1.

$$R=\begin{pmatrix} e_1^T e_1' & e_1^T e_2' & e_1^T e_3' \\ e_2^T e_1' & e_2^T e_2' & e_2^T e_3' \\ e_3^T e_1' & e_3^T e_2' & e_3^T e_3' \end{pmatrix}$$

其中 e_i'，e_i^T 都为向量基底.
省略等号 $\Rightarrow$

① 证明 $R^TR=I$

$$R^T=\begin{pmatrix} e_1^T e_1' & e_2^T e_1' & e_3^T e_1' \\ e_1^T e_2' & e_2^T e_2' & e_3^T e_2' \\ e_1^T e_3' & e_2^T e_3' & e_3^T e_3' \end{pmatrix}$$

现考虑 R^{-1}，由R的定义，几何性质得，R^{-1} 相当于逆向转动

即：(e_1', e_2', e_3') 发生旋转到 (e_1, e_2, e_3).

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} e_1'^T e_1 & e_2'^T e_1 & e_3'^T e_1 \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

观察 $e_1^T e_1' = e_1'^T e_1$

$e_2^T e_1' = e_2'^T e_1$

$\vdots$ 以此类推 $\Rightarrow$ $R^T=R^{-1}$

因为 $R^TR=I$

证明： $\det(R) = +1$

$$\det(I) = \det(RR^T) = \det(R)\det(R^T)$$

$$= \det(R) \cdot \det(R) = 1$$

$$\Longrightarrow \det(R) = \pm\sqrt{1} = \pm 1$$

$+1$ 由定义给出

2.

$\varepsilon \in \mathbb{R}^3$

$\eta \in \mathbb{R}^1$

3.

$$q_1 = (x, y, z, w) = (\vec{v_1}, w)$$

$$q_2 = (a, b, c, d) = (\vec{v_2}, d)$$

$$v_1^\wedge = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

$$q_1 \cdot q_2 = \left[w \cdot d - \vec{v_1}^T \vec{v_2},\ w\vec{v_2} + d \cdot \vec{v_1} + \vec{v_1} \times \vec{v_2} \right]$$

$$= \left[wd - xa - yb - zc,\ w \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} + d \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \vec{v_1} \times \vec{v_2} \right]$$

其中 $\vec{v_1} \times \vec{v_2} = \begin{pmatrix} cy - bz \\ az - cx \\ bx - ay \end{pmatrix}$

$$= \left[wd - xa - yb - zc,\ \begin{pmatrix} aw + dx + cy - bz \\ bw + dy + az - cx \\ cw + dz + bx - ay \end{pmatrix} \right] \quad ①$$

① 计算 $q_1^+ q_2$

$$q_1^+ = \begin{bmatrix} w\vec{1} + \vec{v_1}^\wedge & \vec{v_1} \\ -\vec{v_1}^T & w \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} w & & \\ & w & \\ & & w \end{pmatrix} + \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} & \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{matrix} -x & -y & -z \end{matrix} & w \end{bmatrix}$$

$$= \begin{bmatrix} w & -z & y & x \\ z & w & -x & y \\ -y & x & w & z \\ -x & -y & -z & w \end{bmatrix}$$

$$q_1^{+} q_2 = \begin{pmatrix} wa - bz + cy + dx \\ za + wb - xc + yd \\ -ya + xb + wc + zd \\ -xa - yb - zc + wd \end{pmatrix} \quad ②$$

① 与 ② 相比发现 ① = ②

② 证明 $\quad q_2^{\oplus} q_1 = q_1 q_2$

$$q_2^{\oplus} = \begin{pmatrix} d\mathbb{1} - \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} & \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ -a \quad -b \quad -c & d \end{pmatrix}$$

$$= \begin{pmatrix} d & c & -b & a \\ -c & d & a & b \\ b & -a & d & c \\ -a & -b & -c & d \end{pmatrix}$$

$$q_2^{\oplus} \cdot q_1 = \begin{pmatrix} dx + cy - bz + aw \\ -cx + dy + az + bw \\ bx - ay + dz + cw \\ -ax - by - cz + dw \end{pmatrix} \quad ③$$

发现 ① = ③

综上 $\quad q_1 q_2 = q_1^{+} q_2 = q_2^{\oplus} q_1$