2D Gaussian derivatives

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1 Construction of the expected μ at $[x, y]$

The normalized bivariate Gaussian $\mu = f(x, y \mid x_c, y_c, \sigma_x, \sigma_y, \rho)$ at position $[x, y]$ is:

$$\mu = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-x_c}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-x_c}{\sigma_x}\right)\left(\frac{y-y_c}{\sigma_y}\right) + \left(\frac{y-y_c}{\sigma_y}\right)^2\right]\right\}$$

where $[x_c, y_c]$ is the position of the Gaussian center; $[\sigma_x, \sigma_y]$ is the Gaussian sigma in x-, and y-axis, respectively; and $\rho \in [0, 1)$ is the correlation between x and y.
Define:

$$u(x \mid x_c, \sigma_x) = \frac{x-x_c}{\sigma_x}; \qquad v(y \mid y_c, \sigma_y) = \frac{y-y_c}{\sigma_y}$$

$$\begin{aligned} p(x, y \mid x_c, y_c, \sigma_x, \sigma_y, \rho) &= -\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-x_c}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-x_c}{\sigma_x}\right)\left(\frac{y-y_c}{\sigma_y}\right) + \left(\frac{y-y_c}{\sigma_y}\right)^2\right] \\ &= -\frac{1}{2(1-\rho^2)}\left(u^2 - 2\rho uv + v^2\right) \end{aligned}$$

$$q(\sigma_x, \sigma_y, \rho) = \sigma_x^{-1}\sigma_y^{-1}\left(1-\rho^2\right)^{-1/2}$$

And thus

$$\mu = \frac{1}{2\pi} \cdot q \cdot \exp(p)$$

2 Construction of the Jacobian at $[x, y]$

$$\begin{aligned} u(x \mid x_c, \sigma_x) &= \frac{x-x_c}{\sigma_x}; & \frac{\partial u}{\partial x_c} &= -\frac{1}{\sigma_x}; & \frac{\partial u}{\partial \sigma_x} &= -\frac{x-x_c}{\sigma_x^2} = -\frac{u}{\sigma_x} \\ v(y \mid y_c, \sigma_y) &= \frac{y-y_c}{\sigma_y}; & \frac{\partial v}{\partial y_c} &= -\frac{1}{\sigma_y}; & \frac{\partial v}{\partial \sigma_y} &= -\frac{y-y_c}{\sigma_y^2} = -\frac{v}{\sigma_y} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial p}{\partial x_c} &= \frac{\partial p}{\partial u}\frac{\partial u}{\partial x_c} = -\frac{1}{2(1-\rho^2)}(2u - 2\rho v)\left(-\frac{1}{\sigma_x}\right) = \frac{1}{\sigma_x}\frac{u-\rho v}{1-\rho^2} \\ \frac{\partial p}{\partial y_c} &= \frac{\partial p}{\partial v}\frac{\partial v}{\partial y_c} = -\frac{1}{2(1-\rho^2)}(2v - 2\rho u)\left(-\frac{1}{\sigma_y}\right) = \frac{1}{\sigma_y}\frac{v-\rho u}{1-\rho^2} \\ \frac{\partial p}{\partial \sigma_x} &= \frac{\partial p}{\partial u}\frac{\partial u}{\partial \sigma_x} = -\frac{1}{2(1-\rho^2)}(2u - 2\rho v)\left(-\frac{u}{\sigma_x}\right) = \frac{u}{\sigma_x}\frac{u-\rho v}{1-\rho^2} = \frac{1}{\sigma_x}\frac{u^2-\rho uv}{1-\rho^2} \\ \frac{\partial p}{\partial \sigma_y} &= \frac{\partial p}{\partial v}\frac{\partial v}{\partial \sigma_y} = -\frac{1}{2(1-\rho^2)}(2v - 2\rho u)\left(-\frac{v}{\sigma_y}\right) = \frac{v}{\sigma_y}\frac{v-\rho u}{1-\rho^2} = \frac{1}{\sigma_y}\frac{v^2-\rho uv}{1-\rho^2} \\ \frac{\partial p}{\partial \rho} &= -\frac{1}{2}\left[-\frac{-2\rho}{(1-\rho^2)^2}\left(u^2 - 2\rho uv + v^2\right) + \frac{1}{1-\rho^2}(-2uv)\right] \\ &= \frac{1}{1-\rho^2}\left[-\frac{2\rho}{2(1-\rho^2)}\left(u^2 - 2\rho uv + v^2\right) + uv\right] = \frac{2\rho p + uv}{1-\rho^2} \end{aligned}$$

and

$$\frac{\partial q}{\partial \sigma_x} = -\frac{1}{\sigma_x} q; \qquad \frac{\partial q}{\partial \sigma_y} = -\frac{1}{\sigma_y} q; \qquad \frac{\partial q}{\partial \rho} = -\frac{-2\rho}{2\left(1-\rho^2\right)} q = \frac{\rho}{1-\rho^2} q$$

Thus the Jocobian is:

$$\begin{aligned}
\frac{\partial \mu}{\partial x_c} &= \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) \cdot \frac{\partial p}{\partial x_c} = \mu \frac{1}{\sigma_x} \frac{u - \rho v}{1-\rho^2} \\
\frac{\partial \mu}{\partial y_c} &= \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) \cdot \frac{\partial p}{\partial y_c} = \mu \frac{1}{\sigma_y} \frac{v - \rho u}{1-\rho^2} \\
\frac{\partial \mu}{\partial \sigma_x} &= \frac{1}{2\pi} \cdot \frac{\partial q}{\partial \sigma_x} \cdot \exp\left(p\right) + \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) \cdot \frac{\partial p}{\partial \sigma_x} \\
&= -\frac{1}{\sigma_x} \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) + \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) \cdot \frac{1}{\sigma_x} \frac{u^2 - \rho u v}{1-\rho^2} \\
&= \mu \frac{1}{\sigma_x} \left(\frac{u^2 - \rho u v}{1-\rho^2} - 1 \right) \\
\frac{\partial \mu}{\partial \sigma_y} &= \frac{1}{2\pi} \cdot \frac{\partial q}{\partial \sigma_y} \cdot \exp\left(p\right) + \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) \cdot \frac{\partial p}{\partial \sigma_y} \\
&= -\frac{1}{\sigma_y} \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) + \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) \cdot \frac{1}{\sigma_y} \frac{v^2 - \rho u v}{1-\rho^2} \\
&= \mu \frac{1}{\sigma_y} \left(\frac{v^2 - \rho u v}{1-\rho^2} - 1 \right) \\
\frac{\partial \mu}{\partial \rho} &= \frac{1}{2\pi} \cdot \frac{\partial q}{\partial \rho} \cdot \exp\left(p\right) + \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) \cdot \frac{\partial p}{\partial \rho} \\
&= \frac{\rho}{1-\rho^2} \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) + \frac{1}{2\pi} \cdot q \cdot \exp\left(p\right) \cdot \frac{2\rho p + uv}{1-\rho^2} \\
&= \mu \left(\frac{2\rho p + uv + \rho}{1-\rho^2} \right)
\end{aligned}$$