Mesonic dynamics and QCD phase transition

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We study the finite temperature and density two flavor quark-meson model under the functional renormalisation group. The effect of broken $O(4)-$ symmetry of the wave function renormalisation and expansion point of effective potential on the thermodynamic quantities and baryon number fluctuation are investigate. At the same time, the field dependent Yukawa coupling is also considered. We give results of the pion mass, the quark mass, the trace anomaly, the baryon number fluctuation and the freeze-out curve.

I. INTRODUCTION

The QCD phase structure and the search of the critical end point (CEP) are the most popular research direction in both experimental and theoretical field. The phase transition between the quark gluon plasma (QGP) and hadron is the main research objects. The research of the QGP-hadron phase transition can help us to help us better understand the nature of elementary particles. The experiment to looking for the QGP is being made at the Large Hadron Collider (LHC) and the Relativistic Heavy-Ion Collider (RHIC).

In terms of theoretical research, there many different methods to investigate the QCD phase structure. The most widely studied method is the lattice QCD. A lot of properties of the QCD matter have been discussed under the lattice simulations. Although the lattice theory has the sign problem at high baryon chemical potential, it still gave us plenty of great outcomes. In order to avoid the problem that occurs in lattice calculation, the study of the continuous non-perturbative field theory is in progress at the same time. For example, the Dyson-Schwinger equation. And the Functional Renormalization Group (FRG) is the other good functional approach of the continuous theory. In these ways we can study the behavior of the strong interaction matter under the finite temperature and density better.

This work is done with the low energy effective model under the FRG approach.

The low-energy effective models, e.g. the quark-meson (QM) model [1], Nambu–Jona-Lasinio (NJL) model, and their Polyakov-loop improved variants: PQM and PNJL, are suitable to be employed to study the QCD phase transitions. They have been investigated quite a lot in literatures, see, e.g., [] for more details. In this work, we adopt the scale-dependent effective action for the two-flavor PQM model, as follows

$$\Gamma_k = \int_x \bigg\{ Z_{q,k}\bar{q}\Big[\gamma_\mu\partial_\mu - \gamma_0(\mu + igA_0)\Big]q \\ + \frac{1}{2}\Big[Z^{\parallel}_{\phi,k}(\partial_0\phi)^2 + Z^{\perp}_{\phi,k}(\partial_i\phi)^2\Big] \\ + h_k(\rho)\bar{q}\big(T^0\sigma + i\gamma_5\vec{T}\cdot\vec{\pi}\big)q + V_k(\rho) - c\sigma \bigg\}, \quad (1)$$

with $\mu = (0, 1, ..., 3)$ and $i = (1, 2, 3)$. In Eq. (1) we have used notation $\int_x = \int_0^{1/T} dx_0 \int d^3x$, where the imaginary time formalism for the field theory at finite temperature is used, and the temporal length reads $\beta = 1/T$. Apparently, when the temperature is nonzero, the $O(4)$-symmetry in the Euclidean space is broken into that of $\mathbb{Z}_2 \otimes O(3)$, which leads to the split of the magnetic and electric components of correlations functions. They correspond to the components transversal and longitudinal to the heat bath, respectively. In this work, we take this split into account in the two-point correlation function for the mesons, as shown in the second line on the r.h.s. of Eq. (1), where $Z^{\parallel}_{\phi,k}$ and $Z^{\perp}_{\phi,k}$ indicate the longitudinal and transversal wave function renormalizations for the temporal and spacial components, respectively.

The reason why we concentrate on the split of the wave function renormalization especially for the mesons is due to the facts as follows. Firstly, in comparison to the quark wave function renormalization $Z_{q,k}$ and the scale dependent Yukawa coupling h_k in Eq. (1), it is found that the meson wave function renormalization $Z_{\phi,k}$ plays the most significant role beyond the local potential approximation (LPA) [2, 3]. In the LPA, the propagators are classical, i.e., $Z_{q,k} = Z_{\phi,k} = 1$ and the Yukawa coupling h is a constant and independent of the scale k. Secondly, In Ref. [4] calculations based on the full momentum-dependent two-point correlation functions of mesons are compared with those from LPA and LPA′, and here in LPA′ a momentum-independent $Z_{\phi,k}$ is included, and it is found that there is a good agreement between the full momentum calculation and the LPA′, while not LPA, which indicates that the dispersion relation for the meson, resulting from a scale dependent $Z_{\phi,k}$, have already captured most momentum dependence of the two-point correlation function.

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Considering the importance of the wave function renormalization for the mesons and the success of LPA′, in this work we would like to investigate the effects of the splitting of $Z_{\phi,k}$ in the LPA′ as shown in Eq. (1), which is a natural choice at finite temperature as discussed above. Furthermore, we will also study the interplay between the splitting of $Z_{\phi,k}$ and other truncation approaches, e.g., the field dependent Yukawa coupling $h_k(\rho)$ which encodes higher order quark-meson scattering processes [2], fixed point expansion for the effective potential $V_k(\rho)$ in Eq. (1) versus the physical point expansion, etc. Their influences on the QCD phase transition and observables, e.g. fluctuations of the baryon number, will be investigated in detail.

II. FUNCTIONAL RENORMALIZATION GROUP AND FLOW EQUATIONS

To proceed, we describe other notations in the effective action in Eq. (1). $\phi = (\sigma, \vec{\pi})$ is a meson field with four components. The effective potential $V_k(\rho)$ with $\rho = \phi^2/2$ is $O(4)$ invariant and the c-term $c\sigma$ breaks the chiral symmetry explicitly. The mesons interact with quarks through the scalar and pseudo-scalar channels with a mesonic field dependent Yukawa coupling $h_k(\rho)$, and T^0 and T^i are the generators in the flavor space with the convention as follows: $\mathrm{Tr}(T^iT^j) = \frac{1}{2}\delta^{ij}$ and $T^0 = \frac{1}{\sqrt{2N_f}}\mathbb{1}_{N_f\times N_f}$ with $N_f = 2$. Besides the wave function renormalization for the meson, we also introduce one for the quark, i.e., $Z_{q,k}$. Since it plays a minor role in the chiral dynamics in comparison to $Z_{\phi,k}$, the splitting of $Z_{q,k}$ into the transversal and longitudinal components are neglected for simplicity in calculations. Finally, μ in the first line on the r.h.s. of Eq. (1) denotes the quark chemical potential, and here the temporal gluon background field A_0, which encodes the Polyakov dynamics, are also taken into account.

The renormalization group (RG) scale k in Eq. (1) is an infrared cutoff, below which quantum fluctuations are suppressed in the effective action. The evolution of the effective action with k is described by the Wetterich equation [5], which reads

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\big(G_{\phi\phi,k}\partial_t R_k^{\phi}\big) - \mathrm{Tr}\big(G_{q\bar{q},k}\partial_t R_k^{q}\big)\,, \tag{2}$$

with the RG time $t = \ln(k/\Lambda)$ and the initial ultraviolet (UV) cutoff Λ, where R_k^{ϕ} and R_k^{q} are the regulators for the meson and quark fields, respectively, and they are given in Eqs. (B1) and (B2). The scale dependent meson and quark propagators are given by

$$G_{\phi\phi/q\bar{q}}[\Phi] = \left(\frac{1}{\frac{\delta^2\Gamma_k[\Phi]}{\delta\Phi^2} + R_k^{\Phi}}\right)_{\phi\phi/q\bar{q}}\,, \tag{3}$$

with $\Phi = (q, \bar{q}, \phi)$ denoting all species of fields.

Inserting the effective action in Eq. (1) into the flow equation (2), one arrives at the flow equation for the effective potential, which reads

$$\begin{aligned}\partial_t V_k(\rho) =&\frac{k^4}{4\pi^2}\Big[(N_f^2-1)\,l_0^{(B,4)}(\bar{m}^2_{\pi,k}, \eta^{\perp}_{\phi,k}, z_\phi; T)\\ &+ l_0^{(B,4)}(\bar{m}^2_{\sigma,k}, \eta^{\perp}_{\phi,k}, z_\phi; T)\\ &- 4N_cN_f\, l_0^{(F,4)}(\bar{m}^2_{q,k}, \eta_{q,k}; T, \mu)\Big]\,,\end{aligned} \tag{4}$$

with the RG invariant dimensionless meson and quark masses as follows

$$\bar{m}^2_{\pi,k} = \frac{V_k'(\rho)}{k^2 Z^{\perp}_{\phi,k}}\,, \qquad \bar{m}^2_{\sigma,k} = \frac{V_k'(\rho) + 2\rho V_k''(\rho)}{k^2 Z^{\perp}_{\phi,k}}\,, \tag{5}$$

$$\bar{m}^2_{q,k} = \frac{h_k^2\rho}{2k^2 Z^2_{q,k}}\,. \tag{6}$$

The threshold functions $l_0^{(B)}$ and $l_0^{(F)}$ are presented in **??**, and the anomalous dimensions for the mesons and quark are defined as

$$\eta^{\perp}_{\phi,k} = -\frac{\partial_t Z^{\perp}_{\phi,k}}{Z^{\perp}_{\phi,k}}\,, \quad \eta^{\parallel}_{\phi,k} = -\frac{\partial_t Z^{\parallel}_{\phi,k}}{Z^{\parallel}_{\phi,k}}\,, \quad \eta_{q,k} = -\frac{\partial_t Z_{q,k}}{Z_{q,k}}\,. \tag{7}$$

Note that $z_\phi \equiv Z^{\parallel}_{\phi,k}/Z^{\perp}_{\phi,k}$ enters into the flow of $V_k(\rho)$ through the mesonic fluctuations as shown in Eq. (4). The transversal anomalous dimension for the π-meson is obtained by employing the projection as follows

$$\eta^{\perp}_{\phi,k} = -\frac{1}{3Z^{\perp}_{\phi,k}}\delta_{ij}\frac{\partial}{\partial(|\boldsymbol{p}|^2)}\frac{\delta^2\partial_t\Gamma_k}{\delta\pi_i(-p)\delta\pi_j(p)}\bigg|_{\substack{p_0=0\\ \boldsymbol{p}=0}}\,, \tag{8}$$

and the longitudinal one reads

$$\eta^{\parallel}_{\phi,k} = -\frac{1}{3Z^{\parallel}_{\phi,k}}\delta_{ij}\frac{\partial}{\partial(p_0^2)}\frac{\delta^2\partial_t\Gamma_k}{\delta\pi_i(-p)\delta\pi_j(p)}\bigg|_{\substack{p_0=0\\ \boldsymbol{p}=0}}\,. \tag{9}$$

and we neglect the difference of the anomalous dimension between the π and σ mesons

$$\begin{aligned}\eta_q(p_0,\boldsymbol{p}) =& \frac{1}{4Z_{q,k}(p_0,\boldsymbol{p})}\\ &\times \mathrm{Re}\left[\frac{\partial}{\partial(|\boldsymbol{p}|^2)}\mathrm{tr}\left(i\boldsymbol{\gamma}\cdot\boldsymbol{p}\left(-\frac{\delta^2}{\delta\bar{q}(p)\delta q(p)}\partial_t\Gamma_k\right)\right)\right]\,.\end{aligned} \tag{10}$$

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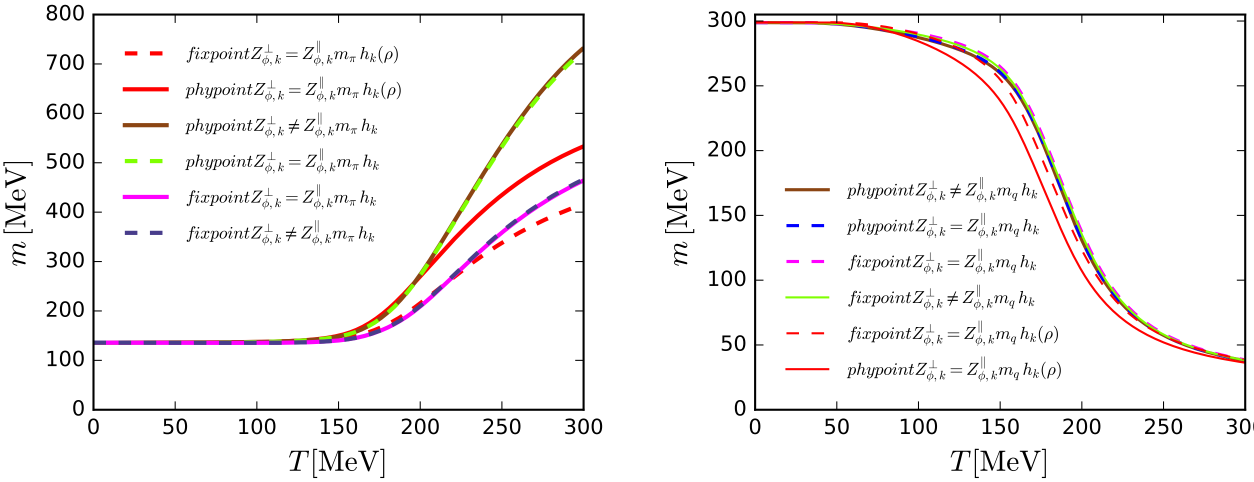

FIG. 1. The left diagram is the mass of the pion as a function of temperature, the right curves are the constituent quark mass. These diagrams are obtained under fix point and physical point expansion of the effective potential.

III. POLYAKOV-QUARK-MESON MODEL UNDER THE FRG APPROXIMATION

As is shown in the previous work, the Polyakov-quark-meson model of two-flavor can be studied with the flow equation of the scale-dependent effective action $\Gamma_k[\Phi]$, Φ is the superfield, it can be written as $\Phi = (q, \bar{q}, \phi, ...)$. The flow equation within the framework of FRG can be written as

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} Tr\big(G_{\phi\phi}\partial_t R_k^{\phi}\big) - Tr\big(G_{q\bar{q}}\partial_t R_k^{q}\big) \tag{11}$$

$$G_{\phi\phi/q\bar{q}}[\Phi] = \left(\frac{1}{\frac{\delta^2\Gamma_k[\Phi]}{\delta\Phi^2} + R_k^{\Phi}}\right)_{\phi\phi/q\bar{q}} \tag{12}$$

is the propagator which is full field-dependent. The effective action that depends on the scale of the quark-meson model can be written like this

$$\Gamma_k = \int_x \big\{ Z_{q,k}\bar{q}[\gamma_\mu\partial_\mu - \gamma_0(\mu + igA_0)]q + \frac{1}{2} Z_{\phi,k}(\partial_\mu\phi)^2 + h_k\bar{q}(T^0\sigma + i\gamma_5\vec{T}\cdot\vec{\pi})q + V_k(\rho) - c\sigma \big\} + \cdots \tag{13}$$

here we omited the higher-order terms. The integral sign can be written as $\int_x = \int_0^{1/T} dx_0 \int d^3x$.The meson field is $\phi = (\sigma, \vec{\pi})$. $V_k(\rho)$ is meson field-dependent effective potential which is $O(4)$ invariant, with $\rho = \phi^2/2$. The k is the infrared cutoff scale in FRG; Λ is the UV cutoff renormalization scale, in this work Λ takes 700Mev ; μ is the chemical potential of quark. T is the generators of flavor space $\mathrm{Tr}(T^iT^j) = \frac{1}{2}\delta^{ij}$ and $T^0 = \frac{1}{\sqrt{2N_f}}\mathbb{1}_{N_f\times N_f}$. In this work, we only consider the situation of two flavors that $N_f = 2$. The linear term $-c\sigma$ breaks the chiral symmetry. At the same time, the linear breaking parameter c is related to the mass of $\vec{\pi}$; h_k is the Yukawa coupling.

In the effective action and the flow equation, we only consider the matter part that is composed of quark loop and meson loop. The glue part is working as a input background. For the purpose to encode the information of de-confinement, we adopted the Polyakov loop. The Polyakov loop plays the role of the back ground field potential which has the same effect as the gluon. It works in the model by the expectation value of the traced Polyakov loop which can be written as

$$L = \frac{1}{N_c}\langle \mathrm{Tr}\,\mathcal{P}\rangle, \qquad \bar{L} = \frac{1}{N_c}\langle \mathrm{Tr}\,\mathcal{P}^{\dagger}\rangle \tag{14}$$

and

$$\mathcal{P}(\vec{x}) = \mathcal{P}\exp\left(ig\int_0^{\beta} d\tau A_0(\vec{x},\tau)\right) \tag{15}$$

L and $\bar{L}$ can be considered as the order-parameter of the phase transition from confinement to the de-confinement.

The bosonic and the fernionic distribution satisfy the equation given below

$$n_B(\bar{m}^2_{\phi,k}, z_\phi; T) = \frac{1}{\exp\{\frac{1}{T}\frac{k}{z_\phi^{1/2}}\sqrt{1+\bar{m}^2_{\phi,k}}\} - 1} \tag{16}$$

and

$$n_F(\bar{m}^2_{q,k}, z_q; T) = \frac{1}{\exp\{\frac{1}{T}\frac{k}{z_q}\sqrt{1+\bar{m}^2_{q,k}} - \mu\} + 1} \tag{17}$$

With the addition of the Polyakov loop, the distribution function of the fermion can be modified by the following form

$$n_F(x, z_q, T, L, \bar{L}) = \frac{1 + 2\bar{L}e^{x/T} + Le^{2x/T}}{1 + 3\bar{L}e^{x/T} + 3Le^{2x/T} + e^{3x/T}} \tag{18}$$

The x stands for

$$\begin{aligned} x &= \frac{k}{z_q}\sqrt{1+\bar{m}_{q,k}^2} - \mu \\ \bar{x} &= \frac{k}{z_q}\sqrt{1+\bar{m}_{q,k}^2} + \mu \end{aligned} \tag{19}$$

The plus and minus sign in front of the chemical potential μ stand for the quark and anti-quark.

A. The flow equations of the effective potential and the Yukawa coupling

We use the three-dimension regulators throughout our calculation of the threshold functions. Through the derivation of the effective action 11 and 12 we can get the flow equation of the effective potential under the constant mesonic fields:

$$\begin{aligned} \partial_t V_k(\rho) =& \frac{k^4}{4\pi^2}[(N_f^2-1)l_0^{(B,4)}(\bar{m}_{\pi,k}^2, \eta_{\phi,k}^{\perp}; T) \\ &+ l_0^{(B,4)}(\bar{m}_{\sigma,k}^2, \eta_{\phi,k}^{\perp}; T) \\ &- 4N_c N_f l_0^{(F,4)}(\bar{m}_{q,k}^2, \eta_{q,k}; T, \mu)] \end{aligned} \tag{20}$$

The $l_0^{(B/F,n)}$ stands for the threshold functions of the boson and fermion. The analytical form of the threshold functions are given at **??**. Below are the quark mass and meson masses which are renormalized and dimensionless

$$\begin{aligned} \bar{m}_{\pi,k}^2 &= \frac{V_k'(\rho)}{k^2 Z_{\phi,k}^{\perp}} \\ \bar{m}_{\sigma,k}^2 &= \frac{V_k'(\rho) + 2\rho V_k''(\rho)}{k^2 Z_{\phi,k}^{\perp}} \\ \bar{m}_{q,k}^2 &= \frac{h_k^2 \rho}{2k^2 Z_{q,k}^2} \end{aligned} \tag{21}$$

In the finite temperature the meson wave function renormalization $Z_{\phi,k}$ is split into $Z^{\parallel}$ and $Z^{\perp}$. In the past calculation we use a approximation that we assume $Z^{\parallel} = Z^{\perp}$. In this work, we abolished this approximation and observe if the change have any influence on the results.By considering the difference between $Z_{\phi,k}^{\perp}$ and $Z_{\phi,k}^{\parallel}$ we can calculate the anomalous dimensions respectively.The definition of the anomalous dimensions is given by

$$\eta_{\phi,k}^{\perp} = -\frac{\partial_t Z_{\phi,k}^{\perp}}{Z_{\phi,k}^{\perp}}, \qquad \eta_{\phi,k}^{\parallel} = -\frac{\partial_t Z_{\phi,k}^{\parallel}}{Z_{\phi,k}^{\parallel}} \tag{22}$$

and the definition of the quark anomalous dimension is

$$\eta_{q,k} = -\frac{\partial_t Z_{q,k}}{Z_{q,k}} \tag{23}$$

The frequency and spatial momentum are independent when we calculating the anomalous dimensions. So we can easily obtain the longitudinal component of the meson anomalous dimension with the same derivation method of the transversal component. The analytic form of the transversal component is consistentwith the anomalous dimension under $Z_{\phi,k}^{\perp} = Z_{\phi,k}^{\parallel}$. The calculation of anomalous dimensions is ultimately the calculation of the flow of the wave-function renormalizations. We can use the Wetterich equation which is mentioned in Eq 12.

Here we consider two kinds of expansion point of the effective potential. On one hand, the Taylor expansion of the effective potential is about a field value κ which is unrenormalised and fixed. The effective potential can be written as

$$\bar{V}_k(\bar{\rho}) = \sum_{n=0}^{N_v} \frac{\bar{\lambda}_{n,k}}{n!}(\bar{\rho} - \bar{\kappa}_k)^n \tag{24}$$

with $\bar{\lambda}_{n,k} = \lambda_{n,k}/Z_{\phi,k}^{\perp n}$ and $\bar{\kappa}_k = Z_{\phi,k}^{\perp}\kappa$. The other hand, we make the expansion point κ depends on the cutoff scales k. Then the $\bar{\kappa}_k$ can be written as

$$\bar{\kappa}_k = Z_{\phi,k}^{\perp}\kappa_k \tag{25}$$

Then we give the flow equation of the derivation of the effective potential

$$\begin{aligned} &\partial_{\bar{\rho}}^n(\partial_t\big|_{\rho}\bar{V}_k(\bar{\rho}))\Big|_{\bar{\rho}=\bar{\kappa}_k} \\ &=(\partial_t\bar{\lambda}_{n,k} - n\eta_{\phi,k}\bar{\lambda}_{n,k}) - (\partial_t\bar{\kappa}_k + \eta_{\phi,k}\bar{\kappa}_k)\bar{\lambda}_{n+1,k} \end{aligned} \tag{26}$$

The definition of the pion decay constant is $f_\pi = \langle\sigma\rangle$ which is equal to the expected value of the σ field.

The Yukawa coupling h_k is the coupling coefficient of the quark and meson. Here we also consider the $\rho-$dependent h and calculate its impact on the decouple speed of the meson mass. Similar to the expansion of the effective potential, the Yukawa coupling can be expanded at the point κ_k. So the h_k can be written as

$$\bar{h}_k(\bar{\rho}) = \sum_{n=0}^{N_h} \frac{\bar{h}_{n,k}}{n!}(\bar{\rho} - \bar{\kappa}_k)^n \tag{27}$$

We only study the impact of the expansion of h under the $Z_{\phi,k}^{\parallel} = Z_{\phi,k}^{\perp}$, so the definition of the $\bar{h}_k(\bar{\rho})$ and the expansion coefficients is

$$\bar{h}_k(\bar{\rho}) = \frac{h_k(\rho)}{Z_{q,k}Z_{\phi,k}^{1/2}} \tag{28}$$

$$\bar{h}_{n,k} = \frac{h_{n,k}}{Z_{q,k} Z_{\phi,k}^{(2n+1)/2}} \tag{29}$$

The expression of the Yukawa coupling is

$$\begin{aligned}
\partial_t \bar{h}_k(\bar{\rho}) = &(\frac{1}{2}\eta_{\phi,k} + \eta_{q,k})\bar{h}_k(\bar{\rho}) \\
&+ 8v_3 \bar{h}_k^3(\bar{\rho}) \Big[L_{(1,1)}^{(4)}(\bar{m}_{q,k}^2, \bar{m}_{\sigma,k}^2, \eta_{q,k}, \eta_{\phi,k}; T, \mu) \\
&- (N_f^2 - 1) L_{(1,1)}^{(4)}(\bar{m}_{q,k}^2, \bar{m}_{\pi,k}^2, \eta_{q,k}, \eta_{\phi,k}; T, \mu) \Big] \\
&+ 4v_3 \bar{h}_k(\bar{\rho}) \bar{h}_k'(\bar{\rho}) \bar{\rho} \Big[\bar{h}_k(\bar{\rho}) + \bar{\rho} \bar{h}_k'(\bar{\rho}) \Big] \\
&\times L_{(1,1)}^{(4)}(\bar{m}_{q,k}^2, \bar{m}_{\sigma,k}^2, \eta_{q,k}, \eta_{\phi,k}; T, \mu) \\
&- 2v_3 k^2 \Big[(3\bar{h}_k'(\bar{\rho}) + 2\bar{\rho}\bar{h}_k''(\bar{\rho}) l_1^{(B,4)}(\bar{m}_{\sigma,k}^2, \eta_{\phi,k}; T) \\
&+ 3\bar{h}_k'(\bar{\rho}) l_1^{(B,4)}(\bar{m}_{\pi,k}^2, \eta_{\phi,k}; T)) \Big],
\end{aligned} \tag{30}$$

Similar to the effective potential, the flow equation of the derivative Yukawa coupling is given by

$$\begin{aligned}
&\partial_{\bar{\rho}}^n (\partial_t \big|_\rho \bar{h}_k(\bar{\rho})) \Big|_{\bar{\rho} = \bar{\kappa}_k} \\
=&(\partial_t \bar{h}_{n,k} - n\eta_{\phi,k} \bar{h}_{n,k}) - (\partial_t \bar{\kappa}_k + \eta_{\phi,k} \bar{\kappa}_k) \bar{h}_{n+1,k}
\end{aligned} \tag{31}$$

B. Baryon number fluctuation and kurtosis

The fluctuation of the baryon number is related to some observables. The kurtosis of the baryon number fluctuation at different collision energy can be observed in the experiment like STAR. So the calculation of the baryon number fluctuation is important work. The definition of the each order baryon number fluctuation can be written as

$$\chi_n^B = \frac{\partial^n}{\partial(\mu_B/T)^n} \frac{p}{T^4} \tag{32}$$

The baryon chemical potential satisfying the relationship of $\mu_B = 3\mu$. The quadratic and quartic fluctuations can be expressed using the average baryon number fluctuation $\langle N_B \rangle$

$$\begin{aligned}
\chi_2^B &= \frac{1}{VT^3} \langle \delta {N_B}^2 \rangle \\
\chi_4^B &= \frac{1}{VT^3} \left(\langle \delta {N_B}^4 \rangle - 3 \langle \delta {N_B}^2 \rangle^2 \right)
\end{aligned} \tag{33}$$

The definition of the kurtosis can be written as

$$\kappa \sigma^2 = \frac{\chi_4^B}{\chi_2^B} \tag{34}$$

At the same time, we calculated the chemical freeze-out line as the function of the collision energy and the kurtosis of the baryon number fluctuation. The value of the kurtosis is determined by the temperature and the baryon chemical potential. We use the correspondence of the collision energy and chemical potential which is give in the Table I. We obtain the freeze-out temperature by the three and two order of the baryon number fluctuation χ_3^B/χ_2^B under the different baryon chemical potential which are corresponding to the different collision energy. Then we can compare the numerical results between the fix point expansion and physical point expansion of the effective potential $V_k(\rho)$ and the results of splitted meson wave function renormalisation. There are many ways to determine the freeze-out temperature.

$\sqrt{s}$ (GeV)	200	62.4	39	27	19.6	11.5	7.7
$\mu_{B,N_f=2}$ (MeV)	25.3	78.1	121	168.7	222.7	343	459.4

TABLE I. The relationship of collision energy and baryon chemical potential

IV. NUMERICAL RESULTS AND CONCLUSION

In this work, we investigated the baryon number fluctuations and some thermodynamic quantities with in the functional renormalization group. We made a comparison of the results between the approximation of the mesonic wave function renormalizations and without the approximation. The calculation are accomplished under the fix point and physical point expansion of the effective potential.Now we give the initial conditions of our calculation. The initial UV scale is $\Lambda = 700$MeV. At $k = \Lambda$ the initial effective potential can be written as

$$V_\Lambda(\rho) = \frac{\lambda_\Lambda}{2}\rho^2 + \nu_\Lambda \rho \tag{35}$$

And the parameter we chosen in the equation above is shown in the table below These four parameters are

	λ_Λ	ν_Λ	c	h
$\kappa\ Z_\phi^\parallel = Z_\phi^\perp\ h_k$	10.15	$8.11(GeV^2)$	$0.2568(GeV^3)$	7.274
$\kappa\ Z_\phi^\parallel \neq Z_\phi^\perp\ h_k$	10.15	$8.17(GeV^2)$	$0.2568(GeV^3)$	7.274
$\kappa_k\ Z_\phi^\parallel = Z_\phi^\perp\ h_k$	5.00	$17.86(GeV^2)$	$0.3696(GeV^3)$	10.110
$\kappa_k\ Z_\phi^\parallel \neq Z_\phi^\perp\ h_k$	11.00	$17.91(GeV^2)$	$0.3684(GeV^3)$	10.198

TABLE II. The setting of the initial condition parameters.

chosen by fitting the observables: $m_\pi = 135.9$MeV, $m_\sigma = 460.0$MeV, $f_\pi = 92.1$MeV, $m_q = 298.8$MeV.

In the end, we give the numerical calculation results of the physical quantity. In this section we will show the results of the comparison in the two flavor PQM model. Two comparisons have been made, one is $Z_\phi^\parallel = Z_\phi^\perp$ and $Z_\phi^\parallel \neq Z_\phi^\perp$, the other one is the fix point and physical point expansion of the effective potential.

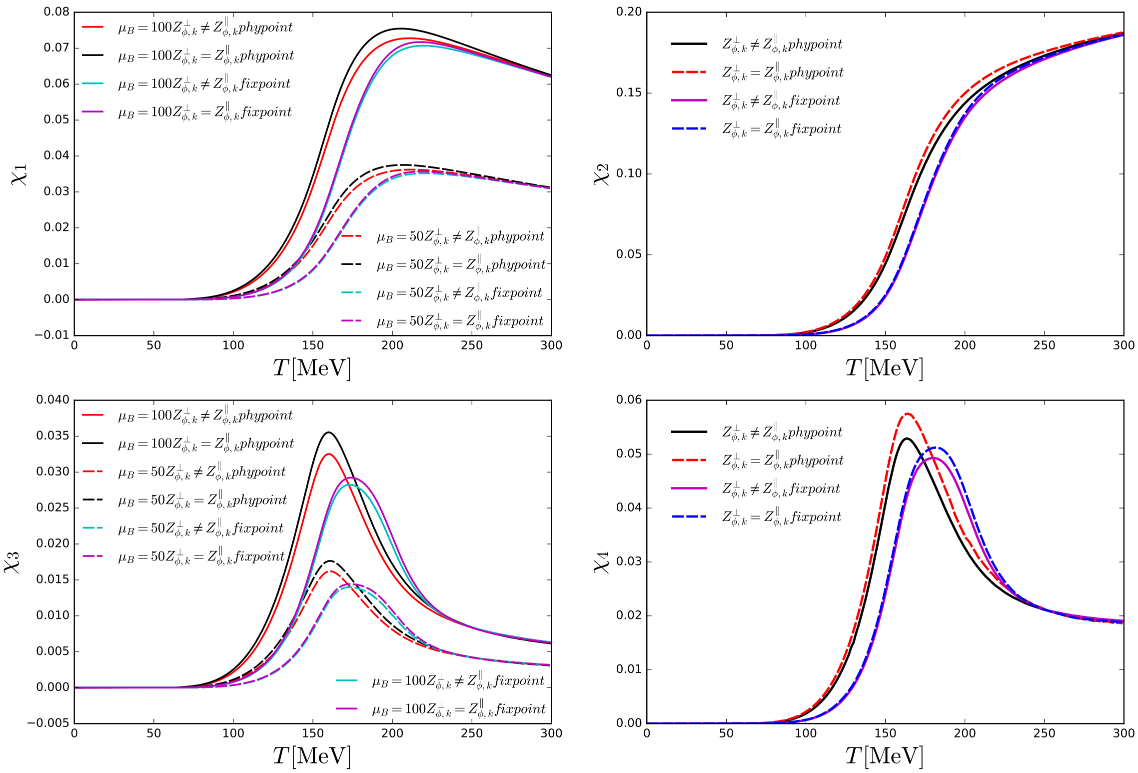

FIG. 2. The first to the fourth order of the baryon number fluctuation, which are obtained under $Z^{\parallel}_{\phi} = Z^{\perp}_{\phi}$, $Z^{\parallel}_{\phi} \neq Z^{\perp}_{\phi}$ and fix point, physical point expansion of the effective potential.

As we can see in the Fig. II, the pion mass and constituent quark mass as the function of temperature. It is clear that the pion mass under the physical point is larger at the high temperature, which means the pion is decoupling quicker. For the meson mass is related to the expansion of the effective potential, the expansion point is running with the cutoff scale k or not will of course influence the value of the pion mass. However, the split of the meson wave function renormalisation make little effect on the mass.

As is shown in the Fig. III B, because the baryon chemical potential has little influence to the numerical result of the χ_2^B and χ_4^B, however the χ_1^B and χ_3^B are more sensitive to the chemical potential. Consequently, we give the odd order of the fluctuation under the chemical potential of 50 MeV and 100 MeV, and the even order under vanishing chemical potential. The results of the quartic fluctuations under two kinds of wave function renormalizations have larger difference at the peak of the curves, the results of $Z^{\perp}_{\phi,k} \neq Z^{\parallel}_{\phi,k}$ are little lower than the results of $Z^{\perp}_{\phi,k} = Z^{\parallel}_{\phi,k}$.

In the Fig. 3 we show $\kappa\sigma^2 = \chi_4^B/\chi_2^B$ the kurtosis of the baryon number fluctuation. We can tell from the curves, the two kinds of Z_ϕ almost no effect on the kurtosis results. On the other hand, the influence of the running expansion point of the effective potential is greater. The physical point expansion suppress the kurtosis quicker as the temperature rises than the fix point expansion. It is expected from the results of the four order of the fluctuation.

In the Fig. IV we give the pion decay constant and trace anomaly as a function of temperature. The left one is the pion decay constant. It is clear, the separation of the meson wave function renormalisation doesn't make much change. However, the running expansion point of the effective potential brings some differences at high temperature. The trace anomaly doesn't change much either under the different wave function renormalisation, and the physical point expansion suppress the value at high temperature.

ACKNOWLEDGMENTS

Thanks

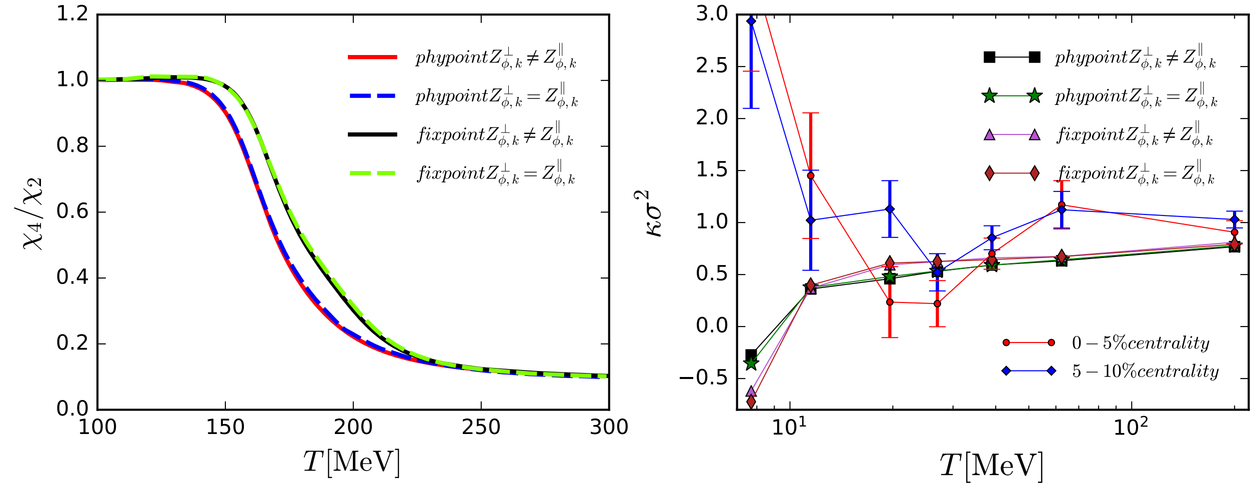

FIG. 3. The left curves are the kurtosis as the function of temperature. The right broken lines are the kurtosis as the function of collision energy.

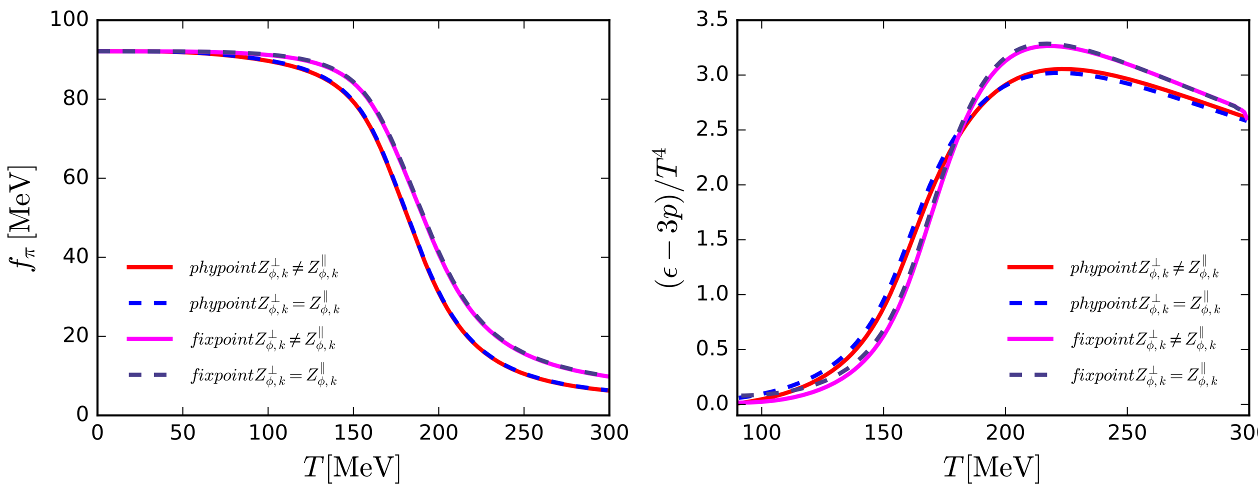

FIG. 4.

Appendix A: Anomalous dimensions

Inserting the effective action in Eq. (1) and its the flow equation in Eq. (2) into Eq. (A1), and employing the $3d$- regulators in Eq. (B1) and Eq. (B2), one obtains the transversal anomalous dimension for the meson, as follows

$$\eta^{\perp}_{\phi,k} = \frac{1}{6\pi^2}\Bigg\{\frac{4}{k^2 z_\phi^4}\bar{\kappa}_k(\bar{V}''_k(\bar{\kappa}_k))^2\mathcal{BB}_{(2,2)}(\bar{m}^2_{\pi,k},\bar{m}^2_{\sigma,k};T) + N_c\bar{h}_k^2\Big[\mathcal{F}_{(2)}(\bar{m}^2_{q,k};T,\mu)(2\eta_{q,k}-3) - 4(\eta_{q,k}-2)\mathcal{F}_{(3)}(\bar{m}^2_{q,k};T,\mu)\Big]\Bigg\}, \tag{A1}$$

and relevant threshold functions are given in Appendix B. In the same way, the longitudinal anomalous dimension for the meson in Eq. (A2) reads

$$\eta^{\parallel}_{\phi,k} = \frac{1}{6\pi^2}\Bigg\{\frac{4}{k^2 z_\phi^4}\bar{\kappa}_k(\bar{V}_k''(\bar{\kappa}_k))^2\Big[-6\mathcal{BB}_{(2,2)}(\bar{m}^2_{\pi,k},\bar{m}^2_{\sigma,k};T)$$
$$+\frac{4}{z_\phi}(1+\bar{m}^2_\sigma)\mathcal{BB}_{(2,3)}(\bar{m}^2_{\pi,k},\bar{m}^2_{\sigma,k};T)$$
$$+\frac{4}{z_\phi}(1+\bar{m}^2_\pi)\mathcal{BB}_{(3,2)}(\bar{m}^2_{\pi,k},\bar{m}^2_{\sigma,k};T)\Big]$$
$$\times(1-\frac{1}{5}\eta^{\perp}_{\phi,k})+\frac{N_c\bar{h}_k^2}{z_\phi}\mathcal{F}_{(3)}(\bar{m}^2_{q,k};T,\mu)(4-\eta_{q,k})\Bigg\}, \tag{A2}$$

The anomalous dimension for the quark in Eq. (10) is given by

$$\eta_{q,k} = \frac{1}{24\pi^2 N_f}(4-\eta^{\perp}_{\phi,k})\bar{h}_k^2$$
$$\times\Big\{(N_f^2-1)\mathcal{FB}_{(1,2)}(\bar{m}^2_{q,k},\bar{m}^2_{\pi,k};T,\mu,p_{0,ex})$$
$$+\mathcal{FB}_{(1,2)}(\bar{m}^2_{q,k},\bar{m}^2_{\sigma,k};T,\mu,p_{0,ex})\Big\}, \tag{A3}$$

where $p_{0,ex}=\pi T$ for the finite temperature part and $\big[k^2+(\pi T)^2\exp\{-2k/(5T)\}\big]^{1/2}$ for the vacuum part in the threshold function $\mathcal{FB}$'s in Eq. (B22). Note that the modification of the lowest-order Matsubara frequency in the vacuum part is employed to suppress the artificial temperature dependence of thermodynamics in the low temperature regime, see e.g. [3, 6] for more discussions.

Appendix B: Threshold functions

In this work we use the $3d$- flat or Litim regulators [7, 8], which are very suited for the computations at finite temperature and densities, since the summation for the Matsubara frequencies is not affected by the $3d$ regulators, and can be performed analytically. The regulators in Eq. (2) read

$$R^\phi_k(q_0,\boldsymbol{q}) = Z^{\perp}_{\phi,k}\boldsymbol{q}^2 r_B(\boldsymbol{q}^2/k^2), \tag{B1}$$
$$R^q_k(q_0,\boldsymbol{q}) = Z_{q,k}i\boldsymbol{\gamma}\cdot\boldsymbol{q}r_F(\boldsymbol{q}^2/k^2), \tag{B2}$$

with

$$r_B(x) = \left(\frac{1}{x}-1\right)\Theta(1-x), \tag{B3}$$
$$r_F(x) = \left(\frac{1}{\sqrt{x}}-1\right)\Theta(1-x), \tag{B4}$$

where $\Theta(x)$ is the Heaviside step function. Note that since $Z^{\perp}_{\phi,k}\neq Z^{\parallel}_{\phi,k}$, it is the transversal wave function renormalization for the meson appearing in Eq. (B1).

In the threshold functions we usually need the dimensionless meson and quark propagators as follows

$$G_\phi(q,\bar{m}^2_{\phi,k}) = \frac{1}{z_\phi\tilde{q}_0^2+1+\bar{m}^2_{\phi,k}}, \tag{B5}$$
$$G_q(q,\bar{m}^2_{q,k}) = \frac{1}{(\tilde{q}_0+i\tilde{\mu})^2+1+\bar{m}^2_{q,k}}, \tag{B6}$$

with $\tilde{\mu}=\mu/k$ and $\tilde{q}_0=q_0/k$, where the Matsubara frequency is $q_0=2n_q\pi T$ for bosons and $(2n_q+1)\pi T$ for fermions with $n_q\in\mathbb{Z}$.

The definition of the threshold functions $\mathcal{B}_{(n)}$ and $\mathcal{F}_{(n)}$ is given by

$$\mathcal{B}_{(n)}(\bar{m}^2_{\phi,k};T) = \frac{T}{k}\sum_{n_q}\Big(G_\phi(q,\bar{m}^2_{\phi,k})\Big)^n, \tag{B7}$$
$$\mathcal{F}_{(n)}(\bar{m}^2_{q,k};T,\mu) = \frac{T}{k}\sum_{n_q}\Big(G_q(q,\bar{m}^2_{q,k})\Big)^n. \tag{B8}$$

After performing the Matsubara sum, one arrives at

$$\mathcal{B}_{(1)}(\bar{m}^2_{\phi,k};T) = \frac{1}{\sqrt{z_\phi(1+\bar{m}^2_{\phi,k})}}\left(\frac{1}{2}+n_B(\bar{m}^2_{\phi,k},z_\phi;T)\right), \tag{B9}$$

and

$$\mathcal{F}_{(1)}(\bar{m}^2_{q,k};T,\mu) = \frac{1}{2\sqrt{1+\bar{m}^2_{q,k}}}$$
$$\times\Big(1-n_F(\bar{m}^2_{q,k};T,\mu)-n_F(\bar{m}^2_{q,k};T,-\mu)\Big), \tag{B10}$$

with the bosonic and fermionic distribution functions being

$$n_B(\bar{m}^2_{\phi,k},z_\phi;T) = \frac{1}{\exp\Big\{\frac{1}{T}\frac{k}{z_\phi^{1/2}}\sqrt{1+\bar{m}^2_{\phi,k}}\Big\}-1}, \tag{B11}$$

and

$$n_F(\bar{m}^2_{q,k};T,\mu) = \frac{1}{\exp\Big\{\frac{1}{T}\Big[k\sqrt{1+\bar{m}^2_{q,k}}-\mu\Big]\Big\}+1}. \tag{B12}$$

Note that when the Polaykov loop is taken into account, the fermionic distribution function is modified as follows

$$n_F(\bar{m}^2_{q,k};T,\mu,L,\bar{L}) = \frac{1+2\bar{L}e^{x/T}+Le^{2x/T}}{1+3\bar{L}e^{x/T}+3Le^{2x/T}+e^{3x/T}}, \tag{B13}$$

with $x = k\sqrt{1+\bar{m}_{q,k}^2} - \mu$, and $n_F(\bar{m}_{q,k}^2;T,-\mu)$ in Eq. (B10) is replaced with $n_F(\bar{m}_{q,k}^2;T,-\mu,\bar{L},L)$ accordingly. With Eq. (B9) and Eq. (B10), high-order threshold functions in Eqs. (B7) and (B8) are readily obtained as

$$\mathcal{B}_{(n+1)}(\bar{m}_{\phi,k}^2;T) = -\frac{1}{n}\frac{\partial}{\partial \bar{m}_{\phi,k}^2}\mathcal{B}_{(n)}(\bar{m}_{\phi,k}^2;T)\,, \qquad \text{(B14)}$$

$$\mathcal{F}_{(n+1)}(\bar{m}_{q,k}^2;T,\mu) = -\frac{1}{n}\frac{\partial}{\partial \bar{m}_{q,k}^2}\mathcal{F}_{(n)}(\bar{m}_{q,k}^2;T,\mu)\,. \qquad \text{(B15)}$$

The threshold functions in the flow of effective potential in Eq. (4) are given by

$$\begin{aligned} & l_0^{(B,d)}(\bar{m}_{\phi,k}^2,\eta_{\phi,k}^{\perp},z_\phi;T) \\ =& \frac{2}{d-1}\left(1-\frac{\eta_{\phi,k}^{\perp}}{d+1}\right)\mathcal{B}_{(1)}(\bar{m}_{\phi,k}^2,z_\phi;T)\,, \end{aligned} \qquad \text{(B16)}$$

and

$$\begin{aligned} & l_0^{(F,d)}(\bar{m}_{q,k}^2,\eta_{q,k};T,\mu) \\ =& \frac{2}{d-1}\left(1-\frac{\eta_{q,k}}{d}\right)\mathcal{F}_{(1)}(\bar{m}_{q,k}^2;T,\mu)\,, \end{aligned} \qquad \text{(B17)}$$

Furthermore, we also need other threshold functions, such as

$$\begin{aligned} & \mathcal{BB}_{(n_1,n_2)}(m_1^2,m_2^2;T) \\ =& \frac{T}{k}\sum_{n_q}\Big(G_\phi(q,\bar{m}_{\phi_a,k}^2)\Big)^{n_1}\Big(G_\phi(q,\bar{m}_{\phi_b,k}^2)\Big)^{n_2}\,. \end{aligned} \qquad \text{(B18)}$$

in the expressions of the mesonic anomalous dimension in Eqs. (A1) and (A2). Inserting Eq. (B5) into Eq. (B18), one is led to

$$\begin{aligned} & \mathcal{BB}_{(1,1)}(\bar{m}_{\phi_a,k}^2,\bar{m}_{\phi_b,k}^2;T) \\ =& -\frac{1}{z_\phi^{1/2}}\Bigg\{\left(\frac{1}{2}+n_B(\bar{m}_{\phi_a,k}^2,z_\phi;T)\right)\frac{1}{(1+\bar{m}_{\phi_a,k}^2)^{1/2}} \\ & \times\frac{1}{\bar{m}_{\phi_a,k}^2-\bar{m}_{\phi_b,k}^2}+\left(\frac{1}{2}+n_B(\bar{m}_{\phi_b,k}^2,z_\phi;T)\right) \\ & \times\frac{1}{(1+\bar{m}_{\phi_b,k}^2)^{1/2}}\frac{1}{\bar{m}_{\phi_b,k}^2-\bar{m}_{\phi_a,k}^2}\Bigg\}\,. \end{aligned} \qquad \text{(B19)}$$

In the same way, one could obtain higher-order ones by employing, e.g.,

$$\begin{aligned} & \mathcal{BB}_{(n_1+1,n_2)}(\bar{m}_{\phi_a,k}^2,\bar{m}_{\phi_b,k}^2;T) \\ =& -\frac{1}{n_1}\frac{\partial}{\partial \bar{m}_{\phi_a,k}^2}\mathcal{BB}_{(n_1,n_2)}(\bar{m}_{\phi_a,k}^2,\bar{m}_{\phi_b,k}^2;T)\,. \end{aligned} \qquad \text{(B20)}$$

In the flow of the Yukawa coupling in (30), the threshold function L is introduced, which reads

$$\begin{aligned} & L_{(1,1)}^{(4)}(\bar{m}_{q,k}^2,\bar{m}_{\phi,k}^2,\eta_{q,k},\eta_{\phi,k};T,\mu,p_0) \\ =& \frac{2}{3}\Bigg[\left(1-\frac{\eta_{\phi,k}^{\perp}}{5}\right)\mathcal{FB}_{(1,2)}(\bar{m}_{q,k}^2,\bar{m}_{\phi,k}^2;T,\mu,p_0) \\ & +\left(1-\frac{\eta_{q,k}}{4}\right)\mathcal{FB}_{(2,1)}(\bar{m}_{q,k}^2,\bar{m}_{\phi,k}^2;T,\mu,p_0)\Bigg]\,, \end{aligned} \qquad \text{(B21)}$$

where the fermionic and bosonic mixing threshold functions $\mathcal{FB}$'s are given by

$$\begin{aligned} & \mathcal{FB}_{(n_f,n_b)}(\bar{m}_{q,k}^2,\bar{m}_{\phi,k}^2;T,\mu,p_0) \\ =& \frac{T}{k}\sum_{n_q}\Big(G_q(q,\bar{m}_{q,k}^2)\Big)^{n_f}\Big(G_\phi(q-p,\bar{m}_{\phi,k}^2)\Big)^{n_b}\,. \end{aligned} \qquad \text{(B22)}$$

The explicit expression for $\mathcal{FB}_{(1,1)}$ is readily obtained after summing the Matsubara frequency, which can be found in, e.g. [3]. Higher-order $\mathcal{FB}$'s are obtained from $\mathcal{FB}_{(1,1)}$ by performing derivatives w.r.t. relevant masses as same as other threshold functions mentioned above.

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