1. $\mathfrak{g} = (\mathbb{R}^3, \mathbb{R}, \times)$

① 封闭性： 设任意 $\vec{a} = (a_1, a_2, a_3)^T \in \mathbb{R}^3$

$\vec{b} = (b_1, b_2, b_3)^T \in \mathbb{R}^3$

$$\vec{a} \times \vec{b} = a^{\wedge} \cdot \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \in \mathbb{R}^3$$

满足

② 双线性： 设有 $\vec{a} = (a_1, a_2, a_3)^T \in \mathbb{R}^3$

$\vec{b} = (b_1, b_2, b_3)^T \in \mathbb{R}^3$

$\vec{c} = (c_1, c_2, c_3)^T \in \mathbb{R}^3$

$x \in \mathbb{R}$ $y \in \mathbb{R}$

1. 验证 $[aX + bY, Z] = a[X, Z] + b[Y, Z]$

$$\text{左式} = (x\vec{a} + y\vec{b}) \times \vec{c}$$

$$= \begin{pmatrix} xa_1 + yb_1 \\ xa_2 + yb_2 \\ xa_3 + yb_3 \end{pmatrix} \times \vec{c}$$

$$= \boxed{\begin{pmatrix} (xa_2 + yb_2)c_3 - (xa_3 + yb_3)c_2 \\ (xa_3 + yb_3)c_1 - (xa_1 + yb_1)c_3 \\ (xa_1 + yb_1)c_2 - (xa_2 + yb_2)c_1 \end{pmatrix}} \quad ①$$

右式 $x(\vec{a}\times\vec{c}) + y(\vec{b}\times\vec{c})$

$$= \boxed{x\begin{pmatrix} a_2c_3 - a_3c_2 \\ a_3c_1 - a_1c_3 \\ a_1c_2 - a_2c_1 \end{pmatrix} + y\begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}} \quad ②$$

经对比 ① = ②

2. 验证 $[Z, aX+bY] = a[Z,X] + b[Z,Y]$

左式 $= \vec{c}\times(x\vec{a}+y\vec{b})$

$= \boxed{-(x\vec{a}+y\vec{b})\times\vec{c}}$ ③

右式 $= x(\vec{c}\times\vec{a}) + y(\vec{c}\times\vec{b})]$

$= \boxed{-[x(\vec{a}\times\vec{c}) + y(\vec{b}\times\vec{c})]}$ ④

由之前验证得 ③ = ④

$\Rightarrow$ 满足

③ 反对称性 $\forall\vec{a}\in\mathbb{R}^3$ $\vec{a} = (a_1, a_2, a_3)^T$

$\vec{a}\times\vec{a} = \vec{0}$

满足

④ 雅可比等价 设有 $\vec{a} = (a_1, a_2, a_3)^T \in\mathbb{R}^3$

$\vec{b} = (b_1, b_2, b_3)^T \in\mathbb{R}^3$

$\vec{c} = (c_1, c_2, c_3)^T \in\mathbb{R}^3$

验证 $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

代入

$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$

$= a^\wedge(b^\wedge\vec{c}) + b^\wedge(c^\wedge\vec{a}) + c^\wedge(a^\wedge\vec{b})$

$= \underbrace{a^\wedge b^\wedge\cdot\vec{c}}_{①} + \underbrace{b^\wedge c^\wedge\vec{a}}_{②} + \underbrace{c^\wedge a^\wedge\vec{b}}_{③}$

①

$$
a \wedge b \wedge \vec{c} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}
$$

$$
= \begin{pmatrix} -a_3(b_3c_1 - b_1c_2) + a_2(b_1c_2 - b_2c_1) \\ a_3(b_2c_3 - b_3c_2) - a_1(b_3c_1 - b_1c_3) \\ -a_3(b_2c_3 - b_3c_2) + a_1(b_3c_1 - b_1c_3) \end{pmatrix}
$$

②, ③ 暴力计算省略

① + ② + ③ = 0 满足.

综上 $g = (\mathbb{R}^3, \mathbb{R}, \times)$ 构成李代数