FRG note

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Contents

1 Wetterich Equation

Consider the generating function which is cutoff at infrared:

$$Z_k[J] = \exp(W_k[J]) = \int [d\phi] \exp\{-S[\phi] - \Delta S_k[\phi]] + J^a \phi_a\} \tag{1}$$

The ϕ stands for every kind of field here. Index a stands for every degree of freedom, including different fields, different components of a field. Such as the space-time coordinates or momentum index. $S[\phi]$ is the classical action, J^a is the source of ϕ_a. $\Delta S_k[\phi]$ is the infrared cutoff, its function is to cutoff the quantum fluctuation at $p^2 \leq k^2$ and keep the fluctuation at $p^2 > k^2$ invariant. We usually choose the form of quadratic term (mass term) to achieve the infrared cutoff.

$$\Delta S_k[\phi] = \frac{1}{2}\phi_a R_k^{ab} \phi_b \tag{2}$$

with $R_k^{ab} = R_k^{ba}$ (a, b is the index of the boson), $R_k^{ab} = -R_k^{ba}$ (a, b is the index of the fermion). Here we give a example of a simple scalar field. In the coordinate space:

$$\Delta S_k[\phi] = \frac{1}{2}\int d^4x d^4y \varphi(x) R_k(x,y) \varphi(y) \tag{3}$$

So

$$\begin{aligned} R_k(x,y) &= \int \frac{d^4p}{(2\pi)^4} R_k(p,-p) e^{ip(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4} R_k(p) e^{ip(x-y)} \end{aligned} \tag{4}$$

$$\varphi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \varphi(p) \tag{5}$$

$$\begin{aligned} \Delta S_k[\varphi] =& \frac{1}{2}\int d^4x d^4y \varphi(x) R_k(x,y) \varphi(y) \\ =& \frac{1}{2}\int d^4x d^4y \int \frac{d^4p_1}{(2\pi)^4} e^{ip_1x}\varphi(p_1) \int \frac{d^4p}{(2\pi)^4} R_k(p) e^{ip(x-y)} \int \frac{d^4p_2}{(2\pi)^4} e^{ip_2y} \varphi(p_2) \\ =& \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \varphi(-p) R_k(p) \varphi(p) \\ =& \frac{1}{2}\int \frac{d^4q}{(2\pi)^4} \varphi(-q) R_k(q) \varphi(q) \end{aligned} \tag{6}$$

For the determined q , the regulator $R_k(q)$ satisfies the following properties

$$R_{k\to\infty}(q) \to \infty \quad R_{k\to 0}(q) \to 0 \tag{7}$$

To suppress the fluctuation at $q^2 < k^2$ and keep the fluctuation at high momentum unchanging, we should choose

$$R_k(q)|_{q^2<k^2} \sim k^2, \quad R_k(q)|_{q^2>k^2} \sim 0 \tag{8}$$

for example

$$R_k(q) \sim \frac{q^2}{e^{\frac{q^2}{k^2}} - 1} \tag{9}$$

of course, we can choose other forms of the regulator. From (1) we can get

$$\begin{aligned}\frac{\delta W_k[J]}{\delta J^a} =& \frac{1}{Z_k}\frac{\delta Z_k[J]}{\delta J^a} \\ =& \frac{1}{Z_k}\int [d\phi]\phi_a \exp\{-S[\phi] - \Delta S_k[\phi] + J^a\phi_a\} \\ =& \langle\phi_a\rangle\end{aligned} \tag{10}$$

In the below discussing, we replace $\langle\phi_a\rangle$ with ϕ_a and

$$\frac{\delta^2 W_k[J]}{\delta J^b \delta J^a} = \langle\phi_b\phi_a\rangle_c \equiv G^k_{ba} \tag{11}$$

index c stands for the connected diagram, G is the propagator that depends on the scale k. Now we do the Legendre transformation on the generated functional of connected diagram, then we can obtain the generated functional of one-particle irreducible diagram, that is the effective action

$$\Gamma_k[\phi] = -W_k[J] + J^a\phi_a - \Delta S_k[\phi] \tag{12}$$

Beware here $\phi_a \equiv \langle\phi_k\rangle$. In order to consider the boson and fermion together, we introduce the following symbols

$$J^a\phi_a = r^a_b\phi_a J^b \tag{13}$$

with

$$r^a_b = (-1)^{ab}\delta^a_b \tag{14}$$

$$(-1)^{ab} \equiv \begin{cases} -1, & for \quad a,b \quad Fermionic \\ 1, & for \quad a,b \quad Bosonic \end{cases} \tag{15}$$

So from (12) we obtain

$$\frac{\delta(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta\phi_a} = r^a_b J^b \tag{16}$$

Differentiate J on both sides of the above formula

$$\frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta J^b \delta\phi_a} = r^a_b \tag{17}$$

$$\begin{aligned} l.h.s =& \frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta J^b \delta\phi_a} \\ =& \frac{\delta\phi_c}{\delta J^b}\frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta\phi_c\delta\phi_a} \\ =& G^k_{bc}\frac{\Gamma_k[\phi] + \Delta S_k[\phi]}{\delta\phi_c\delta\phi_a}\end{aligned} \tag{18}$$

We obtain

$$G^k_{bc}\frac{\Gamma_k[\phi]+\Delta S_k[\phi]}{\delta\phi_c\delta\phi_a}=r^a_b \tag{19}$$

$$\frac{\Gamma_k[\phi]+\Delta S_k[\phi]}{\delta\phi_c\delta\phi_a}=(\Gamma^{(2)}_k[\phi]+R_k)^{ca} \tag{20}$$

then

$$G^k_{bc}(\Gamma^{(2)}_k[\phi]+R_k)^{ca}=r^a_b G^k_{bc}=r^a_b(\Gamma^{(2)}_k[\phi]+R_k)^{-1}_{ac} \tag{21}$$

Now we calculate $\partial_t W_k[J]=k\frac{\partial}{\partial k}W_k[J]$

$$\begin{aligned}\partial_t W_k[J]=&\partial_t lnZ_k[J]\\ =&\frac{1}{Z_k}\partial_t Z_k[J]\\ =&\frac{1}{Z_k}\int[d\phi](-\partial_t\Delta S_k[\phi])e^{-S[\phi]-\Delta S_k[\phi]+J^a\phi_a}\\ =&-\frac{1}{Z_k}\int[d\phi]\frac{1}{2}\phi_a\partial_t R^{ab}_k\phi_b e^{-S[\phi]-\Delta S_k[\phi]+J^a\phi_a}\\ =&-\frac{1}{2}\langle\phi_a\phi_b\rangle\partial_t R^{ab}_k\\ =&-\frac{1}{2}(\langle\phi_a\phi_b\rangle_c+\langle\phi_a\rangle\langle\phi_b\rangle)\partial_t R^{ab}_k\end{aligned} \tag{22}$$

we replace $\langle\phi_a\rangle$ with ϕ_a again

$$\langle\phi_a\phi_b\rangle_c=\begin{cases}\langle\phi_b\phi_a\rangle, & for \quad a,b \quad Bosonic\\ -\langle\phi_b\phi_a\rangle, & for \quad a,b \quad Fermionic\end{cases} \tag{23}$$

Then we obtain

$$\partial_t W_k[J]=-\frac{1}{2}STrG_k(\partial_t R_k)-\frac{1}{2}\phi_a\partial_t R^{ab}_k\phi_b \tag{24}$$

STr is super trace, including the trace of every field and dispersed, continuously degree of freedom. For the fermion a negative sign should be added. Finally, from (12) and (24) we obtain

$$\begin{aligned}\partial_t\Gamma_k[\phi]=&-\partial_t W_k[J]-\partial_t\Delta S_k[\phi]\\ =&\frac{1}{2}\phi_a\partial_t R^{ab}_k\phi_b-\frac{1}{2}\phi_a\partial_t R^{ab}_k\phi_b\\ =&\frac{1}{2}STrG_k(\partial_t R_k)\end{aligned} \tag{25}$$

This is the Wetterich equation.

2 QCD

In this section, we introduce the application of FRG in QCD. We start from the effective action that depends on the infrared cutoff scale k

$$\begin{aligned}\Gamma_k = \int d^4x\{ &\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + Z_{c,k}(\partial_\mu \bar{c}^a)D^{ab}_\mu c^b + \frac{1}{2\xi}(\partial_\mu A^a_\mu) \\ &+ Z_{q,k}\bar{q}(\gamma_\mu D_\mu)q - \lambda_{q,k}[(\bar{q}T^0 q)^2 - (\bar{q}\gamma_5 T^a q)^2] \\ &+ h_k[\bar{q}(i\gamma_5 T^a \pi^a + T^0\sigma)q] + \frac{1}{2}Z_{\phi,k}(\partial_\mu \phi)^2\}\end{aligned} \tag{26}$$

In the effective action

$$\begin{aligned}&\phi = (\sigma, \pi^a), \rho = \frac{1}{2}\phi^2 \\ &D_\mu = \partial_\mu - iZ^{\frac{1}{2}}_{A,k} g_k A^a_\mu t^a \\ &D^{ab}_\mu = \partial_\mu \delta^{ab} + Z^{\frac{1}{2}}_{A,k} g_k f^{acb} A^c_\mu\end{aligned} \tag{27}$$

In the definition above $T^a(a = 1,2,...,N_f^2 - 1)$ is the generator of flavor space with $T^0 = \frac{1}{\sqrt{2N_f}}\mathbb{1}$; $t^a(a = 1,2,...,N_c^2 - 1)$ is the generator of color space.

$$\begin{aligned}&F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = F^a_{\mu\nu}t^a \\ &F^a_{\mu\nu} = Z^{\frac{1}{2}}_{A,k}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + Z^{\frac{1}{2}}_{A,k} g_k f^{abc} A^b_\mu A^c_\nu)\end{aligned} \tag{28}$$

We rewrite the Wetterich equation as below

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} STr \widetilde{\partial}_t ln(\Gamma^{(2)}_k + R_k) \tag{29}$$

the tilde on the ∂_t stands for the derivation only works on the regulator R_k.

$$(\Gamma^{(2)}_k)_{ab} = \frac{\overrightarrow{\delta}}{\delta \Phi^T_a}\Gamma_k \frac{\overleftarrow{\delta}}{\delta \Phi_b} \tag{30}$$

the definition of Φ is

$$\begin{aligned}&\Phi = \begin{pmatrix} A(q) \\ \sigma(q) \\ \pi(q) \\ q(q) \\ \bar{q}^T(q) \end{pmatrix} \\ &\Phi^T = \Big(A^T(-q), \sigma(-q), \pi(-q), q^T(q), \bar{q}(q)\Big)\end{aligned} \tag{31}$$

the fluctuation matrix in the (29) can be rewrite as

$$\Gamma^{(2)}_k + R_k = P + F \tag{32}$$

The matrix P contains the propagators and regulators ; matrix F contains the dependence of all kind of fields. So we can expanse the (29) with the number of the fields

$$\begin{aligned}\partial_t\Gamma_k[\Phi] =&\frac{1}{2}STr\widetilde{\partial}_t ln(P+F)\\ =&\frac{1}{2}STr\widetilde{\partial}_t ln[P(1+\frac{1}{P}F)]\\ \sim&\frac{1}{2}STr\widetilde{\partial}_t ln(1+\frac{1}{P}F)\\ =&\frac{1}{2}STr\left\{\widetilde{\partial}_t[\frac{1}{P}F-\frac{1}{2}(\frac{1}{P}F)^2+\frac{1}{3}(\frac{1}{P}F)^3-\frac{1}{4}(\frac{1}{P}F)^4+\cdots]\right\}\\ =&\frac{1}{2}STr\widetilde{\partial}_t(\frac{1}{P}F)-\frac{1}{4}STr\widetilde{\partial}_t(\frac{1}{P}F)^2+\frac{1}{6}STr\widetilde{\partial}_t(\frac{1}{P}F)^3-\frac{1}{8}STr\widetilde{\partial}_t(\frac{1}{P}F)^4+\cdots\end{aligned}\tag{33}$$

then we take the contribution of the lowest order in

$$\begin{aligned}\partial_t\Gamma_k[\Phi] =&\frac{1}{2}STr\widetilde{\partial}_t ln(P+F)\\ =&\frac{1}{2}STr\widetilde{\partial}_t ln[P(1+\frac{1}{P}F)]\\ =&\frac{1}{2}STr\widetilde{\partial}_t[lnP+ln(1+\frac{1}{P}F)]\\ =&\frac{1}{2}STr\widetilde{\partial}_t lnP+\frac{1}{2}STr\widetilde{\partial}_t ln(1+\frac{1}{P}F)\\ =&\frac{1}{2}STr\widetilde{\partial}_t lnP+\frac{1}{2}STr\widetilde{\partial}_t(\frac{1}{P}F)-\frac{1}{4}STr\widetilde{\partial}_t(\frac{1}{P}F)^2+\frac{1}{6}STr\widetilde{\partial}_t(\frac{1}{P}F)^3-\frac{1}{8}STr\widetilde{\partial}_t(\frac{1}{P}F)^4+\cdots\end{aligned}\tag{34}$$

2.1 Meson field propagator

The meson propagator part of the effective potential and its Fourier transform

$$\begin{aligned}&\int d^4x\frac{1}{2}Z_{\phi,k}(\partial_\mu\phi)^2\\ =&\int\frac{d^4q}{(2\pi)^4}\frac{1}{2}Z_{\phi,k}\phi(-q)q^2\phi(q)\end{aligned}\tag{35}$$

then consider the contribution of the meson mass

$$\begin{aligned}\Gamma_k^{\sigma\sigma}(q',q)\equiv&\frac{\delta^2\Gamma_k}{\delta\sigma(q')\delta\sigma(q)}\\ =&(Z_{\phi,k}q^2+m_\sigma^2)(2\pi)^4\delta^4(q+q')\end{aligned}\tag{36}$$

The corresponding cutoff function is

$$R_k^{\sigma\sigma}(q',q)=Z_{\phi,k}\vec{q}^2 r_B(\frac{\vec{q}^2}{k^2})(2\pi)^4\delta^4(q+q')\tag{37}$$

here we use the 3d regulator which is convenient to the calculation of the finite temperature. The $r_B(x)=(\frac{1}{x}-1)\theta(1-x)$ is an optimized regulator.

Then we treat the Pi meson in the same way

$$\begin{aligned}\Gamma_{k,ij}^{\pi\pi}(q',q)\equiv&\frac{\delta^2\Gamma_k}{\delta\pi_i(q')\delta\pi_j(q)}\\ =&\delta_{ij}(Z_{\phi,k}q^2+m_\pi^2)(2\pi)^4\delta^4(q+q')\end{aligned}\tag{38}$$

The cutoff function is

$$R_{k,ij}^{\pi\pi} = \delta_{ij} R_k^{\sigma\sigma} \tag{39}$$

In the calculation of finite temperature we have $q_0 = 2\pi nT$ in which the n is integer.

2.2 Quark field propagator

The quark propagator part and its Fourier transform

$$\begin{aligned}
&\int d^4x Z_{q,k}\overline{q}(x)(\gamma_\mu\partial_\mu + \frac{m_f}{Z_{q,k}})q(x) \\
=&\int \frac{d^4q}{(2\pi)^4} Z_{q,k}\overline{q}(q)(i\not{q} + \frac{m_f}{Z_{q,k}})q(q)
\end{aligned} \tag{40}$$

The Fourier transform of the quark and anti-quark fields

$$\begin{aligned}
q(x) &= \int \frac{d^4q}{(2\pi)^4} e^{iqx} q(q) \\
\overline{q}(x) &= \int \frac{d^4q}{(2\pi)^4} e^{-iqx} \overline{q}(q)
\end{aligned} \tag{41}$$

The derivation of the effective action amount of the quark fields is

$$\begin{aligned}
\Gamma_{k,ij}^{\overline{q}q}(q',q) &\equiv \frac{\overrightarrow{\delta}}{\delta q_i(q')} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \overline{q}_j(q)} \\
&= (Z_{q,k} i\not{q}_{ij} + m_f\delta_{ij})(2\pi)^4\delta^4(q'-q) \\
&= (Z_{q,k} iq_\mu(\gamma_\mu)_{ij} + m_f\delta_i j)(2\pi)^4\delta^4(q'-q)
\end{aligned} \tag{42}$$

$$\begin{aligned}
\Gamma_{k,ij}^{q\overline{q}} &\equiv \frac{\overrightarrow{\delta}}{\delta q_i(q')} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \overline{q}_j(q)} \\
&= -(Z_{q,k} iq_\mu(\gamma_\mu)_{ji} + m_f\delta_{ji})(2\pi)^4\delta^4(q'-q)
\end{aligned} \tag{43}$$

The corresponding regulator function is

$$R_{k,ij}^{\overline{q}q}(q',q) = Z_{q,k} i\vec{q}\cdot\vec{\gamma}_{ij} r_F(\frac{\vec{q}^2}{k^2})(2\pi)^4\delta^4(q'-q) \tag{44}$$

with

$$r_F(x) = (\frac{1}{\sqrt{x}} - 1)\theta(1-x) \tag{45}$$

2.3 Gluon field propagator

The differential form of the effective action is

$$\begin{aligned}
(\Gamma_k^{AA})_{\mu\nu}^{ab}(q',q) &\equiv \frac{\delta^2\Gamma_k}{\delta A_\mu^a(q')\delta A_\nu^b(q)} \\
&= [Z_{A,k}q^2(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{q^2}{\xi}(\frac{q_\mu q_\nu}{q^2})]\delta^{ab}(2\pi)^4\delta^4(q'+q)
\end{aligned} \tag{46}$$

The regulator function is

$$(R_k^{AA})^{ab}_{\mu\nu}(q',q) = [Z_{A,k}\vec{q}^2 r_B(\frac{\vec{q}^2}{k^2})(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{\vec{q}^2}{\xi} r_B(\frac{\vec{q}}{k^2})(\frac{q_\mu q_\nu}{q^2})]\delta^{ab}(2\pi)^4\delta^4(q'+q) \tag{47}$$

The propagator of the gluon is

$$(G_k^{AA})^{ab}_{\mu\nu}(q',q) = [\frac{1}{Z_{A,k}(q_0^2+\vec{q}^2(1+r_B))}(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{\xi}{q_0^2+\vec{q}^2(1+r_B)}\frac{q_\mu q_\nu}{q^2}]\delta^{ab}(2\pi)^4\delta^4(q'+q) \tag{48}$$

In the following calculation we adopt the Landau gauge $\xi = 0$, then we can obtain the matrix P

$P =$

$$\begin{pmatrix} Z_{A,k}(q_0^2+\vec{q}^2(1+r_B))(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2})\delta_{ab} & 0 & 0 & 0 & 0 \\ 0 & Z_{\phi,k}(q_0^2+\vec{q}^2(1+r_B))+m_\sigma^2 & 0 & 0 & 0 \\ 0 & 0 & [Z_{\phi,k}(q_0^2+\vec{q}^2(1+r_B))+m_\pi^2]\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & -(Z_{q,k}i(q_0r_0+\vec{q}\cdot\vec{r}(1+r_F))+m_f)^T \\ 0 & 0 & 0 & Z_{q,k}i(q_0r_0+\vec{q}\cdot\vec{r}(1+r_F))+m_f & 0 \end{pmatrix} \tag{49}$$

And then we can obtain the propagator matrix

$$\frac{1}{P} = \begin{pmatrix} (G_k^{AA})^{ab}_{\mu\nu} & 0 & 0 & 0 & 0 \\ 0 & G_k^\sigma & 0 & 0 & 0 \\ 0 & 0 & (G_k^\pi)_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_k^{q\bar{q}} \\ 0 & 0 & 0 & G_k^{\bar{q}q} & 0 \end{pmatrix} \tag{50}$$

the definition of the matrix elements are

$$\begin{aligned} (G_k^{AA})^{ab}_{\mu\nu} &= \frac{1}{Z_{A,k}(q_0^2+\vec{q}^2(1+r_B))}(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})\delta_{ab} \\ G_k^\sigma &= \frac{1}{Z_{\phi,k}(q_0^2+\vec{q}^2(1+r_B))+m_\sigma^2} \\ (G_k^\pi)_{ij} &= \frac{1}{Z_{\phi,k}(q_0^2+\vec{q}^2(1+r_B))+m_\pi^2}\delta_{ij} \\ G_k^{q\bar{q}} &= \frac{-Z_{q,k}i(q_0r_0+\vec{q}\cdot\vec{r}(1+r_F))+m_f}{Z_{q,k}^2(q_0^2+\vec{q}^2(1+r_F)^2)+m_f^2} \\ G_k^{\bar{q}q} &= -(G_k^{q\bar{q}})^T \end{aligned} \tag{51}$$

2.4 Gluon vertex

$$\begin{aligned} \Gamma_k &\sim \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \\ &= \frac{1}{4}Z_{A,k}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + Z_{A,k}^{\frac{1}{2}}g_k f^{abc}A^b_\mu A^c_\nu)(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + Z_{A,k}^{\frac{1}{2}}g_k f^{ab'c'}A^{b'}_\mu A^{c'}_\nu) \\ &\sim \frac{1}{2}Z_{A,k}^{\frac{3}{2}}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)g_k f^{abc}A^b_\mu A^c_\nu + \frac{1}{4}Z_{A,k}^2 g_k^2 f^{abc}f^{ab'c'}A^b_\mu A^c_\nu A^{b'}_\mu A^{c'}_\nu \end{aligned} \tag{52}$$

here we let

$$\Gamma_1 = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)f^{abc}A^b_\mu A^c_\nu \tag{53}$$

then we can calculate the derivative

$$\frac{\delta \Gamma_1}{\delta A^a_\mu} = -2(\partial_\nu)_a f^{abc} A^b_\mu A^c_\nu + 2 f^{abc} (\partial_\nu A^b_\mu - \partial_\mu A^b_\nu) A^c_\nu \tag{54}$$

$$\begin{aligned}\frac{\delta^2 \Gamma_1}{\delta A^a_\mu \delta A^b_\nu} &= -2(\partial_\rho)_a f^{abc} \delta_{\mu\nu} A^c_\rho - 2(\partial_\nu)_a f^{acb} A^c_\mu + 2 f^{abc} (\partial_\rho)_b \delta_{\mu\nu} A^c_\rho + 2 f^{acb} \partial_\nu A^c_\mu - 2 f^{abc} (\partial_\mu)_b A^c_\nu - 2 f^{acb} \partial_\mu A^c_\nu \\ &= 2\delta_{\mu\nu} f^{abc} [(\partial_\rho)_b - (\partial_\rho)_a] A^c_\rho + 2 f^{abc} [(\partial_\nu)_a A^c_\mu - (\partial_\mu)_b A^c_\nu] + 2 f^{abc} (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{\delta^3 \Gamma_1}{\delta A^a_\mu \delta A^b_\nu \delta A^c_\rho} &= 2\delta_{\mu\nu} f^{abc} [(\partial_\rho)_b - (\partial_\rho)_a] + 2 f^{abc} [(\partial_\nu)_a \delta_{\mu\rho} - (\partial_\mu)_b \delta_{\rho\nu}] + 2 f^{abc} [(\partial_\mu)_c \delta_{\rho\nu} - (\partial_\nu)_c \delta_{\mu\rho}] \\ &= 2 f^{abc} \Big\{ \delta_{\mu\nu} [(\partial_\rho)_b - (\partial_\rho)_a] + \delta_{\mu\rho} [(\partial_\nu)_a - (\partial_\nu)_c] + \delta_{\rho\nu} [(\partial_\mu)_c - (\partial_\mu)_b] \Big\}\end{aligned} \tag{56}$$

If we set

$$\begin{aligned}\Gamma_2 &= f^{abc} f^{ab'c'} A^b_\mu A^c_\nu A^{b'}_\mu A^{c'}_\nu \\ &= f^{abc} f^{a'b'c} A^a_\mu A^b_\nu A^{a'}_\mu A^{b'}_\nu\end{aligned} \tag{57}$$

Then we can obtain the derivation of the Γ_2

$$\frac{\delta \Gamma_2}{\delta A^a_\mu} = 4 f^{abc} f^{a'b'c} A^b_\nu A^{a'}_\mu A^{b'}_\nu \tag{58}$$

$$\begin{aligned}\frac{\delta^2 \Gamma_2}{\delta A^a_\mu \delta A^b_\nu} =& 4 f^{abc} f^{a'b'c} A^{a'}_\mu A^{b'}_\nu + 4 f^{ab''c} f^{ab'c} A^{b''}_\rho \delta_{\mu\nu} A^{b'}_\rho \\ &+ 4 f^{ab''c} f^{a'bc} A^{b''}_\nu A^{a'}_\mu \\ =& 4 f^{abc} f^{a'b'c} A^{a'}_\mu A^{b'}_\nu + 4\delta_{\mu\nu} f^{ab'c} f^{ab''c} A^{b'}_\rho A^{b''}_\rho - 4 f^{ab''c} f^{ba'c} A^{a'}_\mu A^{b''}_\nu\end{aligned} \tag{59}$$

So we get the F matrix element of the gluon part

A Fourier transform of n-point function

Consider a general n-point function $V(x_1, x_2, ..., x_n)$, Its Fourier transformation can be written as:

$$V(x_1, x_2, ..., x_n) = \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} V(p_1, p_2, ... p_n) e^{i(p_1 x_1 + p_2 x_2 + ... p_n x_n)} \tag{60}$$

if V satisfies the following properties:

$$V(x_1, x_2, ... x_n) = V(x_1 - x_n, x_2 - x_n, ..., 0) \tag{61}$$

The value of V depends only on the relative value of the coordinates, so we can obtain

$$\begin{aligned}
V(p_1,p_2,...p_n) =& \int d^4x_1 d^4x_2 ... d^4x_n V(x_1,x_2,...x_n) e^{-i(p_1x_1+p_2x_2+...p_nx_n)} \\
=& \int d^4x_1 d^4x_2 ... d^4x_n V(x_1-x_n,x_2-x_n,...0) e^{-i(p_1x_1+p_2x_2+...p_nx_n)} \\
=& \int d^4x_1 d^4x_2 ... d^4x_n V(x_1,x_2,...0) e^{-i[p_1(x_1+x_n)p_2(x_2+x_n)+...p_{n-1}(x_{n-1}+x_n)]} \\
=& \int d^4x_1 d^4x_2 ... d^4x_n V(x_1,x_2,...0) e^{-i[p_1x_1+p_2x_2+...p_{n-1}x_{n-1}+(p_1+p_2+...+p_n)x_n]} \\
=& (2\pi)^4\delta^4(p_1+p_2+...p_n) \int d^4x_1 d^4x_2 ... d^4x_{n-1} V(x_1,x_2,...,0) e^{-i(p_1x_1+p_2x_2+...p_{n-1}x_{n-1})}
\end{aligned} \tag{62}$$

So we can redefine $V(p_1,p_2,...p_n)$ as

$$V(p_1,p_2,...p_n) = (2\pi)^4\delta^4(p_1+p_2+...p_n) V(p_1,p_2,...p_{n-1},-(p_1+p_2+...p_{n-1})) \tag{63}$$

$$\begin{aligned}
V(x_1,x_2,...x_n) =& \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}...\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2...+p_n)V(p_1,p_2,...-(p_1,p_2+...p_{n-1}))e^{i(p_1x_1+p_2x_2+...p_nx_n)} \\
=& \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}...\frac{d^4p_{n-1}}{(2\pi)^4}V(p_1,p_2,...-(p_1,p_2+...p_{n-1}))e^{i[p_1(x_1-x_n)+p_2(x_2-x_n)+...p_{n-1}(x_{n-1}-x_n)]}
\end{aligned} \tag{64}$$