FRG note

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Contents

1 Wetterich Equation

Consider the generating function which is cutoff at infrared:

$$Z_k[J] = \exp(W_k[J]) = \int [d\phi] \exp\{-S[\phi] - \Delta S_k[\phi]] + J^a \phi_a\} \tag{1}$$

The ϕ stands for every kind of field here. Index a stands for every degree of freedom, including different fields, different components of a field. Such as the space-time coordinates or momentum index. $S[\phi]$ is the classical action, J^a is the source of ϕ_a. $\Delta S_k[\phi]$ is the infrared cutoff, its function is to cutoff the quantum fluctuation at $p^2 \leq k^2$ and keep the fluctuation at $p^2 > k^2$ invariant. We usually choose the form of quadratic term (mass term) to achieve the infrared cutoff.

$$\Delta S_k[\phi] = \frac{1}{2} \phi_a R_k^{ab} \phi_b \tag{2}$$

with $R_k^{ab} = R_k^{ba}$ (a, b is the index of the boson), $R_k^{ab} = -R_k^{ba}$ (a, b is the index of the fermion). Here we give a example of a simple scalar field. In the coordinate space:

$$\Delta S_k[\phi] = \frac{1}{2} \int d^4x d^4y \varphi(x) R_k(x,y) \varphi(y) \tag{3}$$

So

$$R_k(x,y) = \int \frac{d^4p}{(2\pi)^4} R_k(p,-p) e^{ip(x-y)}$$
$$= \int \frac{d^4p}{(2\pi)^4} R_k(p) e^{ip(x-y)} \tag{4}$$

$$\varphi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \varphi(p) \tag{5}$$

$$\Delta S_k[\varphi] = \frac{1}{2} \int d^4x d^4y \varphi(x) R_k(x,y) \varphi(y)$$
$$= \frac{1}{2} \int d^4x d^4y \int \frac{d^4p_1}{(2\pi)^4} e^{ip_1 x} \varphi(p_1) \int \frac{d^4p}{(2\pi)^4} R_k(p) e^{ip(x-y)} \int \frac{d^4p_2}{(2\pi)^4} e^{ip_2 y} \varphi(p_2)$$
$$= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \varphi(-p) R_k(p) \varphi(p)$$
$$= \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \varphi(-q) R_k(q) \varphi(q) \tag{6}$$

For the determined q, the regulator $R_k(q)$ satisfies the following properties

$$R_{k\to\infty}(q) \to \infty \quad R_{k\to 0}(q) \to 0 \tag{7}$$

To suppress the fluctuation at $q^2 < k^2$ and keep the fluctuation at high momentum unchanging, we should choose

$$R_k(q)|_{q^2 < k^2} \sim k^2, \quad R_k(q)|_{q^2 > k^2} \sim 0 \tag{8}$$

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for example

$$R_k(q) \sim \frac{q^2}{e^{\frac{q^2}{k^2}} - 1} \tag{9}$$

of course, we can choose other forms of the regulator. From (1) we can get

$$\frac{\delta W_k[J]}{\delta J^a} = \frac{1}{Z_k} \frac{\delta Z_k[J]}{\delta J^a}$$

$$= \frac{1}{Z_k} \int [d\phi] \phi_a \exp\{-S[\phi] - \Delta S_k[\phi] + J^a \phi_a\} \tag{10}$$

$$= \langle \phi_a \rangle$$

In the below discussing, we replace $\langle \phi_a \rangle$ with ϕ_a and

$$\frac{\delta^2 W_k[J]}{\delta J^b \delta J^a} = \langle \phi_b \phi_a \rangle_c \equiv G_{ba}^k \tag{11}$$

index c stands for the connected diagram, G is the propagator that depends on the scale k. Now we do the Legendre transformation on the generated functional of connected diagram, then we can obtain the generated functional of one-particle irreducible diagram, that is the effective action

$$\Gamma_k[\phi] = -W_k[J] + J^a \phi_a - \Delta S_k[\phi] \tag{12}$$

Beware here $\phi_a \equiv \langle \phi_k \rangle$. In order to consider the boson and fermion together, we introduce the following symbols

$$J^a \phi_a = r_b^a \phi_a J^b \tag{13}$$

with

$$r_b^a = (-1)^{ab} \delta_b^a \tag{14}$$

$$(-1)^{ab} \equiv \begin{cases} -1, & for \quad a,b \quad Fermionic \\ 1, & for \quad a,b \quad Bosonic \end{cases} \tag{15}$$

So from (12) we obtain

$$\frac{\delta(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta \phi_a} = r_b^a J^b \tag{16}$$

Differentiate J on both sides of the above formula

$$\frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta J^b \delta \phi_a} = r_b^a \tag{17}$$

$$l.h.s = \frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta J^b \delta \phi_a}$$

$$= \frac{\delta \phi_c}{\delta J^b} \frac{\delta^2(\Gamma_k[\phi] + \Delta S_k[\phi])}{\delta \phi_c \delta \phi_a} \tag{18}$$

$$= G_{bc}^k \frac{\Gamma_k[\phi] + \Delta S_k[\phi]}{\delta \phi_c \delta \phi_a}$$

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We obtain

$$G^k_{bc} \frac{\Gamma_k[\phi] + \Delta S_k[\phi]}{\delta\phi_c \delta\phi_a} = r^a_b \tag{19}$$

$$\frac{\Gamma_k[\phi] + \Delta S_k[\phi]}{\delta\phi_c \delta\phi_a} = (\Gamma^{(2)}_k[\phi] + R_k)^{ca} \tag{20}$$

then

$$G^k_{bc}(\Gamma^{(2)}_k[\phi] + R_k)^{ca} = r^a_b G^k_{bc} = r^a_b (\Gamma^{(2)}_k[\phi] + R_k)^{-1}_{ac} \tag{21}$$

Now we calculate $\partial_t W_k[J] = k\frac{\partial}{\partial k}W_k[J]$

$$
\begin{aligned}
\partial_t W_k[J] =& \partial_t \ln Z_k[J] \\[2mm]
=& \frac{1}{Z_k}\partial_t Z_k[J] \\[2mm]
=& \frac{1}{Z_k}\int [d\phi](-\partial_t\Delta S_k[\phi])e^{-S[\phi]-\Delta S_k[\phi]+J^a\phi_a} \\[2mm]
=& -\frac{1}{Z_k}\int [d\phi]\frac{1}{2}\phi_a\partial_t R^{ab}_k\phi_b e^{-S[\phi]-\Delta S_k[\phi]+J^a\phi_a} \\[2mm]
=& -\frac{1}{2}\langle\phi_a\phi_b\rangle\partial_t R^{ab}_k \\[2mm]
=& -\frac{1}{2}(\langle\phi_a\phi_b\rangle_c + \langle\phi_a\rangle\langle\phi_b\rangle)\partial_t R^{ab}_k
\end{aligned}
\tag{22}
$$

we replace $\langle\phi_a\rangle$ with ϕ_a again

$$\langle\phi_a\phi_b\rangle_c = \begin{cases} \langle\phi_b\phi_a\rangle, & for \quad a,b \quad Bosonic \\ -\langle\phi_b\phi_a\rangle, & for \quad a,b \quad Fermionic \end{cases} \tag{23}$$

Then we obtain

$$\partial_t W_k[J] = -\frac{1}{2}STrG_k(\partial_t R_k) - \frac{1}{2}\phi_a\partial_t R^{ab}_k\phi_b \tag{24}$$

STr is super trace, including the trace of every field and dispersed, continuously degree of freedom. For the fermion a negative sign should be added. Finally, from (12) and (24) we obtain

$$
\begin{aligned}
\partial_t \Gamma_k[\phi] =& -\partial_t W_k[J] - \partial_t \Delta S_k[\phi] \\[2mm]
=& \frac{1}{2}\phi_a\partial_t R^{ab}_k\phi_b - \frac{1}{2}\phi_a\partial_t R^{ab}_k\phi_b \\[2mm]
=& \frac{1}{2}STrG_k(\partial_t R_k)
\end{aligned}
\tag{25}
$$

This is the Wetterich equation.

2 QCD

In this section, we introduce the application of FRG in QCD. We start from the effective action that depends on the infrared cutoff scale k

$$
\begin{aligned}
\Gamma_k = \int d^4 x \{ & \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + Z_{c,k} (\partial_\mu \bar{c}^a) D^{ab}_\mu c^b + \frac{1}{2\xi} (\partial_\mu A^a_\mu) \\
& + Z_{q,k} \bar{q} (\gamma_\mu D_\mu) q - \lambda_{q,k} [(\bar{q} T^0 q)^2 - (\bar{q} \gamma_5 T^a q)^2] \\
& + h_k [\bar{q} (i\gamma_5 T^a \pi^a + T^0 \sigma) q] + \frac{1}{2} Z_{\phi,k} (\partial_\mu \phi)^2 \}
\end{aligned}
\tag{26}
$$

In the effective action

$$
\begin{aligned}
& \phi = (\sigma, \pi^a), \rho = \frac{1}{2} \phi^2 \\
& D_\mu = \partial_\mu - i Z^{\frac{1}{2}}_{A,k} g_k A^a_\mu t^a \\
& D^{ab}_\mu = \partial_\mu \delta^{ab} + Z^{\frac{1}{2}}_{A,k} g_k f^{acb} A^c_\mu
\end{aligned}
\tag{27}
$$

In the definition above $T^a (a = 1, 2, ..., N_f^2 - 1)$ is the generator of flavor space with $T^0 = \frac{1}{\sqrt{2N_f}} \mathbb{1}$; $t^a (a = 1, 2, ..., N_c^2 - 1)$ is the generator of color space.

$$
\begin{aligned}
& F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] = F^a_{\mu\nu} t^a \\
& F^a_{\mu\nu} = Z^{\frac{1}{2}}_{A,k} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + Z^{\frac{1}{2}}_{A,k} g_k f^{abc} A^b_\mu A^c_\nu)
\end{aligned}
\tag{28}
$$

We rewrite the Wetterich equation as below

$$
\partial_t \Gamma_k[\Phi] = \frac{1}{2} STr \widetilde{\partial_t} \ln(\Gamma^{(2)}_k + R_k)
\tag{29}
$$

the tilde on the ∂_t stands for the derivation only works on the regulator R_k.

$$
(\Gamma^{(2)}_k)_{ab} = \frac{\overrightarrow{\delta}}{\delta \Phi^T_a} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \Phi_b}
\tag{30}
$$

the definition of Φ is

$$
\Phi = \begin{pmatrix} A(q) \\ \sigma(q) \\ \pi(q) \\ q(q) \\ \bar{q}^T(q) \end{pmatrix}
\tag{31}
$$

$$
\Phi^T = \left(A^T(-q), \sigma(-q), \pi(-q), q^T(q), \bar{q}(q) \right)
$$

the fluctuation matrix in the (29) can be rewrite as

$$
\Gamma^{(2)}_k + R_k = P + F
\tag{32}
$$

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The matrix P contains the propagators and regulators ; matrix F contains the dependence of all kind of fields. So we can expanse the (29) with the number of the fields

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} STr \widetilde{\partial_t} ln(P+F)$$

$$= \frac{1}{2} STr \widetilde{\partial_t} ln[P(1+\frac{1}{P}F)]$$

$$\sim \frac{1}{2} STr \widetilde{\partial_t} ln(1+\frac{1}{P}F) \tag{33}$$

$$= \frac{1}{2} STr\{\widetilde{\partial_t}[\frac{1}{P}F - \frac{1}{2}(\frac{1}{P}F)^2 + \frac{1}{3}(\frac{1}{P}F)^3 - \frac{1}{4}(\frac{1}{P}F)^4 + \cdots]\}$$

$$= \frac{1}{2} STr\widetilde{\partial_t}(\frac{1}{P}F) - \frac{1}{4} STr\widetilde{\partial_t}(\frac{1}{P}F)^2 + \frac{1}{6} STr\widetilde{\partial_t}(\frac{1}{P}F)^3 - \frac{1}{8} STr\widetilde{\partial_t}(\frac{1}{P}F)^4 + \cdots$$

then we take the contribution of the lowest order in

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} STr \widetilde{\partial_t} ln(P+F)$$

$$= \frac{1}{2} STr \widetilde{\partial_t} ln[P(1+\frac{1}{P}F)]$$

$$= \frac{1}{2} STr \widetilde{\partial_t}[lnP + ln(1+\frac{1}{P}F)] \tag{34}$$

$$= \frac{1}{2} STr \widetilde{\partial_t} lnP + \frac{1}{2} STr \widetilde{\partial_t} ln(1+\frac{1}{P}F)$$

$$= \frac{1}{2} STr \widetilde{\partial_t} lnP + \frac{1}{2} STr\widetilde{\partial_t}(\frac{1}{P}F) - \frac{1}{4} STr\widetilde{\partial_t}(\frac{1}{P}F)^2 + \frac{1}{6} STr\widetilde{\partial_t}(\frac{1}{P}F)^3 - \frac{1}{8} STr\widetilde{\partial_t}(\frac{1}{P}F)^4 + \cdots$$

2.1 Meson field propagator

The meson propagator part of the effective potential and its Fourier transform

$$\int d^4x \frac{1}{2} Z_{\phi,k}(\partial_\mu \phi)^2$$

$$= \int \frac{d^4q}{(2\pi)^4} \frac{1}{2} Z_{\phi,k}\phi(-q)q^2\phi(q) \tag{35}$$

then consider the contribution of the meson mass

$$\Gamma_k^{\sigma\sigma}(q',q) \equiv \frac{\delta^2 \Gamma_k}{\delta\sigma(q')\delta\sigma(q)}$$

$$= (Z_{\phi,k}q^2 + m_\sigma^2)(2\pi)^4\delta^4(q+q') \tag{36}$$

The corresponding cutoff function is

$$R_k^{\sigma\sigma}(q',q) = Z_{\phi,k}\vec{q}^2 r_B(\frac{\vec{q}^2}{k^2})(2\pi)^4\delta^4(q+q') \tag{37}$$

here we use the 3d regulator which is convenient to the calculation of the finite temperature. The $r_B(x) = (\frac{1}{x} - 1)\theta(1-x)$ is an optimized regulator.

Then we treat the Pi meson in the same way

$$\Gamma_{k,ij}^{\pi\pi}(q',q) \equiv \frac{\delta^2 \Gamma_k}{\delta\pi_i(q')\delta\pi_j(q)}$$

$$= \delta_{ij}(Z_{\phi,k}q^2 + m_\pi^2)(2\pi)^4\delta^4(q+q') \tag{38}$$

The cutoff function is

$$R_{k,ij}^{\pi\pi} = \delta_{ij}R_k^{\sigma\sigma} \tag{39}$$

In the calculation of finite temperature we have $q_0 = 2\pi nT$ in which the n is integer.

2.2 Quark field propagator

The quark propagator part and its Fourier transform

$$\int d^4x Z_{q,k}\bar{q}(x)(\gamma_\mu\partial_\mu + \frac{m_f}{Z_{q,k}})q(x)$$
$$= \int \frac{d^4q}{(2\pi)^4}Z_{q,k}\bar{q}(q)(i\slashed{q} + \frac{m_f}{Z_{q,k}})q(q) \tag{40}$$

The Fourier transform of the quark and anti-quark fields

$$q(x) = \int \frac{d^4q}{(2\pi)^4}e^{iqx}q(q)$$
$$\bar{q}(x) = \int \frac{d^4q}{(2\pi)^4}e^{-iqx}\bar{q}(q) \tag{41}$$

The derivation of the effective action amount of the quark fields is

$$\Gamma_{k,ij}^{\bar{q}q}(q',q) \equiv \frac{\overrightarrow{\delta}}{\delta q_i(q')}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\bar{q}_j(q)}$$
$$= (Z_{q,k}i\slashed{q}_{ij} + m_f\delta_{ij})(2\pi)^4\delta^4(q'-q)$$
$$= (Z_{q,k}iq_\mu(\gamma_\mu)_{ij} + m_f\delta_i j)(2\pi)^4\delta^4(q'-q) \tag{42}$$

$$\Gamma_{k,ij}^{q\bar{q}} \equiv \frac{\overrightarrow{\delta}}{\delta q_i(q')}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\bar{q}_j(q)}$$
$$= -(Z_{q,k}iq_\mu(\gamma_\mu)_{ji} + m_f\delta_{ji})(2\pi)^4\delta^4(q'-q) \tag{43}$$

The corresponding regulator function is

$$R_{k,ij}^{\bar{q}q}(q',q) = Z_{q,k}i\vec{q}\cdot\vec{\gamma}_{ij}r_F(\frac{\vec{q}^2}{k^2})(2\pi)^4\delta^4(q'-q) \tag{44}$$

with

$$r_F(x) = (\frac{1}{\sqrt{x}} - 1)\theta(1-x) \tag{45}$$

2.3 Gluon field propagator

The differential form of the effective action is

$$(\Gamma_k^{AA})_{\mu\nu}^{ab}(q',q) \equiv \frac{\delta^2\Gamma_k}{\delta A_\mu^a(q')\delta A_\nu^b(q)}$$
$$= [Z_{A,k}q^2(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{q^2}{\xi}(\frac{q_\mu q_\nu}{q^2})]\delta^{ab}(2\pi)^4\delta^4(q'+q) \tag{46}$$

The regulator function is

$$(R_k^{AA})_{\mu\nu}^{ab}(q',q) = [Z_{A,k}\vec{q}^2 r_B(\frac{\vec{q}^2}{k^2})(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{\vec{q}^2}{\xi} r_B(\frac{\vec{q}}{k^2})(\frac{q_\mu q_\nu}{q^2})]\delta^{ab}(2\pi)^4\delta^4(q'+q) \tag{47}$$

The propagator of the gluon is

$$(G_k^{AA})_{\mu\nu}^{ab}(q',q) = [\frac{1}{Z_{A,k}(q_0^2 + \vec{q}^2(1+r_B))}(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{\xi}{q_0^2 + \vec{q}^2(1+r_B)}\frac{q_\mu q_\nu}{q^2}]\delta^{ab}(2\pi)^4\delta^4(q'+q) \tag{48}$$

In the following calculation we adopt the Landau gauge $\xi = 0$, then we can obtain the matrix P

$P =$

$$\begin{pmatrix} Z_{A,k}(q_0^2+\vec{q}^2(1+r_B))(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2})\delta_{ab} & 0 & 0 & 0 & 0 \\ 0 & Z_{\phi,k}(q_0^2+\vec{q}^2(1+r_B))+m_\sigma^2 & 0 & 0 & 0 \\ 0 & 0 & [Z_{\phi,k}(q_0^2+\vec{q}^2(1+r_B))+m_\pi^2]\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & -(Z_{q,k}i(q_0r_0+\vec{q}\cdot\vec{r}(1+r_F))+m_f)^T \\ 0 & 0 & 0 & Z_{q,k}i(q_0r_0+\vec{q}\cdot\vec{r}(1+r_F))+m_f & 0 \end{pmatrix} \tag{49}$$

And then we can obtain the propagator matrix

$$\frac{1}{P} = \begin{pmatrix} (G_k^{AA})_{\mu\nu}^{ab} & 0 & 0 & 0 & 0 \\ 0 & G_k^\sigma & 0 & 0 & 0 \\ 0 & 0 & (G_k^\pi)_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_k^{q\bar{q}} \\ 0 & 0 & 0 & G_k^{\bar{q}q} & 0 \end{pmatrix} \tag{50}$$

the definition of the matrix elements are

$$\begin{aligned} (G_k^{AA})_{\mu\nu}^{ab} &= \frac{1}{Z_{A,k}(q_0^2+\vec{q}^2(1+r_B))}(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})\delta_{ab} \\ G_k^\sigma &= \frac{1}{Z_{\phi,k}(q_0^2+\vec{q}^2(1+r_B))+m_\sigma^2} \\ (G_k^\pi)_{ij} &= \frac{1}{Z_{\phi,k}(q_0^2+\vec{q}^2(1+r_B))+m_\pi^2}\delta_{ij} \\ G_k^{q\bar{q}} &= \frac{-Z_{q,k}i(q_0r_0+\vec{q}\cdot\vec{r}(1+r_F))+m_f}{Z_{q,k}^2(q_0^2+\vec{q}^2(1+r_F)^2)+m_f^2} \\ G_k^{\bar{q}q} &= -(G_k^{q\bar{q}})^T \end{aligned} \tag{51}$$

2.4 Gluon vertex

First is the gauge part

$$\begin{aligned} \Gamma_k &\sim \frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a \\ &= \frac{1}{4}Z_{A,k}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + Z_{A,k}^{\frac{1}{2}}g_k f^{abc}A_\mu^b A_\nu^c)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + Z_{A,k}^{\frac{1}{2}}g_k f^{ab'c'}A_\mu^{b'} A_\nu^{c'}) \\ &\sim \frac{1}{2}Z_{A,k}^{\frac{3}{2}}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)g_k f^{abc}A_\mu^b A_\nu^c + \frac{1}{4}Z_{A,k}^2 g_k^2 f^{abc} f^{ab'c'}A_\mu^b A_\nu^c A_\mu^{b'} A_\nu^{c'} \end{aligned} \tag{52}$$

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here we let

$$\Gamma_1 = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) f^{abc} A_\mu^b A_\nu^c \tag{53}$$

then we can calculate the derivative

$$\frac{\delta \Gamma_1}{\delta A_\mu^a} = -2(\partial_\nu)_a f^{abc} A_\mu^b A_\nu^c + 2 f^{abc} (\partial_\nu A_\mu^b - \partial_\mu A_\nu^b) A_\nu^c \tag{54}$$

$$\frac{\delta^2 \Gamma_1}{\delta A_\mu^a \delta A_\nu^b} = -2(\partial_\rho)_a f^{abc} \delta_{\mu\nu} A_\rho^c - 2(\partial_\nu)_a f^{acb} A_\mu^c + 2 f^{abc} (\partial_\rho)_b \delta_{\mu\nu} A_\rho^c + 2 f^{acb} \partial_\nu A_\mu^c - 2 f^{abc} (\partial_\mu)_b A_\nu^c - 2 f^{acb} \partial_\mu A_\nu^c$$

$$= 2\delta_{\mu\nu} f^{abc} [(\partial_\rho)_b - (\partial_\rho)_a] A_\rho^c + 2 f^{abc} [(\partial_\nu)_a A_\mu^c - (\partial_\mu)_b A_\nu^c] + 2 f^{abc} (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c) \tag{55}$$

$$\frac{\delta^3 \Gamma_1}{\delta A_\mu^a \delta A_\nu^b \delta A_\rho^c} = 2\delta_{\mu\nu} f^{abc} [(\partial_\rho)_b - (\partial_\rho)_a] + 2\delta_{\mu\nu} f^{abc} [(\partial_\nu)_a \delta_{\mu\rho} - (\partial_\mu)_b \delta_{\nu\rho}] + 2 f^{abc} [(\partial_\mu)_c \delta_{\rho\nu} - (\partial_\nu)_c \delta_{\mu\rho}]$$

$$= 2 f^{abc} \left\{ \delta_{\mu\nu} [(\partial_\rho)_b - (\partial_\rho)_a] + \delta_{\mu\rho} [(\partial_\nu)_a - (\partial_\nu)_c] + \delta_{\rho\nu} [(\partial_\mu)_c - (\partial_\mu)_b] \right\} \tag{56}$$

Then we assume

$$\Gamma_2 = f^{abc} f^{ab'c'} A_\mu^b A_\nu^c A_\mu^{b'} A_\nu^{c'}$$

$$= f^{abc} f^{a'b'c} A_\mu^a A_\nu^b A_\mu^{a'} A_\nu^{b'} \tag{57}$$

And the derivation can be written as

$$\frac{\delta \Gamma_2}{\delta A_\mu^a} = 4 f^{abc} f^{a'b'c} A_\nu^b A_\mu^{a'} A_\nu^{b'} \tag{58}$$

$$\frac{\delta \Gamma_2}{\delta A_\mu^a A_\nu^b} = 4 f^{abc} f^{a'b'c} A_\mu^{a'} A_\nu^{b'} + 4 f^{ab''c} f^{bb'c} A_\rho^{b''} \delta_{\mu\nu} A_\rho^{b'} + 4 f^{ab''c} f^{a'bc} A_\nu^{b''} A_\mu^{a'}$$

$$= 4 f^{abc} f^{a'b'c} A_\mu^{a'} A_\nu^{b'} + 4\delta_{\mu\nu} f^{ab'c} f^{bb''c} A_\rho^{b'} A_\rho^{b''} - 4 f^{ab''c} f^{ba'c} A_\mu^{a'} A_\nu^{b''} \tag{59}$$

So now we can obtain the F matrix elements of the glon part

$$(F_k^{AA})_{\mu\nu}^{ab} = \frac{\delta^2}{A_\mu^a A_\nu^b} \left(\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \right)$$

$$= Z_{A,k}^{\frac{3}{2}} g_k f^{abc} \left\{ \delta_{\mu\nu} [(\partial_\rho)_b - (\partial_\rho)_a] A_\rho^c + [(\partial_\nu)_a A_\mu^c - (\partial_\mu)_b A_\nu^c] + (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c) \right\} \tag{60}$$

$$+ Z_{A,k}^2 g_k^2 [f^{abc} f^{a'b'c} A_\mu^{a'} A_\nu^{b'} + \delta_{\mu\nu} f^{ab'c} f^{bb''c} A_\rho^{b'} A_\rho^{b''} - f^{ab''c} f^{ba'c} A_\mu^{a'} A_\nu^{b''}]$$

From the expression of the effective action we find that the interaction between the gluon and meson vanished

$$F_k^{A\sigma} = 0 \tag{61}$$

$$F_k^{A\pi} = 0 \tag{62}$$

Now we derive the gluon-quark vertex. The gluon-quark part of the effective action reads

$$\Gamma_k \sim Z_{q,k} \bar{q} \gamma_\mu D_\mu q \tag{63}$$

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The expression of the convariant derivative is

$$D_\mu = \partial_\mu - iZ_{A,k}^{\frac{1}{2}} g_k A_\mu^a t^a \tag{64}$$

So we get

$$\Gamma_k \sim Z_{q,k} \bar{q} \gamma_\mu (-iZ_{A,k}^{\frac{1}{2}} g_k A_\mu^a t^a) q$$
$$= -iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k A_\mu^a \bar{q} \gamma_\mu t^a q \tag{65}$$

The gluon-quark vertex can be obtained from the action above

$$(F_k^{Aq})_{\mu j}^a = \frac{\overrightarrow{\delta}}{\delta A_\mu^a} \Gamma_k \frac{\overleftarrow{\delta}}{\delta q_j} = -iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\bar{q} \gamma_\mu t^a)_j$$
$$\Longrightarrow (F_k^{Aq})_\mu^a = -iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\bar{q} \gamma_\mu t^a) \tag{66}$$

$$(F_k^{A\bar{q}})_{\mu j}^a = \frac{\overrightarrow{\delta}}{\delta A_\mu^a} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \bar{q}_j} = iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k {}_j (\gamma_\mu t^a q)$$
$$\Longrightarrow (F_k^{A\bar{q}})_\mu^a = iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k q^T (\gamma_\mu t^a)^T = -(F_k^{\bar{q}A})_\mu^{a\,T} \tag{67}$$

$$(F_k^{qA})_{i\nu}^b = \frac{\overrightarrow{\delta}}{\delta q_i} \Gamma_k \frac{\overleftarrow{\delta}}{\delta A_\nu^b} = iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\bar{q} \gamma_\nu t^b)_i$$
$$\Longrightarrow (F_k^{qA})_\mu^a = iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\gamma_\mu t^a)^T \bar{q}^T = -(F_k^{Aq})_\mu^{a\,T} \tag{68}$$

$$(F_k^{\bar{q}A})_{i\nu}^a = \frac{\overrightarrow{\delta}}{\delta \bar{q}_i} \Gamma_k \frac{\overleftarrow{\delta}}{\delta A_\nu^b} = iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k {}_i (\gamma_\nu t^b q)$$
$$\Longrightarrow (F_k^{\bar{q}A})_\mu^a = -iZ_{q,k} Z_{A,k}^{\frac{1}{2}} g_k (\gamma_\mu t^a q) \tag{69}$$

2.5   Meson vertex

The corresponding effective action of the meson part is

$$\Gamma_k \sim V_k(\rho) - c\sigma \tag{70}$$

the ρ stands for the square of the meson field

$$\rho = \frac{1}{2} \phi_i^2 \tag{71}$$

and the definition of the meson field are

$$\phi_{i=0} = \sigma, \ \ \phi_i = \pi_i \ \ i = 1,2,3 \tag{72}$$

Then we define

$$\begin{cases} \phi_i = \bar{\phi}_0 + \sigma, & for \ i = 0 \\ \phi_i = \pi_i, & for \ i = 1,2,3 \end{cases} \tag{73}$$

The derivation of the effective action

$$\frac{\delta \Gamma_k}{\delta \phi_i} = V_k'(\rho) \phi_i - c\delta_{i,0} \tag{74}$$

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$$\frac{\delta^2 \Gamma_k}{\delta \phi_i \delta \phi_j} = V_k^{'}(\rho)\delta_{i,j} + V_k^{''}(\rho)\phi_i\phi_j \tag{75}$$

Now expand the equation above at the point $\phi_i = \bar{\phi}_i$

$$\bar{\phi}_i = \begin{cases} \bar{\phi}_0 & i = 0 \\ 0 & i = 1, 2, 3 \end{cases} \tag{76}$$

$$(\Gamma_k^{\phi\phi})_{ij} = (\Gamma_k^{\phi\phi})_{ij}\bigg|_{\bar{\phi}} + \frac{\delta^3 \Gamma_k}{\delta \phi_i \delta \phi_j \delta \phi_k}\bigg|_{\bar{\phi}} (\phi_k - \bar{\phi}_k) + \frac{1}{2}\frac{\delta^4 \Gamma_k}{\delta \phi_i \delta \phi_j \delta \phi_k \delta \phi_l}\bigg|_{\bar{\phi}} (\phi_k - \bar{\phi}_k)(\phi_l - \bar{\phi}_l) + \ldots \tag{77}$$

Here we only consider to the two derivative of the effective potential

$$\frac{\delta^3 \Gamma_k}{\delta \phi_i \delta \phi_j \delta \phi_k} = V_k^{''}(\rho)\phi_k\delta_{ij} + V_k^{''}(\rho)(\delta_{ik}\phi_i + \delta_{jk}\phi_i) + V_k^{'''}(\rho)\phi_i\phi_j\phi_k$$
$$= V_k^{''}(\rho)(\delta_{ij}\phi_k + \delta_{ik}\phi_j + \delta_{jk}\phi_i) + V_k^{'''}(\rho)\phi_i\phi_j\phi_k \tag{78}$$

$$\frac{\delta^4 \Gamma_k}{\delta \phi_i \delta \phi_j \delta \phi_k \delta \phi_l} = V_k^{''}(\rho)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\phi_{il})$$
$$+ V_k^{'''}(\rho)(\delta_{ij}\phi_k\phi_l + \delta_{ik}\phi_j\phi_l + \delta_{jk}\phi_i\phi_l)$$
$$+ V_k^{'''}(\rho)(\delta_{il}\phi_j\phi_k + \delta_{jl}\phi_i\phi_k + \delta_{kl}\phi_i\phi_j)$$
$$+ V_k^{''''}(\rho)\phi_i\phi_j\phi_k\phi_l$$
$$= V_k^{''}(\rho)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\phi_{il}) \tag{79}$$
$$+ V_k^{(3)}(\rho)(\delta_{ij}\phi_k\phi_l + \delta_{ik}\phi_j\phi_l + \delta_{jk}\phi_i\phi_l + \delta_{il}\phi_j\phi_k + \delta_{jl}\phi_i\phi_k + \delta_{kl}\phi_i\phi_j)$$
$$+ V_k^{(4)}(\rho)\phi_i\phi_j\phi_k\phi_l$$

The definition of the Sigma meson mass is

$$m_\sigma^2 = (\Gamma_k^{\sigma\sigma})\big|_{\bar{\phi}} = (\Gamma_k^{\phi\phi})_{00}\big|_{\bar{\phi}} = V_k^{'}(\rho) + V_k^{''}(\rho)\bar{\phi}_0\bar{\phi}_0$$
$$= V_k^{'}(\rho) + 2\rho V_k^{''}(rho) \tag{80}$$

And the Pion mass is

$$m_\pi^2 = (\Gamma_k^{\pi\pi})_{ij}\big|_{\bar{\phi}} = (\Gamma_k^{\phi\phi})_{ij}\big|_{\bar{\phi}} = V_k^{'}(\rho)\delta_{ij} \tag{81}$$

Then the other derivation part of the Pion and Sigma meson can be written as

$$\left(\Gamma_k^{\sigma\pi}\right)\big|_{\bar{\phi}} = 0$$

$$\frac{\delta^3 \Gamma_k}{\delta\sigma\delta\sigma\delta\sigma}\bigg|_{\bar{\phi}} = V_k''(\rho)(3\bar{\phi}_0) + V_k^{(3)}(\rho)\bar{\phi}_0^3$$

$$= 3(2\rho)^{\frac{1}{2}} V_k''(\rho) + (2\rho)^{\frac{3}{2}} V_k^{(3)}(\rho)$$

$$\frac{\delta^3 \Gamma_k}{\delta\sigma\delta\sigma\delta\pi_k}\bigg|_{\bar{\phi}} = 0$$

$$\frac{\delta^3 \Gamma_k}{\delta\sigma\delta\pi_j\delta\sigma}\bigg|_{\bar{\phi}} = 0 \tag{82}$$

$$\frac{\delta^3 \Gamma_k}{\delta\sigma\delta\pi_j\delta\pi_k}\bigg|_{\bar{\phi}} = V_k''(\rho)\delta_{jk}\bar{\phi}_0 = \delta_{jk}(2\rho)^{\frac{1}{2}} V_k''(\rho)$$

$$\frac{\delta^3 \Gamma_k}{\delta\pi_i\delta\pi_j\delta\sigma}\bigg|_{\bar{\phi}} = \delta_{ij}(2\rho)^{\frac{1}{2}} V_k''(\rho)$$

$$\frac{\delta^3 \Gamma_k}{\delta\pi_i\delta\pi_j\delta\pi_k}\bigg|_{\bar{\phi}} = 0$$

If we only take the lowest derivation into account

$$F_k^{\sigma\sigma} = \frac{\delta^3 \Gamma_k}{\delta\sigma^3}\bigg|_{\bar{\phi}} \sigma + \frac{\delta^3 \Gamma_k}{\delta\sigma^2\delta\pi_k}\bigg|_{\bar{\phi}} \pi_k \tag{83}$$

$$= \left[3(2\rho)^{\frac{1}{2}} V_k^{(2)}(\rho) + (2\rho)^{\frac{3}{2}} V_k^{(3)}(\rho)\right]\sigma$$

$$\left(F_k^{\sigma\pi}\right)_j = \frac{\delta^3 \Gamma_k}{\delta\sigma\delta\pi_j\delta\sigma}\bigg|_{\bar{\phi}} \sigma + \frac{\delta^3 \Gamma_k}{\delta\sigma\delta\pi_j\delta\pi_k}\bigg|_{\bar{\phi}} \pi_k \tag{84}$$

$$= \delta_{jk}(2\rho)^{\frac{1}{2}} V_k^{(2)}(\rho)\pi_k = (2\rho)^{\frac{1}{2}} V_k^{(2)}(\rho)\pi_j$$

$$\left(F_k^{\pi\sigma}\right)_i = (2\rho)^{\frac{1}{2}} V_k^{(2)}(\rho)\pi_i \tag{85}$$

$$\left(F_k^{\pi\pi}\right)_{ij} = \frac{\delta^3 \Gamma_k}{\delta\pi_i\delta\pi_j\delta\sigma}\bigg|_{\bar{\phi}} = \delta_{ij}(2\rho)^{\frac{1}{2}} V_k^{(2)}(\rho)\sigma \tag{86}$$

And we define

$$V_{3\sigma}(\rho) = 3(2\rho)^{\frac{1}{2}} V_k^{(2)}(\rho) + (2\rho)^{\frac{3}{2}} V_k^{(3)}(\rho)$$

$$V_{2\pi\sigma} = (2\rho)^{\frac{1}{2}} V_k^{(2)}(\rho) \tag{87}$$

Then we can write the matrix elements as

$$F_k^{\sigma\sigma} = V_{3\sigma}(\rho)\sigma, \quad \left(F_k^{\sigma\pi}\right)_j = V_{2\pi\sigma}(\rho)\pi_j, \quad \left(F_k^{\pi\sigma}\right)_i = V_{2\pi\sigma}(\rho)\pi_i, \quad \left(F_k^{\pi\pi}\right)_{ij} = \delta_{ij}V_{2\pi\sigma}(\rho)\sigma \tag{88}$$

2.6 Quark-Meson Vertex

We can get the quark-meson interaction vertex by the derivation of the Yukawa part of the effective action

$$\Gamma_k \sim h_k\left[\bar{q}(i\gamma_5 T^a\pi^a + T^0\sigma)q\right] \tag{89}$$

$$(F_k^{\sigma q})_j = \frac{\overrightarrow{\delta}}{\delta\sigma}\Gamma_k\frac{\overleftarrow{\delta}}{\delta q_j} = h_k(\bar{q}T^0)_j \quad \Rightarrow \quad F_k^{\sigma q} = h_k\bar{q}T^0$$

$$(F_k^{\sigma\bar{q}})_j = \frac{\overrightarrow{\delta}}{\delta\sigma}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\bar{q}_j} = -h_{k\,j}(T^0 q) \quad \Rightarrow \quad F_k^{\sigma\bar{q}} = -h_k q^T T^{0T}$$

$$(F_k^{q\sigma})_i = \frac{\overrightarrow{\delta}}{\delta q_i}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\sigma} = -h_k(\bar{q}T^0)_i \quad \Rightarrow \quad F_k^{q\sigma} = -h_k T^{0T}\bar{q}^T$$

$$(F_k^{\bar{q}\sigma})_i = \frac{\overrightarrow{\delta}}{\delta\bar{q}_i}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\sigma} = h_{k\,i}(T^0 q) \quad \Rightarrow \quad F_k^{\bar{q}\sigma} = h_k T^0 q$$

$$(90)$$

A   Fourier transform of n-point function

Consider a general n-point function $V(x_1, x_2, ..., x_n)$, Its Fourier transformation can be written as:

$$V(x_1, x_2, ..., x_n) = \int \frac{d^4 p_1}{(2\pi)^4}\frac{d^4 p_2}{(2\pi)^4}...\frac{d^4 p_n}{(2\pi)^4}V(p_1, p_2, ...p_n)e^{i(p_1 x_1 + p_2 x_2 + ... p_n x_n)} \tag{91}$$

if V satisfies the following properties:

$$V(x_1, x_2, ...x_n) = V(x_1 - x_n, x_2 - x_n, ..., 0) \tag{92}$$

The value of V depends only on the relative value of the coordinates, so we can obtain

$$\begin{aligned}
V(p_1, p_2, ...p_n) &= \int d^4 x_1 d^4 x_2 ...d^4 x_n V(x_1, x_2, ...x_n)e^{-i(p_1 x_1 + p_2 x_2 + ... p_n x_n)}\\
&= \int d^4 x_1 d^4 x_2 ...d^4 x_n V(x_1 - x_n, x_2 - x_n, ...0)e^{-i(p_1 x_1 + p_2 x_2 + ... p_n x_n)}\\
&= \int d^4 x_1 d^4 x_2 ...d^4 x_n V(x_1, x_2, ...0)e^{-i[p_1(x_1 + x_n)p_2(x_2 + x_n) + ... p_{n-1}(x_{n-1} + x_n)]}\\
&= \int d^4 x_1 d^4 x_2 ...d^4 x_n V(x_1, x_2, ...0)e^{-i[p_1 x_1 + p_2 x_2 + ... p_{n-1}x_{n-1} + (p_1 + p_2 + ... + p_n)x_n]}\\
&= (2\pi)^4\delta^4(p_1 + p_2 + ...p_n)\int d^4 x_1 d^4 x_2 ...d^4 x_{n-1}V(x_1, x_2, ..., 0)e^{-i(p_1 x_1 + p_2 x_2 + ... p_{n-1}x_{n-1})}
\end{aligned} \tag{93}$$

So we can redefine $V(p_1, p_2, ...p_n)$ as

$$V(p_1, p_2, ...p_n) = (2\pi)^4\delta^4(p_1 + p_2 + ...p_n)V(p_1, p_2, ...p_{n-1}, -(p_1 + p_2 + ...p_{n-1})) \tag{94}$$

$$\begin{aligned}
V(x_1, x_2, ...x_n) &= \int \frac{d^4 p_1}{(2\pi)^4}\frac{d^4 p_2}{(2\pi)^4}...\frac{d^4 p_n}{(2\pi)^4}(2\pi)^4\delta^4(p_1 + p_2... + p_n)V(p_1, p_2, ... - (p_1, p_2 + ...p_{n-1}))e^{i(p_1 x_1 + p_2 x_2 + ... p_n x_n)}\\
&= \int \frac{d^4 p_1}{(2\pi)^4}\frac{d^4 p_2}{(2\pi)^4}...\frac{d^4 p_{n-1}}{(2\pi)^4}V(p_1, p_2, ... - (p_1, p_2 + ...p_{n-1}))e^{i[p_1(x_1 - x_n) + p_2(x_2 - x_n) + ... p_{n-1}(x_{n-1} - x_n)]}
\end{aligned} \tag{95}$$

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