1. $\underline{A} \in \mathbb{R}^{m\times n}$ $\quad \underline{x} \in \mathbb{R}^{n}$

$$\underset{m\times 1}{y} = \underset{m\times n}{\underline{A}} \cdot \underset{n\times 1}{\underline{x}}$$

$$\frac{dy}{d\underline{x}^T} = \begin{pmatrix} \frac{\partial y_1}{\partial \underline{x}^T} \\ \vdots \\ \frac{\partial y_m}{\partial \underline{x}^T} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}}_{m\times n} = \underline{A}$$

则 $\dfrac{d(\underline{A}\,\underline{x})}{d\underline{x}^T} = \underline{A}$

2. $\dfrac{d(\underline{x}^T \underline{A}\,\underline{x})}{d\underline{x}}$ $\quad \underline{x}^T \underline{A}\,\underline{x} \rightarrow \mathbb{R}^{1\times 1}$, $\quad d\underline{x} \rightarrow \mathbb{R}^{N\times 1}$

为标量对列向量求导.
结果应为 $\mathbb{R}^{N\times 1}$

$$= \frac{d\underline{x}^T \underline{A}\,\underline{x} + \underline{x}^T \underline{A}\, d\underline{x}}{d\underline{x}}$$

标量转置为本身

$$= \frac{d\underline{x}^T \underline{A}\,\underline{x} + d\underline{x}^T \underline{A}^T \underline{x}}{d\underline{x}}$$

$$= \underline{A}\,\underline{x} + \underline{A}^T \underline{x} = (\underline{A} + \underline{A}^T)\,\underline{x}$$

3. ① $tr(\underline{A}\,\underline{x}\,\underline{x}^T) = tr\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} [x_1 \cdots x_n]\right)$

$$= tr\left(\underset{n\times 1}{\begin{bmatrix} \sum_{i=1}^{n} a_{1i}\cdot x_i \\ \vdots \\ \sum_{i=1}^{n} a_{ni}\cdot x_i \end{bmatrix}} \underset{1\times n}{[x_1 \cdots x_n]} \right)$$

$$= tr\begin{pmatrix} x_1\cdot\sum_{i=1}^{n} a_{1i}\cdot x_i & & & * \\ & x_2\cdot\sum_{i=1}^{n} a_{2i}\cdot x_i & & \\ & & \ddots & \\ * & & & x_n\cdot\sum_{i=1}^{n} a_{ni}\cdot x_i \end{pmatrix}$$

$$= \boxed{\sum_{i=1}^{n}\left[x_i\cdot\left(\sum_{j=1}^{n} a_{ij}\cdot x_j\right)\right]} \quad ①$$

② $\underline{x}^T\underline{A}\,\underline{x} = \underset{1\times n}{[x_1 \cdots x_n]} \underset{n\times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}$

$$= \underset{1\times n}{[x_1 \cdots x_n]} \underset{n\times n.}{\begin{bmatrix} \sum_{i=1}^{n} a_{1i}\cdot x_i \\ \\ \sum_{i=1}^{n} a_{ni}\cdot x_i \end{bmatrix}}$$

$$= \boxed{\sum_{i=1}^{n}\left[x_i\left(\sum_{j=1}^{n}a_{ij}\cdot x_j\right)\right]} \quad ②$$

$$① = ② \implies \underline{x}^T\underline{A}\,\underline{x} = tr(\underline{A}\,\underline{x}\,\underline{x}^T)$$

证毕