A Conceptual Framework for Public Education Reform Analysis

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Abstract

This framework describes a plan to study different channels through which public education subsidy scheme has impacts on the aggregated economy. We intend to figure out relative importance of each channel through perturbation method. Then, we plan to find a formula of optimal regressivity of education subsidy expressed by sufficient statistics. Based on some evidence that we've got from calibration and welfare decomposition, we suppose the income effect may affect the most.

Keywords perturbation method; education subsidy; welfare decomposition; optimal public education;

1 Introduction

The distribution of human capital is very unbalanced, which is not conducive to social progress and development. To reallocate human capital, nowadays, the governments often adopt some non-linear subsidized redistribution education policies, where the redistribution of public education is skewed towards the lower income. That is, if a household has a low level of income/human capital, the corresponding public education investment will be more. These non-linear subsidized redistribution policies can be interpreted as "poverty alleviation through education". This may correspond to a policy of school funding equalization across local communities, such as poverty subsidy policies, grants and scholarships, or more generally of subsidizing differentially the education of rich and poor children. This realistic kind of public education scheme needs investigation, both on the equality and efficiency perspectives. Specifically, it is worthy to study how education subsidy influence human capital accumulation via income distribution and social welfare. Moreover, in the context of income taxes, since income is a proxy variable of human capital, it provides great convenience for our research.

In our previous work [1] , we mainly study how education subsidy influence human capital accumulation via income distribution. However, the quantifiable impact of education subsidy on social welfare and other aggregates like total public education expenditure remains unclear. If possible, the channels of impact also need to be identified. This is the main problem to be solved.

[1] "Poverty Alleviation through Public Education from an Income Inequality Perspective". See related work and codes on Github: https://github.com/yinuonino/Public-Education-Subsidy

1.1 Background and Research Purpose

Here we give some descriptions about our study on equality. To sum up, we investigates the education elasiticity and education subsidy scheme in an infinite-horizon Aiyagari-Bewley Huggett(ABH) economy with idiosyncratic human capital investment risk under linear income taxation. Through Mean-Field-Games (MFG) system combining Hamilton-Jacobi-Bellman (HJB) and Kolmogorov Forward (KF) equations, we theoretically show that the stationary human capital distribution displays a Pareto tail and the tail exponent depends on education elasiticity and taxation. Particularly, we show it follows an Inverse Gamma pattern. Futher, we prove consumption and saving behaviors' asymptotical linearity propositions of the high-income households.

Table 1: Models Fitting based on Top1% Income Distribution

	Income Partition							
Percentile	$0-20$	$20-40$	$40-60$	$60-80$	$80-90$	$90-95$	$95-99$	$99-100$
2007 SCF data	0.028	0.067	0.113	0.183	0.138	0.102	0.159	0.210
Subsidy model	0.008	0.020	0.045	0.121	0.157	0.163	0.276	0.210

Note: Data of income share (2007 SCF) is from Díaz-Giménez, Glover, and Ríos-Rull (2011).

Table 1 displays the wealth share moments we use and compares the moments in the data with those obtained simulating the model in terms of income distribution ranking. [1] The income distribution of our calibrated model matches the target moments in 2007 U.S. data [2] reasonably well. In this process, We find that idiosyncratic investment risk would bring more precautionary savings and there are both income effects and substitution effects of educational elasticity variation.

The purpose of our conceptual research is to study different channels through which the regressivity of public education subsidy has impacts on the economy and the relative importance of each channel. Then, we intend to find a formula of optimal regressivity of education subsidy expressed by sufficient statistics.

1.2 Main Plan and Rationale

We plan to do subsidy incidence analysis by implementing perturbation methods by taking derivatives of endogenous variables with respect to the regressivity of education subsidy. The basic idea of perturbation method is giving a minimal perturbation (for examle, 0.01) to a stationary state of the system mechanism you want to study(here it is education subsidy). Then calculate the effect derivatives through deviations.

The objective of subsidy incidence analysis is to characterize the first-order effects of locally reforming a given, potentially suboptimal, subsidy system on the distribution of income and social welfare. Existing literatures of incidence analysis include Saez(2001)[10] and Sachs et al.(2020) [11] studing nonlinear income taxation in an economy with endogenous wages, Park (2021)[7] studing

[1] Compared with the traditional tax-tranfer economy, our subsidy economy generate income distributions where increases in education subsidy scheme and compulsory public education would, as expected, make society more equal.

[2] Calculated From *Facts on the distributions of earnings, income, and wealth in the united states: 2007 update.* based on the income tail and find out the optimal linear tax rate.

taxation with Gateaux derivative for labor supply. [1]

Based on the perturbation method proposed by Schmitt-Grohé and Uribe (2001)[14], there are two groups of the endogenous variables in our incidence analysis. The first group consists of individual variables, such as consumption and savings, reflecting the behavior effect and the intertemporal effects. We plan to obtain these effects through finding derivatives of individual variables of the HJB equation with respect to the regressivity of education subsidy. The second group includes the aggregate variable (the aggregate human capital and the income distribution), reflecting the pecuniary externality and the inequality. We plan to obtain these effects through finding derivatives of the density function of the KF equation with respect to the regressivity of education subsidy. [2]

An operational plan of our research is as follows. From HJB and KF equations with governement budget balanced, we solve conumption and saving policy functions $c(h), s(h)$ and income density $f(h)$. Then, We manage to use an algorithm for numerically computing the subsidy-incidence on key variables $\hat{c}, \hat{s}, \hat{\beta}, \hat{v}$, and $\hat{f}$. [3] Based on these subsidy incidences $\hat{c}, \hat{s}$, and $\hat{\beta}$, we could solve out aggregate incidence on diffierent channels of social welfare decomposition. These channel-incidence results would be used to find the optimal regressivity of education subsidy is promoted by Uribe (2001) [14].

2 Research Design and Preliminary Conjectures

2.1 Set-up

Time is continuous, indexed by $t \in [0, \infty)$. There is a continuum of infinitely-lived households of unit mass, heterogeneous in human capital/income $h > 0$ which is exogenously determined ex ante. Households have identical constant relative risk aversion (CRRA) preference [4]. All uncertainty is purely idiosyncratic and, therefore, all aggregates are deterministic. Given linear income tax rate τ, the government taxes gross income $y(h)$ at tax rate τ and makes redistribution throught public education subsidy under a equilibrium state-dependently transfer scheme.

$$\max_{c(h),e(h)} E_0 \int_{t_0}^{\infty} e^{-\rho t} u(c) dt \tag{1}$$

s.t.

$$y(h) = h \tag{2}$$

$$c(h) + e(h) = (1 - \tau) y(h), \quad e(h) > 0 \tag{3}$$

[1]From former studies (like Chetty (2009) on tax rate) we know that the optimal solution can be derived in closed form in terms of some sufficient statistics, like labor supply elasticity, the elasticity of substitution, or the Pareto parameter of the tail of the income distribution.

[2]KF equation is the technique to connect the macro description of the economy to the individual income accumulation process since the coeffcient of the KF equation comes from the household's policy functions.

[3]The meaning of these symbols will be explained later. In what follows as well as else where in the proposal, we use the following notation: for any variable $\hat{m}$ in a short-hand notation for $\frac{dm}{d\zeta}$, i.e. $\hat{m}$ gives the change in m in response to the regressive education subsidy ζ; for any variable $\Lambda\left(c(h)\right)$ is a short-hand notation for $\hat{c}(h)/c(h)$, i.e. $\Lambda\left(c(h)\right)$ gives the percentage deviation in $c(h)$ in response to the regressivity of education subsidy ζ.

[4]$u(c_t) = \frac{c_t^{1-\eta}-1}{1-\eta}$ with relative risk aversion η over utility flows from future consumption discounted by factor (interest rate) ρ

The HJB equation reads $\rho v(h) = \max_{c(h)} u(c(h)) + v'(h)s(h) + \frac{\sigma^2 h^2}{2} v''(h)$, where first-order condition (F.O.C.) solves $c(h)^{-\eta} = v'(h)\gamma_1\epsilon_1\big[(1-\tau)h - c(h)\big]^{\epsilon_1 - 1}$. The KF equation reads $0 = -\frac{\partial}{\partial h}\big(s(h)f(h)\big) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial h^2}\big(h^2 f(h)\big)$. [1]

The government offers a constant payment π while there is also a subsidised transfer payment decreasing with household's income-level. Our public education system $\tilde{T}(h,\tau)$ for individuals reads[2]

$$\tilde{T}(h,\tau) = \pi + \beta(\zeta)h^{-\zeta}, \quad b > 0 \tag{4}$$

Equilibrium is solved from the government budget constraint

$$\int_{\underline{h}}^{\infty} \left(\pi + \beta(\zeta)h^{-\zeta}\right) f(h)dh = \tau \int_{\underline{h}}^{\infty} hf(h)dh \tag{5}$$

Human capital is the product of three inputs: innate ability, stable quality of the home environment and educational investment. The only source of additive idiosyncratic shock θ is i.i.d. across households and time and is drawn from normal distribution $\theta \sim N(0,\sigma^2)$. [3] As for education, all materials are counted into two the public e T and the private $e(h)$, where the marginal returns on education are $\gamma_1, \gamma_2 > 0$ respectively, while $0 < \epsilon_1, \epsilon_2 \leq 1$ capture the education elasticities. The human capital motion reads $dh = s(h)dt + h\sigma dZ$, where the saving function reads[4]

$$\begin{aligned} s(h) &= F_e(e(h), T) - \delta h \\ &= \gamma_1\left((1-\tau)h - c(h)\right)^{\epsilon_1} + \gamma_2\left(\pi + \beta(\zeta)h^{-\zeta}\right)^{\epsilon_2} - \delta h + w \end{aligned} \tag{6}$$

2.2 The Subsidy Incidence Analysis

The effect of idiosyncratic human capital investment risk σ on saving, welfare, investment, and total human capital in the education subsidy economy is displayed in table 2. There are income effects. A higher idiosyncratic risk would cause increase in precautionary savings and less private investment in education. Although the government needs to provide more education to compensate for the lack of private education, it cannot reverse the decline in overall output. The increasing elasticity of private education clearly has an incentive effect on education investment, inducing a better output. [5] There are also substitution effects, because aggregate savings also goes up. But we can't tell how big these two effects are.

Define social welfare function $\max_{\tau,\zeta} W(\tau) = \max \int_{\underline{h}}^{\infty} v(h)f(h,\tau)dh$. To derive an analytical result, here we guess a linear form of educational return, i.e. setting $\epsilon_1 = \epsilon_2 = 1$. Let's give some notes for

[1] Let $s(.)$ denote policy saving function. Let $f(.)$ denote the density function of stationary distribution which not change over time.

[2] The constant part in total public education expenditure represents the degree of equalization of school resources, which can be thought of as the standard compulsory education for all citizens. The regressive subsidy part in total public education expenditure represents the average rate of education subsidization, reflects a policy adjustment made by the gadministration to ensure the fairness of education.

[3] The scalar σ represents the undiversified idiosyncratic investment risk due to market incompleteness.

[4] Denote a constant endowment value w, human capital's social discount rate δ.

[5] Some similar phenomena occur in the calculation of gini coefficients like that a higher tax rate does not means a more equal economy. A reasonable explaination for this anomaly is that although an increase in linear taxation would reduce household's privated education investment, the decrease of private education return is much less due to the setting of the exponential educational elasticity.

Table 2: Aggregate Analysis

	$\sigma = .56, \epsilon_1 = .7$	$\sigma = .56, \epsilon_1 = .9$	$\sigma = 0.70, \epsilon_1 = .7$	$\sigma = .70, \epsilon_1 = .9$
H	$112.03(-37.49\%)$	179.22	$54.10(-69.81\%)$	$161.37(-9.95\%)$
W	-4304.01	-2577.25	-6609.95	-5842.91
S	$3.081(-69.59\%)$	10.133	$3.329(-67.14\%)$	$15.045(48.47\%)$
B	$0.8683(41.42\%)$	0.6140	$0.9786(59.38\%)$	$0.873(42.18\%)$

simplicity here :

$$\begin{cases} p_1 :=\gamma_2\beta\zeta h^{-\zeta-1} \\ p_2 :=w + (\gamma_1(1-\tau)-\delta)\,h + \gamma_2\beta h^{-\zeta} \\ p_3 := \displaystyle\int_{\underline{h}}^{\infty} hf(h)dh - \pi \\ p_4 := \displaystyle\int_{\underline{h}}^{\infty} h^{-\zeta} f(h)dh \\ \\ p_5 :=c(h)^{-\eta}\left\{\dfrac{\gamma_2}{\gamma_1}\left(\Lambda(\beta)-\ln h\right)\beta h^{-\zeta}\right\} \\ \qquad + c(h)^{-\eta}\left\{-\dfrac{\eta}{\gamma_1}\Lambda(c(h))s(h) + \dfrac{\sigma^2 h^2}{2}\dfrac{\eta}{\gamma_1}\dfrac{c'(h)}{c(h)}\left[(\eta+1)\Lambda(c(h)) - \Lambda(c'(h))\right]\right\} \\ p_7 := \left[\dfrac{\sigma^2}{2}h^2\hat{f}''(h) + 2\sigma^2 h\hat{f}'(h)\right] + \left[-\hat{s}'(h)f(h) - \hat{s}(h)f'(h) - s(h)\hat{f}'(h)\right] \end{cases} \tag{7}$$

Based on the derivative of Euler equation and KF equation w.r.t the regressivity of subsidy, we have conjecture 2.1 on steps to solve out subsidy incidences as follow

Conjecture 2.1 *we can solve $\hat{c}(h)$, $\hat{s}(h)$, $\hat{v}(h)$,$\hat{f}(h)$ firstly, then $\hat{\beta}$ throught $\hat{f(h)}$.*

$$\begin{aligned} &\hat{v}(h) = \frac{p_5}{\rho} \\ &\hat{f}(h) = \frac{p_7}{s'(h)-\sigma^2} \\ &\hat{\beta} = \tau\left(\frac{\hat{p_3}}{p_4} - \frac{p_3-\pi}{p_4}\frac{\hat{p_4}}{p_4}\right), \quad \Lambda(\beta) = \frac{p_3}{p_3-\pi}\Lambda\left(p_3\right) - \Lambda\left(p_4\right) \end{aligned} \tag{8}$$

See computation in APPENDIX.

We can see that $\hat{f}(h)$ is affected by the idiosyncratic investment risk effect, as well as drift-term effect. In p_7, the first term $\left[\frac{\sigma^2}{2}h^2\hat{f}''(h) + 2\sigma^2 h\hat{f}'(h)\right]$ represents the idiosyncratic investment risk effect. The second term $\left[-\hat{s}'(h)f(h) - \hat{s}(h)f'(h) - s(h)\hat{f}'(h)\right]$ denotes the drift-term effect.

2.3 Channel Decomposition of Social Welfare

The social welfare decomposition reads $\hat{W} = \int_{\underline{h}}^{\infty} \left[\hat{v}(h)f(h) + v(h)\hat{f}(h)\right] dh$. We implement a detailed decomposition of tax incidence on social welfare[1]

$$\hat{W} = \hat{W}_v + \hat{W}_f = \hat{W}_e + \hat{W}_s + \hat{W}_f \tag{9}$$

where

$$\begin{cases} \hat{W}_e = \displaystyle\int_{\underline{h}}^{\infty} \frac{u'\left(c(h)\right)}{\rho} \left[\Lambda(\beta) - \ln h\right] \beta h^{-\zeta} f(h) dh \\ \hat{W}_s = \displaystyle\int_{\underline{h}}^{\infty} \frac{u'\left(c(h)\right)}{\rho} \left\{ -\frac{\eta}{\gamma_1} \Lambda\left(c(h)\right) s(h) \right\} f(h) dh \\ \qquad + \displaystyle\int_{\underline{h}}^{\infty} \frac{u'\left(c(h)\right)}{\rho} \left\{ \frac{\sigma^2 h^2}{2} \frac{\eta}{\gamma_1} \frac{c'(h)}{c(h)} \left[(\eta+1)\Lambda\left(c(h)\right) - \Lambda\left(c'(h)\right)\right] \right\} f(h) dh \\ \hat{W}_f = \displaystyle\int_{\underline{h}}^{\infty} v(h) f(h) \Lambda\left(f(h)\right) dh \end{cases} \tag{10}$$

$\hat{W}_e$ and $\hat{W}_s$ are divied from the first term $\int_{\underline{h}}^{\infty} \hat{v}(h)f(h)dh$ in social welfare decomposition equation. $\hat{W}_e$ reflects part of the change in net public-education transfer [2]. Then $\hat{W}_s$ represents the intertemporal substitution effects, containing part of the change in precautionary saving and capital-investment risk. Finally, the last item $\hat{W}_f$ shows the distribution effects, i.e. the second term in social welfare decomposition 9. We give conjecture 2.2 on channel composition and conjecture 2.3 on its computation.

Conjecture 2.2 *There are three parts in channel composition, education transfer effects $\hat{W}_e$, intertemporal substitution effects $\hat{W}_s$ and distribution effects $\hat{W}_f$.*

Conjecture 2.3 *Based on incidence $\hat{c}$, $\hat{s}$, and $\hat{\beta}$, we can get aggregate incidence on diffierent channels of social welfare decomposition $\hat{W}_e, \hat{W}_s, \hat{W}_f$ by using equations 10.*

We plan to perform conjecture 2.3 through perturbation method. See detailed steps of our preliminary perturbation method 3.1 in Section 3.1.

2.4 Find the Optimal Subsidy Formula

We plan to find a formula of optimal regressivity of education subsidy expressed by sufficient statistics. We use the KF equation to find the stationary human capital distribution. Comparing it before and after the education subsidy reform, we can exactly calculate the change of the distribution. Thus, we can find the households-percentage of those financially binding before reform and are not binding

[1]The aggregate human capital is $H = \int_0^{\infty} hf(h)dh$, and the subsidy incidence on the aggregate human capital is $\Lambda(H) = \frac{1}{H}\int_0^{\infty} \Lambda\left(f(h)\right) hf(h)dh$, since $\frac{\hat{H}}{H} = \int_{\underline{h}}^{\infty} \frac{\hat{f}(h)}{f(h)} \frac{hf(h)}{H} dh$.

[2]$\hat{c}(h) + \frac{1}{\gamma_1}\hat{s}(h) = \frac{\gamma_2}{\gamma_1}(\,\hat{\hat{T}})$.

after the reform. [1] With the help of KF equation we can find this group of household directly and calculate this term numerically without simulations. This is the advantage of the KF equation. By comparing $\hat{W}_v$ and $\hat{W}_f$, we can observe the balance between efficiency and equity. Then, we solve the optimal subsidy as conjecture 2.4

Conjecture 2.4 *When the sum of $\hat{W}_v$ and $\hat{W}_f$ close to 0 enough, the progressivity of income taxation is optimal.*

$$\hat{W}_e(\zeta^*) + \hat{W}_s(\zeta^*) + \hat{W}_f(\zeta^*) = \hat{W}(\zeta^*) = 0 \tag{11}$$

A preliminary algorithm (iterative) has been developed to search the optimal regressivity of education subsidy ζ^*. See detailed algorithm 3.2 in Section 3.1.

3 Methodology

Algorithm 3.1 *(Perturbation Method)*

- *Step 1: Caculating $\{\hat{c}, \hat{s}, \hat{\beta}, \hat{v}, \hat{f}\}$: Specializing $\zeta = \zeta_x$. First, when human capital stationary distribution is reached under current public education scheme through MFG system, obtain policy, value function and density $\{c(\zeta_x), s(\zeta_x), \beta(\zeta_x), v(\zeta_x), f(\zeta_x)\}$. Next, give a minimal perturbation (for examle, 0.01) to education subsidy (donated as ζ_x^p), and obtain new composite $\{c(\zeta_x^p), s(\zeta_x^p), \beta(\zeta_x^p), v(\zeta_x^p), f(\zeta_x^p)\}$. Finally, $\{\hat{c}, \hat{s}, \hat{\beta}, \hat{v}, \hat{f}\}$ is derived from $\{c(\zeta_x^p) - c(\zeta_x), s(\zeta_x^p) - s(\zeta_x), \beta(\zeta_x^p) - \beta(\zeta_x), v(\zeta_x^p) - v(\zeta_x), f(\zeta_x^p) - f(\zeta_x)\}$.*
- *Step 2: Caculating $c'(h)$ and $\hat{c}'(h)$: Calculate $c'(h)$ first. [2] Using the same perturbation method we did in Step 1 , we obtain $\hat{c}'(h)$. The results of $c'(h)$ and $\hat{c}'(h)$ are prepared for calculating $\hat{W}_s$.*
- *Step 3: Caculating $\hat{W}_s, \hat{W}_e$, and $\hat{W}_f$: Based on steps 1-2 , we now have $\hat{c}, \hat{s}, \hat{\beta}, \hat{v}, \hat{f}, c'(h)$, and $\hat{c}'(h)$. We can calculate $\hat{W}_e, \hat{W}_s, \hat{W}_f$ by equations 10.*

Algorithm 3.2 *(Iterative Algorithm: Find the Optimal Regressivity)*

- *Step (1). Guess ζ^n.*
- *Step (2). Compute $\hat{W}_e(\zeta^n), \hat{W}_s(\zeta^n)$, and $\hat{W}_f(\zeta^n)$. We obtain ζ^{n+1} from equation 11.*
- *Step (3). Calculate the difference between ζ^{n+1} and ζ^n. If ζ^{n+1} is close to ζ^n : stop. Otherwise, if the difference is positive, ζ^n need to be increased, and conversely, if the difference is negative, ζ^n need to be decreased. Then go back to step (2).*

[1] Chang and Park (2021) [13] uses a simulation method to calculate private insurance effect represented by the tax incidence of this specific group of households. It is an important term in the optimal-tax formula issue, however, we will apply it to its education subsidy counterpart.

[2] The algorithm I'm using here is to use the forward difference approximation $c'(h_i) \approx \frac{c_{i+1} - c_i}{\Delta h}$. However, we need to deal specifically with the upper end of the state space $h_{\max}$. When calculting $c'(h_I)$, $c(h_{I+1})$ remains uncharted, so there we take it as $c'(h_I) = c'(h_{I-1})$. After some trials, this substitution has been tested to be feasible, since in numerical results, the difference $c'(h_{I-2}) - c'(h_{I-3})$ and $c'(h_{I-1}) - c'(h_{I-2})$ are less than 10^{-4}. This error can be ignored.

APPENDIX: Computation of Channel Decomposition

To derive an analytical result, here we guess a linear form of educational return [1], i.e. setting $\epsilon_1 = \epsilon_2 = 1$. Recalling the Euler equation[2], we have

$$\begin{aligned}
&\left[\rho + \delta - \gamma_1(1-\tau) + \gamma_2\beta\zeta h^{-\zeta-1}\right]u'\left(c(h)\right) \\
&= u''\left(c(h)\right)c'(h)\left[\left(\gamma_1(1-\tau)-\delta\right)h + \gamma_2\beta h^{-\zeta} + w - \gamma_1 c(h) + \sigma^2 h\right] \\
&+ \left[u'''\left(c(h)\right)\left(c'(h)\right)^2 + u''\left(c(h)\right)c''(h)\right]\frac{\sigma^2h^2}{2}
\end{aligned} \tag{12}$$

Denote the first part p_1, the second part p_2. Differentiating the Euler equation w.r.t ζ.

$$\begin{aligned}
&p_1 := \gamma_2\beta\zeta h^{-\zeta-1}, \quad \hat{p}_1 = \gamma_2\beta\zeta h^{-\zeta-1}\left[\frac{\hat{\beta}}{\beta} + \frac{1}{\zeta} - \ln h\right] \\
&p_2 := w + \left(\gamma_1(1-\tau)-\delta\right)h + \gamma_2\beta h^{-\zeta}, \quad \hat{p_2} = \gamma_2\beta h^{-\zeta}\left[\frac{\hat{\beta}}{\beta} - \ln h\right].
\end{aligned}$$

We get the derivative of Euler equation [3]

$$\begin{aligned}
&-\left(\rho+\delta-\gamma_1(1-\tau)-p_1\right)\hat{c}(h) - \frac{\hat{p}_1}{\eta}c(h) \\
&= \left[(\eta+1)\Lambda(c(h))c'(h) - (\hat{c}(h))'\right]\left[p_2 - \gamma_1 c(h) + \sigma^2 h\right] - c'(h)\left(\hat{p}_2 - \hat{c}(h)\right) \\
&+ \frac{\sigma^2h^2}{2}\left[-(\eta+1)(\eta+2)\Lambda(c(h))\frac{c'(h)^2}{c(h)} + 2(\eta+1)\frac{c'(h)}{c(h)}\left(\hat{c}(h)\right)' + (\eta+1)\Lambda(c(h))c''(h) - \left(\hat{c}(h)\right)''\right]
\end{aligned}$$

Owing to the budget constraint, we solve $\hat{\beta}$ throught $f(\hat{h})$, $\hat{p_3} = \int_{\underline{h}}^{\infty} h\Lambda(f(h))f(h)dh$ and $\hat{p}_4 = \int_{\underline{h}}^{\infty} h^{-\zeta}\left[-\ln h + \Lambda(f(h))\right]f(h)dh$

$$\beta = \frac{\tau\int_{\underline{h}}^{\infty} hf(h)dh - \pi}{\int_{\underline{h}}^{\infty} h^{-\zeta}f(h)dh} = \tau\frac{p_3-\pi}{p_4}, \quad \Lambda(\beta) = \frac{p_3}{p_3-\pi}\Lambda\left(p_3\right) - \Lambda\left(p_4\right) \tag{13}$$

To get $\hat{v}(h)$, differentiate the HJB equation with respect to ζ[4], $\rho\hat{v}(h) = u'\left(c(h)\right)\hat{c}(h) + \left(\hat{v}(h)\right)'s(h) + v'(h)\hat{s}(h) + \frac{\sigma^2h^2}{2}\left(\hat{v}(h)\right)''$.

$$\begin{aligned}
\rho\hat{v}(h) &= c(h)^{-\eta}\left\{\hat{c}(h) + \frac{1}{\gamma_1}\hat{s}(h)\right\} \\
&+ c(h)^{-\eta}\left\{-\frac{\eta}{\gamma_1}\Lambda(c(h))s(h) + \frac{\sigma^2h^2}{2}\frac{\eta}{\gamma_1}\frac{c'(h)}{c(h)}\left[(\eta+1)\Lambda(c(h)) - \Lambda(c'(h))\right]\right\}
\end{aligned} \tag{14}$$

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[1] $s(h) = \left(\gamma_1(1-\tau)-\delta\right)h + \gamma_2\pi + \gamma_2\beta h^{-\zeta} - \gamma_1 c(h) + w$

[2] Using $v'(h) = \frac{1}{\gamma_1}u'\left(c(h)\right), v''(h) = \frac{1}{\gamma_1}u''\left(c(h)\right)c'(h), v'''(h) = \frac{1}{\gamma_1}u'''\left(c(h)\right)\left(c'(h)\right)^2 + \frac{1}{\gamma_1}u''\left(c(h)\right)c''(h)$, the Euler equation reads $\left[\rho+\delta-\gamma_1(1-\tau)+\gamma_2\beta\zeta h^{-\zeta-1}\right]v'(h) = u'(c)c'(h) - \gamma_1 c'(h)v'(h) + v''(h)\left(s(h)+\sigma^2h\right) + v'''(h)\frac{\sigma^2h^2}{2}$.

[3] The utility function is in CRRA form, so $u'(h) = c(h)^{-\eta}, u''(h) = -\eta c(h)^{-\eta-1}, u'''(h) = \eta(\eta+1)c(h)^{-\eta-2}$ and $u''''(h) = -\eta(\eta+1)(\eta+2)c(h)^{-\eta-3}$, substituting them into the derivative of Euler equation. Divide both sides by $c(h)^{-\eta-1}$ rearrange it.

[4] Thanks to the linearity of education return, we can use derivatives $v'(h) = \frac{1}{\gamma_1}u'\left(c(h)\right), v''(h) = \frac{1}{\gamma_1}u''\left(c(h)\right)c'(h), v'''(h) = \frac{1}{\gamma_1}u'''\left(c(h)\right)\left(c'(h)\right)^2 + \frac{1}{\gamma_1}u''\left(c(h)\right)c''(h)$.

Denote the RHS of equation 14 p_5 and get $\hat{v}(h) = \frac{p_5}{\rho}$,

$$p_5 = c(h)^{-\eta} \left\{ \frac{\gamma_2}{\gamma_1} \left(\Lambda(\beta) - \ln h \right) \beta h^{-\zeta} \right\} + c(h)^{-\eta} \left\{ -\frac{\eta}{\gamma_1} \Lambda(c(h)) s(h) + \frac{\sigma^2 h^2}{2} \frac{\eta}{\gamma_1} \frac{c'(h)}{c(h)} \left[(\eta+1)\Lambda(c(h)) - \Lambda(c'(h)) \right] \right\}$$

Differentiate KF equation w.r.t. ζ [1] and rearrange it

$$\left[s'(h) - \sigma^2 \right] \hat{f}(h) = \left[\frac{\sigma^2}{2} h^2 \hat{f}''(h) + 2\sigma^2 h \hat{f}'(h) \right] + \left[-\hat{s}'(h) f(h) - \hat{s}(h) f'(h) - s(h) \hat{f}'(h) \right] \tag{15}$$

Denoting the RHS of equation 15 p_7, we obtain $\hat{f}(h)$

$$\hat{f}(h) = \frac{p_7}{s'(h) - \sigma^2} \tag{16}$$

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[1] Recalling KF equation, we have $0 = -s(h)f'(h) + \frac{\sigma^2}{2}\left[2f(h) + 4hf'(h) + h^2 f''(h)\right] - s'(h)f(h)$. Original differential reads$0 = \sigma^2 \hat{f}(h) + 2\sigma^2 h \hat{f}'(h) + \frac{\sigma^2}{2} h^2 \hat{f}''(h) - \hat{s}'(h) f(h) - s'(h)\hat{f}(h) - \hat{s}(h) f'(h) - s(h)\hat{f}'(h)$.

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