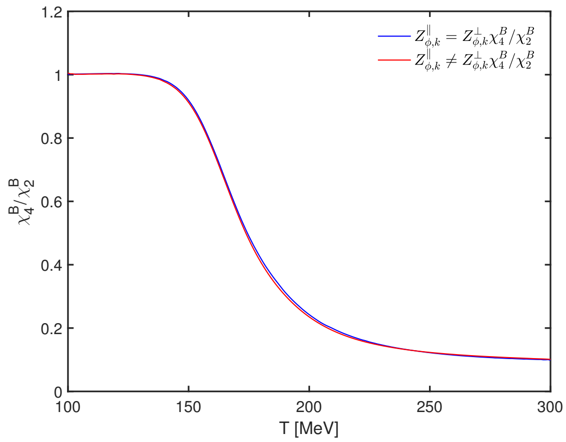

FIG. 4. blue color stands for the results with $Z^{\perp}_{\phi,k} = Z^{\parallel}_{\phi,k}$ and the red color stands for the $Z^{\perp}_{\phi,k} \neq Z^{\parallel}_{\phi,k}$.

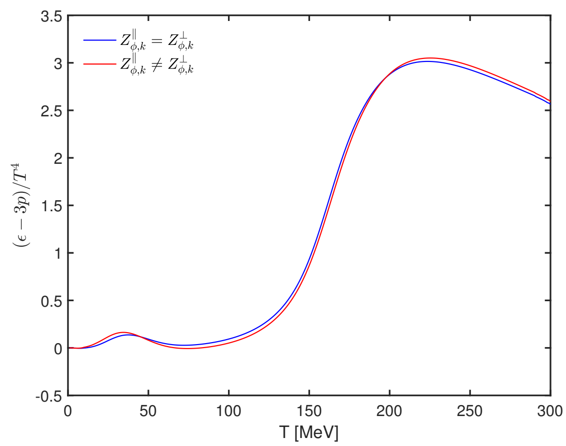

FIG. 5.

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$$
\begin{aligned}
G_\phi(q, \bar{m}^2_{\phi,k}) &= \frac{1}{z_\phi \tilde{q}_0^2 + 1 + \bar{m}^2_{\phi,k}}, \\
G_q(q, \bar{m}^2_{q,k}) &= \frac{1}{z_q^2(\tilde{q}_0 + i\tilde{\mu})^2 + 1 + \bar{m}^2_{q,k}}
\end{aligned}
\tag{24}
$$

in the equation above we have $\tilde{q}_0 = q_0/k$, $\tilde{\mu} = \mu/k$ and for the fermions we have $q_0 = (2n_q + 1)\pi T (n_q \in \mathbb{Z})$ for the bosons we have $q_0 = 2n_q \pi T$. Here $z = Z^{\parallel}/Z^{\perp}$ is the ratio of the two components of the wave function renormalizations. In this work we choose $z_q = 1$ which means $Z^{\parallel}_q = Z^{\perp}_q$ and $z_\phi \neq 1$ which means $Z^{\parallel}_\phi \neq Z^{\perp}_\phi$. To obtain the threshold functions, we define

in the equations above the form of the distribution functions are

$$
n_B(\bar{m}^2_{\phi,k}, z_\phi; T) = \frac{1}{\exp\{\frac{1}{T}\frac{k}{z_\phi^{1/2}}\sqrt{1 + \bar{m}^2_{\phi,k}}\} - 1} \tag{25}
$$

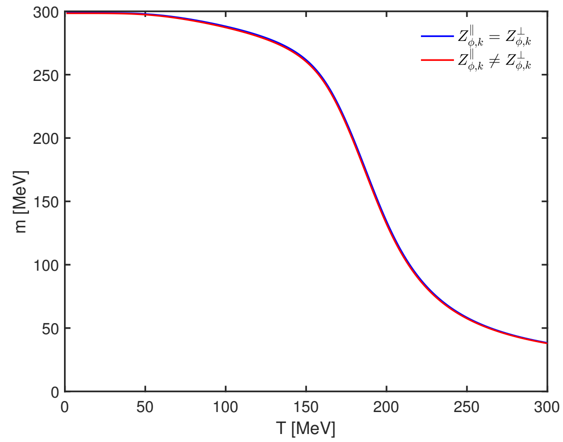

FIG. 6. blue color stands for the results with $Z^{\perp}_{\phi,k} = Z^{\parallel}_{\phi,k}$ and the red color stands for the $Z^{\perp}_{\phi,k} \neq Z^{\parallel}_{\phi,k}$.

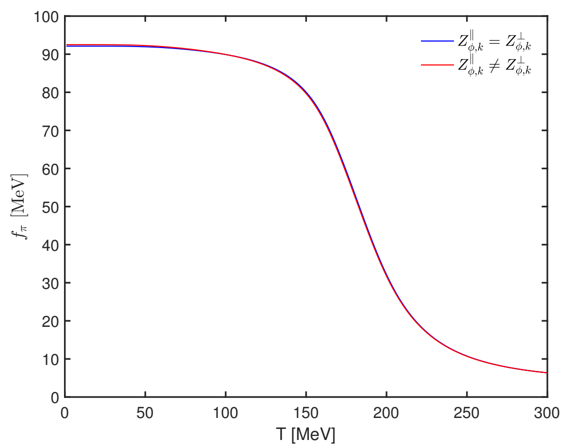

FIG. 7. blue color stands for the results with $Z^{\perp}_{\phi,k} = Z^{\parallel}_{\phi,k}$ and the red color stands for the $Z^{\perp}_{\phi,k} \neq Z^{\parallel}_{\phi,k}$.

and

$$
n_F(\bar{m}^2_{q,k}, z_\phi; T) = \frac{1}{\exp\{\frac{1}{T}\frac{k}{z_q}\sqrt{1 + \bar{m}^2_{q,k}} - \mu\} + 1} \tag{26}
$$

In the effective potential's flow equation, there are bosonic and fermionic threshold functions. The anomalous dimension of the mesonic field has change into the transverse component of it. The form of the threshold functions are

$$
l_0^{(B,d)}(\bar{m}^2_{\phi,k}, \eta^{\perp}_{\phi,k}; T) = \frac{2}{d-1}\left(1 - \frac{\eta^{\perp}_{\phi,k}}{d+1}\right)\mathcal{B}_{(1)}(\bar{m}^2_{\phi,k}, z_\phi; T) \tag{27}
$$

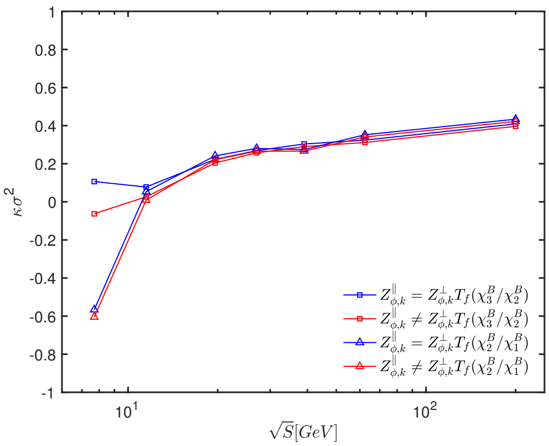

FIG. 8. blue color stands for the results with $Z^{\perp}_{\phi,k} = Z^{\parallel}_{\phi,k}$ and the red color stands for the $Z^{\perp}_{\phi,k} \neq Z^{\parallel}_{\phi,k}$.

and

$$\begin{aligned} & l_0^{(F,d)}(\bar{m}^2_{q,k}, \eta_{q,k}; T, \mu) \\ & \quad = \frac{2}{d-1}\left(1 - \frac{\eta_{q,k}}{d}\right) \mathcal{F}_{(1)}(\bar{m}^2_{q,k}, z_q = 1; T, \mu) \end{aligned} \tag{28}$$

The definition of the threshold function $\mathcal{BB}_{(1,1)}$ is

$$\begin{aligned} & \mathcal{BB}_{(1,1)}(\bar{m}^2_{\phi_a,k}, \bar{m}^2_{\phi_b,k}, z_\phi; T) \\ & \quad = -\frac{T}{k}\sum_{n_q} G_\phi(q, \bar{m}^2_{\phi_a,k}) G_\phi(q, \bar{m}^2_{\phi_b,k}) \end{aligned} \tag{29}$$

then we can obtain the $\mathcal{BB}_{(2,2)}$ in a same way

$$\begin{aligned} & \mathcal{BB}_{(2,2)}(\bar{m}^2_{\phi_a,k}, \bar{m}^2_{\phi_b,k}, z_\phi; T) \\ & \quad = \frac{\partial^2}{\partial \bar{m}^2_{\phi_a,k} \partial \bar{m}^2_{\phi_b,k}} \mathcal{BB}_{(1,1)}(\bar{m}^2_{\phi_a,k} \bar{m}^2_{\phi_b,k}, z_\phi; T) \end{aligned} \tag{30}$$

The expression of the $\mathcal{BB}_{(1,1)}$ is

$$\begin{aligned} \mathcal{BB}_{(1,1)}(\bar{m}^2_{\phi_a,k}, \bar{m}^2_{\phi_b,k}, z_\phi; T) = & -\frac{1}{z_\phi^{1/2}}\left\{\left(\frac{1}{2} + n_B(\bar{m}^2_{\phi_a,k}, z_\phi; T)\right) \frac{1}{(1+\bar{m}^2_{\phi_a,k})^{1/2}}\right. \\ & \times \frac{1}{\bar{m}^2_{\phi_a,k} - \bar{m}^2_{\phi_b,k}} + \left(\frac{1}{2} + n_B(\bar{m}^2_{\phi_b,k}, z_\phi; T)\right) \\ & \left. \times \frac{1}{(1+\bar{m}^2_{\phi_b,k})^{1/2}} \frac{1}{\bar{m}^2_{\phi_b,k} - \bar{m}^2_{\phi_a,k}}\right\} \end{aligned} \tag{31}$$

then we can get the expression of the threshold functions of any n. At the same time, in our calculation there are also some other kind of threshold functions.

$$\begin{aligned} & \mathcal{BB}\tilde{q}^2_{0\,(1,1)}(\bar{m}^2_{\phi_a,k}, \bar{m}^2_{\phi_b,k}, z_\phi; T) \\ & \quad = -\frac{T}{k}\sum_{n_q} G_\phi(q, \bar{m}^2_{\phi_a,k}) G_\phi(q, \bar{m}^2_{\phi_b,k}) \tilde{q}_0^2 \end{aligned} \tag{32}$$

The form of the threshold functions $\mathcal{F}\tilde{q}_0^2$, $\mathcal{F}\tilde{q}_0^4$ and $\mathcal{BB}\tilde{q}_0^2$ is like

$$\begin{aligned} & \mathcal{BB}\tilde{q}^2_{0\,(1,1)}(\bar{m}^2_{\phi_a,k}, \bar{m}^2_{\phi_b,k}, z_\phi; T) = \\ & - Z_\phi^{1/2}\left\{\left(\frac{1}{2} + n_B(\bar{m}^2_{\phi_a,k}, z_\phi; T)\right) \frac{(1+\bar{m}^2_{\phi_a,k})^{-1/2}}{\bar{m}^2_{\phi_b,k} - \bar{m}^2_{\phi_a,k}}\right. \\ & \left. + \left(\frac{1}{2} + n_B(\bar{m}^2_{\phi_b,k}, z_\phi; T)\right) \frac{(1+\bar{m}^2_{\phi_a,k})^{-1/2}}{\bar{m}^2_{\phi_a,k} - \bar{m}^2_{\phi_b,k}}\right\} \end{aligned} \tag{33}$$

In order to obtain the flow equation of the effective potential, the mesonic anomalous dimensions are needed. Because we have divided the wave function renormalizations into the transverse and longitudinal components of it, so the anomalous dimensions should be divided either. The analytical form of the mesonic anomalous dimensions can be written like

$$\begin{aligned} \eta^{\parallel}_{\phi,k} = & \frac{1}{6\pi^2}\left\{\frac{4}{k^2 z_\phi^4}\bar{\kappa}_k (\bar{V}_k''(\bar{\kappa}_k))^2 \left[2\mathcal{BB}_{(2,2)}(\bar{m}^2_{\pi,k}, \bar{m}^2_{\sigma,k}; T)\right.\right. \\ & \left. -4\mathcal{BB}\tilde{q}^2_{0\,(2,3)}(\bar{m}^2_{\pi,k}, \bar{m}^2_{\sigma,k}; T) - 4\mathcal{BB}\tilde{q}^2_{0\,(3,2)}(\bar{m}^2_{\pi,k}, \bar{m}^2_{\sigma,k}; T)\right] \\ & \left. \times(1 - \frac{1}{5}\eta^{\perp}_{\phi,k}) + \frac{N_c \bar{h}_k^2}{z_\phi}\mathcal{F}_{(3)}(\bar{m}^2_{q,k}; T, \mu)(4 - \eta_{q,k})\right\} \end{aligned} \tag{34}$$

and the other component

$$\begin{aligned} \eta^{\perp}_{\phi,k} = & \frac{1}{6\pi^2}\left\{\frac{4}{k^2 z_\phi^4}\bar{\kappa}_k (\bar{V}_k''(\bar{\kappa}_k))^2 \mathcal{BB}_{(2,2)}(\bar{m}^2_{\pi,k}, \bar{m}^2_{\sigma,k}; T)\right. \\ & + N_c \bar{h}_k^2 \left[\mathcal{F}_{(2)}(\bar{m}^2_{q,k}; T, \mu)(2\eta_{q,k} - 3)\right. \\ & \left.\left. -4(\eta_{q,k} - 2)\mathcal{F}_{(3)}(\bar{m}^2_{q,k}; T, \mu)\right]\right\} \end{aligned} \tag{35}$$

The flow of the quark wave function renormalization are given by

$$\eta_{q,k}(p_0,\vec{p}) = \frac{1}{Z_{q,k}(p_0,\vec{p})}\frac{1}{4N_cN_f} Re\left[\frac{\partial^2}{\partial|p|^2}Tr\left(i\vec{\gamma}\cdot\vec{p}\left(-\frac{\delta^2\partial_t\Gamma_k}{\delta\bar{q}(-p)\delta q(p)}\right)\right)\Big|_{\rho=\kappa}\right] \tag{36}$$

and the flow of the Yukawa coupling can be written as

$$\partial_t h_k(p_0,\vec{p}) = \frac{\sqrt{2N_f}}{\sigma}\frac{1}{4N_cN_f} \times Re\left[Tr\left(-\frac{\delta^2\partial_t\Gamma_k}{\delta\bar{q}(-p)\delta q(p)}\right)\Big|_{\rho=\kappa}\right] \tag{37}$$

Now we can obtain the form of the quark anomalous dimension

$$\eta_{q,k} = \frac{1}{24\pi^2 N_f}(4-\eta_{\phi,k})\bar{h}_k^2 \times \{(N_f^2-1)\mathcal{FB}_{(1,2)}(\bar{m}_{q,k}^2,\bar{m}_{\pi,k}^2;T,\mu,p_{0,ex}) + \mathcal{FB}_{(1,2)}(\bar{m}_{q,k}^2,\bar{m}_{\sigma,k}^2;T,\mu,p_{0,ex})\} \tag{38}$$