

4.

$$\exp(\xi^1) = \sum_{n=0}^{\infty} \frac{1}{n!} (\xi^1)^n \quad , \quad \text{其中} \quad \xi^1 = \begin{bmatrix} \phi^1 & p \\ \underline{0}^T & 0 \end{bmatrix}$$

$$= \underline{1} + \xi^1 + \frac{1}{2} (\xi^1)^2 + \frac{1}{6} (\xi^1)^3 + \dots + \frac{1}{n!} (\xi^1)^n + \dots$$

找规律 $\xi^1 = \begin{bmatrix} \phi^1 & p \\ \underline{0}^T & 0 \end{bmatrix} \quad , \quad (\xi^1)^2 = \begin{bmatrix} \phi^1 & p \\ \underline{0}^T & 0 \end{bmatrix} \begin{bmatrix} \phi^1 & p \\ \underline{0}^T & 0 \end{bmatrix}$

$$= \begin{bmatrix} \phi^1 \phi^1 & \phi^1 p \\ \underline{0}^T & 0 \end{bmatrix}$$

$$(\xi^1)^3 = \begin{bmatrix} \phi^1 \phi^1 \phi^1 & \phi^1 \phi^1 p \\ \underline{0}^T & 0 \end{bmatrix}$$

$$\Rightarrow \text{因此} \quad (\xi^1)^n = \begin{bmatrix} (\phi^1)^n & (\phi^1)^{n-1} p \\ \underline{0}^T & 0 \end{bmatrix}$$

ϕ^1 为向量, 其角度与模长分别为 $\phi = \theta \cdot a$, $\phi^1 = \theta \alpha^1$

其中 $\begin{cases} \alpha^1 \alpha^1 = a a^T - I \\ \alpha^1 \alpha^1 \alpha^1 = -\alpha^1 \end{cases}$

$$\exp(\xi^1) = \underline{1} + \begin{bmatrix} \theta \cdot \underline{a}^1 & p \\ \underline{0}^T & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \theta^2 \underline{a}^1 \underline{a}^1 & \theta \underline{a}^1 \cdot p \\ \underline{0}^T & 0 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} \underline{a} \underline{a}^T - \underline{a}^1 \underline{a}^1 & 0 \\ \underline{0}^T & 1 \end{bmatrix} + \begin{bmatrix} \theta \cdot \underline{a}^1 & p \\ \underline{0}^T & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \theta^2 \underline{a}^1 \underline{a}^1 & \theta \underline{a}^1 p \\ \underline{0}^T & 0 \end{bmatrix}$$

$$+ \frac{1}{3!} \begin{bmatrix} \theta^3 (-\underline{a}^1) & \theta^2 \underline{a}^1 \underline{a}^1 p \\ \underline{0}^T & 0 \end{bmatrix} + \frac{1}{4!} \begin{bmatrix} \theta^4 \alpha^1 \alpha^1 & -\theta^3 \alpha^1 p \\ \underline{0}^T & 0 \end{bmatrix} + \dots$$

$$\begin{aligned}
 &= \begin{bmatrix} \underline{a} \underline{a}^T & \underline{p} \\ \underline{0}^T & 1 \end{bmatrix} + \begin{bmatrix} \underbrace{\left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) \underline{a}^1}_{\sin \theta} & \underbrace{\left(0 + \frac{1}{2!} \theta^2 - \frac{1}{4!} \theta^4 + \dots \right) \underline{a}^1 \underline{a}^1 \underline{p}}_{\textcircled{1}} \\ \underline{0}^T & 0 \end{bmatrix} \\
 &+ \begin{bmatrix} \underbrace{\left(-1 + \frac{1}{2!} \theta^2 - \frac{1}{4!} \theta^4 + \dots \right) \underline{a}^1 \underline{a}^1}_{-\cos \theta} & \underbrace{\left(\frac{1}{2!} \theta - \frac{1}{4!} \theta^3 + \dots \right) \underline{a}^1 \underline{p}}_{\textcircled{2}} \\ \underline{0}^T & 0 \end{bmatrix} \\
 &= \begin{bmatrix} \underline{a}^1 \underline{a}^1 + \underline{I} + \sin \theta \underline{a}^1 - \cos \theta \underline{a}^1 \underline{a}^1 & \\ & \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{1} &= \left(-0 + \frac{1}{2!} \theta^2 - \frac{1}{4!} \theta^4 + \dots \right) \underline{a}^1 \underline{a}^1 \underline{p} \\
 &= \left(-1 + \frac{1}{2!} \theta^2 - \frac{1}{4!} \theta^4 + \dots \right) \underline{a}^1 \underline{a}^1 \underline{p} + \underline{a}^1 \underline{a}^1 \underline{p} \\
 &= \frac{\left(-\theta + \frac{1}{3!} \theta^3 - \frac{1}{5!} \theta^5 + \dots \right)}{\theta} \underline{a}^1 \underline{a}^1 \underline{p} + \underline{a}^1 \underline{a}^1 \underline{p} \\
 &= \left(1 - \frac{\sin \theta}{\theta} \right) \underline{a}^1 \underline{a}^1 \underline{p}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{2} &= \left(\frac{1}{2!} \theta - \frac{1}{4!} \theta^3 + \dots \right) \underline{a}^1 \underline{p} \\
 &= \frac{\left(\frac{1}{2!} \theta^2 - \frac{1}{4!} \theta^4 + \dots \right)}{\theta} \underline{a}^1 \underline{p} \\
 &= \frac{\left(-1 + \frac{1}{2!} \theta^2 - \frac{1}{4!} \theta^4 + \dots \right)}{\theta} \underline{a}^1 \underline{p} + \frac{1}{\theta} \underline{a}^1 \underline{p} \\
 &= \left(\frac{1}{\theta} - \frac{\cos \theta}{\theta} \right) \underline{a}^1 \underline{p} \\
 &= \left(\frac{1 - \cos \theta}{\theta} \right) \underline{a}^1 \underline{p}
 \end{aligned}$$

$$\begin{aligned}
 \text{右半部分} &= \underline{p} + \left(1 - \frac{\sin\theta}{\theta}\right) \underline{a}^T \underline{a}^T \underline{p} + \left(\frac{1 - \cos\theta}{\theta}\right) \underline{a}^T \underline{p} \\
 &= \underline{p} + \left(1 - \frac{\sin\theta}{\theta}\right) (\underline{a} \underline{a}^T - \underline{I}) \underline{p} + \left(\frac{1 - \cos\theta}{\theta}\right) \underline{a}^T \underline{p} \\
 &= \underline{p} + \left(1 - \frac{\sin\theta}{\theta}\right) \underline{a} \underline{a}^T \underline{p} - \left(1 - \frac{\sin\theta}{\theta}\right) \underline{p} + \left(\frac{1 - \cos\theta}{\theta}\right) \underline{a}^T \underline{p} \\
 &= \left[\frac{\sin\theta}{\theta} + \left(1 - \frac{\sin\theta}{\theta}\right) \underline{a} \underline{a}^T + \left(\frac{1 - \cos\theta}{\theta}\right) \underline{a}^T \right] \underline{p}
 \end{aligned}$$

$$\text{综上} \quad \exp(\underline{\xi}^1) = \begin{bmatrix} \exp(\phi^1) & \underline{I} \cdot \underline{p} \\ \underline{0}^T & 1 \end{bmatrix}$$