4.

$$\exp(\xi^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!} (\xi^\wedge)^n \quad , 其中 \; \xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ \underline{0}^T & 0 \end{bmatrix}$$

$$= \underline{I} + \xi^\wedge + \frac{1}{2}(\xi^\wedge)^2 + \frac{1}{6}(\xi^\wedge)^3 + \cdots \frac{1}{n!}(\xi^\wedge)^n + \cdots$$

找规律 $\xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ \underline{0}^T & 0 \end{bmatrix}$, $(\xi^\wedge)^2 = \begin{bmatrix} \phi^\wedge & \rho \\ \underline{0}^T & 0 \end{bmatrix}\begin{bmatrix} \phi^\wedge & \rho \\ \underline{0}^T & 0 \end{bmatrix}$

$$= \begin{bmatrix} \phi^\wedge \phi^\wedge & \phi^\wedge \rho \\ \underline{0}^T & 0 \end{bmatrix}$$

$$(\xi^\wedge)^3 = \begin{bmatrix} \phi^\wedge \phi^\wedge \phi^\wedge & \phi^\wedge \phi^\wedge \rho \\ \underline{0}^T & 0 \end{bmatrix}$$

$\Longrightarrow$ 因此 $(\xi^\wedge)^n = \begin{bmatrix} (\phi^\wedge)^n & (\phi^\wedge)^{n-1} \cdot \rho \\ \underline{0}^T & 0 \end{bmatrix}$

$\phi^\wedge$ 为向量，其角度与模长分别为 $\phi = \theta \cdot a$, $\phi^\wedge = \theta a^\wedge$

其中 $\begin{cases} a^\wedge a^\wedge = a a^T - I \\ a^\wedge a^\wedge a^\wedge = -a^\wedge \end{cases}$

$$\exp(\xi^\wedge) = \underline{I} + \begin{bmatrix} \theta \cdot \underline{a}^\wedge & \rho \\ \underline{0}^T & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \theta^2 \underline{a}^\wedge \underline{a}^\wedge & \theta \underline{a}^\wedge \cdot \rho \\ \underline{0}^T & 0 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} \underline{a}\underline{a}^T - \underline{a}^\wedge \underline{a}^\wedge & 0 \\ \underline{0}^T & 1 \end{bmatrix} + \begin{bmatrix} \theta \cdot \underline{a}^\wedge & \rho \\ \underline{0}^T & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \theta^2 \underline{a}^\wedge \underline{a}^\wedge & \theta \underline{a}^\wedge \rho \\ \underline{0}^T & 0 \end{bmatrix}$$

$$+ \frac{1}{3!}\begin{bmatrix} \theta^3 (-\underline{a}^\wedge) & \theta^2 \underline{a}^\wedge \underline{a}^\wedge \rho \\ \underline{0}^T & 0 \end{bmatrix} + \frac{1}{4!}\begin{bmatrix} -\theta^4 a^\wedge a^\wedge & -\theta^3 a^\wedge \rho \\ \underline{0}^T & \underline{0} \end{bmatrix} + \cdots$$

$$
= \begin{bmatrix} a a^T & \rho \\ 0^T & 1 \end{bmatrix} + \begin{bmatrix} \underbrace{\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right)}_{\sin\theta} a^\wedge & \overbrace{\left(-0 + \frac{1}{3!}\theta^2 - \frac{1}{5!}\theta^4 + \cdots\right)}^{①} a^\wedge a^\wedge \rho \\ 0^T & 0 \end{bmatrix}
$$

$$
+ \begin{bmatrix} \underbrace{\left(-1 + \frac{1}{2!}\theta^2 - \frac{1}{4!}\theta^4 + \cdots\right)}_{-\cos\theta} a^\wedge a^\wedge & \overbrace{\left(\frac{1}{2!}\theta - \frac{1}{4!}\theta^3 + \cdots\right)}^{②} a^\wedge \rho \\ 0^T & 0 \end{bmatrix}
$$

$$
= \Big[a^\wedge a^\wedge + I + \sin\theta\, a^\wedge - \cos\theta\, a^\wedge a^\wedge
$$

$$
① = \left(-0 + \frac{1}{3!}\theta^2 - \frac{1}{5!}\theta^4 + \cdots\right) a^\wedge a^\wedge \rho
$$

$$
= \left(-1 + \frac{1}{3!}\theta^2 - \frac{1}{5!}\theta^4 + \cdots\right) a^\wedge a^\wedge \rho + a^\wedge a^\wedge \rho
$$

$$
= \frac{\left(-\theta + \frac{1}{3!}\theta^3 - \frac{1}{5!}\theta^5 + \cdots\right)}{\theta} a^\wedge a^\wedge \rho + a^\wedge a^\wedge \rho
$$

$$
= \left(1 - \frac{\sin\theta}{\theta}\right) a^\wedge a^\wedge \rho
$$

$$
② = \left(\frac{1}{2!}\theta - \frac{1}{4!}\theta^3 + \cdots\right) a^\wedge \rho
$$

$$
= \frac{\left(\frac{1}{2!}\theta^2 - \frac{1}{4!}\theta^4 + \cdots\right)}{\theta} a^\wedge \rho
$$

$$
= \frac{\left(-1 + \frac{1}{2!}\theta^2 - \frac{1}{4!}\theta^4 + \cdots\right)}{\theta} a^\wedge \rho + \frac{1}{\theta} a^\wedge \rho
$$

$$
= \left(\frac{1}{\theta} - \frac{\cos\theta}{\theta}\right) a^\wedge \rho
$$

$$
= \left(\frac{1 - \cos\theta}{\theta}\right) a^\wedge \rho
$$

$$
\begin{aligned}
\text{右上部分} &= \rho + \left(1-\frac{\sin\theta}{\theta}\right) a^{\wedge} a^{\wedge} \rho + \left(\frac{1-\cos\theta}{\theta}\right) a^{\wedge} \rho \\
&= \rho + \left(1-\frac{\sin\theta}{\theta}\right)\left(a a^{T} - I\right) \rho + \left(\frac{1-\cos\theta}{\theta}\right) a^{\wedge} \rho \\
&= \rho + \left(1-\frac{\sin\theta}{\theta}\right) a a^{T} \rho - \left(1-\frac{\sin\theta}{\theta}\right) \rho + \left(\frac{1-\cos\theta}{\theta}\right) a^{\wedge} \rho \\
&= \underbrace{\left[\frac{\sin\theta}{\theta} + \left(1-\frac{\sin\theta}{\theta}\right) a a^{T} + \left(\frac{1-\cos\theta}{\theta}\right) a^{\wedge}\right]}_{J} \rho
\end{aligned}
$$

综上 $$\exp(\xi^{\wedge}) = \begin{bmatrix} \exp(\phi^{\wedge}) & J \cdot \rho \\ 0^{T} & 1 \end{bmatrix}$$