

FRG note

November 19, 2019

Contents

1	Wetterich Equation	3
2	QCD	6
2.1	Meson field propagator	7
2.2	Quark field propagator	8
2.3	Gluon field propagator	8
2.4	Gluon vertex	9
2.5	Meson vertex	11
2.6	Quark-Meson Vertex	13
2.7	Four-fermion interaction	14
A	Fourier transform of n-point function	14

1 Wetterich Equation

Consider the generating function which is cutoff at infrared:

$$Z_k[J] = \exp(W_k[J]) = \int [d\phi] \exp\{-S[\phi] - \Delta S_k[\phi] + J^a \phi_a\} \quad (1)$$

The ϕ stands for every kind of field here. Index a stands for every degree of freedom, including different fields, different components of a field. Such as the space-time coordinates or momentum index. $S[\phi]$ is the classical action, J^a is the source of ϕ_a . $\Delta S_k[\phi]$ is the infrared cutoff, its function is to cutoff the quantum fluctuation at $p^2 \leq k^2$ and keep the fluctuation at $p^2 > k^2$ invariant. We usually choose the form of quadratic term (mass term) to achieve the infrared cutoff.

$$\Delta S_k[\phi] = \frac{1}{2} \phi_a R_k^{ab} \phi_b \quad (2)$$

with $R_k^{ab} = R_k^{ba}$ (a, b is the index of the boson), $R_k^{ab} = -R_k^{ba}$ (a, b is the index of the fermion). Here we give a example of a simple scalar field. In the coordinate space:

$$\Delta S_k[\phi] = \frac{1}{2} \int d^4x d^4y \phi(x) R_k(x, y) \phi(y) \quad (3)$$

So

$$R_k(x, y) = \int \frac{d^4p}{(2\pi)^4} R_k(p, -p) e^{ip(x-y)} \quad (4)$$

$$= \int \frac{d^4p}{(2\pi)^4} R_k(p) e^{ip(x-y)}$$

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \phi(p) \quad (5)$$

$$\begin{aligned} \Delta S_k[\phi] &= \frac{1}{2} \int d^4x d^4y \phi(x) R_k(x, y) \phi(y) \\ &= \frac{1}{2} \int d^4x d^4y \int \frac{d^4p_1}{(2\pi)^4} e^{ip_1x} \phi(p_1) \int \frac{d^4p}{(2\pi)^4} R_k(p) e^{ip(x-y)} \int \frac{d^4p_2}{(2\pi)^4} e^{ip_2y} \phi(p_2) \\ &= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \phi(-p) R_k(p) \phi(p) \\ &= \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \phi(-q) R_k(q) \phi(q) \end{aligned} \quad (6)$$

For the determined q , the regulator $R_k(q)$ satisfies the following properties

$$R_{k \rightarrow \infty}(q) \rightarrow \infty \quad R_{k \rightarrow 0}(q) \rightarrow 0 \quad (7)$$

To suppress the fluctuation at $q^2 < k^2$ and keep the fluctuation at high momentum unchanging, we should choose

$$R_k(q)|_{q^2 < k^2} \sim k^2, \quad R_k(q)|_{q^2 > k^2} \sim 0 \quad (8)$$

for example

$$R_k(q) \sim \frac{q^2}{e^{\frac{q^2}{k^2}} - 1} \quad (9)$$

of course, we can choose other forms of the regulator. From (1) we can get

$$\begin{aligned} \frac{\delta W_k[J]}{\delta J^a} &= \frac{1}{Z_k} \frac{\delta Z_k[J]}{\delta J^a} \\ &= \frac{1}{Z_k} \int [d\phi] \phi_a \exp\{-S[\phi] - \Delta S_k[\phi] + J^a \phi_a\} \\ &= \langle \phi_a \rangle \end{aligned} \quad (10)$$

In the below discussing, we replace $\langle \phi_a \rangle$ with ϕ_a and

$$\frac{\delta^2 W_k[J]}{\delta J^b \delta J^a} = \langle \phi_b \phi_a \rangle_c \equiv G_{ba}^k \quad (11)$$

index c stands for the connected diagram, G is the propagator that depends on the scale k . Now we do the Legendre transformation on the generated functional of connected diagram, then we can obtain the generated functional of one-particle irreducible diagram, that is the effective action

$$\Gamma_k[\phi] = -W_k[J] + J^a \phi_a - \Delta S_k[\phi] \quad (12)$$

Beware here $\phi_a \equiv \langle \phi_k \rangle$. In order to consider the boson and fermion together, we introduce the following symbols

$$J^a \phi_a = r_b^a \phi_a J^b \quad (13)$$

with

$$r_b^a = (-1)^{ab} \delta_b^a \quad (14)$$

$$(-1)^{ab} \equiv \begin{cases} -1, & \text{for } a, b \text{ Fermionic} \\ 1, & \text{for } a, b \text{ Bosonic} \end{cases} \quad (15)$$

So from (12) we obtain

$$\frac{\delta(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta \phi_a} = r_b^a J^b \quad (16)$$

Differentiate J on both sides of the above formula

$$\frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta J^b \delta \phi_a} = r_b^a \quad (17)$$

$$\begin{aligned} l.h.s &= \frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta J^b \delta \phi_a} \\ &= \frac{\delta \phi_c}{\delta J^b} \frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta \phi_c \delta \phi_a} \\ &= G_{bc}^k \frac{\Gamma_k[\phi] + \Delta S_k[\phi]}{\delta \phi_c \delta \phi_a} \end{aligned} \quad (18)$$

We obtain

$$G_{bc}^k \frac{\Gamma_k[\phi] + \Delta S_k[\phi]}{\delta\phi_c \delta\phi_a} = r_b^a \quad (19)$$

$$\frac{\Gamma_k[\phi] + \Delta S_k[\phi]}{\delta\phi_c \delta\phi_a} = (\Gamma_k^{(2)}[\phi] + R_k)^{ca} \quad (20)$$

then

$$G_{bc}^k (\Gamma_k^{(2)}[\phi] + R_k)^{ca} = r_b^a G_{bc}^k = r_b^a (\Gamma_k^{(2)}[\phi] + R_k)_{ac}^{-1} \quad (21)$$

Now we calculate $\partial_t W_k[J] = k \frac{\partial}{\partial k} W_k[J]$

$$\begin{aligned} \partial_t W_k[J] &= \partial_t \ln Z_k[J] \\ &= \frac{1}{Z_k} \partial_t Z_k[J] \\ &= \frac{1}{Z_k} \int [d\phi] (-\partial_t \Delta S_k[\phi]) e^{-S[\phi] - \Delta S_k[\phi] + J^a \phi_a} \\ &= -\frac{1}{Z_k} \int [d\phi] \frac{1}{2} \phi_a \partial_t R_k^{ab} \phi_b e^{-S[\phi] - \Delta S_k[\phi] + J^a \phi_a} \\ &= -\frac{1}{2} \langle \phi_a \phi_b \rangle \partial_t R_k^{ab} \\ &= -\frac{1}{2} (\langle \phi_a \phi_b \rangle_c + \langle \phi_a \rangle \langle \phi_b \rangle) \partial_t R_k^{ab} \end{aligned} \quad (22)$$

we replace $\langle \phi_a \rangle$ with ϕ_a again

$$\langle \phi_a \phi_b \rangle_c = \begin{cases} \langle \phi_b \phi_a \rangle, & \text{for } a, b \text{ Bosonic} \\ -\langle \phi_b \phi_a \rangle, & \text{for } a, b \text{ Fermionic} \end{cases} \quad (23)$$

Then we obtain

$$\partial_t W_k[J] = -\frac{1}{2} \text{STr} G_k (\partial_t R_k) - \frac{1}{2} \phi_a \partial_t R_k^{ab} \phi_b \quad (24)$$

STr is super trace, including the trace of every field and dispersed, continuously degree of freedom. For the fermion a negative sign should be added. Finally, from (12) and (24) we obtain

$$\begin{aligned} \partial_t \Gamma_k[\phi] &= -\partial_t W_k[J] - \partial_t \Delta S_k[\phi] \\ &= \frac{1}{2} \phi_a \partial_t R_k^{ab} \phi_b - \frac{1}{2} \phi_a \partial_t R_k^{ab} \phi_b \\ &= \frac{1}{2} \text{STr} G_k (\partial_t R_k) \end{aligned} \quad (25)$$

This is the Wetterich equation.

2 QCD

In this section, we introduce the application of FRG in QCD. We start from the effective action that depends on the infrared cutoff scale k

$$\begin{aligned}\Gamma_k = \int d^4x \{ & \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + Z_{c,k} (\partial_\mu \bar{c}^a) D_\mu^{ab} c^b + \frac{1}{2\xi} (\partial_\mu A_\mu^a)^2 \\ & + Z_{q,k} \bar{q} (\gamma_\mu D_\mu) q - \lambda_{q,k} [(\bar{q} T^0 q)^2 - (\bar{q} \gamma_5 T^a q)^2] \\ & + h_k [\bar{q} (i \gamma_5 T^a \pi^a + T^0 \sigma) q] + \frac{1}{2} Z_{\phi,k} (\partial_\mu \phi)^2 \}\end{aligned}\quad (26)$$

In the effective action

$$\begin{aligned}\phi &= (\sigma, \pi^a), \rho = \frac{1}{2} \phi^2 \\ D_\mu &= \partial_\mu - i Z_{A,k}^{\frac{1}{2}} g_k A_\mu^a t^a \\ D_\mu^{ab} &= \partial_\mu \delta^{ab} + Z_{A,k}^{\frac{1}{2}} g_k f^{acb} A_\mu^c\end{aligned}\quad (27)$$

In the definition above $T^a (a = 1, 2, \dots, N_f^2 - 1)$ is the generator of flavor space with $T^0 = \frac{1}{\sqrt{2N_f}} \mathbb{1}$; $t^a (a = 1, 2, \dots, N_c^2 - 1)$ is the generator of color space.

$$\begin{aligned}F_{\mu\nu} &= \frac{i}{g} [D_\mu, D_\nu] = F_{\mu\nu}^a t^a \\ F_{\mu\nu}^a &= Z_{A,k}^{\frac{1}{2}} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + Z_{A,k}^{\frac{1}{2}} g_k f^{abc} A_\mu^b A_\nu^c\end{aligned}\quad (28)$$

We rewrite the Wetterich equation as below

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \text{STr} \tilde{\partial}_t \ln(\Gamma_k^{(2)} + R_k) \quad (29)$$

the tilde on the ∂_t stands for the derivation only works on the regulator R_k .

$$(\Gamma_k^{(2)})_{ab} = \frac{\overrightarrow{\delta}}{\delta \Phi_a^T} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \Phi_b} \quad (30)$$

the definition of Φ is

$$\begin{aligned}\Phi &= \begin{pmatrix} A(q) \\ \sigma(q) \\ \pi(q) \\ q(q) \\ \bar{q}^T(q) \end{pmatrix} \\ \Phi^T &= (A^T(-q), \sigma(-q), \pi(-q), q^T(q), \bar{q}(q))\end{aligned}\quad (31)$$

the fluctuation matrix in the (29) can be rewrite as

$$\Gamma_k^{(2)} + R_k = P + F \quad (32)$$

The matrix P contains the propagators and regulators ; matrix F contains the dependence of all kind of fields. So we can expanse the (29) with the number of the fields

$$\begin{aligned}
\partial_t \Gamma_k[\Phi] &= \frac{1}{2} \text{STr} \tilde{\partial}_t \ln(P + F) \\
&= \frac{1}{2} \text{STr} \tilde{\partial}_t \ln[P(1 + \frac{1}{P}F)] \\
&\sim \frac{1}{2} \text{STr} \tilde{\partial}_t \ln(1 + \frac{1}{P}F) \\
&= \frac{1}{2} \text{STr} \{ \tilde{\partial}_t [\frac{1}{P}F - \frac{1}{2}(\frac{1}{P}F)^2 + \frac{1}{3}(\frac{1}{P}F)^3 - \frac{1}{4}(\frac{1}{P}F)^4 + \dots] \} \\
&= \frac{1}{2} \text{STr} \tilde{\partial}_t (\frac{1}{P}F) - \frac{1}{4} \text{STr} \tilde{\partial}_t (\frac{1}{P}F)^2 + \frac{1}{6} \text{STr} \tilde{\partial}_t (\frac{1}{P}F)^3 - \frac{1}{8} \text{STr} \tilde{\partial}_t (\frac{1}{P}F)^4 + \dots
\end{aligned} \tag{33}$$

then we take the contribution of the lowest order in

$$\begin{aligned}
\partial_t \Gamma_k[\Phi] &= \frac{1}{2} \text{STr} \tilde{\partial}_t \ln(P + F) \\
&= \frac{1}{2} \text{STr} \tilde{\partial}_t \ln[P(1 + \frac{1}{P}F)] \\
&= \frac{1}{2} \text{STr} \tilde{\partial}_t [\ln P + \ln(1 + \frac{1}{P}F)] \\
&= \frac{1}{2} \text{STr} \tilde{\partial}_t \ln P + \frac{1}{2} \text{STr} \tilde{\partial}_t \ln(1 + \frac{1}{P}F) \\
&= \frac{1}{2} \text{STr} \tilde{\partial}_t \ln P + \frac{1}{2} \text{STr} \tilde{\partial}_t (\frac{1}{P}F) - \frac{1}{4} \text{STr} \tilde{\partial}_t (\frac{1}{P}F)^2 + \frac{1}{6} \text{STr} \tilde{\partial}_t (\frac{1}{P}F)^3 - \frac{1}{8} \text{STr} \tilde{\partial}_t (\frac{1}{P}F)^4 + \dots
\end{aligned} \tag{34}$$

2.1 Meson field propagator

The meson propagator part of the effective potential and its Fourier transform

$$\begin{aligned}
&\int d^4x \frac{1}{2} Z_{\phi,k} (\partial_\mu \phi)^2 \\
&= \int \frac{d^4q}{(2\pi)^4} \frac{1}{2} Z_{\phi,k} \phi(-q) q^2 \phi(q)
\end{aligned} \tag{35}$$

then consider the contribution of the meson mass

$$\begin{aligned}
\Gamma_k^{\sigma\sigma}(q', q) &\equiv \frac{\delta^2 \Gamma_k}{\delta \sigma(q') \delta \sigma(q)} \\
&= (Z_{\phi,k} q^2 + m_\sigma^2) (2\pi)^4 \delta^4(q + q')
\end{aligned} \tag{36}$$

The corresponding cutoff function is

$$R_k^{\sigma\sigma}(q', q) = Z_{\phi,k} \vec{q}^2 r_B(\frac{\vec{q}^2}{k^2}) (2\pi)^4 \delta^4(q + q') \tag{37}$$

here we use the 3d regulator which is convenient to the calculation of the finite temperature. The $r_B(x) = (\frac{1}{x} - 1)\theta(1 - x)$ is an optimized regulator.

Then we treat the Pi meson in the same way

$$\begin{aligned}
\Gamma_{k,ij}^{\pi\pi}(q', q) &\equiv \frac{\delta^2 \Gamma_k}{\delta \pi_i(q') \delta \pi_j(q)} \\
&= \delta_{ij} (Z_{\phi,k} q^2 + m_\pi^2) (2\pi)^4 \delta^4(q + q')
\end{aligned} \tag{38}$$

The cutoff function is

$$R_{k,ij}^{\pi\pi} = \delta_{ij} R_k^{\sigma\sigma} \quad (39)$$

In the calculation of finite temperature we have $q_0 = 2\pi nT$ in which the n is integer.

2.2 Quark field propagator

The quark propagator part and its Fourier transform

$$\begin{aligned} & \int d^4x Z_{q,k} \bar{q}(x) (\gamma_\mu \partial_\mu + \frac{m_f}{Z_{q,k}}) q(x) \\ &= \int \frac{d^4q}{(2\pi)^4} Z_{q,k} \bar{q}(q) (i\not{q} + \frac{m_f}{Z_{q,k}}) q(q) \end{aligned} \quad (40)$$

The Fourier transform of the quark and anti-quark fields

$$\begin{aligned} q(x) &= \int \frac{d^4q}{(2\pi)^4} e^{iqx} q(q) \\ \bar{q}(x) &= \int \frac{d^4q}{(2\pi)^4} e^{-iqx} \bar{q}(q) \end{aligned} \quad (41)$$

The derivation of the effective action amount of the quark fields is

$$\begin{aligned} \Gamma_{k,ij}^{\bar{q}q}(q', q) &\equiv \frac{\overrightarrow{\delta}}{\delta q_i(q')} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \bar{q}_j(q)} \\ &= (Z_{q,k} i\not{q}_{ij} + m_f \delta_{ij}) (2\pi)^4 \delta^4(q' - q) \\ &= (Z_{q,k} i q_\mu (\gamma_\mu)_{ij} + m_f \delta_{ij}) (2\pi)^4 \delta^4(q' - q) \end{aligned} \quad (42)$$

$$\begin{aligned} \Gamma_{k,ij}^{q\bar{q}} &\equiv \frac{\overrightarrow{\delta}}{\delta q_i(q')} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \bar{q}_j(q)} \\ &= -(Z_{q,k} i q_\mu (\gamma_\mu)_{ji} + m_f \delta_{ji}) (2\pi)^4 \delta^4(q' - q) \end{aligned} \quad (43)$$

The corresponding regulator function is

$$R_{k,ij}^{\bar{q}q}(q', q) = Z_{q,k} i \vec{q} \cdot \vec{\gamma}_{ij} r_F(\frac{\vec{q}^2}{k^2}) (2\pi)^4 \delta^4(q' - q) \quad (44)$$

with

$$r_F(x) = (\frac{1}{\sqrt{x}} - 1) \theta(1 - x) \quad (45)$$

2.3 Gluon field propagator

The differential form of the effective action is

$$\begin{aligned} (\Gamma_k^{AA})_{\mu\nu}^{ab}(q', q) &\equiv \frac{\delta^2 \Gamma_k}{\delta A_\mu^a(q') \delta A_\nu^b(q)} \\ &= [Z_{A,k} q^2 (\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{q^2}{\xi} (\frac{q_\mu q_\nu}{q^2})] \delta^{ab} (2\pi)^4 \delta^4(q' + q) \end{aligned} \quad (46)$$

The regulator function is

$$(R_k^{AA})_{\mu\nu}^{ab}(q', q) = [Z_{A,k}\vec{q}^2 r_B(\frac{\vec{q}^2}{k^2})(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{\vec{q}^2}{\xi} r_B(\frac{\vec{q}}{k^2})(\frac{q_\mu q_\nu}{q^2})] \delta^{ab} (2\pi)^4 \delta^4(q' + q) \quad (47)$$

The propagator of the gluon is

$$(G_k^{AA})_{\mu\nu}^{ab}(q', q) = [\frac{1}{Z_{A,k}(q_0^2 + \vec{q}^2(1+r_B))}(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{\xi}{q_0^2 + \vec{q}^2(1+r_B)} \frac{q_\mu q_\nu}{q^2}] \delta^{ab} (2\pi)^4 \delta^4(q' + q) \quad (48)$$

In the following calculation we adopt the Landau gauge $\xi = 0$, then we can obtain the matrix P

$P =$

$$\begin{pmatrix} Z_{A,k}(q_0^2 + \vec{q}^2(1+r_B))(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})\delta_{ab} & 0 & 0 & 0 & 0 \\ 0 & Z_{\phi,k}(q_0^2 + \vec{q}^2(1+r_B)) + m_\sigma^2 & 0 & 0 & 0 \\ 0 & 0 & [Z_{\phi,k}(q_0^2 + \vec{q}^2(1+r_B)) + m_\pi^2]\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_{q,k}i(q_0 r_0 + \vec{q} \cdot \vec{r}(1+r_F)) + m_f & 0 \end{pmatrix} \quad (49)$$

And then we can obtain the propagator matrix

$$\frac{1}{P} = \begin{pmatrix} (G_k^{AA})_{\mu\nu}^{ab} & 0 & 0 & 0 & 0 \\ 0 & G_k^\sigma & 0 & 0 & 0 \\ 0 & 0 & (G_k^\pi)_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_k^{q\bar{q}} \\ 0 & 0 & 0 & G_k^{\bar{q}q} & 0 \end{pmatrix} \quad (50)$$

the definition of the matrix elements are

$$\begin{aligned} (G_k^{AA})_{\mu\nu}^{ab} &= \frac{1}{Z_{A,k}(q_0^2 + \vec{q}^2(1+r_B))}(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})\delta_{ab} \\ G_k^\sigma &= \frac{1}{Z_{\phi,k}(q_0^2 + \vec{q}^2(1+r_B)) + m_\sigma^2} \\ (G_k^\pi)_{ij} &= \frac{1}{Z_{\phi,k}(q_0^2 + \vec{q}^2(1+r_B)) + m_\pi^2}\delta_{ij} \\ G_k^{q\bar{q}} &= \frac{-Z_{q,k}i(q_0 r_0 + \vec{q} \cdot \vec{r}(1+r_F)) + m_f}{Z_{q,k}^2(q_0^2 + \vec{q}^2(1+r_F)^2) + m_f^2} \\ G_k^{\bar{q}q} &= -(G_k^{q\bar{q}})^T \end{aligned} \quad (51)$$

2.4 Gluon vertex

First is the gauge part

$$\begin{aligned} \Gamma_k &\sim \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \\ &= \frac{1}{4} Z_{A,k}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + Z_{A,k}^{\frac{1}{2}} g_k f^{abc} A_\mu^b A_\nu^c)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + Z_{A,k}^{\frac{1}{2}} g_k f^{ab'c'} A_\mu^{b'} A_\nu^{c'}) \\ &\sim \frac{1}{2} Z_{A,k}^{\frac{3}{2}}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g_k f^{abc} A_\mu^b A_\nu^c + \frac{1}{4} Z_{A,k}^2 g_k^2 f^{abc} f^{ab'c'} A_\mu^b A_\nu^c A_\mu^{b'} A_\nu^{c'} \end{aligned} \quad (52)$$

here we let

$$\Gamma_1 = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) f^{abc} A_\mu^b A_\nu^c \quad (53)$$

then we can calculate the derivative

$$\frac{\delta \Gamma_1}{\delta A_\mu^a} = -2(\partial_\nu)_a f^{abc} A_\mu^b A_\nu^c + 2f^{abc} (\partial_\nu A_\mu^b - \partial_\mu A_\nu^b) A_\nu^c \quad (54)$$

$$\frac{\delta^2 \Gamma_1}{\delta A_\mu^a \delta A_\nu^b} = -2(\partial_\rho)_a f^{abc} \delta_{\mu\nu} A_\rho^c - 2(\partial_\nu)_a f^{acb} A_\mu^c + 2f^{abc} (\partial_\rho)_b \delta_{\mu\nu} A_\rho^c + 2f^{acb} \partial_\nu A_\mu^c - 2f^{abc} (\partial_\mu)_b A_\nu^c - 2f^{acb} \partial_\mu A_\nu^c \quad (55)$$

$$\begin{aligned} &= 2\delta_{\mu\nu} f^{abc} [(\partial_\rho)_b - (\partial_\rho)_a] A_\rho^c + 2f^{abc} [(\partial_\nu)_a A_\mu^c - (\partial_\mu)_b A_\nu^c] + 2f^{abc} (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c) \\ \frac{\delta^3 \Gamma_1}{\delta A_\mu^a \delta A_\nu^b \delta A_\rho^c} &= 2\delta_{\mu\nu} f^{abc} [(\partial_\rho)_b - (\partial_\rho)_a] + 2\delta_{\mu\nu} f^{abc} [(\partial_\nu)_a \delta_{\mu\rho} - (\partial_\mu)_b \delta_{\nu\rho}] + 2f^{abc} [(\partial_\mu)_c \delta_{\rho\nu} - (\partial_\nu)_c \delta_{\mu\rho}] \\ &= 2f^{abc} \left\{ \delta_{\mu\nu} [(\partial_\rho)_b - (\partial_\rho)_a] + \delta_{\mu\rho} [(\partial_\nu)_a - (\partial_\nu)_c] + \delta_{\rho\nu} [(\partial_\mu)_c - (\partial_\mu)_b] \right\} \end{aligned} \quad (56)$$

Then we assume

$$\begin{aligned} \Gamma_2 &= f^{abc} f^{ab'c'} A_\mu^b A_\nu^c A_\mu^{b'} A_\nu^{c'} \\ &= f^{abc} f^{a'b'c} A_\mu^a A_\nu^b A_\mu^{a'} A_\nu^{b'} \end{aligned} \quad (57)$$

And the derivation can be written as

$$\frac{\delta \Gamma_2}{\delta A_\mu^a} = 4f^{abc} f^{a'b'c} A_\nu^b A_\mu^{a'} A_\nu^{b'} \quad (58)$$

$$\begin{aligned} \frac{\delta \Gamma_2}{\delta A_\mu^a \delta A_\nu^b} &= 4f^{abc} f^{a'b'c} A_\mu^{a'} A_\nu^{b'} + 4f^{ab''c} f^{bb'c} A_\rho^{b''} \delta_{\mu\nu} A_\rho^{b'} + 4f^{ab''c} f^{a'bc} A_\nu^{b''} A_\mu^{a'} \\ &= 4f^{abc} f^{a'b'c} A_\mu^{a'} A_\nu^{b'} + 4\delta_{\mu\nu} f^{ab'c} f^{bb''c} A_\rho^{b'} A_\rho^{b''} - 4f^{ab''c} f^{ba'c} A_\mu^{a'} A_\nu^{b''} \end{aligned} \quad (59)$$

So now we can obtain the F matrix elements of the glon part

$$\begin{aligned} (F_k^{AA})_{\mu\nu}^{ab} &= \frac{\delta^2}{A_\mu^a A_\nu^b} \left(\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \right) \\ &= Z_{A,k}^{\frac{3}{2}} g_k f^{abc} \left\{ \delta_{\mu\nu} [(\partial_\rho)_b - (\partial_\rho)_a] A_\rho^c + [(\partial_\nu)_a A_\mu^c - (\partial_\mu)_b A_\nu^c] + (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c) \right\} \\ &\quad + Z_{A,k}^2 g_k^2 [f^{abc} f^{a'b'c} A_\mu^{a'} A_\nu^{b'} + \delta_{\mu\nu} f^{ab'c} f^{bb''c} A_\rho^{b'} A_\rho^{b''} - f^{ab''c} f^{ba'c} A_\mu^{a'} A_\nu^{b''}] \end{aligned} \quad (60)$$

From the expression of the effective action we find that the interaction between the gluon and meson vanished

$$F_k^{A\sigma} = 0 \quad (61)$$

$$F_k^{A\pi} = 0 \quad (62)$$

Now we derive the gluon-quark vertex. The gluon-quark part of the effective action reads

$$\Gamma_k \sim Z_{q,k} \bar{q} \gamma_\mu D_\mu q \quad (63)$$

The expression of the covariant derivative is

$$D_\mu = \partial_\mu - iZ_{A,k}^{\frac{1}{2}} g_k A_\mu^a t^a \quad (64)$$

So we get

$$\begin{aligned} \Gamma_k &\sim Z_{q,k} \bar{q} \gamma_\mu (-iZ_{A,k}^{\frac{1}{2}} g_k A_\mu^a t^a) q \\ &= -iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k A_\mu^a \bar{q} \gamma_\mu t^a q \end{aligned} \quad (65)$$

The gluon-quark vertex can be obtained from the action above

$$\begin{aligned} (F_k^{Aq})_{\mu j}^a &= \frac{\overrightarrow{\delta}}{\delta A_\mu^a} \Gamma_k \frac{\overleftarrow{\delta}}{\delta q_j} = -iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\bar{q} \gamma_\mu t^a)_j \\ \Rightarrow (F_k^{Aq})_\mu^a &= -iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\bar{q} \gamma_\mu t^a) \end{aligned} \quad (66)$$

$$\begin{aligned} (F_k^{A\bar{q}})_{\mu j}^a &= \frac{\overrightarrow{\delta}}{\delta A_\mu^a} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \bar{q}_j} = iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k j (\gamma_\mu t^a q) \\ \Rightarrow (F_k^{A\bar{q}})_\mu^a &= iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k q^T (\gamma_\mu t^a)^T = -(F_k^{\bar{q}A})_\mu^a{}^T \end{aligned} \quad (67)$$

$$\begin{aligned} (F_k^{qA})_{i\nu}^b &= \frac{\overrightarrow{\delta}}{\delta q_i} \Gamma_k \frac{\overleftarrow{\delta}}{\delta A_\nu^b} = iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\bar{q} \gamma_\nu t^b)_i \\ \Rightarrow (F_k^{qA})_\mu^a &= iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\gamma_\mu t^a)^T \bar{q}^T = -(F_k^{Aq})_\mu^a{}^T \end{aligned} \quad (68)$$

$$\begin{aligned} (F_k^{\bar{q}A})_{i\nu}^a &= \frac{\overrightarrow{\delta}}{\delta \bar{q}_i} \Gamma_k \frac{\overleftarrow{\delta}}{\delta A_\nu^a} = iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k i (\gamma_\nu t^a q) \\ \Rightarrow (F_k^{\bar{q}A})_\mu^a &= -iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\gamma_\mu t^a q) \end{aligned} \quad (69)$$

2.5 Meson vertex

The corresponding effective action of the meson part is

$$\Gamma_k \sim V_k(\rho) - c\sigma \quad (70)$$

the ρ stands for the square of the meson field

$$\rho = \frac{1}{2} \phi_i^2 \quad (71)$$

and the definition of the meson field are

$$\phi_{i=0} = \sigma, \quad \phi_i = \pi_i \quad i = 1, 2, 3 \quad (72)$$

Then we define

$$\begin{cases} \phi_i = \bar{\phi}_0 + \sigma, & \text{for } i = 0 \\ \phi_i = \pi_i, & \text{for } i = 1, 2, 3 \end{cases} \quad (73)$$

The derivation of the effective action

$$\frac{\delta \Gamma_k}{\delta \phi_i} = V'_k(\rho) \phi_i - c \delta_{i,0} \quad (74)$$

$$\frac{\delta^2 \Gamma_k}{\delta \phi_i \delta \phi_j} = V'_k(\rho) \delta_{i,j} + V''_k(\rho) \phi_i \phi_j \quad (75)$$

Now expand the equation above at the point $\phi_i = \bar{\phi}_i$

$$\bar{\phi}_i = \begin{cases} \bar{\phi}_0 & i = 0 \\ 0 & i = 1, 2, 3 \end{cases} \quad (76)$$

$$(\Gamma_k^{\phi\phi})_{ij} = (\Gamma_k^{\phi\phi})_{ij} \Big|_{\bar{\phi}} + \frac{\delta^3 \Gamma_k}{\delta \phi_i \delta \phi_j \delta \phi_k} \Big|_{\bar{\phi}} (\phi_k - \bar{\phi}_k) + \frac{1}{2} \frac{\delta^4 \Gamma_k}{\delta \phi_i \delta \phi_j \delta \phi_k \delta \phi_l} \Big|_{\bar{\phi}} (\phi_k - \bar{\phi}_k)(\phi_l - \bar{\phi}_l) + \dots \quad (77)$$

Here we only consider to the two derivative of the effective potential

$$\begin{aligned} \frac{\delta^3 \Gamma_k}{\delta \phi_i \delta \phi_j \delta \phi_k} &= V''_k(\rho) \phi_k \delta_{ij} + V''_k(\rho) (\delta_{ik} \phi_i + \delta_{jk} \phi_j) + V'''_k(\rho) \phi_i \phi_j \phi_k \\ &= V''_k(\rho) (\delta_{ij} \phi_k + \delta_{ik} \phi_j + \delta_{jk} \phi_i) + V'''_k(\rho) \phi_i \phi_j \phi_k \end{aligned} \quad (78)$$

$$\begin{aligned} \frac{\delta^4 \Gamma_k}{\delta \phi_i \delta \phi_j \delta \phi_k \delta \phi_l} &= V''_k(\rho) (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) \\ &\quad + V'''_k(\rho) (\delta_{ij} \phi_k \phi_l + \delta_{ik} \phi_j \phi_l + \delta_{jk} \phi_i \phi_l) \\ &\quad + V'''_k(\rho) (\delta_{il} \phi_j \phi_k + \delta_{jl} \phi_i \phi_k + \delta_{kl} \phi_i \phi_j) \\ &\quad + V''''_k(\rho) \phi_i \phi_j \phi_k \phi_l \\ &= V''_k(\rho) (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) \\ &\quad + V_k^{(3)}(\rho) (\delta_{ij} \phi_k \phi_l + \delta_{ik} \phi_j \phi_l + \delta_{jk} \phi_i \phi_l + \delta_{il} \phi_j \phi_k + \delta_{jl} \phi_i \phi_k + \delta_{kl} \phi_i \phi_j) \\ &\quad + V_k^{(4)}(\rho) \phi_i \phi_j \phi_k \phi_l \end{aligned} \quad (79)$$

The definition of the Sigma meson mass is

$$\begin{aligned} m_\sigma^2 &= (\Gamma_k^{\sigma\sigma})|_{\bar{\phi}} = (\Gamma_k^{\phi\phi})_{00}|_{\bar{\phi}} = V'_k(\rho) + V''_k(\rho) \bar{\phi}_0 \bar{\phi}_0 \\ &= V'_k(\rho) + 2\rho V''_k(\rho) \end{aligned} \quad (80)$$

And the Pion mass is

$$m_\pi^2 = (\Gamma_k^{\pi\pi})_{ij}|_{\bar{\phi}} = (\Gamma_k^{\phi\phi})_{ij}|_{\bar{\phi}} = V'_k(\rho) \delta_{ij} \quad (81)$$

Then the other derivation part of the Pion and Sigma meson can be written as

$$\begin{aligned}
(\Gamma_k^{\sigma\pi})|_{\bar{\phi}} &= 0 \\
\frac{\delta^3\Gamma_k}{\delta\sigma\delta\sigma\delta\sigma}|_{\bar{\phi}} &= V_k''(\rho)(3\bar{\phi}_0) + V_k^{(3)}(\rho)\bar{\phi}_0^3 \\
&= 3(2\rho)^{\frac{1}{2}}V_k''(\rho) + (2\rho)^{\frac{3}{2}}V_k^{(3)}(\rho) \\
\frac{\delta^3\Gamma_k}{\delta\sigma\delta\sigma\delta\pi_k}|_{\bar{\phi}} &= 0 \\
\frac{\delta^3\Gamma_k}{\delta\sigma\delta\pi_j\delta\sigma}|_{\bar{\phi}} &= 0 \\
\frac{\delta^3\Gamma_k}{\delta\sigma\delta\pi_j\delta\pi_k}|_{\bar{\phi}} &= V_k''(\rho)\delta_{jk}\bar{\phi}_0 = \delta_{jk}(2\rho)^{\frac{1}{2}}V_k''(\rho) \\
\frac{\delta^3\Gamma_k}{\delta\pi_i\delta\pi_j\delta\sigma}|_{\bar{\phi}} &= \delta_{ij}(2\rho)^{\frac{1}{2}}V_k''(\rho) \\
\frac{\delta^3\Gamma_k}{\delta\pi_i\delta\pi_j\delta\pi_k}|_{\bar{\phi}} &= 0
\end{aligned} \tag{82}$$

If we only take the lowest derivation into account

$$\begin{aligned}
F_k^{\sigma\sigma} &= \frac{\delta^3\Gamma_k}{\delta\sigma^3}|_{\bar{\phi}}\sigma + \frac{\delta^3\Gamma_k}{\delta\sigma^2\delta\pi_k}|_{\bar{\phi}}\pi_k \\
&= [3(2\rho)^{\frac{1}{2}}V_k^{(2)}(\rho) + (2\rho)^{\frac{3}{2}}V_k^{(3)}(\rho)]\sigma
\end{aligned} \tag{83}$$

$$\begin{aligned}
(F_k^{\sigma\pi})_j &= \frac{\delta^3\Gamma_k}{\delta\sigma\delta\pi_j\delta\sigma}|_{\bar{\phi}}\sigma + \frac{\delta^3\Gamma_k}{\delta\sigma\delta\pi_j\delta\pi_k}|_{\bar{\phi}}\pi_k \\
&= \delta_{jk}(2\rho)^{\frac{1}{2}}V_k^{(2)}(\rho)\pi_k = (2\rho)^{\frac{1}{2}}V_k^{(2)}(\rho)\pi_j
\end{aligned} \tag{84}$$

$$(F_k^{\pi\sigma})_i = (2\rho)^{\frac{1}{2}}V_k^{(2)}(\rho)\pi_i \tag{85}$$

$$(F_k^{\pi\pi})_{ij} = \frac{\delta^3\Gamma_k}{\delta\pi_i\delta\pi_j\delta\sigma}|_{\bar{\phi}} = \delta_{ij}(2\rho)^{\frac{1}{2}}V_k^{(2)}(\rho)\sigma \tag{86}$$

And we define

$$\begin{aligned}
V_{3\sigma}(\rho) &= 3(2\rho)^{\frac{1}{2}}V_k^{(2)}(\rho) + (2\rho)^{\frac{3}{2}}V_k^{(3)}(\rho) \\
V_{2\pi\sigma} &= (2\rho)^{\frac{1}{2}}V_k^{(2)}(\rho)
\end{aligned} \tag{87}$$

Then we can write the matrix elements as

$$F_k^{\sigma\sigma} = V_{3\sigma}(\rho)\sigma, \quad (F_k^{\sigma\pi})_j = V_{2\pi\sigma}(\rho)\pi_j, \quad (F_k^{\pi\sigma})_i = V_{2\pi\sigma}(\rho)\pi_i, \quad (F_k^{\pi\pi})_{ij} = \delta_{ij}V_{2\pi\sigma}(\rho)\sigma \tag{88}$$

2.6 Quark-Meson Vertex

We can get the quark-meson interaction vertex by the derivation of the Yukawa part of the effective action

$$\Gamma_k \sim h_k [\bar{q}(i\gamma_5 T^a \pi^a + T^0 \sigma)q] \tag{89}$$

$$\begin{aligned}
(F_k^{\sigma q})_j &= \frac{\overrightarrow{\delta}}{\delta \sigma} \Gamma_k \frac{\overleftarrow{\delta}}{\delta q_j} = h_k (\bar{q} T^0)_j \Rightarrow F_k^{\sigma q} = h_k \bar{q} T^0 \\
(F_k^{\sigma \bar{q}})_j &= \frac{\overrightarrow{\delta}}{\delta \sigma} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \bar{q}_j} = -h_k{}_j (T^0 q) \Rightarrow F_k^{\sigma \bar{q}} = -h_k q^T T^{0T} \\
(F_k^{q\sigma})_i &= \frac{\overrightarrow{\delta}}{\delta q_i} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \sigma} = -h_k (\bar{q} T^0)_i \Rightarrow F_k^{q\sigma} = -h_k T^{0T} \bar{q}^T \\
(F_k^{\bar{q}\sigma})_i &= \frac{\overrightarrow{\delta}}{\delta \bar{q}_i} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \sigma} = h_k{}_i (T^0 q) \Rightarrow F_k^{\bar{q}\sigma} = h_k T^0 q \\
(F_k^{\pi q})_{aj} &= \frac{\overrightarrow{\delta}}{\delta \pi_a} \Gamma_k \frac{\overleftarrow{\delta}}{\delta q_j} = h_k (\bar{q} i \gamma_5 T^a)_j \Rightarrow F_k^{\pi q} = h_k \bar{q} i \gamma_5 T^a \\
(F_k^{\pi \bar{q}})_{aj} &= \frac{\overrightarrow{\delta}}{\delta \pi_a} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \bar{q}_j} = -h_k{}_j (i \gamma_5 T^a q) \Rightarrow F_k^{\pi \bar{q}} = -h_k q^T (i \gamma_5 T^a)^T = -(F_k^{\bar{q}\pi})^T \\
(F_k^{q\pi})_{ia} &= \frac{\overrightarrow{\delta}}{\delta q_i} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \pi_a} = -h_k (\bar{q} i \gamma_5 T^a)_i \Rightarrow F_k^{q\pi} = -h_k (i \gamma_5 T^a)^T \bar{q}^T = -(F_k^{\pi q})^T \\
(F_k^{\bar{q}\pi})_{ia} &= \frac{\overrightarrow{\delta}}{\delta \bar{q}_i} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \pi_a} = h_k{}_i (i \gamma_5 T^a q) \Rightarrow F_k^{\bar{q}\pi} = h_k i \gamma_5 T^a q
\end{aligned} \tag{90}$$

2.7 Four-fermion interaction

Here is the vertex of the four-quark interaction. Be careful of the exchange of the fermion field which obey the Grassman algebra. The effective action of the four-fermion part is

$$\Gamma_k \sim -\lambda_{q,k} \left[(\bar{q} T^0 q)^2 - (\bar{q} \gamma_5 T^a q)^2 \right] \tag{91}$$

$$\begin{aligned}
\frac{\overrightarrow{\delta}}{\delta \bar{q}_i} \Gamma_k &= -2\lambda_{q,k} \left[{}_i(T^0 q)(\bar{q} T^0 q) - {}_i(\gamma_5 T^a q)(\bar{q} \gamma_5 T^a q) \right] \\
\frac{\overrightarrow{\delta}}{\delta \bar{q}_i} \Gamma_k \frac{\overleftarrow{\delta}}{\delta q_j} &= -2\lambda_{q,k} \left\{ T_{ij}^0 (\bar{q} T^0 q) + {}_i(T^0 q)(\bar{q} T^0)_j - (\gamma_5 T^a)_{ij} (\bar{q} \gamma_5 T^a q) - {}_i(\gamma_5 T^a q)(\bar{q} \gamma_5 T^a)_j \right\}
\end{aligned} \tag{92}$$

A Fourier transform of n-point function

Consider a general n-point function $V(x_1, x_2, \dots, x_n)$, Its Fourier transformation can be written as:

$$V(x_1, x_2, \dots, x_n) = \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \dots \frac{d^4 p_n}{(2\pi)^4} V(p_1, p_2, \dots, p_n) e^{i(p_1 x_1 + p_2 x_2 + \dots + p_n x_n)} \tag{93}$$

if V satisfies the following properties:

$$V(x_1, x_2, \dots, x_n) = V(x_1 - x_n, x_2 - x_n, \dots, 0) \tag{94}$$

The value of V depends only on the relative value of the coordinates, so we can obtain

$$\begin{aligned}
V(p_1, p_2, \dots, p_n) &= \int d^4 x_1 d^4 x_2 \dots d^4 x_n V(x_1, x_2, \dots, x_n) e^{-i(p_1 x_1 + p_2 x_2 + \dots p_n x_n)} \\
&= \int d^4 x_1 d^4 x_2 \dots d^4 x_n V(x_1 - x_n, x_2 - x_n, \dots, 0) e^{-i(p_1 x_1 + p_2 x_2 + \dots p_n x_n)} \\
&= \int d^4 x_1 d^4 x_2 \dots d^4 x_n V(x_1, x_2, \dots, 0) e^{-i[p_1(x_1 + x_n) p_2(x_2 + x_n) + \dots p_{n-1}(x_{n-1} + x_n)]} \\
&= \int d^4 x_1 d^4 x_2 \dots d^4 x_n V(x_1, x_2, \dots, 0) e^{-i[p_1 x_1 + p_2 x_2 + \dots p_{n-1} x_{n-1} + (p_1 + p_2 + \dots + p_n) x_n]} \\
&= (2\pi)^4 \delta^4(p_1 + p_2 + \dots p_n) \int d^4 x_1 d^4 x_2 \dots d^4 x_{n-1} V(x_1, x_2, \dots, 0) e^{-i(p_1 x_1 + p_2 x_2 + \dots p_{n-1} x_{n-1})}
\end{aligned} \tag{95}$$

So we can redefine $V(p_1, p_2, \dots, p_n)$ as

$$V(p_1, p_2, \dots, p_n) = (2\pi)^4 \delta^4(p_1 + p_2 + \dots p_n) V(p_1, p_2, \dots, p_{n-1}, -(p_1 + p_2 + \dots p_{n-1})) \tag{96}$$

$$\begin{aligned}
V(x_1, x_2, \dots, x_n) &= \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \dots \frac{d^4 p_n}{(2\pi)^4} (2\pi)^4 \delta^4(p_1 + p_2 + \dots + p_n) V(p_1, p_2, \dots, -(p_1 + p_2 + \dots p_{n-1})) e^{i(p_1 x_1 + p_2 x_2 + \dots p_n x_n)} \\
&= \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \dots \frac{d^4 p_{n-1}}{(2\pi)^4} V(p_1, p_2, \dots, -(p_1 + p_2 + \dots p_{n-1})) e^{i[p_1(x_1 - x_n) + p_2(x_2 - x_n) + \dots p_{n-1}(x_{n-1} - x_n)]}
\end{aligned} \tag{97}$$