# Block Coordinate Descent BFGS Method on Convex Non-smooth Problems

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## Abstract

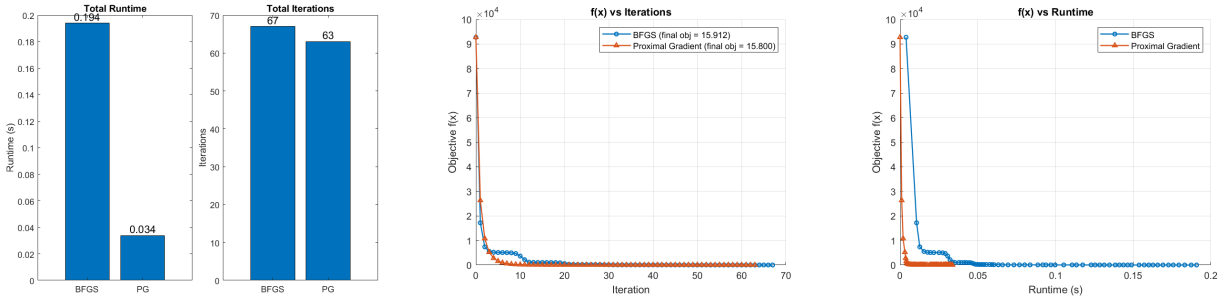
We introduce Block BFGS, a block coordinate descent adaptation of BFGS for composite objectives with overlapping groups. Curvature is refreshed only on the active block, and a composite directional-derivative Wolfe rule threads steps through non-smoothness—no proximal detours. We prove global convergence under standard assumptions and under a milder smooth extension around the iterates. In head-to-head experiments, Block BFGS matches the accuracy of proximal-gradient methods while delivering substantial runtime gains.

## 1 Background

Quasi-Newton methods, such as the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm, have been used for long as powerful tools for smooth optimization due to their efficient use of gradient information, yet its potential in non-smooth optimization—a realm traditionally dominated by proximal methods—has long been overlooked. In recent years, an increasing number of research indicates that quasi-Newton methods can also provide significant computational advantages for non-smooth problems. For example, in [1], the authors discovered that BFGS outperforms the subgradient methods on non-smooth problems in terms of both solution accuracy and runtime efficiency. We illustrate this on the classical LASSO:

$$\min_{x\in\mathbb{R}^p} F(x) \;=\; \tfrac{1}{2}\|Ax-b\|_2^2 \;+\; \lambda\|x\|_1,$$

with $A \in \mathbb{R}^{n\times p}$, $b \in \mathbb{R}^n$, and $\lambda > 0$. The objective is non-smooth due to the $\ell_1$ term. As a baseline we use a standard Proximal Gradient (PG) method, and we then made a simple comparison between BFGS and PG.




(a) Totals (runtime & iterations). (b) $F(x)$ vs. iterations. (c) $F(x)$ vs. runtime.

Figure 1: Comparison of BFGS and PG on LASSO.

Preliminary computational result in Figure 1 confirms that BFGS does handle the non-smooth LASSO and converges in a comparable number of steps to PG, but its per-iteration cost makes it much slower in total runtime, as shown in (a), for one typical instance (1000 samples, 100 features). Specifically, both algorithms descend rapidly and converge at the same point within 100 iterations, but BFGS is about 5.7 times slower than PG, with only 4 more iterations, to converge to a very close objective function value as PG. These numerical results indicate significant gains in both speed and solution quality, motivating deeper theoretical and algorithmic exploration.

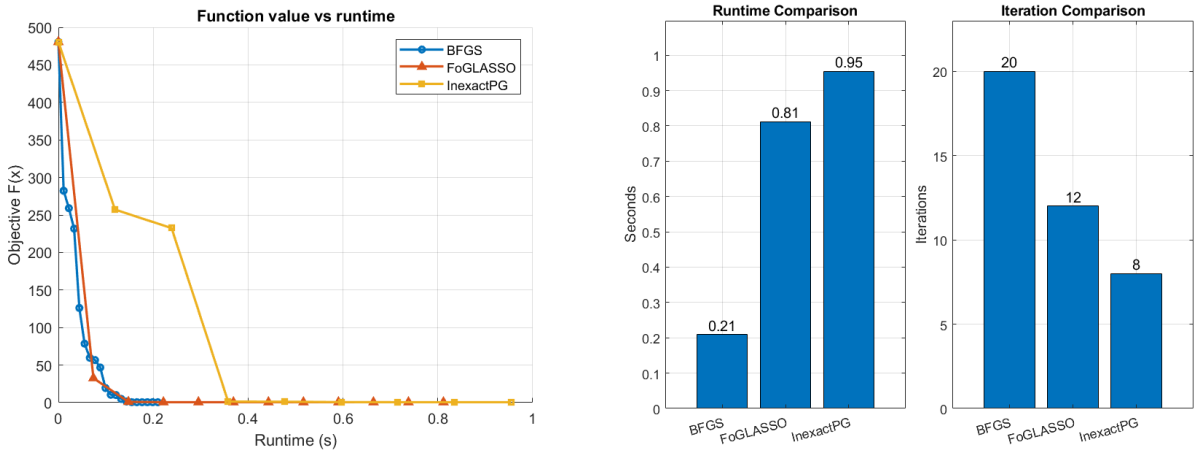
## GroupLASSO Problems

GroupLasso is widely used to impose structured sparsity in real data; for example, a protein-complex-based GroupLasso logistic model (PCLassoLog) has been applied to proteomic data for pan-cancer classification, selecting entire protein complexes rather than isolated proteins [14]. For overlapping groups, Yuan, Liu, and Ye's *FoGLasso* computes the overlapping-group proximal map by solving a smooth convex dual and embeds it in an accelerated gradient scheme, yielding strong runtime improvements [2]. Dai and Robinson's *InexactPG* studies proximal-gradient methods when the proximal operator lacks a closed form; for the overlapping group $\ell_1$ regularizer, they design an inner solver with adaptive stopping while preserving $\mathcal{O}(\tau^{-2})$ iteration complexity and proving support identification [3].

In what follows, we compare these proximal-gradient approaches with BFGS applied to the same non-smooth objectives, focusing on computation time and solution quality. Concretely, we consider the composite overlapping-group problem

$$\min_{x\in\mathbb{R}^p} \; F(x) \;=\; \frac{1}{2}\|Ax-b\|^2 \;+\; \Big(\lambda_1\|x\|_1 + \lambda_2 \sum_{i=1}^{m} w_i \|x_{G_i}\|_2\Big), \tag{1.1}$$

where $A \in \mathbb{R}^{n\times p}$ and $b \in \mathbb{R}^n$ define the data-fidelity term, $\lambda_1, \lambda_2 \geq 0$ are regularization parameters, $\{G_i\}_{i=1}^m$ are (possibly overlapping) index sets specifying groups, $x_{G_i}$ is the subvector of $x$ on group $G_i$, and $w_i > 0$ are optional group weights (e.g., to normalize by group size). The penalty combines elementwise sparsity ($\ell_1$) with structured sparsity across groups ($\ell_2$), and remains convex even when the $G_i$ overlap.




(a) Objective vs. runtime.

(b) Runtime and iteration counts.

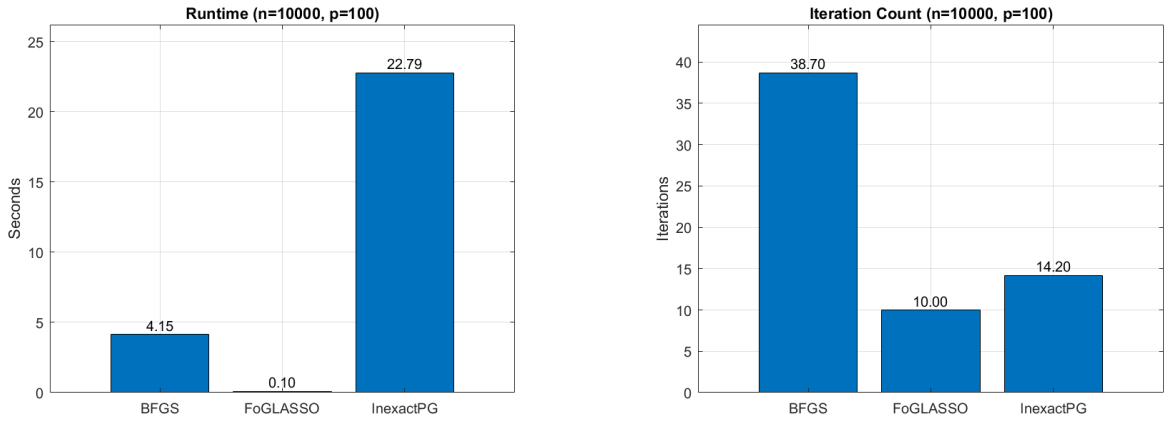
Figure 2: Comparison of BFGS, FoGLASSO, and InexactPG on $n = 100$ samples, $p = 100$ features, and $m = 4$ groups.

From Figure 2, BFGS is the fastest on this instance. It reaches the target tolerance in 0.21 s, whereas FoGLASSO and InexactPG take 0.81 s and 0.95 s, respectively—about 3.9× and 4.5× longer. Panel (a) shows that all three methods attain the same final objective value (within $10^{-4}$), while panel (b) illustrates that iteration counts are not directly comparable: BFGS requires 20 iterations versus 12 (FoGLASSO) and 8 (InexactPG) yet still finishes earlier, reflecting differences in per-iteration work and progress per step. These observations motivate a closer examination of when quasi-Newton updates are competitive for non-smooth composite objectives with overlapping group structure and when proximal-gradient variants prevail at larger scales.

# 2 Block Coordinate Descent BFGS

## 2.1 Limitations of BFGS

Although BFGS is able to solve non-smooth problems with a comparable accuracy as the proximal gradient methods, however, for large-scale optimization problems, the BFGS algorithm faces significant computational and storage challenges due to the complexity of full Hessian updates [6]. Block coordinate descent methods naturally address these scalability issues by decomposing problems into smaller, more manageable subproblems. Moreover, block coordinate descent techniques can exploit problem structure to reduce per-iteration computational burden and enable parallel or distributed implementations, thereby significantly enhancing efficiency in high-dimensional settings [7]. We scale up the instance from Figure 2:




(a) Runtime (seconds).

(b) Iteration counts.

Figure 3: $n = 10{,}000$ and $p = 100$. BFGS becomes much slower than FoGLASSO.

Compared with Figure 2, the large-scale case in Figure 3 shows a reversal: BFGS becomes significantly slower—about 41.5× slower than FoGLASSO, highlighting that with larger datasets, the algebraic per-iteration cost scales chiefly for full BFGS. Building on these observations and the promising small-scale results, we develop a *Block Coordinate Descent BFGS* method tailored to medium-to-large-scale composite, non-smooth structured problems. By maintaining a block-diagonal Hessian approximation, our approach retains key quasi-Newton properties while reducing per-iteration cost and improving scalability.

## 2.2 Problem Definition

Throughout this paper, we consider a composite function

$$F(x) \;=\; f(x) + g(x), \tag{2.1}$$

where $f$ is smooth and convex, and $g$ is convex (possibly non-smooth). We make the following assumptions and notational conventions.

## 2.3 Level Set

Given an initial point $x_0 \in \mathbb{R}^n$, define the level set

$$\mathcal{D} \;=\; \{\, x \in \mathbb{R}^p \;:\; F(x) \;\le\; F(x_0) \}. \tag{2.2}$$

All iterates $\{x_k\}$ generated by our algorithm remain in $\mathcal{D}$.

## 2.4 Strong Convexity of the Smooth Part $f$

We assume $f$ is twice continuously differentiable on $\mathcal{D}$. Moreover, there exist constants $0 < m \le M$ such that

$$\begin{gathered} m\,\|z\|^2 \;\le\; z^\top \nabla^2 f(x)\, z \;\le\; M\,\|z\|^2, \\ \forall\, z \in \mathbb{R}^p, \;\; \forall\, x \in \mathcal{D}. \end{gathered} \tag{2.3}$$

Since $f$ is $m$-strongly convex, and $g$ is convex, the overall function $F = f + g$ is also $m$-strongly convex on $\mathcal{D}$.

### 2.5 Block Partition of the Variables

We partition $x \in \mathbb{R}^p$ into $n_B$ blocks, writing

$$x \;=\; \big(x^{(1)},\, x^{(2)},\, \ldots,\, x^{(n_B)}\big). \tag{2.4}$$

At each iteration $k$, we select one block index $i_k \in \{1, \ldots, n_B\}$ to update (e.g. in random fashion). The other blocks remain temporarily fixed.

### 2.6 Blockwise Hessian Approximation

We maintain a positive definite matrix $B_k \in \mathbb{R}^{p\times p}$ that is often block-diagonal with one block for each $x^{(i)}$. In practice, $B_k$ is only modified in the rows and columns corresponding to the updated block $i_k$ at iteration $k$. The submatrix $B_k^{(i_k,i_k)}$ thus serves as a local Hessian approximation for the $i_k$-th block.

### 2.7 Algorithm Description

**Partial Gradient and Search Direction.** At iteration $k$, we compute the partial gradient of $f$ with respect to block $i_k$:

$$g_k^{(i_k)} \;=\; \nabla_{i_k} f\big(x_k\big). \tag{2.5}$$

We then form a search direction $p_k^{(i_k)}$ for this block by

$$p_k^{(i_k)} \;=\; -\Big[B_k^{(i_k,i_k)}\Big]^{-1} g_k^{(i_k)}. \tag{2.6}$$

**Wolfe-Type Line Search** We seek a step length $\alpha_k > 0$ such that $x_{k+1}$ satisfies the usual Wolfe conditions. Specifically, define $u_{i_k}(\cdot)$ to embed a block vector into the full dimension (all other components zero), and require

$$F\big(x_k + \alpha_k\, u_{i_k}(p_k^{(i_k)})\big) \;\le\; F(x_k) \;+\; c_1\, \alpha_k\, \big(g_k^{(i_k)}\big)^\top p_k^{(i_k)}, \tag{2.7}$$

$$\nabla_{i_k} f\big(x_k + \alpha_k\, u_{i_k}(p_k^{(i_k)})\big)^\top p_k^{(i_k)} \;\ge\; c_2\, \big(g_k^{(i_k)}\big)^\top p_k^{(i_k)}, \tag{2.8}$$

where $0 < c_1 < c_2 < 1$ are fixed constants.

**Block Update** Once $\alpha_k$ is chosen, we update the $i_k$-th block by

$$x_{k+1}^{(i_k)} \;=\; x_k^{(i_k)} + \alpha_k\, p_k^{(i_k)}, \quad \text{and } x_{k+1}^{(j)} = x_k^{(j)} \text{ for all } j \neq i_k. \tag{2.9}$$

We then define

$$s_k^{(i_k)} \;=\; x_{k+1}^{(i_k)} - x_k^{(i_k)}, \qquad y_k^{(i_k)} \;=\; g_{k+1}^{(i_k)} - g_k^{(i_k)}, \tag{2.10}$$

where $g_{k+1}^{(i_k)} = \nabla_{i_k} f(x_{k+1})$.

**Wolfe Conditions and $\cos\theta_k$** Under the Wolfe conditions (2.7)–(2.8), we obtain two useful inequalities:

1. *(Sufficient Decrease)*
$$F(x_{k+1}) \;\le\; F(x_k) \;+\; c_1\,\alpha_k\,\big(g_k^{(i_k)}\big)^\top p_k^{(i_k)}. \tag{2.11}$$

2. *(Curvature)*
$$\nabla_{i_k} f(x_{k+1})^\top p_k^{(i_k)} \;\ge\; c_2\,\big(g_k^{(i_k)}\big)^\top p_k^{(i_k)}. \tag{2.12}$$

We also introduce the angle $\theta_k$ between $-g_k^{(i_k)}$ and $s_k^{(i_k)}$:
$$\cos\theta_k^{(i_k)} \;=\; \frac{\big(s_k^{(i_k)}\big)^\top\big(-\,g_k^{(i_k)}\big)}{\|s_k^{(i_k)}\|\,\|\,g_k^{(i_k)}\|}. \tag{2.13}$$

## 2.8 Algorithm Pseudocode

**Algorithm 1** Block Coordinate Descent BFGS (Block BFGS)

**Given:**

- Block partition $\{x^{(1)},\dots,x^{(n_B)}\}$ and initial $x_0\in\mathbb{R}^p$,
- Initial block Hessians $\{B_0^{(i,i)}\}$,
- Wolfe parameters $0<c_1<c_2<1$,
- Tolerance $\varepsilon>0$,
- Maximum iterations $K_{\max}$.

**function** BLOCK BFGS($f, g, x_0, \{B_0^{(i,i)}\}, \varepsilon, K_{\max}, c_1, c_2$)
  $x \leftarrow x_0$;  $k \leftarrow 0$;  flag $\leftarrow 0$
  **while** $k < K_{\max}$ **do**
    $S_k \leftarrow 0$ ▷ Accumulate total norm of per-block steps this iteration
    $\pi_k \leftarrow \text{RandPerm}(\{1,\dots,n_B\})$
    **for** $i \in \pi_k$ **do** ▷ Random order of blocks in this sweep
      **Compute partial gradient:** $r_k^{(i)} \leftarrow \nabla_i f(x) + \partial_i g(x)$
      **Search direction:** $p_k^{(i)} = -\big[B_k^{(i,i)}\big]^{-1} r_k^{(i)}$
      **Wolfe line-search:** Find $\alpha^{(i)}$ such that:
        **Armijo:** $F\big(x+\alpha^{(i)}u_i(p_k^{(i)})\big) \le F(x) + c_1\alpha^{(i)}\big(g_k^{(i)}\big)^\top p_k^{(i)}$
        **Curvature:** $\nabla_i F\big(x+\alpha^{(i)}u_i(p_k^{(i)})\big)^\top p_k^{(i)} \ge c_2\big(g_k^{(i)}\big)^\top p_k^{(i)}$
      **Update block** $i$**:** $x^{(i)} \leftarrow x^{(i)} + \alpha^{(i)}p_k^{(i)}$; $s_k^{(i)} \leftarrow \alpha^{(i)}p_k^{(i)}$; $S_k \leftarrow S_k + \|s_k^{(i)}\|$
      **Compute** $y_k^{(i)}$ **and update BFGS:** $y_k^{(i)} = \nabla_i f(x) - \nabla_i f\big(x - s_k^{(i)}\big)$
      **if** $\big(y_k^{(i)}\big)^\top s_k^{(i)} > 0$ **then**
        $$B_{k+1}^{(i,i)} = B_k^{(i,i)} - \frac{B_k^{(i,i)} s_k^{(i)} (s_k^{(i)})^\top B_k^{(i,i)}}{(s_k^{(i)})^\top B_k^{(i,i)} s_k^{(i)}} + \frac{y_k^{(i)} (y_k^{(i)})^\top}{(y_k^{(i)})^\top s_k^{(i)}}$$
      **end if**
    **end for**
    **if** $S_k \le \varepsilon$ **then**
      flag $\leftarrow 1$;  **break**
    **end if**

$k \leftarrow k+1$
**end while**
**return** $(x, \{B_k^{(i,i)}\}, \text{flag})$
**end function**

## 2.9 Computational Efficiency

We follow the standard coordinate descent for the smooth part of the composite function $F(x)$, and extend it to variable-metric block updates in the composite setting. For example, the design for least squares follows exactly from [13]: Let $x_k$ be the current iterate and $\{\mathcal{I}_i\}$ a block partition (or cover). We cache the smooth image $w_k = Ax_k$ and, per block, $V_i = A_{:,\mathcal{I}_i}^\top b$, so that a trial on block $\mathcal{I}_i$ updates without re-multiplying $Ax$:

$$w_k^{\text{trial}}(\Delta x_{\mathcal{I}_i}) = w_k + A_{:,\mathcal{I}_i}\,\Delta x_{\mathcal{I}_i} \tag{2.14}$$

$$\nabla_{\mathcal{I}_i} f(x_k) = A_{:,\mathcal{I}_i}^\top (w_k - b) = A_{:,\mathcal{I}_i}^\top w_k - V_i. \tag{2.15}$$

Meanwhile, assume the non-smooth (possibly overlapping) group decomposition $g(x) = \sum_{\gamma\in\mathcal{G}} \psi_\gamma\big(x_{G_\gamma}\big)$, where $\psi_\gamma : \mathbb{R}^{|G_\gamma|} \to \mathbb{R}$ is convex. For a current block $\mathcal{I}_i$, define the set of *affected groups*

$$\mathcal{H}_i \;=\; \{\gamma \in \mathcal{G} :\; G_\gamma \cap \mathcal{I}_i \neq \emptyset\}. \tag{2.16}$$

Only these groups can change during a block move. The block subdifferential of $g$ obeys

$$\partial_{\mathcal{I}_i} g(x) \;=\; \sum_{\gamma\in\mathcal{H}_i} \big[\partial\psi_\gamma(x_{G_\gamma})\big]_{\mathcal{I}_i}, \tag{2.17}$$

so assembling a block (sub)gradient touches only $\mathcal{H}_i$. Here, we propose two practices:

- *Sequential:* pick a consistent selection of subgradients $v_\gamma(x) \in \partial\psi_\gamma(x_{G_\gamma})$ once per sweep and slice $\big[\sum_\gamma v_\gamma(x)\big]_{\mathcal{I}_i}$.
- *Pre-Index (we used):* pre-index variables to groups $\mathcal{G}(j) = \{\gamma :\; j \in G_\gamma\}$ and update only $\{x_{G_\gamma}\}_{\gamma\in\mathcal{H}_i}$, costing $O\big(\sum_{\gamma\in\mathcal{H}_i} |G_\gamma \cap \mathcal{I}_i|\big)$.

**Block line search.** Let $u_i : \mathbb{R}^{|\mathcal{I}_i|} \to \mathbb{R}^n$ embed a block vector with zeros elsewhere. Form a composite block residual

$$g^{(i)} \in \nabla_{\mathcal{I}_i} f(x_k) \;+\; \partial_{\mathcal{I}_i} g(x_k),$$

choose a SPD block metric $B_i \succ 0$, and set the search direction $p^{(i)} = -B_i^{-1} g^{(i)}$. Define $\phi(\alpha) = F\big(x_k + \alpha\, u_i(p^{(i)})\big)$ for $\alpha \geq 0$. We use weak Wolfe conditions stated with the *one-sided* directional derivative

$$\phi'_+(0) = \big\langle \nabla f(x_k),\, u_i(p^{(i)})\big\rangle + Dg\big(x_k;\, u_i(p^{(i)})\big), \quad Dg(x;d) := \lim_{\tau\downarrow 0} \frac{g(x+\tau d) - g(x)}{\tau},$$

and analogously $\phi'_+(\alpha)$ at $x_k + \alpha u_i(p^{(i)})$. Given $0 < c_1 < c_2 < 1$, accept $\alpha > 0$ if

$$\phi(\alpha) \;\leq\; \phi(0) + c_1\,\alpha\,\phi'_+(0) \quad \text{and} \quad \phi'_+(\alpha) \;\geq\; c_2\,\phi'_+(0). \tag{2.18}$$

All quantities in (2.18) are computed *block-locally*:

$$\begin{aligned}
\text{(smooth)} \quad & w_k^{\text{trial}}(\alpha p^{(i)}) = w_k + A_{:,\mathcal{I}_i}(\alpha p^{(i)}), \quad f\big(x_k + \alpha u_i(p^{(i)})\big) = \tfrac{1}{2}\|w_k^{\text{trial}}(\alpha p^{(i)}) - b\|_2^2, \\
\text{(non-smooth)} \quad & g\big(x_k + \alpha u_i(p^{(i)})\big) = \sum_{\gamma \in \mathcal{H}_i} \psi_\gamma\Big(x_{G_\gamma} + \alpha\, p^{(i)}_{G_\gamma}\Big) + \text{const.}, \\
& Dg\big(x_k + \alpha u_i(p^{(i)});\ u_i(p^{(i)})\big) = \sum_{\gamma \in \mathcal{H}_i} D\psi_\gamma\Big(x_{G_\gamma} + \alpha p^{(i)}_{G_\gamma};\ p^{(i)}_{G_\gamma}\Big).
\end{aligned}$$

*Cheap backtracking for group norms.* For $\psi_\gamma(z) = \lambda w_\gamma \|z\|_2$, precompute on entry to block $\mathcal{I}_i$

$$a_\gamma = \|x_{G_\gamma}\|_2^2, \quad b_\gamma = 2\langle x_{G_\gamma}, p^{(i)}_{G_\gamma}\rangle, \quad c_\gamma = \|p^{(i)}_{G_\gamma}\|_2^2,$$

so each trial updates only scalars:

$$\|x_{G_\gamma} + \alpha p^{(i)}_{G_\gamma}\|_2 = \sqrt{a_\gamma + b_\gamma \alpha + c_\gamma \alpha^2}, \qquad D\psi_\gamma = \begin{cases} \lambda w_\gamma \dfrac{\langle x_{G_\gamma} + \alpha p^{(i)}_{G_\gamma}, p^{(i)}_{G_\gamma}\rangle}{\|x_{G_\gamma} + \alpha p^{(i)}_{G_\gamma}\|_2}, & \text{if } \big\|x_{G_\gamma} + \alpha p^{(i)}_{G_\gamma}\big\|_2 > 0, \\ \lambda w_\gamma \|p^{(i)}_{G_\gamma}\|_2, & \text{otherwise.} \end{cases}$$

**Block-diagonal variable metric.** We maintain a block SPD metric and update only the active block using *smooth* curvature:

$$s^{(i)} = x^{k+1}_{\mathcal{I}_i} - x^k_{\mathcal{I}_i}, \qquad y^{(i)} = \nabla_{\mathcal{I}_i} f(x^{k+1}) - \nabla_{\mathcal{I}_i} f(x^k).$$

Whenever $(s^{(i)})^\top y^{(i)} > 0$, apply the BFGS update to $B_i$. This preserves descent compatibility with a non-smooth $g$ while reducing storage/solves from $O(n^2)$ to $O\big(\sum_i |\mathcal{I}_i|^2\big)$ . Therefore, a single trial in the line search on block $\mathcal{I}_i$ costs

$$O\left( \underbrace{\operatorname{nnz}\big(A_{:,\mathcal{I}_i}\big)}_{\text{smooth update of } w_k} + \underbrace{\sum_{\gamma \in \mathcal{H}_i} |G_\gamma \cap \mathcal{I}_i|}_{\text{one-time group setup}} + \underbrace{|\mathcal{I}_i|^2}_{\text{solve/apply } B_i^{-1}} \right), \tag{2.19}$$

independent of $n$ and of groups disjoint from $\mathcal{I}_i$. Each *backtrack* thereafter touches $\operatorname{nnz}(A_{:,\mathcal{I}_i})$ plus $O(|\mathcal{H}_i|)$ scalar updates $(a_\gamma, b_\gamma, c_\gamma)$. In separable cases, $\mathcal{H}_i$ reduces to groups contained in $\mathcal{I}_i$, recovering the coordinate descent economy in [13]; with overlap, restricting work to $\mathcal{H}_i$ ensures we revisit only genuinely coupled terms, retaining low per-iteration cost even when coordinates are not perfectly separable.

# 3 Global Convergence Analysis

# 4 Experiments with Leastsquares + Overlapping GroupLASSO

We consider the following composite optimization problem:

$$\min_{x \in \mathbb{R}^n} \ F(x) \ = \ \underbrace{\frac{1}{2}\|Ax - b\|^2}_{f(x)} \ + \ \underbrace{\Big(\lambda_1 \|x\|_1 + \lambda_2 \sum_{i=1}^m w_i \|x_{G_i}\|_2\Big)}_{g(x)},$$

where

- $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^n$ define the least-squares loss term,
- $G = \{G_1, G_2, \ldots, G_m\}$ is a collection of index subsets corresponding to *overlapping groups* of variables in $x$,
- $\|x_{G_i}\|_2$ denotes the Euclidean norm of the subvector of $x$ indexed by $G_i$,
- $\lambda_1, \lambda_2 > 0$ are regularization parameters, and $w_i > 0$ is a weight for group $G_i$.

This problem consists of a smooth component $f(x)$ and a non-smooth component $g(x)$. Although each group norm is non-smooth at zero, it is differentiable when $x_{G_i} \neq 0$. We solve this problem by treating the smooth part with a quasi-Newton strategy while incorporating subgradient information for the non-smooth component. The algorithm is terminated when either (i) the cumulative change in $x$ across all blocks falls below a threshold (set to $\varepsilon = 10^{-6}$) or (ii) a maximum number of iterations is reached.

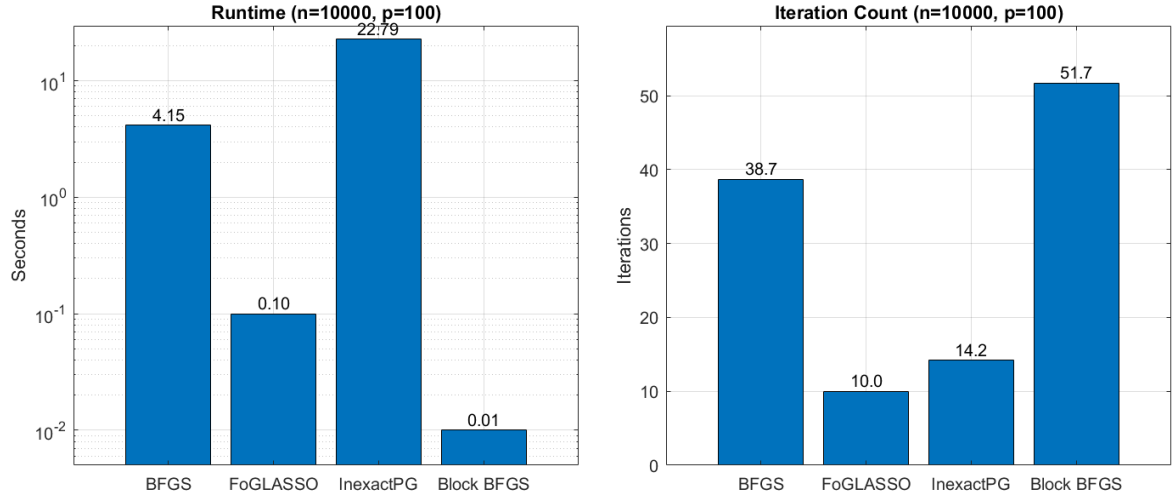



Figure 4: Scaling comparison of BFGS, FoGLASSO, InexactPG, and Block BFGS on an overlapping GroupLasso least-squares instance ($n = 10{,}000$, $p = 100$).

Recall the comparison from Figure 3, we now add the Block BFGS to this problem as shown in Figure 4. Its runtime is about 0.01 s, compared with 4.15 s for full BFGS and 0.10 s for FoGLASSO. Thus, Block BFGS is roughly 415× faster than full BFGS and 10× faster than FoGLASSO, highlighting how block-diagonal curvature updates sharply reduce per-iteration cost while maintaining strong progress per step. In the following sections, we introduce the formal computational experiments.

## 4.1 Simulated Experiments

**Dataset Generation and Experimental Protocol:** For our simulated experiments, datasets are generated as follows: given the number of samples $n$, the number of features $p$, and the number of groups $m$, we partition the $n$ features into overlapping groups. A sparse ground truth vector $x^\star$ is constructed by randomly activating a subset (typically 20%) of groups and assigning them random nonzero coefficients. The design matrix $A \in \mathbb{R}^{p \times n}$ is drawn from a multivariate Gaussian distribution. We compute the noise-free response $Ax^\star$, and then add Gaussian noise to obtain the

observed vector $b$. In these simulated experiments, we use a least-squares loss combined with GroupLasso regularization. (For our experiments on LIBSVM datasets, we employ logistic regression, so the experimental setup is different.)

To ensure robust statistical evaluation, each algorithm is run 10 times using seeds 1 to 10. This repetition allows us to compute both the average runtime per iteration and the overall average runtime across 10 independent runs, thereby reducing the impact of random initialization and noise variability.

We vary the dimensions of $A$ and $b$ to explore different regimes, including settings where the number of samples is less than the number of features ($p > n$). One notable observation is that the *Block BFGS* method sometimes fails after several hundred iterations when the dataset is underdetermined. This failure can be attributed to the fact that, in such cases, the Hessian matrix $A^\top A$ becomes rank deficient or nearly singular. Since the Block BFGS update relies on curvature information extracted from this Hessian, an insufficient sample size causes the approximated Hessian to lose its positive definiteness. In particular, small eigenvalues lead to numerical instabilities in the quasi-Newton update, eventually destabilizing the algorithm. In contrast, first-order methods such as *InexactPG* and *FoGLASSO* [2] are less dependent on curvature information and tend to be more robust in underdetermined settings, though they may exhibit slower convergence.

**Results of Simulated Experiments** Tables 1 and 2 summarize the experimental results in terms of the average runtime (in seconds) and runtime per iteration, computed over 10 runs:

Table 1: Solution Accuracy for Simulated Datasets

| $n$ | $p$ | Block BFGS | | BFGS | | FoGLASSO | | InexactPG | |
|---|---|---|---|---|---|---|---|---|---|
| | | Max Error | Mean Error | Max Error | Mean Error | Max Error | Mean Error | Max Error | Mean Error |
| 10 | 10 | 0.308 | 0.398 | 0.308 | 0.398 | 0.308 | 0.399 | 0.308 | 0.398 |
| 10 | 100 | 0.607 | 0.745 | 0.607 | 0.745 | 0.357 | 0.437 | 0.357 | 0.437 |
| 10 | 1000 | 0.111 | 0.237 | 0.120 | 0.243 | 0.114 | 0.235 | 0.114 | 0.235 |
| 100 | 10 | 1.009 | 1.009 | 1.008 | 1.098 | 1.019 | 1.114 | 1.019 | 1.114 |
| 100 | 100 | 1.241 | 1.824 | 1.242 | 1.825 | 0.984 | 1.064 | 0.984 | 1.064 |
| 100 | 1000 | 0.855 | 0.895 | 0.855 | 0.895 | 0.860 | 0.896 | 0.860 | 0.896 |
| 100 | 10000 | 0.111 | 0.237 | 0.685 | 0.708 | 2.403 | 2.729 | 2.403 | 2.729 |
| 1000 | 10 | 0.198 | 0.198 | 3.150 | 3.289 | 3.159 | 3.304 | 3.159 | 3.304 |
| 1000 | 100 | 1.073 | 1.073 | 3.017 | 3.112 | 3.175 | 3.276 | 3.174 | 3.180 |
| 1000 | 1000 | 4.160 | 4.160 | 2.544 | 2.969 | 3.129 | 3.235 | 3.129 | 3.233 |
| 1000 | 10000 | 0.325 | 0.331 | 0.743 | 0.754 | 0.316 | 0.321 | 0.315 | 0.316 |
| 10000 | 10 | 0.005 | 0.005 | 0.003 | 0.005 | 0.003 | 0.005 | 0.003 | 0.005 |
| 10000 | 100 | 0.092 | 0.092 | 0.011 | 0.013 | 0.010 | 0.012 | 0.010 | 0.012 |
| 10000 | 1000 | 0.031 | 0.031 | 0.033 | 0.034 | 0.033 | 0.034 | 0.033 | 0.034 |
| 10000 | 10000 | 0.399 | 0.440 | 0.396 | 0.435 | 0.444 | 0.501 | 0.511 | 0.534 |

**Solution Accuracy.** As seen in Table 1, Block BFGS delivers accuracy competitive with full BFGS and exceeds it in several regimes. It matches or improves upon BFGS on 13 of 15 instances and is numerically indistinguishable on most small-to-medium settings; on higher-dimension problems it can be substantially tighter—for example, at ($n$=100, $p$=10,000) the mean/max errors are

Table 2: Runtime and Iteration Count for Simulated Datasets

| $n$ | $p$ | $m$ | Block BFGS | | | BFGS | | FoGLASSO | | InexactPG | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | # Blocks | RunTime | # Iters | RunTime | # Iters | RunTime | # Iters | RunTime | # Iters |
| 10 | 10 | 4 | 2 | 0.02 | 31.1 | 0.14 | 41.8 | 0.44 | 7.6 | 1.20 | 10.2 |
| 10 | 100 | 10 | 3 | 0.08 | 155.5 | 4.17 | 83.1 | 19.14 | 113.4 | 131.11 | 11.9 |
| 10 | 1000 | 150 | 10 | 8.89 | 843.3 | 403.63 | 933.8 | 1960.47 | 3739.6 | 4451.53 | 15.3 |
| 100 | 10 | 4 | 3 | 0.01 | 21.8 | 0.16 | 27.9 | 0.33 | 6.4 | 2.23 | 10.1 |
| 100 | 100 | 10 | 3 | 0.09 | 278.6 | 18.01 | 86.4 | 0.14 | 86.6 | 676.71 | 50.1 |
| 100 | 1000 | 150 | 4 | 55.02 | 4662.2 | 361.18 | 816.4 | 108880.11 | 21940.4 | 96401.15 | 77 |
| 100 | 10000 | 1000 | 10 | 46659.11 | 47584.7 | 639785.56 | 2475.17 | 320305.87 | 39486.1 | 682905.56 | 182.4 |
| 1000 | 10 | 4 | 2 | 0.01 | 13.3 | 0.31 | 16.3 | 0.05 | 8.0 | 2.99 | 15.6 |
| 1000 | 100 | 10 | 6 | 0.04 | 30.3 | 5.95 | 56.1 | 0.07 | 16.7 | 77.88 | 22.3 |
| 1000 | 1000 | 150 | 8 | 11.93 | 1686.8 | 84.74 | 270.4 | 6.90 | 300.3 | 48.91 | 13.9 |
| 1000 | 10000 | 1000 | 20 | 20423.46 | 47213.6 | 243311.12 | 783.4 | 23683.81 | 40859.1 | 491808.14 | 140.0 |
| 10000 | 10 | 4 | 2 | 0.02 | 9.7 | 0.07 | 14.8 | 0.04 | 7.1 | 1.86 | 12.7 |
| 10000 | 100 | 10 | 2 | 0.01 | 51.7 | 4.15 | 38.7 | 0.10 | 10.0 | 22.79 | 14.2 |
| 10000 | 1000 | 150 | 12 | 6.55 | 100.6 | 40.43 | 117.5 | 14.53 | 23.2 | 28540.04 | 40.9 |
| 10000 | 10000 | 1000 | 20 | 8449.07 | 7424.4 | 290364.52 | 1205.6 | 12989.23 | 16027.2 | 221076.19 | 112.7 |

0.111/0.237 for Block BFGS versus 0.685/0.708 for BFGS. Relative to the proximal-gradient baselines, Block BFGS is consistently competitive and often markedly more accurate at high dimensionality—for instance, at ($n$=100, $p$=10,000) FoGLASSO and InexactPG yield mean/max error at 2.403/2.729 versus 0.111/0.237 for Block BFGS. Taken together with its runtime advantages, these results indicate that Block BFGS achieves reliable, BFGS-level (or better) accuracy while avoiding the variability of the proximal methods, reinforcing its suitability for large-scale structured optimization.

**Runtime and Iteration Efficiency.** According to Table 2, Block-BFGS attains the shortest runtime in 14/15 cases against FoGLASSO, InexactPG, and BFGS; the only exception is ($n$=1000, $p$=1000), where FoGLASSO is faster (6.90 s vs. 11.93 s, i.e., Block-BFGS is 1.73× slower, −72.8%). Relative to BFGS, Block-BFGS is faster in all 15/15 settings and typically by one-two orders of magnitude—for example, at ($n$=10, $p$=1000) the runtime drops from 403.6 s to 8.89 s ($\sim 45\times$), and at ($n$=10,000, $p$=10,000) from 290,365 s to 8,449 s ($\sim 34\times$). A direct comparison with BFGS clarifies why Block-BFGS remains faster even when it takes more steps: each Block-BFGS iteration updates only a small coordinate block and reuses within-block curvature, thereby avoiding the full $p \times p$ quasi-Newton update and dense search-direction computation required by BFGS. For instance, at ($n$=100, $p$=100) Block-BFGS uses 278.6 block steps versus 86.4 BFGS iterations yet runs in 0.092 s versus 18.01 s. Among the two proximal-gradient methods, FoGLASSO is substantially faster (and in some instances faster than BFGS), but it offers no runtime advantage over Block-BFGS—especially at high dimensionality. For example, at ($n$=100, $p$=10,000) FoGLASSO takes 320,306 s versus 46,659 s for Block-BFGS, a difference of $\approx 2.74 \times 10^5$ seconds ($\sim$ 76 hours).

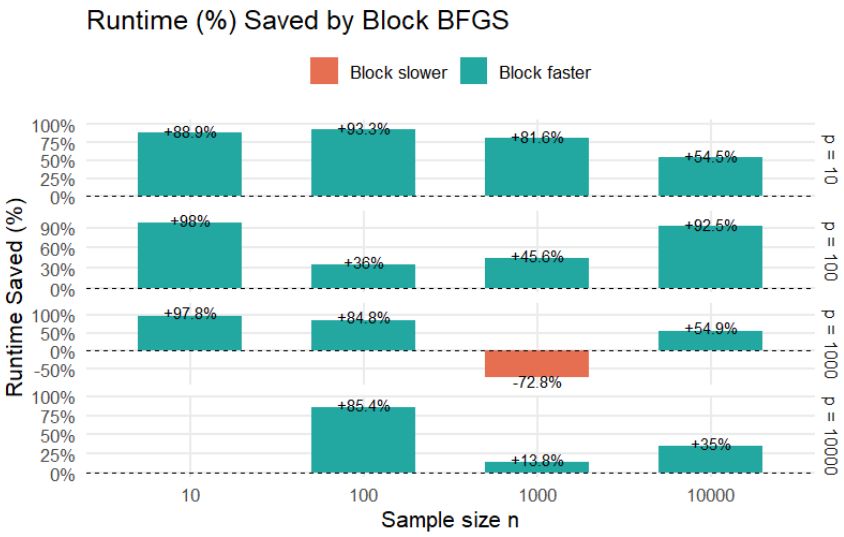



Figure 5: Savings of Block-BFGS relative to the best competing method (BFGS, FoGLASSO, or InexactPG) on each instance. Bars show % savings = $100(1 - t_{\text{Block}}/t_{\text{BestOther}})$. Panels are grouped by feature count $p$; within each panel, bars are indexed by sample size $n$. Positive values indicate Block-BFGS is faster; the single negative bar corresponds to ($n$=1000, $p$=1000).

## 4.2 Number of Blocks vs. Runtime

Table 1 and Table 2 display the optimal performance of Block BFGS versus three competitive methods for various problems. We now examine how the number of blocks $B$ affects runtime and solution quality. Two settings are considered: (i) $n$=100, $p$=1000, $m$=150 groups with tolerance $10^{-9}$; and (ii) $n$=2000, $p$=1000, $m$=200 groups with tolerance $10^{-6}$.

Table 3: Effect of block count on runtime and solution quality ($n$=100, $p$=1000, $m$=150, tol = $10^{-9}$).

| Blocks | Blocks/m ratio | Time (s) | Iters | $\|Ax_{\text{final}} - b\|$ (mean) | $\|x_{\text{final}} - x^\star\|$ (mean) | $\|x_{\text{final}} - x^\star\|$ (max) |
|---|---|---|---|---|---|---|
| 2 | 0.013 | 273.2637 | 16990.5 | 0.8552 | 44.2368 | 56.2936 |
| 4 | 0.027 | 185.4530 | 9448.6 | 0.8552 | 44.2369 | 56.2939 |
| 5 | 0.033 | 148.6696 | 9901.7 | 0.8826 | 44.2370 | 56.2938 |
| 6 | 0.040 | 207.3646 | 10853.7 | 0.8552 | 44.2369 | 56.2937 |
| **8** | **0.053** | **111.6893** | **10698.1** | **0.8552** | **44.2368** | **56.2937** |
| 15 | 0.100 | 417.2210 | 16298.6 | 0.8552 | 44.2368 | 56.2937 |
| 23 | 0.153 | 574.9661 | 15651.5 | 0.8552 | 44.2368 | 56.2937 |
| 45 | 0.300 | 2912.3897 | 32636.4 | 0.8552 | 44.2370 | 56.2904 |
| 75 | 0.500 | 1885.3600 | 15500.2 | 0.8552 | 44.2373 | 56.2959 |
| 120 | 0.800 | 3799.7954 | 29754.2 | 0.8552 | 44.2372 | 56.2940 |

Table 4: Effect of block count on runtime and solution quality ($n$=2000, $p$=1000, $m$=200, tol = $10^{-6}$).

| Blocks | Blocks/m ratio | Time (s) | Iters | $\|Ax_{\text{final}} - b\|$ (mean) | $\|x_{\text{final}} - x^\star\|$ (mean) | $\|x_{\text{final}} - x^\star\|$ (max) |
|---|---|---|---|---|---|---|
| 2 | 0.01 | 44.14 | 564.8 | 3.2197 | 0.0984 | 0.1053 |
| 4 | 0.02 | 14.94 | 307.3 | 3.2197 | 0.0984 | 0.1053 |
| 6 | 0.03 | 11.24 | 225.9 | 3.2197 | 0.0984 | 0.1053 |
| 8 | 0.04 | 9.12 | 189.7 | 3.2197 | 0.0984 | 0.1053 |
| 10 | 0.05 | 9.12 | 174.6 | 3.2197 | 0.0984 | 0.1053 |
| 12 | 0.06 | 8.55 | 149.8 | 3.2197 | 0.0984 | 0.1053 |
| 15 | 0.075 | 9.83 | 140.5 | 3.2197 | 0.0984 | 0.1053 |
| 18 | 0.09 | 9.97 | 129.5 | 3.2197 | 0.0984 | 0.1053 |
| **20** | **0.10** | **8.31** | **123.4** | **3.2197** | **0.0984** | **0.1053** |
| 25 | 0.125 | 9.76 | 117.3 | 3.2197 | 0.0984 | 0.1053 |
| 30 | 0.15 | 9.56 | 116.7 | 3.2197 | 0.0984 | 0.1053 |
| 36 | 0.18 | 12.77 | 124.5 | 3.2197 | 0.0984 | 0.1053 |
| 45 | 0.225 | 16.25 | 111.0 | 3.2197 | 0.0984 | 0.1053 |
| 50 | 0.25 | 16.24 | 105.8 | 3.2197 | 0.0984 | 0.1053 |
| 80 | 0.40 | 29.03 | 123.9 | 3.2197 | 0.0984 | 0.1053 |
| 100 | 0.50 | 28.47 | 102.7 | 3.2197 | 0.0984 | 0.1053 |
| 120 | 0.60 | 42.35 | 130.9 | 3.2197 | 0.0984 | 0.1053 |
| 150 | 0.75 | 59.44 | 166.6 | 3.2197 | 0.0984 | 0.1053 |

In both regimes, solution quality is essentially invariant to $B$: the residual $\|Ax_{\text{final}} - b\|$, the distance to the planted solution $\|x_{\text{final}} - x^\star\|$, $\|x_{\text{final}}\|$, and the final objective remain stable across all block counts. Runtime, however, is *not* monotone in $B$. For $n$=100, the fastest configuration is $B$=8 (Table 3), while very small $B$ (few, very large blocks) and very large $B$ (many tiny blocks) both increase runtime. For $n$=2000, the minimum runtime occurs near $B$=20 (Table 4); iteration counts continue to drift slightly with $B$, but per-iteration cost grows when blocks become too small or too large, yielding a clear U-shaped trade-off. Practically, a *moderate* number of blocks (here, roughly $B/G$ between 5% and 10%) provides the best accuracy-efficiency balance.

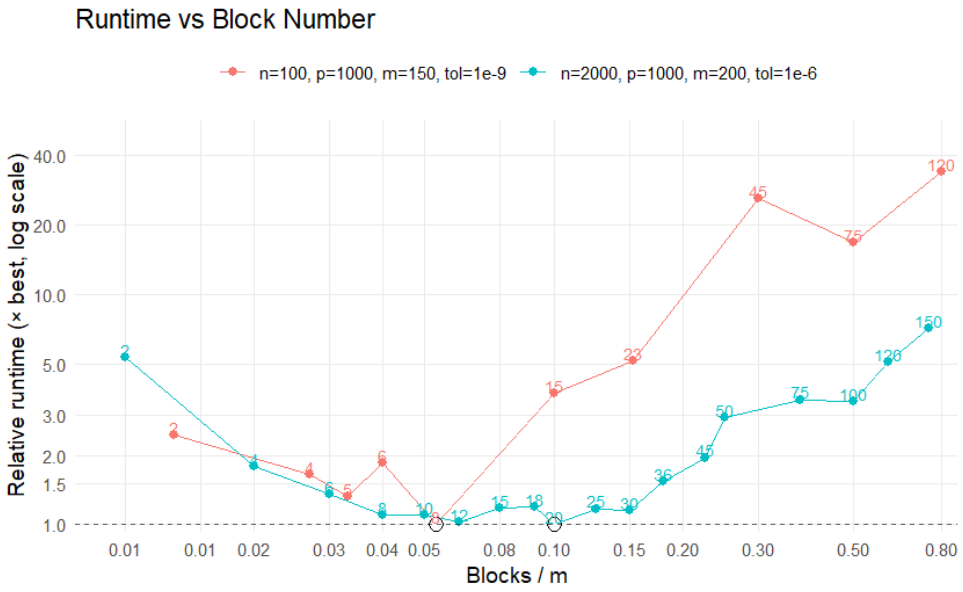



Figure 6: Runtime vs. block ratio (Blocks/$m$) normalized by each experiment's best runtime. Curves show relative runtime ($\times$ best; log scale on both axes), overlaid for $n$=100, $p$=1000, $m$=150 and $n$=2000, $p$=1000, $m$=200. Hollow markers denote the minimizing block count for each experiment; numeric labels give the actual number of blocks. The dashed line at 1$\times$ marks the per-experiment optimum.

### 4.3 Gene Expression Experiments

We evaluate the proposed algorithm on the breast cancer gene expression dataset originally from [10], and used in [2]. The dataset consists of 8,141 genes measured on 295 tumors, and is organized into two settings: (i) *pathways* from MSigDB canonical pathways: 639 groups are formed, and the average group size is 23.7 genes [12]; (ii) *edges* from the network in [11], yielding 42,594 overlapping size-2 groups. The objective is identical to the simulated experiments, and for both pathway and edge settings, we standardize features by columnwise $z$-scoring of $A$ (gene expression), leaving $b$ (labels) unchanged.

**Protocol and hyperparameters.** We run Block BFGS with Wolfe parameters $c_1 = 10^{-3}$, $c_2 = 0.3$, tolerance $\varepsilon = 10^{-8}$. We initialize at $x_0 = \mathbf{0}$ and use the provided weights and groups. For both edge and pathways, we run 10 independent trials with seeds 1-10.

- **Edges:** $A = X_{\text{edge}}$, $b = y$, $G = G_{\text{edge}}$, $w = w_{\text{edge}}$, $(\lambda_1, \lambda_2) = (10^{-3}, 10^{-1})$.
- **Pathways:** $A = X_{\text{path}}$, $b = y$, $G = G_{\text{path}}$, $w = w_{\text{path}}$, $(\lambda_1, \lambda_2) = (10^{-3}, 1)$.

For each run we record the runtime, number of iterations, the final iterate $x$, and the objective history $F(x_k)$. In the tables below, we report (i) *solution accuracy* and (ii) *runtime/efficiency* summaries across the repeated trials for the *edge* and *pathway* groupings.

**Results** On the *edge* groups (Table 5), a moderate block count yields the best efficiency. With 10 blocks, Block BFGS attains $F = 33.1074$, essentially matching FoGLASSO (gap $\approx 5 \times 10^{-4}$, and in fact slightly lower), while significantly reducing runtime from $1.68 \times 10^5$ s (FoGLASSO) to $2.43 \times 10^4$ s ($\approx 6.9\times$ faster). The estimated parameter is also closer to the FoGLASSO solution ($\|x_{\text{final}} - x_{\text{FoGLASSO}}\| = 0.1593$), in contrast to other algorithms. L-BFGS is the fastest in absolute

Table 5: Breast cancer gene expression (edge groups, 42,594 pairs): performance summary.

| Method | Blocks | Iterations | Runtime | $\|x_{\text{final}} - x_{\text{FoGLASSO}}\|$ | $F_{\text{final}} - F_{\text{FoGLASSO}}$ |
|---|---|---|---|---|---|
| Block BFGS | 5 | 25582 | 16769.86 | 0.1809 | 0.9144 |
| **Block BFGS** | **10** | **27221** | **24337.61** | **0.1593** | **0.0005** |
| Block BFGS | 15 | 22679 | 28995.83 | 0.1647 | 0.0006 |
| Block BFGS | 20 | 32729 | 52654.99 | 0.1702 | 0.0006 |
| Block BFGS | 25 | 28497 | 58196.33 | 0.1786 | 0.0006 |
| Block BFGS | 30 | 25634 | 63709.73 | 0.1744 | 0.0006 |
| Block BFGS | 50 | 112681 | 432996.02 | 0.1792 | 0.0006 |
| L-BFGS | N.A. | 5316 | **727.25** | 0.4542 | 0.0733 |
| BFGS | N.A. | 250 | 549710.83 | 0.7106 | 0.0724 |
| FoGLASSO | N.A. | 3217 | 168075.74 | N.A. | N.A. |
| InexactPG | N.A. | 45.9 | 22934.20 | 0.9597 | 7.4170 |

Table 6: Breast cancer gene expression (pathway groups, 637 groups): performance summary.

| Method | Blocks | Iterations | Runtime | $\|x_{\text{final}} - x_{\text{FoGLASSO}}\|$ | $f_{\text{final}} - F_{\text{FoGLASSO}}$ |
|---|---|---|---|---|---|
| Block BFGS | 5 | 2798 | 3056.13 | 0.0097 | 0.0007 |
| Block BFGS | 10 | 1727 | 463.36 | 0.0099 | 0.0007 |
| Block BFGS | 15 | 1571 | 222.14 | 0.0097 | 0.0007 |
| Block BFGS | 20 | 1414 | 134.44 | 0.0098 | 0.0007 |
| **Block BFGS** | **25** | **1144** | **134.08** | **0.0098** | **0.0007** |
| Block BFGS | 30 | 1117 | 187.32 | 0.0097 | 0.0007 |
| Block BFGS | 40 | 1026 | 226.40 | 0.0099 | 0.0007 |
| Block BFGS | 50 | 1020 | 437.33 | 0.0097 | 0.0007 |
| Block BFGS | 100 | 1616 | 615.55 | 0.0097 | 0.0007 |
| Block BFGS | 200 | 2409 | 1998.70 | 0.0097 | 0.0007 |
| L-BFGS | N.A. | 2397 | **20.04** | 0.3944 | 1.5076 |
| BFGS | N.A. | 5942 | 91417.53 | 0.0098 | 0.0007 |
| FoGLASSO | N.A. | 634 | 173745.33 | N.A. | N.A. |
| InexactPG | N.A. | 37 | 66371.84 | 0.1332 | 0.3629 |

time ($7.27 \times 10^2$ s) but settles at a higher objective ($F = 33.18$, +0.22% vs. FoGLASSO). Full BFGS reaches a slightly worse objective ($F = 33.32$) at prohibitive cost ($5.50 \times 10^5$ s). Using too few blocks (e.g., 5) speeds computations but degrades the objective ($F = 34.02$); using many blocks (e.g., 50) does not improve $F$ but inflates runtime (to $4.33 \times 10^5$ s).

On the *pathway* groups (Table 6), Block BFGS with 15-25 blocks converges to the FoGLASSO solution quality *three orders of magnitude faster*. For example, with 20 blocks, we obtain $F \approx 64.79533$ (gap $\approx 7 \times 10^{-4}$) and $\|x_{\text{final}} - x_{\text{FoGLASSO}}\| \approx 0.0098$ in $1.34 \times 10^2$ s, compared to $1.74 \times 10^5$ s for FoGLASSO ($\sim$ 1,300$\times$ speedup). Full BFGS attains the same objective level but requires $9.14 \times 10^4$ s ($\sim$ 680$\times$ slower than Block BFGS). InexactPG is both slower ($6.64 \times 10^4$ s) and less accurate ($F = 65.16$, +0.56%; distance 0.1332). L-BFGS is again fastest ($2.00 \times 10^1$ s) but with a noticeably higher objective ($F = 66.30$, +2.33%) and a much larger parameter deviation (0.3944). Increasing blocks far beyond 25 yields no meaningful accuracy gains but raises runtime.

## 4.4 Post-Processing via FoGLASSO Warm-Starts

Across both gene-expression and simulated experiments, *Block BFGS* attains FoGLASSO-level solutions at far lower cost than FoGLASSO, full BFGS, or InexactPG. Although *L-BFGS* is the fastest in runtime, its solution consistently deviates further from FoGLASSO. To compare solution quality across methods more directly, we *warm-start FoGLASSO* from the terminal iterate of each BFGS variant. For each method $M \in \{\text{Block BFGS}, \text{BFGS}, \text{L-BFGS}\}$, we rerun FoGLASSO with the same hyperparameters and stopping criterion (tolerance $10^{-9}$ before and after), but initialize

at $x_0 = x^{(M)}_{\text{final}}$. If $x^{(M)}_{\text{final}}$ is already near the FoGLASSO solution, the warm-started run should require few iterations and produce negligible changes. We assess post-processing quality by the *parameter change* $\|x^{(M)}_{\text{post}} - x^{(M)}_{\text{final}}\|$ and the *objective change* $F\big(x^{(M)}_{\text{post}}\big) - F\big(x^{(M)}_{\text{final}}\big)$, together with the warm-start iterations and runtime. Because the model induces group sparsity, we also compare recovered group support: letting $\mathcal{Z}(x)$ be the set of groups with zero norm in $x$ and $\mathcal{G}$ the collection of all groups, we report *Agreement*% $= 100 \times |\{g : \mathbf{1}_{\{g \in \mathcal{Z}(x^{(M)}_{\text{post}})\}} = \mathbf{1}_{\{g \in \mathcal{Z}(x_{\text{FoGLASSO}})\}}\}|/|\mathcal{G}|$ and *OnlyFoZero*% $= 100 \times |\mathcal{Z}(x_{\text{FoGLASSO}}) \setminus \mathcal{Z}(x^{(M)}_{\text{post}})|/|\mathcal{G}|$.

Table 7: FoGLASSO warm-starts (simulated data; $n$=100, $p$=1000, groups = 150; tol = $10^{-9}$).

| Method | Runtime (s) | Iterations | $\|x^{(M)}_{\text{post}} - x^{(M)}_{\text{final}}\|$ (mean) | $F(x^{(M)}_{\text{post}}) - F(x^{(M)}_{\text{final}})$ (mean) | OnlyFoZero (%) | Agreement (%) |
|---|---|---|---|---|---|---|
| Block BFGS | 369.0993 | 21.4 | $2.42 \times 10^{-12}$ | $4.26 \times 10^{-14}$ | 0.5333 | 99.22 |
| BFGS | 372.5878 | 21.1 | $2.42 \times 10^{-12}$ | $4.26 \times 10^{-14}$ | 0.5333 | 99.22 |
| L-BFGS | 2410.9961 | 156.2 | $2.51 \times 10^{-10}$ | $3.64 \times 10^{-12}$ | 1.8660 | 96.66 |

**Post-processing: Simulated Experiments** For the simulated setting, warm-starting FoGLASSO from *Block BFGS* (or full BFGS) requires only $\approx 21$ iterations and $\approx 3.7 \times 10^2$ s, with changes at numerical precision: $\|x^{(M)}_{\text{post}} - x^{(M)}_{\text{final}}\| \approx 2.4 \times 10^{-12}$ and $F(x^{(M)}_{\text{post}}) - F(x^{(M)}_{\text{final}}) \approx 4.3 \times 10^{-14}$. Support metrics are likewise aligned (OnlyFoZero $\approx 0.53\%$, Agreement $\approx 99.2\%$). By contrast, warm-starting from *L-BFGS* takes substantially longer (156 iterations and $\approx 3.41 \times 10^3$ s) and corrects a noticeably larger discrepancy ($\|x^{(M)}_{\text{post}} - x^{(M)}_{\text{final}}\| \approx 2.5 \times 10^{-10}$, $F$ change $\approx 3.6 \times 10^{-12}$), with lower support agreement (96.66%) and more missed zeros (OnlyFoZero = 1.866%). These results confirm that Block BFGS arrives extremely close to the FoGLASSO solution—both in parameters and in group support—while L-BFGS endpoints deviate more and require substantial correction despite their standalone speed. In fact, the total runtime for L-BFGS plus post-processing is $16.515 + 2410.9961 = 2427.5111$ s, which still exceeds the combined time for Block BFGS plus post-processing ($111.6893 + 369.0993 = 480.7886$s, block=8).

Table 8: FoGLASSO warm-starts on gene-expression data. *Iters* is the number of FoGLASSO iterations taken from each warm start. **Total Runtime** = (algorithm runtime) + (warm-start runtime).

| Dataset | Method | Iters | Runtime (s) | **Total Runtime (s)** | $\|x_{\text{final}} - x_{\text{prev}}\|$ (mean) | $F_{\text{PG}} - F_{\text{prev}}$ (final) | OnlyFoZero (%) | **Agreement (%)** |
|---|---|---|---|---|---|---|---|---|
| Edge | Block BFGS | 20 | 40814.83 | 65152.44 | 0.1593 | 33.1074 | 8.0281 | 89.64 |
| | Block BFGS | 50 | 47280.52 | 71618.13 | 0.1592 | 33.1074 | 2.45 | **97.05** |
| | BFGS | 20 | 50754.29 | 600465.12 | 0.5706 | 33.2928 | 57.2003 | 83.64 |
| | BFGS | 50 | 53432.14 | 603142.97 | 0.5689 | 33.2710 | 2.45 | **97.05** |
| | L-BFGS | 120 | 52304.18 | 53031.42 | 0.7028 | 33.1760 | 27.7234 | 81.56 |
| | L-BFGS | 200 | 55510.22 | **56237.46** | 0.6904 | 33.1701 | 24.43 | **82.32** |
| Pathway | Block BFGS | 2 | 3652.48 | 3874.62 | 0.0097 | 0.0006 | 21.1823 | 93.25 |
| | Block BFGS | 5 | 4711.62 | 4933.76 | 0.0097 | 0.0007 | 1.4778 | 99.53 |
| | Block BFGS | 10 | 6155.50 | 6377.64 | 0.0097 | 0.0007 | 0.4926 | 99.84 |
| | Block BFGS | 15 | 7652.49 | **7874.62** | 0.0097 | 0.0007 | 0 | **100.00** |
| | BFGS | 2 | 3614.51 | 95032.04 | 0.0098 | 0.0006 | 21.1823 | 92.62 |
| | BFGS | 5 | 4830.48 | 96248.01 | 0.0098 | 0.0007 | 0.9852 | 99.69 |
| | BFGS | 10 | 6357.08 | 97774.61 | 0.0098 | 0.0007 | 0 | **100.00** |
| | BFGS | 15 | 7672.34 | 99089.87 | 0.0098 | 0.0007 | 0 | **100.00** |
| | L-BFGS | 50 | 16652.05 | 16672.08 | 0.2448 | 0.4636 | 23.6453 | 84.93 |
| | L-BFGS | 100 | 32085.20 | 32105.24 | 0.0883 | 0.0484 | 15.2709 | 94.66 |
| | L-BFGS | 150 | 45621.52 | 45641.56 | 0.0327 | 0.0084 | 2.9556 | **99.06** |

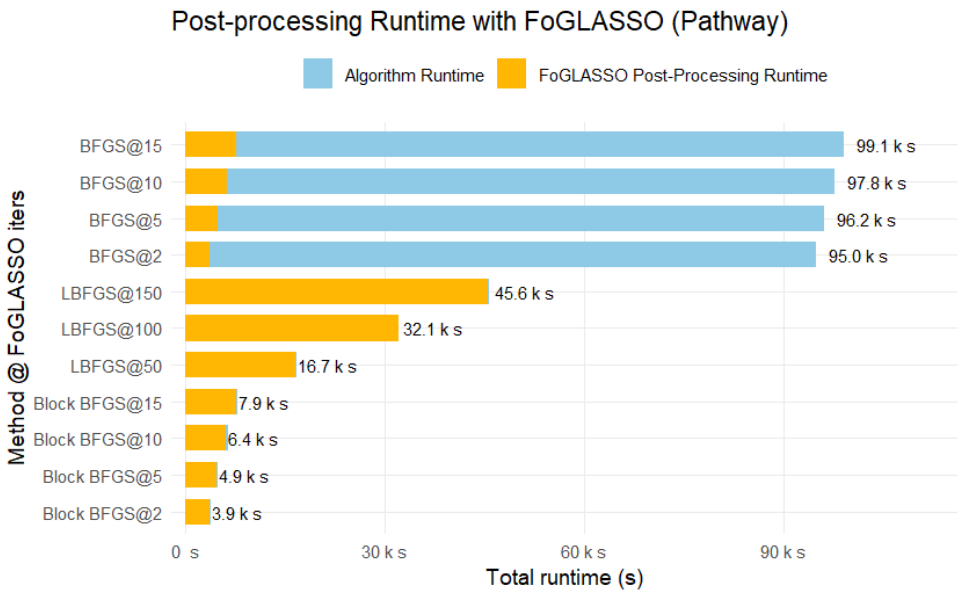



Figure 7: Pathway dataset: total runtime decomposition for BFGS plus FoGLASSO. Bars show the warm–start algorithm time (blue) and the FoGLASSO post–processing time (yellow); labels use `Method@iters`. Block–BFGS attains the same final accuracy after post–processing as the other warm starts but with substantially lower total runtime, whereas L-BFGS requires large FoGLASSO clean–up despite a short initial runtime.

**Post-processing: Gene Expression Experiment** From Table 8 and Figure 7, we can see that on the *Pathway* dataset, warm-starting FoGLASSO from *Block BFGS* requires only 5-15 iterations and makes vanishing adjustments ($\|x_{\text{post}} - x_{\text{final}}\| \approx 9.7 \times 10^{-3}$, $F(x_{\text{post}}) - F(x_{\text{final}}) \approx 7 \times 10^{-4}$). The recovered group support essentially matches FoGLASSO (Agreement = 99.53% $\rightarrow$ 100%; OnlyFoGLASSOZero = 1.4778% $\rightarrow$ 0% as the warm-start iterations increase). Crucially, when we compare *total runtime* (algorithm runtime + post-processing), Block BFGS finishes in $4.93 \times 10^3$-$7.87 \times 10^3$ s, which is an order of magnitude faster than full BFGS ($9.62 \times 10^4$-$9.91 \times 10^4$ s) and also substantially faster than L-BFGS ($1.67 \times 10^4$-$3.21 \times 10^4$ s). Although L-BFGS is fast in isolation, its endpoints need 50-100 warm-start iterations and sizeable corrections ($\|x_{\text{post}} - x_{\text{final}}\| = 2.448 \times 10^{-1} \rightarrow 8.83 \times 10^{-2}$, objective change = 0.4636 $\rightarrow$ 0.0484), with markedly lower support agreement (84.93-94.66%) and many more missed zero groups (OnlyFoGLASSOZero = 23.6453% $\rightarrow$ 15.2709%). Full BFGS attains FoGLASSO-quality solutions after post-processing but incurs prohibitive total runtime due to its expensive pre-run. Overall, Block BFGS delivers FoGLASSO-level accuracy—both numerically and in group support—at the lowest overall cost once post-processing is accounted for.

The same pattern holds on *Edge*. Block BFGS warm-starts with 20-50 iterations finish in $\mathbf{6.52 \times 10^4}$ – $\mathbf{7.16 \times 10^4}$ s with improving support (Agreement 89.64% $\rightarrow$ 97.05%, OnlyFoGLASSOZero 8.03% $\rightarrow$ 2.45%). BFGS ultimately attains comparable agreement at 50 iterations, but its total runtime is $\mathbf{6.03 \times 10^5}$ s, roughly $\mathbf{8-9}\times$ slower than Block BFGS; with only 20 iterations its clean-up is even more expensive ($6.00 \times 10^5$ s; OnlyFoGLASSOZero 57.2%). L-BFGS has small algorithm times but remains inaccurate even after 120-200 iterations (Agreement 81.56%-82.32%; OnlyFoGLASSOZero 27.72%-24.43%), so matching the Block-/full-BFGS accuracy would require many more FoGLASSO steps and a larger total runtime than reported.

# 5 Experiments with Logistic Regression+Overlapping GroupLASSO

We also evaluate the performance of the Block BFGS algorithm on several real-world classification problems from the LIBSVM repository [8]. In these experiments, the smooth component of the objective function is based on logistic regression rather than least-squares. Specifically, for a given dataset the loss function is defined as

$$f(x) = \sum_{i=1}^{N} \log\Big(1 + \exp\big(-b_i\,(A_{i,\cdot}\,x)\big)\Big),$$

where $A \in \mathbb{R}^{N\times n}$ is the feature matrix, $b \in \{-1, 1\}^N$ are the binary class labels (with any label originally 0 converted to $-1$), and $N$ denotes the number of samples. The computational setup for these experiments followed the idea from [3] with adjustments. Each of 8 datasets is loaded using the `libsvmread` function in MATLAB. The feature matrix is converted to a full matrix and the labels are adjusted so that they are in the set $\{-1, 1\}$. Overlapping groups are then constructed by partitioning the feature indices into a fixed number (in our experiments, 10) of groups. Each group is formed by selecting a block of consecutive features, with consecutive groups overlapping by a quarter of the group size. This grouping strategy is designed to capture feature correlations and to reflect realistic scenarios in high-dimensional classification problems.

The overall regularized objective function is given by

$$F(x) = f(x) + \lambda_1\|x\|_1 + \lambda_2 \sum_{g\in G} w_g\|x_g\|,$$

where $\lambda_1$ and $\lambda_2$ are regularization parameters (set to 0.1 and 1, respectively) and the group weights $w_g$ are set uniformly to one. The Block BFGS algorithm is then applied with EBLS line search ($\texttt{c1} = 10^{-3}$ and $\texttt{c2} = 0.3$), and a tolerance of $10^{-6}$. We record performance metrics such as the overall runtime, iteration count, final objective value, misclassification rate, and logistic loss for each dataset. Also, the LIBSVM datasets span a wide range of dimensions. This variety allows us to assess the scalability and robustness of the Block BFGS method in realistic, high-dimensional classification scenarios.

**Results of LIBSVM Experiments** Tables 9 and 10 summarize the experimental results in terms of logistic loss, the average runtime (in seconds), and runtime per iteration:

Table 9: Runtime and Iteration Count for LIBSVM Datasets

| Dataset | Block BFGS | | | FoGLASSO | | | InexactPG | | | BFGS | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | # Iters | RunTime | Runtime/Iter | # Iters | RunTime | Runtime/Iter | # Iters | RunTime | Runtime/Iter | # Iters | RunTime | Runtime/Iter |
| a9a | 156 | 30.9802 | 0.1986 | 310 | 5.2872 | 0.0171 | 40 | **1.7809** | 0.0445 | 96 | 25.3251 | 0.2638 |
| colon-cancer | 905 | **58.4914** | 0.3323 | 176 | 92.4293 | 0.5252 | 43 | 110.3279 | 2.5658 | 430 | 810.5214 | 1.8849 |
| duke | 7 | **3.3119** | 0.4731 | 1369 | 1248.6311 | 0.9121 | 51 | 330.01 | 6.47 | 1107 | 2650.17 | 2.39 |
| leukemia | 904 | 937.4091 | 1.03696 | 139 | **199.7885** | 1.4373 | 40 | 351.30 | 8.76 | 1034 | 2411.06 | 2.33 |
| mushrooms | 295 | 11.0477 | 0.0374 | 121 | 22.8852 | 0.1891 | 41 | **0.3494** | 0.0085 | 236 | 12.6167 | 0.0535 |
| w8a | 410 | 337.9433 | 0.8243 | 710 | 46.0110 | 0.0648 | 39 | **5.2049** | 0.1335 | 182 | 40.3435 | 0.2217 |

**Solution Accuracy.** Block BFGS achieves logistic loss values that are generally competitive with those obtained by BFGS and FoGLASSO. For instance, on the a9a dataset, Block BFGS obtains an average loss of about 0.322, matching the performance of the full-matrix method and closely tracking FoGLASSO. Although in some cases—such as on the colon-cancer dataset—FoGLASSO presents marginally lower loss values, Block BFGS still delivers reliable and high-quality solutions across all tested datasets.

Table 10: Logistic Loss for LIBSVM Datasets

| Dataset | Block BFGS | | FoGLASSO | | InexactPG | | BFGS | |
|---|---|---|---|---|---|---|---|---|
| | Total Loss | Average Loss | Total Loss | Average Loss | Total Loss | Average Loss | Total Loss | Average Loss |
| a9a | 10511.2410 | **0.3219** | 10517.6898 | 0.3230 | 17151.2897 | 0.5267 | 10511.2367 | 0.3228 |
| colon-cancer | 3.8928 | 0.0628 | 1.3063 | **0.0301** | 30.8809 | 0.4981 | 1.8903 | 0.0304 |
| duke | 0.7103 | **0.0115** | 1.9964 | 0.0454 | 4.4505 | 0.1001 | 0.7100 | 0.0122 |
| leukemia | 0.9929 | 0.0261 | 0.5755 | **0.0151** | 4.0100 | 0.1100 | 0.9900 | 0.0302 |
| mushrooms | 1.5939 | 0.0002 | 1.6024 | 0.0002 | 1768.8982 | 0.2177 | 1.5932 | **0.0001** |
| w8a | 5585.8749 | 0.1123 | 5615.0068 | 0.1129 | 23075.0890 | 0.4639 | 5585.8582 | **0.1122** |

**Runtime and Iteration Efficiency** In terms of computational efficiency, Block BFGS requires a moderate number of iterations. While its per-iteration cost is somewhat higher than that of FoGLASSO (which benefits from simpler proximal updates), the overall runtime remains competitive, particularly on larger datasets like w8a where curvature information helps accelerate convergence. Datasets with lower sample sizes (e.g., colon-cancer or leukemia) necessitate more iterations, yet Block BFGS consistently converges in a reasonable total runtime compared to the alternatives.

Overall, Block BFGS strikes a solid balance between leveraging second-order information and maintaining computational efficiency. Its performance on LIBSVM datasets demonstrates robustness across different problem scales and highlights its potential for handling high-dimensional structured classification tasks. Although iteration counts and runtime may vary with data characteristics, the method consistently delivers competitive solution quality, making it a promising approach for real-world applications.

# 6 Conclusion

We proposed Block BFGS, a block-coordinate-descent adaptation of BFGS for composite objectives with overlapping groups. The method refreshes curvature only on the active block, maintains a block-diagonal Hessian approximation, and employs a composite directional-derivative Wolfe rule to navigate non-smoothness without proximal operators. On the theory side, we proved global convergence under standard assumptions and extended the guarantees to a milder smooth-extension setting on neighborhoods containing the iterates, using angle and trace/determinant controls to prevent degeneracy. Empirically, across least-squares and logistic models with structured sparsity, Block BFGS consistently matches the solution quality of proximal-gradient baselines while delivering substantial runtime savings, particularly when curvature is informative and blocks are moderately sized.

## 6.1 Limitations

Block BFGS inherits sensitivity to line-search quality and may be less advantageous when blocks are extremely small, overlap is dense enough to negate block locality, or curvature is poorly conditioned within blocks. Our analysis focuses on convex composites; extending guarantees to broadly nonconvex settings remains open.

## 6.2 Outlook

The evidence in this paper indicates that *Block BFGS* is especially effective when

1. problems exhibit clear block structure (e.g., group or overlapping penalties) so that within-block curvature accelerates progress;

2. dimensions are medium–to–large, making full BFGS prohibitive while first-order methods underuse curvature;

3. objective evaluations dominate cost, allowing block-local quasi-Newton updates to amortize their overhead.

These observations motivate several extensions:

1. **Limited-memory and sketching.** Develop per-block L-BFGS/sketched updates to cut storage and Hessian-approximation cost without sacrificing curvature quality.

2. **Adaptive blocks and screening.** Learn block selection and block sizes on the fly; integrate safe/strong screening to tighten active sets and reduce unnecessary updates.

3. **Stochastic and distributed variants.** Combine block quasi-Newton steps with variance-reduced gradients and asynchronous, multi-node implementations for streaming and large-scale settings.

Together, these directions aim to preserve quasi-Newton accuracy while further reducing per-iteration cost and broadening Block BFGS's applicability.

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# A Notation Dictionary

| Symbol | Meaning | Type / Dimension |
|---|---|---|
| **Data, dimensions, sets** | | |
| $n$ | number of samples (rows of $A$) | integer |
| $p$ | number of features/variables (columns of $A$) | integer |
| $m$ | number of (possibly overlapping) groups | integer |
| $n_B$ | number of coordinate blocks | integer |
| $A$ | design matrix | $\mathbb{R}^{n\times p}$ |
| $b$ | response / labels | $\mathbb{R}^n$ or $\{-1,1\}^n$ |
| $x$ | parameter vector | $\mathbb{R}^p$ |
| $\mathcal{I}_i$ | index set of block $i$; $\lvert\mathcal{I}_i\rvert = p_i$ | subset of $\{1,\dots,p\}$ |
| $\mathcal{G} = \{G_1,\dots,G_m\}$ | group index sets (may overlap) | $G_\gamma \subset \{1,\dots,p\}$ |
| $\mathcal{H}_i$ | groups affected by block $\mathcal{I}_i$ | $\{\gamma :\ G_\gamma \cap \mathcal{I}_i \neq \emptyset\}$ |
| $w_\gamma$ | group weight for $G_\gamma$ | $w_\gamma > 0$ |
| $\mathcal{D}$ | level set $\{x : F(x) \le F(x_0)\}$ | subset of $\mathbb{R}^p$ |
| $[p]$ | shorthand for $\{1,\dots,p\}$ | set |
| **Objective and derivatives** | | |
| $F$ | composite objective $F(x) = f(x) + g(x)$ | $\mathbb{R}^p \to \mathbb{R}$ |
| $f$ | smooth convex loss (LS or logistic) | $\mathbb{R}^p \to \mathbb{R}$ |
| $g$ | convex (possibly non-smooth) penalty | $\mathbb{R}^p \to \mathbb{R}$ |
| $\lambda_1, \lambda_2$ | $\ell_1$ and group penalties | nonnegative scalars |
| $\nabla f(x)$ | gradient of $f$ | $\mathbb{R}^p$ |
| $\nabla_i f(x)$ | block gradient on $\mathcal{I}_i$ | $\mathbb{R}^{p_i}$ |
| $\nabla^2 f(x)$ | Hessian of $f$ | $\mathbb{R}^{p\times p}$ |
| $\partial g(x)$ | subdifferential of $g$ | set in $\mathbb{R}^p$ |
| $\partial_i g(x)$ | block subdifferential on $\mathcal{I}_i$ | set in $\mathbb{R}^{p_i}$ |
| $Dg(x;d)$ | one-sided directional derivative of $g$ at $x$ along $d$ | scalar |
| $m$, $M$ | curvature bounds for $f$ on $\mathcal{D}$ | $0 < m \le M$ |
| **Group penalty helpers** | | |
| $\psi_\gamma(z)$ | group term on $G_\gamma$ (e.g., $\lambda w_\gamma \lVert z\rVert_2$) | $\mathbb{R}^{\lvert G_\gamma\rvert} \to \mathbb{R}$ |
| $a_\gamma, b_\gamma, c_\gamma$ | cache for $\lVert x_{G_\gamma} + \alpha p_{G_\gamma}\rVert_2 = \sqrt{a_\gamma + b_\gamma\alpha + c_\gamma\alpha^2}$ | scalars |
| **Block structure and operators** | | |
| $u_i$ | embedding: inserts a block vector into $\mathbb{R}^p$ | $u_i : \mathbb{R}^{p_i} \to \mathbb{R}^p$ |
| $B_k$ | (typically block-diagonal) SPD metric at iter. $k$ | $\mathbb{R}^{p\times p}$ |
| $B_k^{(i,i)}$ | block-$i$ submatrix (updated only when $i = i_k$) | $\mathbb{R}^{p_i\times p_i}$ |
| **Per-iteration quantities (active block $i_k$)** | | |
| $i_k$ | active block index at iteration $k$ | $\{1,\dots,n_B\}$ |
| $x_k$ | iterate before updating block $i_k$ | $\mathbb{R}^p$ |
| $h_k^{(i)}$ | composite residual $h_k^{(i)} \in \nabla_i f(x_k) + \partial_i g(x_k)$ | $\mathbb{R}^{p_i}$ |
| $p_k^{(i)}$ | search direction on block $i$ | $-\left[B_k^{(i,i)}\right]^{-1} h_k^{(i)}$ |
| $\alpha_k$ | accepted stepsize for block move | scalar $> 0$ |
| $s_k^{(i)}$ | block step $= \alpha_k\, p_k^{(i)}$ | $\mathbb{R}^{p_i}$ |
| $y_k^{(i)}$ | smooth curvature diff. $= \nabla_i f(x_{k+1}) - \nabla_i f(x_k)$ | $\mathbb{R}^{p_i}$ |

*continued on next page*

| Symbol | Meaning | Type / Dimension |
|---|---|---|
| $\theta_k^{(i)}$ | angle between $-\nabla_i f(x_k)$ and $s_k^{(i)}$ | $\cos\theta_k^{(i)} = \frac{(s_k^{(i)})^\top(-\nabla_i f(x_k))}{\|s_k^{(i)}\|\,\|\nabla_i f(x_k)\|}$ |
| $\pi_k$ | random permutation of blocks in a sweep | permutation of $\{1,\dots,n_B\}$ |
| $S_k$ | sum of block step norms in a sweep | scalar |
| **Line search (two variants used)** | | |
| $c_1$, $c_2$ | Wolfe constants with $0 < c_1 < c_2 < 1$ | scalars |
| $\phi(\alpha)$ | 1D restriction $\phi(\alpha) = F\big(x_k + \alpha\, u_{i_k}(p_k^{(i_k)})\big)$ | scalar fn. of $\alpha$ |
| $\phi'_+(\alpha)$ | one-sided derivative of $\phi$ at $\alpha$ | scalar |
| Composite Wolfe (impl.) | accept $\alpha > 0$ if $\phi(\alpha) \le \phi(0) + c_1\,\alpha\,\phi'_+(0)$ and $\phi'_+(\alpha) \ge c_2\,\phi'_+(0)$ | rule on $F$ |
| Hybrid Wolfe (proofs) | Armijo on $F$, curvature on $f$: $F(x_k + \alpha u_i p) \le F(x_k) + c_1\alpha\,\nabla_i f(x_k)^\top p$, $\nabla_i f(x_k + \alpha u_i p)^\top p \ge c_2\,\nabla_i f(x_k)^\top p$ | mixed rule |
| **Quasi-Newton updates** | | |
| BFGS (block $i$) | $B_{k+1}^{(i,i)} = B_k^{(i,i)} - \frac{B_k^{(i,i)} s s^\top B_k^{(i,i)}}{s^\top B_k^{(i,i)} s} + \frac{y y^\top}{y^\top s}$ (when $y^\top s > 0$) | SPD update |
| $\mathrm{tr}(\cdot), \det(\cdot)$ | trace and determinant | scalars |
| **Cached LS quantities** | | |
| $w_k$ | cached smooth image $Ax_k$ | $\mathbb{R}^n$ |
| $V_i$ | $A_{:,\mathcal{I}_i}^\top b$ (constant in block $i$) | $\mathbb{R}^{p_i}$ |
| *Note* | $w_k$ (cache) vs. $w_\gamma$ (group weight) — meaning is by context | — |
| **Sampling / counting** | | |
| $\mathbf{1}_k^{(i)}$ | indicator $\mathbf{1}\{i_k = i\}$ | $\{0,1\}$ |
| $N_i(K)$ | number of updates of block $i$ up to $K$ | integer |
| **Angles, stepsize, and summability (analysis)** | | |
| $\alpha_{\min}$ | positive lower bound on stepsizes on a subsequence (if exists) | scalar |
| Zoutendijk sum | $\sum_k \frac{(\nabla f(x_k)^\top p_k)^2}{\|p_k\|^2} < \infty$ under Wolfe | series |
| **Miscellaneous** | | |
| $\varepsilon$ | termination tolerance (e.g., $\sum_i \|s_k^{(i)}\|$) | scalar |
| $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ | Euclidean norm and inner product | — |

# B Bounds on the Stepsize $\alpha_k$

In our analysis of the Block BFGS method applied to composite non-smooth optimization problems, an essential ingredient is the control of the stepsize $\alpha_k$ [4]. The line search in our algorithm (via the Wolfe-type conditions) plays a crucial role in ensuring sufficient decrease in the objective function while maintaining adequate progress along each block update. In particular, bounding $\alpha_k$ both from below and above allows us to relate the curvature information—encoded by the local Hessian approximation—to the progress made in the descent step. The following bounds on the stepsize $\alpha_k$, trace analysis, and the subsequent global convergence follow the framework developed in [4], where Byrd, Nocedal, and Yuan established convergence properties for a class of quasi-Newton methods on smooth convex problems.

In this section, we establish bounds on the stepsize $\alpha_k$ in terms of the ratio

$$\frac{(s_k^{(i_k)})^T B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|s_k^{(i_k)}\|^2},$$

where the constants depend on $c_1, c_2, m, M$. This analysis assumes one iteration of a block BFGS update, with the partial gradient $g_k^{(i_k)}$, search direction $p_k^{(i_k)} = -\big[B_k^{(i_k,i_k)}\big]^{-1} g_k^{(i_k)}$, and a Wolfe line search satisfying the conditions $0 < c_1 < c_2 < 1$.

**Lemma B.1.** *There exist positive constants $c_1'$ and $c_2'$ (depending on $m, M, c_1, c_2$) such that*

$$c_1' \,\frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|s_k^{(i_k)}\|^2} \;\;\le\;\; \alpha_k \;\;\le\;\; c_2' \,\frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|s_k^{(i_k)}\|^2} \tag{B.1}$$

*Proof.* Define the block average Hessian

$$\bar{G}^{(i_k)} \;=\; \int_0^1 \Big[\nabla^2 f\big(x_k + \tau\, u_{i_k}(s_k^{(i_k)})\big)\Big]_{i_k,i_k}\, d\tau. \tag{B.2}$$

By the mean-value theorem in the $i_k$-th block direction, we have

$$y_k^{(i_k)} \;=\; \nabla_{i_k} f(x_{k+1}) - \nabla_{i_k} f(x_k) \;=\; \Big(\int_0^1 \nabla_{i_k}^2 f\big(x_k + \tau\, u_{i_k}(s_k^{(i_k)})\big)\, d\tau\Big)\; s_k^{(i_k)},$$

which upon restricting to rows and columns $i_k$ gives

$$y_k^{(i_k)} \;=\; \bar{G}^{(i_k)}\, s_k^{(i_k)}. \tag{B.3}$$

From (2.9) and $p_k^{(i_k)} = -[B_k^{(i_k,i_k)}]^{-1} g_k^{(i_k)}$, we have

$$s_k^{(i_k)} \;=\; \alpha_k\, p_k^{(i_k)} \;=\; -\,\alpha_k \left[B_k^{(i_k,i_k)}\right]^{-1} g_k^{(i_k)}. \tag{B.4}$$

Multiplying on the left by $B_k^{(i_k,i_k)}$ yields

$$B_k^{(i_k,i_k)}\, s_k^{(i_k)} \;=\; -\,\alpha_k\, g_k^{(i_k)}, \quad\Longrightarrow\quad \alpha_k\, g_k^{(i_k)} \;=\; -\,B_k^{(i_k,i_k)}\, s_k^{(i_k)}. \tag{B.5}$$

**i. Lower bound on $\alpha_k$**
Using (B.3) and the second Wolfe condition (2.12), we obtain

$$(s_k^{(i_k)})^\top\, \bar{G}^{(i_k)}\, s_k^{(i_k)} \;=\; (s_k^{(i_k)})^\top\, y_k^{(i_k)} \;\ge\; -(1-c_2)\,\big(g_k^{(i_k)}\big)^\top\, s_k^{(i_k)}.$$

On the other hand, from (B.5),

$$(s_k^{(i_k)})^\top\, B_k^{(i_k,i_k)}\, s_k^{(i_k)} \;=\; -\,\alpha_k\,\big(g_k^{(i_k)}\big)^\top s_k^{(i_k)}.$$

Combining these gives

$$(1-c_2)\,(s_k^{(i_k)})^\top\, B_k^{(i_k,i_k)}\, s_k^{(i_k)} \;\le\; \alpha_k\,\big(s_k^{(i_k)}\big)^\top\, \bar{G}^{(i_k)}\, s_k^{(i_k)}.$$

Dividing by $(s_k^{(i_k)})^\top\, \bar{G}^{(i_k)}\, s_k^{(i_k)}$ yields

$$\alpha_k \;\ge\; (1-c_2)\,\frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{(s_k^{(i_k)})^\top \bar{G}^{(i_k)}\, s_k^{(i_k)}}.$$

Since $\nabla^2 f(x) \preceq M\, I$ implies $\bar{G}^{(i_k)} \preceq M\, I$, we get

$$(s_k^{(i_k)})^\top \bar{G}^{(i_k)}\, s_k^{(i_k)} \;\leq\; M\, \|s_k^{(i_k)}\|^2,$$

thus

$$\alpha_k \;\geq\; \frac{(1-c_2)}{M}\, \frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|s_k^{(i_k)}\|^2}. \tag{B.6}$$

**ii. Upper bound on $\alpha_k$**

Using Taylor's theorem on $f$ between $x_k$ and $x_{k+1} = x_k + \alpha_k\, u_{i_k}(p_k^{(i_k)})$, we have

$$f(x_{k+1}) - f(x_k) \;=\; \nabla f(x_k)^\top s_k^{(i_k)} \;+\; \tfrac{1}{2}\, \big(s_k^{(i_k)}\big)^\top \nabla^2 f(\xi_k)\, s_k^{(i_k)},$$

for some $\xi_k$ on the segment. Rearrange (2.11) gives

$$c_1\, \alpha_k\, (g_k^{(i_k)})^\top s_k^{(i_k)} \;\geq\; (g_k^{(i_k)})^\top s_k^{(i_k)} \;+\; \tfrac{1}{2}\, \big(s_k^{(i_k)}\big)^\top \nabla^2 f(\xi_k)\, s_k^{(i_k)}.$$

Thus

$$\tfrac{1}{2}\, (s_k^{(i_k)})^\top \nabla^2 f(\xi_k)\, s_k^{(i_k)} \;\leq\; -(1-c_1)\, \big(g_k^{(i_k)}\big)^\top s_k^{(i_k)} \;=\; (1-c_1)\, \frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\alpha_k},$$

where the last step again uses (B.5). Solving for $\alpha_k$ yields

$$\alpha_k \;\leq\; 2\,(1-c_1)\, \frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{(s_k^{(i_k)})^\top \nabla^2 f(\xi_k)\, s_k^{(i_k)}}.$$

Since $\nabla^2 f(\xi_k) \succeq m\, I$ by strong convexity, we have

$$(s_k^{(i_k)})^\top \nabla^2 f(\xi_k)\, s_k^{(i_k)} \;\geq\; m\, \|s_k^{(i_k)}\|^2, \tag{B.7}$$

thus

$$\alpha_k \;\leq\; \frac{2\,(1-c_1)}{m}\, \frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|s_k^{(i_k)}\|^2}. \tag{B.8}$$

Let

$$c_1' \;=\; \frac{(1-c_2)}{M}, \quad c_2' \;=\; \frac{2\,(1-c_1)}{m}.$$

Then (B.6) and (B.8) imply

$$c_1'\, \frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|s_k^{(i_k)}\|^2} \;\leq\; \alpha_k \;\leq\; c_2'\, \frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|s_k^{(i_k)}\|^2},$$

completing the proof of (B.1). □

# C BFGS Update and Trace

In this section, we analyze the structure and properties of the BFGS update as applied to the active block in our block coordinate framework [4]. The BFGS update modifies the Hessian approximation for the selected block using curvature information derived from the previous and current iterates. A key aspect of our analysis is to study how the update affects the trace and determinant of the block Hessian submatrix.

For general Broyden's algorithms, Hessian approximation $B_k$ is updated by

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k} + \phi(s_k^T B_k s_k) v_k v_k^T \tag{C.1}$$

However, for block BFGS update, $\phi = 0$, so a simplified version is:

$$B_{k+1}^{(i_k,i_k)} \;=\; B_k^{(i_k,i_k)} \;-\; \frac{B_k^{(i_k,i_k)}\, s_k^{(i_k)} \big(s_k^{(i_k)}\big)^T B_k^{(i_k,i_k)}}{\big(s_k^{(i_k)}\big)^T B_k^{(i_k,i_k)}\, s_k^{(i_k)}} \;+\; \frac{y_k^{(i_k)} \big(y_k^{(i_k)}\big)^T}{(y_k^{(i_k)})^T s_k^{(i_k)}}. \tag{C.2}$$

Taking the trace on both sides, we get

$$\begin{aligned}\operatorname{tr}\big(B_{k+1}^{(i_k,i_k)}\big) \;&=\; \operatorname{tr}\big(B_k^{(i_k,i_k)}\big) \;-\; \operatorname{tr}\left(\frac{B_k^{(i_k,i_k)}\, s_k^{(i_k)} \big(s_k^{(i_k)}\big)^T B_k^{(i_k,i_k)}}{\big(s_k^{(i_k)}\big)^T B_k^{(i_k,i_k)}\, s_k^{(i_k)}}\right) \;+\; \operatorname{tr}\left(\frac{y_k^{(i_k)} \big(y_k^{(i_k)}\big)^T}{(y_k^{(i_k)})^T s_k^{(i_k)}}\right). \\ &= \operatorname{tr}\big(B_k^{(i_k,i_k)}\big) \;-\; \frac{(s_k^{(i_k)})^T (B_k^{(i_k,i_k)})^2 s_k^{(i_k)}}{(s_k^{(i_k)})^T B_k^{(i_k,i_k)} s_k^{(i_k)}} \;+\; \frac{\|y_k^{(i_k)}\|^2}{(y_k^{(i_k)})^T\, s_k^{(i_k)}}. \end{aligned} \tag{C.3}$$

We now examine how to bound the two terms on the right-hand side.

**Bound on $\frac{\|y_k^{(i_k)}\|^2}{\left(y_k^{(i_k)}\right)^T s_k^{(i_k)}}$**

**Lemma C.1.** *The following inequality holds:*

$$\frac{\|y_k^{(i_k)}\|^2}{(y_k^{(i_k)})^T s_k^{(i_k)}} \le M \tag{C.4}$$

*Proof.* From (B.2), let $\bar{G}^{(i_k),1/2} \bar{G}^{(i_k),1/2} = \bar{G}^{(i_k)}$. Then set

$$z_k^{(i_k)} = \bar{G}^{(i_k),1/2} s_k^{(i_k)}. \tag{C.5}$$

Rewrite the ratio:

$$\frac{\|y_k^{(i_k)}\|^2}{(y_k^{(i_k)})^T s_k^{(i_k)}} = \frac{(s_k^{(i_k)})^T \bar{G}^{(i_k),2} s_k^{(i_k)}}{(s_k^{(i_k)})^T \bar{G}^{(i_k)} s_k^{(i_k)}} = \frac{(z_k^{(i_k)})^T \bar{G}^{(i_k)} z_k^{(i_k)}}{(z_k^{(i_k)})^T z_k^{(i_k)}} \le M.$$

□

**Bound on** $\dfrac{\left\|B_k^{(i_k,i_k)} s_k^{(i_k)}\right\|^2}{\left(s_k^{(i_k)}\right)^\top B_k^{(i_k,i_k)} s_k^{(i_k)}}$

**Lemma C.2.** *The following inequality holds:*

$$\frac{\left\|B_k^{(i_k,i_k)}\, s_k^{(i_k)}\right\|^2}{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}} \;\geq\; \frac{\alpha_k}{c_2'\,\cos^2\!\big(\theta_k^{(i_k)}\big)}. \tag{C.6}$$

*Proof.* From (B.5), we have:

$$\left\|B_k^{(i_k,i_k)} s_k^{(i_k)}\right\|^2 = \big(\alpha_k\big)^2 \|g_k^{(i_k)}\|^2.$$

and

$$(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)} = -\alpha_k (g_k^{(i_k)})^\top s_k^{(i_k)}.$$

Thus,

$$\frac{\left\|B_k^{(i_k,i_k)} s_k^{(i_k)}\right\|^2}{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)}} = \alpha_k \frac{\|g_k^{(i_k)}\|^2}{\left|(g_k^{(i_k)})^\top s_k^{(i_k)}\right|}.$$

From (2.13),

$$(g_k^{(i_k)})^\top s_k^{(i_k)} = -\|g_k^{(i_k)}\|\|s_k^{(i_k)}\|\cos\theta_k^{(i_k)} \quad \Rightarrow \quad \frac{\left\|B_k^{(i_k,i_k)} s_k^{(i_k)}\right\|^2}{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)}} = \alpha_k \frac{\|g_k^{(i_k)}\|}{\|s_k^{(i_k)}\|\cos\theta_k^{(i_k)}}. \tag{C.7}$$

From (B.7) and (C.7),

$$\begin{aligned}
&\frac{1}{2} m \|s_k^{(i_k)}\|^2 \leq -(1-c_1)(g_k^{(i_k)})^\top s_k^{(i_k)} \\
&\quad\Rightarrow \quad \frac{1}{2} m \|s_k^{(i_k)}\|^2 \leq (1-c_1)\|g_k^{(i_k)}\|\|s_k^{(i_k)}\|\cos\theta_k^{(i_k)} \\
&\quad\Rightarrow \quad \|s_k^{(i_k)}\| \leq \frac{2(1-c_1)}{m}\|g_k^{(i_k)}\|\cos\theta_k^{(i_k)} \\
&\quad\Rightarrow \quad \|s_k^{(i_k)}\| \leq c_2'\|g_k^{(i_k)}\|\cos\theta_k^{(i_k)} \\
&\quad\Rightarrow \quad \frac{1}{\|s_k^{(i_k)}\|\cos\theta_k^{(i_k)}} \geq \frac{1}{c_2'\|g_k^{(i_k)}\|\cos^2\theta_k^{(i_k)}}.
\end{aligned}$$

Thus,

$$\frac{\left\|B_k^{(i_k,i_k)} s_k^{(i_k)}\right\|^2}{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)}} \geq \frac{\alpha_k}{c_2'\cos^2\!\big(\theta_k^{(i_k)}\big)}.$$

□

Hence, from (C.4) and (C.6), we have

$$\mathrm{tr}\big(B_{k+1}^{(i_k,i_k)}\big) \leq \mathrm{tr}\big(B_k^{(i_k,i_k)}\big) + M - \frac{\alpha_k}{c_2'\cos^2\!\big(\theta_k^{(i_k)}\big)} \tag{C.8}$$

With the information obtained so far, we now analyze the block-angle $\cos\theta_k^{(i_k)}$. Recall from (B.5) that

$$g_k^{(i_k)} = -\frac{1}{\alpha_k} B_k^{(i_k,i_k)}\, s_k^{(i_k)} \quad \text{and} \quad \|g_k^{(i_k)}\| = \frac{\|B_k^{(i_k,i_k)}\, s_k^{(i_k)}\|}{\alpha_k}.$$

Then, by the definition (2.13) of the block cosine,

$$\cos\theta_k^{(i_k)} = \frac{\frac{1}{\alpha_k}\,(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\frac{1}{\alpha_k}\,\|B_k^{(i_k,i_k)}\, s_k^{(i_k)}\|\,\|s_k^{(i_k)}\|} = \frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|B_k^{(i_k,i_k)}\, s_k^{(i_k)}\|\,\|s_k^{(i_k)}\|}. \tag{C.9}$$

We can factor this expression as

$$\cos\theta_k^{(i_k)} = \left(\frac{\|s_k^{(i_k)}\|}{\|B_k^{(i_k,i_k)}\, s_k^{(i_k)}\|}\right)\left(\frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|s_k^{(i_k)}\|^2}\right).$$

It is easy to lower-bound each factor by known information: By Lemma B.1 , we have

$$\frac{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)}\, s_k^{(i_k)}}{\|s_k^{(i_k)}\|^2} \geq \frac{\alpha_k}{c_2'}. \tag{C.10}$$

Also, for any symmetric positive definite matrix, we have

$$\|B_k^{(i_k,i_k)}\, s_k^{(i_k)}\| \leq \|B_k^{(i_k,i_k)}\|\,\|s_k^{(i_k)}\| \leq \operatorname{tr}\big(B_k^{(i_k,i_k)}\big)\,\|s_k^{(i_k)}\|.$$

Hence,

$$\frac{\|s_k^{(i_k)}\|}{\|B_k^{(i_k,i_k)}\, s_k^{(i_k)}\|} \geq \frac{1}{\operatorname{tr}\big(B_k^{(i_k,i_k)}\big)}. \tag{C.11}$$

Multiplying the two lower bounds from (C.10) and (C.11), we obtain

$$\cos\theta_k^{(i_k)} \geq \frac{\alpha_k}{c_2'\,\operatorname{tr}\big(B_k^{(i_k,i_k)}\big)}, \tag{C.12}$$

thus $\cos\theta_k^{(i_k)}$ can be small only if $\alpha_k$ is small or $\operatorname{tr}\big(B_k^{(i_k,i_k)}\big)$ is large. Therefore, we will have to consider two cases: (1) suppose the stepsize $\alpha_k$ are bounded away from zero; (2) suppose the stepsize $\alpha_k$ tend to zero.

# D Global Convergence

Inequality (C.12) shows that the block cosine can be small only if either the stepsize $\alpha_k$ is small or the trace $\operatorname{tr}\big(B_k^{(i_k,i_k)}\big)$ is large. Consequently, following [4], we distinguish two cases:

(I) The stepsizes $\alpha_k$ tend to zero;

(II) The stepsizes $\alpha_k$ are bounded away from zero.

## D.1 Random block sampling

At iteration $k$, a single block index $i_k \in \{1, \ldots, n_B\}$ is drawn i.i.d. uniformly at each iteration, independently of the past (i.e., conditionally on the filtration $\mathcal{F}_k := \sigma(x_0, \ldots, x_k, B_0, \ldots, B_k)$). For a fixed block $i$, define the block-activity indicator

$$\mathbf{1}_k^{(i)} := \mathbf{1}\{\, i_k = i \,\}, \qquad N_i(K) := \sum_{k=0}^{K} \mathbf{1}_k^{(i)},$$

the number of times block $i$ is updated up to iteration $K$.

## D.2 $\alpha_k$ tend to zero

In this case we show that the stepsize cannot be arbitrarily small on average. This is accomplished via a determinant argument adapted to the Block BFGS update. Recall that the BFGS update for the active block is

$$B_{k+1}^{(i_k,i_k)} = B_k^{(i_k,i_k)} - \frac{B_k^{(i_k,i_k)} s_k^{(i_k)} (s_k^{(i_k)})^\top B_k^{(i_k,i_k)}}{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)}} + \frac{y_k^{(i_k)} (y_k^{(i_k)})^\top}{(y_k^{(i_k)})^\top s_k^{(i_k)}}.$$

and by applying the Sherman-Morrison Determinant formula, we obtain

$$\det(B_{k+1}) = \det(B_k) \frac{(y_k^{(i_k)})^\top s_k^{(i_k)}}{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)}}. \tag{D.1}$$

Note that when $(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)}$ is small relative to $(y_k^{(i_k)})^\top s_k^{(i_k)}$, the determinant increases, reflecting the fact that the small curvature of our model is corrected, thus increasing some eigenvalues.

**Lemma D.1.** *There is a constant $c_4 \in (0,1)$ and $N_0$ such that for all $k \geq N_0$*

$$\prod_{j=1}^{k} \alpha_j \geq c_4^k. \tag{D.2}$$

*Proof.* By the curvature condition (2.12), we have

$$(y_k^{(i_k)})^\top s_k^{(i_k)} \geq (1-c_2) \left[-g_k^{(i_k)}\right]^\top s_k^{(i_k)}.$$

Moreover, from (B.5), we deduce that

$$(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)} = \alpha_k \left[-g_k^{(i_k)}\right]^\top s_k^{(i_k)}.$$

Dividing the two inequalities gives

$$\frac{(y_k^{(i_k)})^\top s_k^{(i_k)}}{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)}} \geq \frac{1-c_2}{\alpha_k}. \tag{D.3}$$

Substituting this bound into (D.1) yields

$$\begin{aligned} \det(B_{k+1}) &= \det(B_k) \frac{(y_k^{(i_k)})^\top s_k^{(i_k)}}{(s_k^{(i_k)})^\top B_k^{(i_k,i_k)} s_k^{(i_k)}} \\ &\geq \det(B_k) \frac{1-c_2}{\alpha_k}. \end{aligned} \tag{D.4}$$

Iterating this inequality from $j=1$ to $k$ gives

$$\det(B_{k+1}) \geq \det(B_1) \prod_{j=1}^{k} \frac{1-c_2}{\alpha_j}.$$

That is,

$$\prod_{j=1}^{k} \frac{1}{\alpha_j} \leq \frac{1}{(1-c_2)^k} \frac{\det(B_{k+1})}{\det(B_1)}.$$

For Block BFGS only block $i_k$ changes at iteration $k$, so

$$\operatorname{tr}(B_{k+1}) - \operatorname{tr}(B_k) = \operatorname{tr}\big(B_{k+1}^{(i_k,i_k)}\big) - \operatorname{tr}\big(B_k^{(i_k,i_k)}\big).$$

Hence, by summing over $k = 1, 2, \ldots$ on the block trace recursion (C.8) yields

$$\operatorname{tr}(B_{k+1}) \le \operatorname{tr}(B_1) + k\,M - \sum_{j=1}^{k} \frac{\alpha_j}{c_2' \cos^2 \theta^{(i_j)}}. \tag{D.5}$$

Hence,

$$\det(B_{k+1}) \le \left(\frac{\operatorname{tr}(B_1) + kM}{n}\right)^n.$$

Substituting into the previous inequality gives

$$\prod_{j=1}^{k} \frac{1}{\alpha_j} \le \frac{1}{(1-c_2)^k \det(B_1)} \left(\frac{\operatorname{tr}(B_1) + kM}{n}\right)^n.$$

Taking reciprocals yields

$$\prod_{j=1}^{k} \alpha_j \ge (1-c_2)^k \frac{\det(B_1)}{(\frac{\operatorname{tr}(B_1)+kM}{n})^n}$$

Since $\det(B_1)$, $n$, $M$, and $(1-c_2)$ are independent of $k$, we may combine them into a single constant $c_4 \in (0,1)$ and $N_0$ such that for all $k \ge N_0$

$$\prod_{j=1}^{k} \alpha_j \ge c_4^k. \tag{D.6}$$

□

**Per-block determinant growth.** For the active block, the determinant update refines to

$$\det\big(B_{k+1}^{(i_k,i_k)}\big) = \det\big(B_k^{(i_k,i_k)}\big) \frac{\big(y_k^{(i_k)}\big)^\top s_k^{(i_k)}}{\big(s_k^{(i_k)}\big)^\top B_k^{(i_k,i_k)} s_k^{(i_k)}} \;\ge\; \det\big(B_k^{(i_k,i_k)}\big) \frac{1-c_2}{\alpha_k}. \tag{D.7}$$

For a fixed block $i$, define the active-$i$ index sets

$$\mathcal{K}_i(K) := \{\, 0 \le k \le K : \; i_k = i \,\}, \qquad \mathcal{K}_i := \{\, k \ge 0 : \; i_k = i \,\}, \qquad N_i(K) := |\mathcal{K}_i(K)|.$$

Let $n_i := |I_i|$ denote the size of block $i$. Iterating (D.7) over the updates in $\mathcal{K}_i(K)$ gives

$$\det\big(B_{K+1}^{(i,i)}\big) \;=\; \det\big(B_0^{(i,i)}\big) \prod_{k \in \mathcal{K}_i(K)} \frac{\big(y_k^{(i)}\big)^\top s_k^{(i)}}{\big(s_k^{(i)}\big)^\top B_k^{(i,i)} s_k^{(i)}} \;\ge\; \det\big(B_0^{(i,i)}\big) \prod_{k \in \mathcal{K}_i(K)} \frac{1-c_2}{\alpha_k},$$

where the inequality follows from (D.3). Next, apply AM–GM to the $n_i \times n_i$ SPD matrix $B_{K+1}^{(i,i)}$, and then invoke the per-block trace bound—obtained by summing (C.8) over the active-$i$ indices $\mathcal{K}_i(K)$ and dropping the nonpositive term:

$$\det\big(B_{K+1}^{(i,i)}\big) \;\le\; \Big(\tfrac{1}{n_i} \operatorname{tr}\big(B_{K+1}^{(i,i)}\big)\Big)^{n_i} \;\le\; \Big(\tfrac{1}{n_i}\big[\operatorname{tr}\big(B_0^{(i,i)}\big) + M\,N_i(K)\big]\Big)^{n_i}. \tag{D.8}$$

Combining the two displays and rearranging,

$$\prod_{k\in\mathcal{K}_i(K)} \alpha_k \;\geq\; (1-c_2)^{N_i(K)} \,\frac{\det(B_0^{(i,i)})}{\left(\frac{1}{n_i}\left[\mathrm{tr}(B_0^{(i,i)}) + M\,N_i(K)\right]\right)^{n_i}}.$$

Since the denominator grows only polynomially in $N_i(K)$ whereas $(1-c_2)^{N_i(K)}$ decays geometrically, there exist $c_4^{(i)} \in (0,1)$ and $K_0$ such that, for all $K \geq K_0$,

$$\prod_{\substack{0\leq k\leq K\\ i_k=i}} \alpha_k \;\geq\; \left(c_4^{(i)}\right)^{N_i(K)}. \tag{D.9}$$

Consequently, for any $\varepsilon \in (0, c_4^{(i)})$, there exists a subsequence of active-$i$ indices $\{k_\ell^{(i)}\}_{\ell\geq 1}$ with $\alpha_{k_\ell}(i) \geq \varepsilon$ for all $\ell$.

### D.3 $\alpha_k$ bounded away from zero

Assume there exists $\alpha_{\min} > 0$ and an infinite subsequence $\mathcal{K} \subset \mathbb{N}$ such that $\alpha_k \geq \alpha_{\min}$ for all $k \in \mathcal{K}$. We show, *per block*, that the cosine cannot vanish.

**Per-block trace recursion.** Fix a block $i$. Sum the block-trace inequality (B.8) only over the updates of block $i$, i.e., over the active-$i$ index set,

$$\mathrm{tr}\big(B_{K+1}^{(i,i)}\big) \;\leq\; \mathrm{tr}\big(B_0^{(i,i)}\big) \;+\; M\,N_i(K) \;-\; \sum_{\substack{0\leq k\leq K\\ i_k=i}} \frac{\alpha_k}{c_2'\,\cos^2\theta_k^{(i)}}. \tag{D.10}$$

**Theorem D.1** (Global convergence of Block BFGS)**.** *Under the standing assumptions stated earlier and i.i.d. uniform block sampling, the iterates $\{x_k\}$ converge almost surely to the unique minimizer $x^*$. Moreover, $f(x_k) \downarrow f(x^*)$.*

*Proof.* **Fix a block** $i$ and choose $\varepsilon \in (0, c_4^{(i)})$ from (D.9). Define the $\alpha$-large active-$i$ set

$$\mathcal{K}_i^{\geq\varepsilon} := \{\, k \geq 0 :\; i_k = i,\; \alpha_k \geq \varepsilon \,\}.$$

By contradiction, we consider the following disjunction:

(I) $N_i(K) \not\to \infty$.

(II) $\cos\theta^{(i)}_{k_\ell^{(i)}} \to 0$ along $\{k_\ell^{(i)}\}_{\ell\geq 1}$.

**(I)**: Let $E_k^{(i)} := \{i_k = i\}$; the events $E_k^{(i)}$ are independent with $\mathbb{P}(E_k^{(i)}) = 1/n_B$. Since $\sum_{k=0}^{\infty}\mathbb{P}(E_k^{(i)}) = \infty$, the second Borel–Cantelli lemma yields

$$\mathbb{P}(E_k^{(i)} \text{ i.o.}) = 1, \quad \text{i.e.,} \quad N_i(K) \to \infty \quad \text{almost surely.}$$

Thus (I) cannot occur.

**(II)**: Because $N_i(K) \to \infty$ a.s. and $\varepsilon \in (0, c_4^{(i)})$, the product bound (D.9) implies that $\mathcal{K}_i^{\geq\varepsilon}$ is infinite a.s.; enumerate it as $\{k_\ell^{(i)}\}_{\ell\geq 1}$. Suppose, for contradiction, that

$$\cos\theta^{(i)}_{k_\ell^{(i)}} \;\longrightarrow\; 0 \qquad (\ell \to \infty).$$

Evaluate (D.10) at $K_\ell := k_\ell^{(i)}$. Since

$$\{\, k \in \mathbb{N} :\ 0 \le k \le K_\ell,\ i_k = i \,\} \quad \text{contains the first } \ell \text{ indices of } \quad \{k_t^{(i)}\}_{t\ge 1},$$

we obtain

$$\operatorname{tr}\big(B_{K_\ell+1}^{(i,i)}\big) \ \le\ \operatorname{tr}\big(B_0^{(i,i)}\big) \ +\ M\, N_i(K_\ell) \ -\ \frac{1}{c_2'} \sum_{t=1}^{\ell} \frac{\alpha_{k_t^{(i)}}}{\cos^2 \theta^{(i)}_{k_t^{(i)}}}. \tag{D.11}$$

Because $\alpha_{k_t^{(i)}} \ge \varepsilon$ and $\cos\theta^{(i)}_{k_t^{(i)}} \to 0$, the summands $\alpha_{k_t^{(i)}} / \cos^2 \theta^{(i)}_{k_t^{(i)}}$ tend to $+\infty$, hence

$$\sum_{t=1}^{\ell} \frac{\alpha_{k_t^{(i)}}}{\cos^2 \theta^{(i)}_{k_t^{(i)}}} \ =\ +\infty \quad (\ell \to \infty),$$

contradicting boundedness of the left-hand side of (D.10), and therefore (II) cannot occur.

From the failure of (I) and (II) we conclude that, for this block $i$, there exist $\alpha_{\min}^{(i)} > 0$, $\delta_i > 0$, and an infinite subsequence of active-$i$ indices $\{k_\ell^{(i)}\} \subset \mathcal{K}_i$ such that

$$\alpha_{k_\ell^{(i)}} \ge \alpha_{\min}^{(i)} \qquad \text{and} \qquad \cos\theta^{(i)}_{k_\ell^{(i)}} \ge \delta_i \quad \text{for all } \ell,$$

and moreover $N_i(K) \to \infty$ almost surely. Because there are finitely many blocks, intersecting the corresponding probability-one events over $i = 1, \dots, n_B$ yields an event $\Omega$ with $\mathbb{P}(\Omega) = 1$ on which the conclusion above holds for *every* block.

**Generalization.** Recall $p_k$ denote the embedded search direction ($s_k = \alpha_k p_k$ with $p_k$ supported on $i_k$). By the Armijo–Wolfe line search, we know $\nabla f$ is M-Lipschitz on $\mathcal{D}$, hence [4]:

$$\sum_{k=0}^{\infty} \frac{\big(\nabla f(x_k)^\top p_k\big)^2}{\|p_k\|^2} \ <\ \infty. \tag{D.12}$$

Since

$$\nabla f(x_k)^\top p_k = \nabla_{i_k} f(x_k)^\top p_k^{(i_k)} = -\|\nabla_{i_k} f(x_k)\|\, \|p_k^{(i_k)}\| \cos\theta_k^{(i_k)}$$

and

$$\|p_k\| = \|p_k^{(i_k)}\|,$$

(D.12) becomes

$$\sum_{k=0}^{\infty} \|\nabla_{i_k} f(x_k)\|^2 \, \cos^2 \theta_k^{(i_k)} \ <\ \infty.$$

Along the subsequence $\{k_\ell\}$ we have $\cos\theta_{k_\ell}^{(i)} \ge \delta$, hence

$$\sum_{\ell} \|\nabla_i f(x_{k_\ell})\|^2 \ \le\ \frac{1}{\delta^2} \sum_{\ell} \|\nabla_i f(x_{k_\ell})\|^2 \, \cos^2 \theta_{k_\ell}^{(i)} \ <\ \infty,$$

so $\|\nabla_i f(x_{k_\ell})\| \to 0$.

Armijo implies $f(x_{k+1}) \le f(x_k)$, so $\{f(x_k)\}$ is nonincreasing and bounded below by $f(x^*)$. Any accumulation point $\bar{x}$ of $\{x_k\}$ satisfies $\nabla_i f(\bar{x}) = 0$ for the block $i$ above. If some block $j$ admits a similar subsequence, we get $\nabla_j f(\bar{x}) = 0$ as well; iterating across the finitely many blocks and extracting a diagonal subsequence yields $\nabla f(\bar{x}) = 0$. By strong convexity, $\bar{x} = x^*$. Therefore $f(x_k) \downarrow f(x^*)$ and $x_k \to x^*$ a.s. □

### D.4 2-block case

By contradiction, assume that there exists $\varepsilon > 0$ with

$$\|\nabla f(x_k)\| \;\geq\; \varepsilon \qquad \text{for all } k. \tag{D.13}$$

With two blocks, define the maximum block

$$J_k \;\in\; \arg\max\big\{\|\nabla_1 f(x_k)\|,\; \|\nabla_2 f(x_k)\|\big\},$$

so that

$$\|\nabla_{J_k} f(x_k)\| \;\geq\; \frac{\varepsilon}{\sqrt{2}} \quad \text{for all } k. \tag{D.14}$$

Recall $i_k \in \{1, 2\}$ is the active block at iteration $k$, and i.i.d. uniform and independent of $\mathcal{F}_k$.

**Step bound through tr** Recall (C.8)

$$\mathrm{tr}\big(B_{k+1}^{(i_k,i_k)}\big) \leq \mathrm{tr}\big(B_k^{(i_k,i_k)}\big) + M - \frac{\alpha_k}{c_2' \cos^2\!\big(\theta_k^{(i_k)}\big)},$$

Summing over $k = 0, \ldots, K$ (only the active block changes per iteration) yields

$$\mathrm{tr}(B_{K+1}) \;\leq\; \mathrm{tr}(B_0) \;+\; M\,(K+1) \;-\; \sum_{k=0}^{K} \frac{\alpha_k}{c_2'\,\cos^2\!\theta_k^{(i_k)}}. \tag{D.15}$$

Fix any $\tau > M$ and define the index sets S such that

$$\mathcal{S}_{K,\tau} = \{0 \leq k \leq K :\; \frac{\alpha_k}{c_2'\,\cos^2\!\theta_k^{(i_k)}} \leq \tau\}.$$

From (D.15) and $\mathrm{tr}(B_{K+1}) \geq 0$ we obtain

$$|\mathcal{S}_{K,\tau}| \;\geq\; (K+1)\Big(1 - \frac{M}{\tau}\Big) \;-\; \frac{\mathrm{tr}(B_0)}{\tau} \quad \text{for all } K \geq 0. \tag{D.16}$$

Thus, for any $\tau > M$, good-geometry steps occur with positive asymptotic frequency.

**maximum block updates a.s.** Define

$$\mathcal{A}(K) := \{\, 0 \leq k \leq K :\; i_k = J_k \,\}.$$

By the strong law of large numbers for independent $\{i_k\}$, almost surely

$$\frac{|\mathcal{A}(K)|}{K+1} \;\longrightarrow\; \frac{1}{2}. \tag{D.17}$$

By Lemma D.1 (product bound), there exists $c_4 \in (0, 1)$ and $K_0$ such that $\prod_{j=1}^{K} \alpha_j \geq c_4^K$ for all $K \geq K_0$. In particular, for any fixed $\underline{\alpha} \in (0, c_4)$, the set

$$\mathcal{L}(\underline{\alpha}) := \{\, k \geq 0 :\; \alpha_k \geq \underline{\alpha} \,\}$$

is infinite. Therefore, choose $\tau > 4M$ and any $\underline{\alpha} \in (0, c_4)$. By (D.16) and (D.17), almost surely there exists $K_1$ such that for all $K \geq K_1$,

$$|\mathcal{S}_{K,\tau}| \;\geq\; \Big(\frac{3}{4}\Big)(K+1) - C_1, \qquad |\mathcal{A}(K)| \;\geq\; \Big(\frac{1}{2} - \eta\Big)(K+1)$$

for some constants $C_1 < \infty$ and any fixed $\eta \in (0, 1/4)$. Hence, by the pigeonhole bound $|A \cap B| \geq |A| + |B| - (K+1)$,

$$|\mathcal{S}_{K,\tau} \cap \mathcal{A}(K)| \;\geq\; \Big(\frac{1}{4} - \eta\Big)(K+1) - C_1,$$

so $\mathcal{S}_{\infty;\tau} \cap \mathcal{A}(\infty)$ is infinite almost surely. Intersecting with the infinite set $\mathcal{L}(\underline{\alpha})$ yields an infinite index set

$$\mathcal{K}^\star_\infty \;:=\; \mathcal{S}_{\infty;\tau} \cap\; \mathcal{A}(\infty) \;\cap\; \mathcal{L}(\underline{\alpha}).$$

For any $k \in \mathcal{K}^\star_\infty$, we have $\frac{\alpha_k}{c_2' \cos^2\theta_k^{(i_k)}} \leq \tau$ and $\alpha_k \geq \underline{\alpha}$, hence

$$\cos^2\theta_k^{(i_k)} \;\geq\; \frac{\underline{\alpha}}{c_2'\,\tau} \;=\; \delta \;> 0. \tag{D.18}$$

Since $k \in \mathcal{A}$, we also have $i_k = J_k$, so (D.14) applies:

$$\|\nabla_{i_k} f(x_k)\| \geq \frac{\varepsilon}{\sqrt{2}}.$$

Recall the Zoutendijk sum

$$\sum_{k=0}^{\infty} \frac{\big(\nabla f(x_k)^\top p_k\big)^2}{\|p_k\|^2} \;=\; \sum_{k=0}^{\infty} \|\nabla_{i_k} f(x_k)\|^2 \,\cos^2\theta_k^{(i_k)} \;<\; \infty,$$

and along the infinite set $\mathcal{K}^\star_\infty$, (D.14) and (D.18) give

$$\|\nabla_{i_k} f(x_k)\|^2 \,\cos^2\theta_k^{(i_k)} \;\geq\; \Big(\frac{\varepsilon^2}{2}\Big)\delta \;> 0,$$

so the series would diverge, a contradiction. Therefore (D.13) is false and

$$\liminf_{k\to\infty} \;\|\nabla f(x_k)\| \;=\; 0.$$

## D.5 Global Convergence Under Relaxed Assumptions

In this subsection, we prove global convergence for the block–coordinate BFGS algorithm under a relaxed assumption on the smooth part $f$. In particular, we assume:

**Assumption D.1.** *The function $f$ is twice continuously differentiable, convex, and bounded below. Moreover, its Hessian is uniformly bounded on the level set $\mathcal{D}$, i.e., there exists a constant $M > 0$ such that*

$$\|\nabla^2 f(x)\| \leq M, \quad \forall\, x \in \mathcal{D}.$$

Note that Assumption D.1 is strictly weaker than strong convexity. However, the convexity of $f$ together with the Wolfe line search guarantees sufficient descent. We now show the global convergence under the new assumption.

**Theorem D.2** (Relaxed global convergence for block BFGS)**.** *Assume D.1 and Armijo–Wolfe. Then*

$$\liminf_{k\to\infty} \|\nabla f(x_k)\| = 0.$$

*If, in addition, $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_\ell}\}$ with $x_{k_\ell} \to x_\star$, where $x_\star$ is a global minimizer, $\nabla f(x_\star) = 0$, and $f(x_k) \downarrow f_\star$.*

*Proof.* Assume that there exist $\gamma > 0$ and $K_0$ with $\|g_k\| \geq \gamma$ for all $k \geq K_0$. Let the embedded step be $s_k := u_{i_k}(s_k^{(i_k)})$. From (2.11), $g_k^\top s_k \to 0$.

Using (B.5), $B_k^{(i_k,i_k)} s_k^{(i_k)} = -\alpha_k g_k^{(i_k)}$. Embedding gives

$$B_k s_k = -\alpha_k \, u_{i_k}(g_k^{(i_k)}) \quad \text{and} \quad s_k^\top B_k s_k = -\alpha_k g_k^\top s_k.$$

Define

$$\tilde{\eta}_k := -\frac{\|g_k\|^2}{-\, g_k^\top s_k} \; (< 0), \qquad \lambda_k := \alpha_k,$$

so the negative term in (C.3) equals $-\|B_k s_k\|^2/(s_k^\top B_k s_k) = \alpha_k \tilde{\eta}_k = \lambda_k \tilde{\eta}_k$. Since $\|g_k\| \geq \gamma$ and $g_k^\top s_k \to 0$, we have $\tilde{\eta}_k \to -\infty$.

Summing (C.3) from $K_0$ to $K$ and using (C.4),

$$\operatorname{tr}(B_{K+1}) \;\leq\; \operatorname{tr}(B_{K_0}) + M\,(K - K_0 + 1) + \sum_{k=K_0}^{K} \lambda_k \tilde{\eta}_k. \tag{D.19}$$

By Lemma D.1, there exist $c_4 \in (0,1)$, $N_0$ such that for any $\varepsilon \in (0, c_4)$ there is a subsequence $\{k_\ell\}$ with $\lambda_{k_\ell} = \alpha_{k_\ell} \geq \varepsilon$. Thin it so that $\tilde{\eta}_{k_\ell} \leq -\ell$. Then (D.19) gives

$$\operatorname{tr}(B_{K+1}) \;\leq\; \operatorname{tr}(B_{K_0}) + M\,(K - K_0 + 1) - \varepsilon \sum_{\ell:\, k_\ell \leq K} \ell \;\longrightarrow\; -\infty,$$

contradicting $B_{K+1} \succ 0$. Hence $\liminf_k \|g_k\| = 0$.

□

If $\{x_k\}$ is bounded, Armijo implies $f(x_k)$ is monotone and bounded below, so $f(x_k) \to f_\star$. Choose $k_\ell$ with $\|g_{k_\ell}\| \to 0$ and pass to a further subsequence (still $k_\ell$) with $x_{k_\ell} \to x_\star$. Continuity gives $\nabla f(x_\star) = 0$; convexity implies $x_\star$ is a global minimizer. Thus $f(x_k) \downarrow f_\star$.

# E Global Convergence for Non-smooth Functions

## E.1 Smooth Neighborhood for Non-smoothness

Having established global convergence for the smooth case of the block BFGS algorithm under the assumption that the smooth part $f$ is twice continuously differentiable on the entire level set $\mathcal{D}$, we now extend the analysis to the non-smooth setting. In this section we analyze a block BFGS algorithm applied to a composite convex objective. Our goal is to prove that, under suitable assumptions, the sequence generated by the algorithm converges in function value to the global minimum. We rely on techniques outlined in [5], which adapts Powell's convergence results to handle non-smoothness.

Recall the composite function from (1.1), where $f : \mathbb{R}^n \to \mathbb{R}$ is smooth and strongly convex and $g : \mathbb{R}^n \to \mathbb{R}$ is convex. And then, we assume the level set defined in (2.2) to be compact. Then, the following assumptions are made based on Assumption 2.1:

**Assumption E.1** (Non-smooth Component)**.** *The function $g : \mathbb{R}^n \to \mathbb{R}$ is convex. Moreover, there exists an open set $V \subset U$ (with $U$ as in Assumption 2.1) such that:*

1. *All accumulation points of the iterates $\{x_k\} \subset \mathcal{D}$ lie in $V$.*

2. *The composite function $F(x) = f(x) + g(x)$ is twice continuously differentiable on $V$.*

3. $\inf_{x \in V} F(x) = \min F$.

**Assumption E.2.** *The block–coordinate BFGS algorithm uses an Armijo–Wolfe line search on $F$ and i.i.d. uniform block sampling: $i_k \in \{1, \dots, n_B\}$ are i.i.d. uniform and independent of the past $\mathcal{F}_k = \sigma(x_0, \dots, x_k, B_0, \dots, B_k)$.*

**Theorem E.1** (Global convergence in the non-smooth strongly convex case)**.** *Under Assumptions E.1–E.2, the block BFGS iterates satisfy, almost surely,*

$$F(x_k) \downarrow F_\star := \min F \qquad \textit{and} \qquad x_k \to x_\star,$$

*where $x_\star$ is the unique minimizer of $F$.*

*Proof.* **Construct smooth neighborhood.** By compactness of $\mathcal{D}$ and Assumption E.1(i), the set of accumulation points of $\{x_k\}$ is nonempty, compact, and contained in the open set $V$. Hence there exists a compact set $\Omega$ with $\operatorname{accum}(\{x_k\}) \subset \Omega \subset \operatorname{int}(V)$. Therefore, there is $K_0$ such that $x_k \in \Omega$ for all $k \geq K_0$.

Following [5], shrink $V$ slightly and intersect with a level set to obtain a compact set on which $F$ is $C^2$ and strongly convex; then apply the smooth convex extension tool to obtain a function $F_e \in C^2(U_e)$ on a convex open neighborhood $U_e \supset \operatorname{conv}(\Omega)$ such that

$$F_e(x) = F(x), \quad \nabla F_e(x) = \nabla F(x) \qquad \forall\, x \in \Omega,$$

and $\nabla^2 F_e(x)$ is positive definite and continuous on $U_e$. Because $\operatorname{conv}(\Omega)$ is compact, there exist constants $0 < \hat{m} \leq \hat{M} < \infty$ with

$$\hat{m} I \preceq \nabla^2 F_e(x) \preceq \hat{M} I \qquad \forall\, x \in \operatorname{conv}(\Omega).$$

Moreover, $\min F_e = \min F$ and every minimizer lies in $V \subset U_e$.

Fix $k \geq K_0$. Since $x_k, x_{k+1} \in \Omega \subset U_e$ and $F_e = F$, $\nabla F_e = \nabla F$ on $\Omega$, the accepted stepsize $\alpha_k$ satisfies the Armijo–Wolfe conditions for $F_e$ if and only if it does for $F$. Hence the realized pair

$$s_k = u_{i_k}(s_k^{(i_k)}), \qquad y_k = \nabla F(x_{k+1}) - \nabla F(x_k) = \nabla F_e(x_{k+1}) - \nabla F_e(x_k)$$

is the same for $F$ and $F_e$. Consequently, the block BFGS update computed from $(s_k, y_k)$ is identical under $F$ and $F_e$. Therefore, $\{x_k\}_{k \geq K_0}$ is *also* a block-BFGS sequence for the smooth, strongly convex function $F_e$ with the same i.i.d. block sampling.

Since $F_e$ is $C^2$ with $\hat{m} I \preceq \nabla^2 F_e \preceq \hat{M} I$ on $\operatorname{conv}(\Omega)$ and the line segments $[x_k, x_{k+1}] \subset \operatorname{conv}(\Omega)$, the hypotheses of our smooth strong-convexity result (Theorem D.1) hold for $F_e$ on the tail under the same i.i.d. sampling. Hence, almost surely,

$$F_e(x_k) \downarrow \min F_e \qquad \text{and} \qquad x_k \to x_e^\star \quad (k \to \infty).$$

Because $F_e = F$ on $\Omega$ and $\min F_e = \min F$, we obtain $F(x_k) = F_e(x_k) \downarrow \min F =: F_\star$ and $x_k \to x_e^\star =: x_\star$. Uniqueness follows from strong convexity of $F$. □

## E.2 A Weaker Smoothness Condition

In many practical problems—especially when the non-smooth component is prominent—the full smoothness on a neighborhood of the whole level set may fail. Following Section 4 in [5], we assume only that on each fixed "value band" above the minimum there exists a smooth convex extension that agrees with the objective. This allows us to reuse the smooth block-BFGS argument on the realized sequence without altering the steps taken by the algorithm.

**Assumption E.3** (Local Smooth Extension)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}$ be convex with compact level set $L_0 := \{x : \ F(x) \le F(x_0)\}$ and $F_\star := \min F$. For every $\delta > 0$ there exist*

1. *a convex open neighborhood $U_\delta \supset L_0$, and*
2. *a function $F_\delta \in C^2(U_\delta)$ that is convex on $U_\delta$ and satisfies*

$$F_\delta(x) = F(x) \qquad \text{whenever } F(x_0) \ge F(x) \ge F_\star + \delta.$$

**Theorem E.2** (Global Convergence Under Local Extension)**.** *Assume E.3 and the Armijo–Wolfe line search with i.i.d. uniform block sampling (as in Section D). Then, almost surely,*

$$F(x_k) \downarrow F_\star.$$

*Proof.* Suppose by contradiction that $F(x_k) \not\downarrow F_\star$. Then there exists $\delta > 0$ such that

$$F(x_k) \ \ge \ F_\star + 2\delta \qquad \text{for all } k. \tag{E.1}$$

By Assumption E.3, pick $U_\delta$ and a $C^2$ convex $F_\delta$ on $U_\delta$ with $F_\delta(x) = F(x)$ on the band $\{x : F_\star + \delta \le F(x) \le F(x_0)\}$. Since $L_0$ is compact and $U_\delta$ is open with $L_0 \subset U_\delta$, the set $L_0$ is contained in a compact $K_\delta \Subset U_\delta$. By continuity of $\nabla^2 F_\delta$ on $K_\delta$, there exists $M_\delta < \infty$ with $\|\nabla^2 F_\delta(x)\| \le M_\delta$ for all $x \in K_\delta$.

*Claim:* Along the realized sequence, the block steps, Wolfe acceptance, and BFGS updates are identical for $F$ and $F_\delta$. Indeed, by (E.1) each $x_k$ lies in the band where $F_\delta = F$; hence $F_\delta(x_k) = F(x_k)$ and $\nabla F_\delta(x_k) = \nabla F(x_k)$ for all $k$. Therefore the accepted stepsize $\alpha_k$ satisfies Armijo–Wolfe for $F_\delta$ iff it does for $F$, and with $s_k := u_{i_k}(s_k^{(i_k)})$ we have

$$y_k = \nabla F(x_{k+1}) - \nabla F(x_k) = \nabla F_\delta(x_{k+1}) - \nabla F_\delta(x_k),$$

so the block BFGS update computed from $(s_k, y_k)$ is the same for both problems.

Consequently, the realized sequence $\{x_k\}$ is also a block-BFGS sequence for the smooth convex function $F_\delta$ on $K_\delta$ with bounded Hessian. The hypotheses of Theorem D.2, our relaxed smooth result, therefore hold for $F_\delta$: the level set $\{x : \ F_\delta(x) \le F_\delta(x_0)\}$ is contained in $K_\delta$ (hence compact), $\|\nabla^2 F_\delta\| \le M_\delta$ on it, and Armijo–Wolfe holds along the realized steps. Thus, almost surely,

$$F_\delta(x_k) \downarrow \min F_\delta.$$

Finally, since $F_\delta = F$ on the band, $F_\delta(x_k) = F(x_k)$ for all $k$ by (E.1). Moreover, by continuity of $F$ and convexity, there exists $y$ with $F(y) = F_\star + \delta$, whence $F_\delta(y) = F(y)$ and so $\min F_\delta \le F_\delta(y) = F_\star + \delta$. Therefore $F(x_k) = F_\delta(x_k) \downarrow \min F_\delta \le F_\star + \delta$, contradicting (E.1).

Hence $F(x_k) \downarrow F_\star$ almost surely.

□

If $F$ is strongly convex, uniqueness of the minimizer implies $x_k \to x_\star$.

**Corollary E.1** (Block BFGS Analogue of Corollary 3.4 from [5])**.** *Suppose $F$ is semialgebraic, strongly convex on an open semialgebraic convex set $U \supset L_0$, and has compact level sets. Let $V \subset U$ be the (open) locus where $F$ is $C^2$. If $x_0$ is drawn from a distribution absolutely continuous w.r.t. Lebesgue measure on $U$, and the algorithm uses Armijo–Wolfe with i.i.d. uniform block sampling, then almost surely either*

(i) *$F(x_k) \to \min F$ (success), or*

(ii) *a subsequence $\{x_{k_j}\}$ converges to a point where $F$ is not smooth and not minimized (failure).*

In semialgebraic geometry $V$ has full measure, so with probability one the realized iterates and line segments avoid the nonsmooth set; then Theorem E.1 (or Theorem E.2 when the extension exists) yields (i). If (i) fails, any limit point must be outside $V$ and nonoptimal, giving (ii). This is the block-coordinate analogue of [5, Cor. 3.4].

## E.3 Examples

In this section, we illustrate the theory on two concrete convex objectives: a 2D $\ell_1$ model, and the Overlapping Group LASSO. Throughout, the block index $i_k$ is sampled i.i.d. uniformly, and the line search is Armijo–Wolfe as specified earlier.

**Corollary E.2** (Two-block Block BFGS on 2D $\ell_1$)**.** *Let $F(u, v) = |u| + |v|$ on $\mathbb{R}^2$, with two coordinate blocks $I_1 = \{u\}$ and $I_2 = \{v\}$. Assume i.i.d. uniform block sampling and an Armijo–Wolfe line search applied to $F$. Then, almost surely,*

$$F(x_k) \downarrow 0 \qquad \text{and} \qquad x_k = (u_k, v_k) \to (0, 0).$$

*Proof.* The level set $\{F \le F(x_0)\}$ is compact and convex. For every $\delta > 0$, we can construct a convex $C^2$ function $F_\delta$ that agrees with $F$ on the "value band" $\{(u, v) : \delta \le |u| + |v| \le F(x_0)\}$ and is $C^2$ on a convex open neighborhood of the level set ( Assumption E.3). Hence the hypotheses of Theorem E.2 hold, and we conclude $F(x_k) \downarrow \min F = 0$ almost surely.

Because the level set $\{F \le F(x_0)\}$ is compact and $\arg\min F = \{(0, 0)\}$ is a singleton, every accumulation point of $\{x_k\}$ must be $(0, 0)$; hence the whole sequence converges: $x_k \to (0, 0)$. □

**Corollary E.3** (Overlapping Group LASSO)**.** *Consider*

$$F(x) \;=\; \tfrac{1}{2}\|Ax - b\|^2 \;+\; \lambda \sum_{g \in \mathcal{G}} w_g \, \|x_{(g)}\|,$$

*with $A \in \mathbb{R}^{m \times n}$, $\lambda > 0$, weights $w_g > 0$, and overlapping groups $\mathcal{G}$. Assume:*

1. *the level set $\mathcal{D} := \{x : \; F(x) \le F(x_0)\}$ is compact;*
2. *blocks are sampled i.i.d. uniformly (independently of the past);*
3. *the line search is the composite Armijo–Wolfe (Armijo on $F$, curvature on $f(x) = \frac{1}{2}\|Ax-b\|^2$);*
4. *$F$ satisfies Assumption E.3.*

*Then, almost surely,*

$$F(x_k) \downarrow F_\star := \min F,$$

*and, moreover, $\mathbb{E}\big[F(x_k)\big] \downarrow F_\star$.*

*Proof.* By Assumption E.3, for each $\delta > 0$ there exist an open convex $U_\delta \supset \mathcal{D}$ and a convex $C^2$ function $F_\delta$ on $U_\delta$ such that $F_\delta(x) = F(x)$ whenever $F_\star + \delta \le F(x) \le F(x_0)$. Applying Theorem E.2 to the realized sequence, we conclude $F(x_k) \downarrow F_\star$ almost surely.

Along each path the sequence $\{F(x_k)\}$ is nonincreasing by Armijo and bounded below by $F_\star$, and we just proved $F(x_k) \to F_\star$ a.s. Since $0 \le F(x_k) - F_\star \le F(x_0) - F_\star$ for all $k$, the bounded convergence theorem yields $\mathbb{E}[F(x_k)] \to F_\star$. □