Critical Juncture Ahead!

Proceed with Caution to Introduce the Concept of Function

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There are specific points in the teaching and learning of mathematics, critical junctures where students have great difficulty understanding a new and fundamental concept that affects to a great extent their success with future mathematical concepts and ideas. Take, for example, the concept of function: Its importance was established in mathematics centuries ago and still is acknowledged in national and state standards as a key topic of study. Kalchman and Koedinger (2005) affirm that "the new and very central concept introduced with function is that of a dependent relationship, the value of one thing depends on, is determined by, or is a function of another" (p. 352). Cunningham (2005) adds that "representing functions in multiple ways is critical to student understanding of functions and success in mathematics" (p. 73). A key goal, Eisenberg (1992) states succinctly, is "to develop in students *a sense for functions*" (p. 154). Numerous articles have been written offering strategies that target such difficulties (Bossé 2005; Davis 2005; Fernandez 2005; Kersaint 2006; Rivera and Becker 2005; Van Dyke and White 2004). Incorporating new instructional strategies into the classroom, however, is not always a straightforward matter.

One big challenge that teachers face is finding the right balance: Can students achieve conceptual understanding and procedural fluency (Fuson, Kalchman, and Bransford 2005) while being

Coordinate system	Plotting points	Deciding if two lines are parallel	Defining linear functions	Deciding if a set of ordered pairs is a function
Solving direct variation problems	Defining a function	Finding coordinates of points	Graphing using a point and the slope	Defining quadratic equations
Equations of horizontal lines and their graphs	Finding the domain of graphs	Finding joint variation equations	Finding values of a function	Graphing linear equations
Using a t-table to graph an equation	Defining the slope/intercept form of an equation	Fitting equations to real data	Defining perpendicular lines	Identifying the quadrants given specific ordered pairs
Finding the range of a function	Deciding if two lines are perpendicular	Graphing linear equations using intercepts	Writing an equation in point-slope form	Finding an equation given its graph
Deciding when a graph represents a function	Deciding if a point is a solution of an equation	Finding the domain of a function	Defining intercepts	Finding corresponding solutions
Writing a linear equation in standard form	Defining slope	Finding the intercepts of quadratic equations	Finding the slope of a line given an equation	Defining a relation
Function notation	Finding the y-intercept of an equation	Deciding if a relation is a function	Finding inverse variation equations	Using the vertical line test
Finding the equation of a line using two points	Graphing quadratic equations	Finding the slope given two points	Ways to represent a function	Equations of vertical lines and their graphs
Finding combined variation equations	Solving for y when working with linear equations	Solving word problems with linear equations	Finding the equation of a line using the slope and a point	Graphing using the slope and the y-intercept
Defining the constant of variation	The domain and range of a quadratic equation	Graphing functions	Using slope to decide if two lines are perpendicular	Writing a quadratic equation in standard form

Fig. 1 Typical topics covered in an algebra course related to the concept of function

prepared to excel in examinations that test comprehension of state/district content standards? A cursory review of the typical exercises and topics covered in a traditional algebra textbook associated with the concept of function (**fig. 1**) clearly illustrates the sheer volume of information that needs to be taught and learned in the classroom and also helps explain why so many teachers are hard pressed to find such a balance with their students. One way to address this issue is to emphasize key mathematical concepts and their corresponding core ideas (Ma 1999). In this article, I share how language and communication can help establish the core ideas that define a function and then suggest activities to introduce the concept of function through the use of language and the core ideas.

A rule that assigns to each number x (the input) a single value y (the output)	A set of ordered pairs that assigns to each x-value exactly one y-value
A relation for which to each domain value there corresponds exactly one range value	A rule or correspondence that assigns to each element of the set X one and only one element of the set Y
A set of ordered pairs in which each first component in the ordered pairs corresponds to exactly one second component	A relation that matches each element of a first set to an element in a second set in such a way that no element in the first set is assigned two different elements in the second set
A relation in which for each value of x there is a unique value of y; x is an independent variable, y is the dependent variable.	A rule that takes certain numbers as inputs and assigns to each a definite output
A special type of relation in which no two ordered pairs have the same first coordinate and different second coordinates	A set of operations that are performed on each value that is put into it and results in one answer

Fig. 2 Mathematical definitions of a function

WHAT DOES $f(x)$ MEAN ANYWAY?

When asked to share what $f(x) = 2x + 30$ means to them, mathematics teachers will tap into their prior knowledge and expertise and state that it is a function, or a rule to find the values of y, or a linear function with slope of 2 and y-intercept of 30, or a representation of a dependent relationship, and so on. Most teachers will also readily agree that the majority of their students would not have the same insights about the function. The issue thus becomes how to introduce the concept of function to students in a way that taps into *their* prior knowledge and experiences. If, according to Usiskin (2005), "learning algebra should be no more difficult than learning a new language" (p. 13), then the first step in this process is to discuss the everyday meaning of the word *function* because "this provides a base on which to build a connection to formal mathematical language" (NCTM 2000, p. 63).

Setting mathematics aside for a moment, what does the word *function* mean to you in everyday language? What is your function (purpose) as a teacher? Do you find it helpful to change your function (role) as you teach mathematics to your students? Does your school have a special function (social event) to recognize the mathematical achievement of your students at the end of each school year? Is there a particular function (task) that your students need to complete in the first five minutes of class? In all four cases, the word *function* is used as a noun; it is some*thing.* On the other hand, one could have asked you to think about the time of day that your students function (work) best when learning mathematics, or perhaps how students function (perform) when presenting their solution strategies in front of the whole class. Do your students function (act) differently when they are taught by a substitute teacher? Taken as a verb, a function *does* something. One could say, then, that in mathematics one studies what this *thing* called a function looks like (graphically, numerically, symbolically) to shed more light on what it *does* (establishes a dependent relationship). It is important to note that considering the word as a noun and as a verb also helps to identify the corresponding core ideas of a function.

Setting mathematics back at center stage: How does one decide whether the definitions in **figure 2** correctly define the concept of function? What helps you sift through such varied definitions of the same concept? Using the discussion above as a starting point, if a function is a thing, then the definitions convey what it is: a set of operations, a relation, a rule, a set of ordered pairs, a correspondence. If a function does something, then the list clarifies what it does: it assigns, it matches, it corresponds, it performs, it operates, it relates. There are three other instances where the definitions describe the same idea but with different words. One group of words includes the terms *input, x, first, x-value, domain,* and *independent;* by implication, the related terms *output, y, second, y-value, range,* and *dependent* would form another group. The final group of words, which at times have to be phrases, includes *single, definite, one and only one, exactly one, unique,* and *the first element cannot have different second elements.* The key here is to illustrate how language connects the everyday and mathematical meanings of the word *function* and how language helps bring to the forefront the core ideas that define this important concept. The intent is to help make obvious for students what is self-evident to many mathematics teachers.

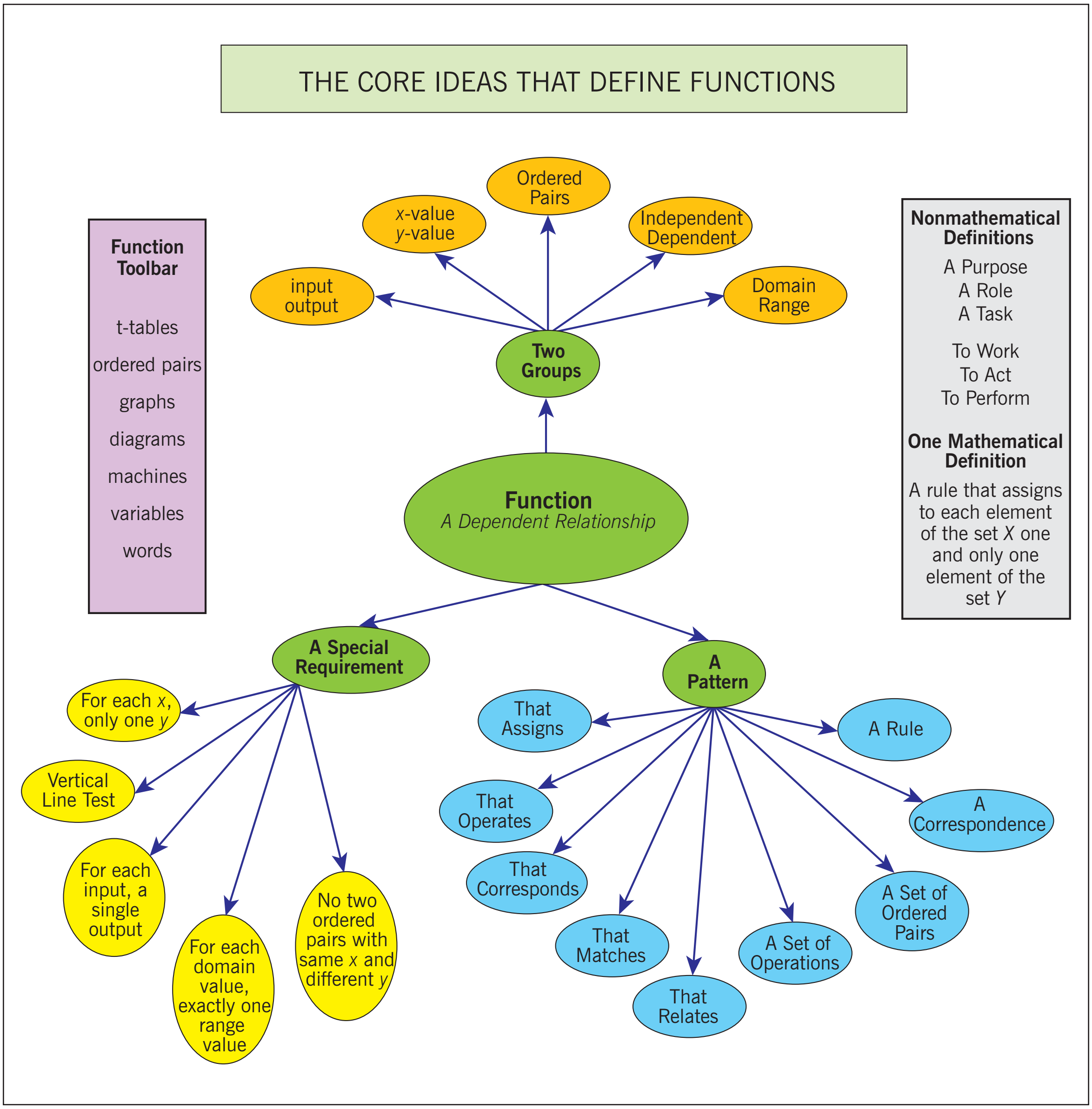

Fig. 3 Core ideas that define functions

THE CORE IDEAS THAT DEFINE A FUNCTION

The following approach has been used successfully with students to introduce the core ideas of a function and to illustrate the usefulness of a function outside the classroom. Given the significance of communication as a standard of mathematics (NCTM 2000), in particular as it relates to the use of language that keeps the target audience in mind, e.g., algebra students, I have named each core idea so that it makes sense to students who are starting to learn about functions. Thus, a bridge is formed to the mathematical meaning and application of those same ideas. This approach, as presented in the classroom to students, is more of a conversation driven by questions than a lecture, because it is "seeing the connections between the different points of view that is important, not simply seeing the concept represented in a different context" (Eisenberg 1992, p. 159).

First core idea

When was the last time that you went to a gas station because your car, or your parent's car, was running low on fuel? To fill up the tank you need to pay for the gas. To buy the gas, you need money. Nothing can occur unless there is money and gasoline. These two things, money and gasoline, point to the first core idea of a function: You need *two groups* of something that, preferably, can be represented with numbers. Each of the two groups represents a set or collection of objects. Typically, the first set is called the domain, and the second set is called the range. Keep in mind, though, that there are other names, such as input and output, which also refer to the same two sets (shown in orange in **fig. 3**). Height and age, hours and wages, distance and speed, are other examples associated with this core idea.

Second core idea

The act of purchasing gasoline is a process that defines a relationship between these two sets, money and gas. In this example, a gallon of gasoline costs $3.00, so $18.00 can buy 6 gallons, $27.00 can buy 9 gallons, and $36.00 can buy 12 gallons. How the numbers relate to each other illustrates the second core idea of a function: *A pattern* exists between the two groups of numbers or objects (shown in blue in **fig. 3**). To uncover a pattern means to find out how numbers in the domain, the first set, are related or paired with numbers in the range, the second set. There are many mathematical tools, such as t-tables, ordered pairs, equations, and graphs, to help make the pattern, or relationship, more obvious and understandable (shown in purple in **fig. 3**). Some patterns are so common that they have specific names, e.g., linear, exponential, and quadratic.

Third core idea

Finally, on any given day, if two people were to drive up in their own cars to fill up their tanks and each paid $18.00, one would expect that both would receive the same amount of gasoline. That is, regardless of who is purchasing the gas, $18.00 should buy, according to this example, 6 and only 6 gallons. If this is the case, then the relationship addresses the third core idea of a function: The pattern between the two groups of numbers must meet a special requirement—that is, each number from the domain is matched or assigned to one and only one number from the range (shown in yellow in **fig. 3**). If this condition is not met, then one does not have a function, one has a relation.

In order to bring closure to this introduction on functions, a PowerPoint presentation (available at www.ivcampus.sdsu.edu/math_ed/index_files/Functions_PPS.pps) helps facilitate a summative discussion about the core ideas and their usefulness. A key advantage to those who use this type of medium is that it provides an ability to describe, in a dynamic manner, the same function with numbers, visuals, graphics, and equations. This can help students "understand that these are different ways of describing the same relationship" (Kalchman and Koedinger 2005, p. 352). These representations also provide students with more ways in which to understand the nature of functions. This introductory activity is likely to raise more questions than answers from students. For example, they will want to know how to find the equation, thus setting the stage for discussions about functions, their usefulness, and different representations. Then again, this is the point of the activity: to create in students an initial sense of wonder about and appreciation for the concept of function.

THE CRITICAL JUNCTURE: INTRODUCING THE CONCEPT OF FUNCTION

Too often, students have little sense of what to do when given functions to explore, whether they are presented in word problems or numerically, graphically, or symbolically. When teachers help students identify the core ideas within the problems, students can have a sense of what to look for as they sift through the details of the scenarios and/or the specific skills in their textbooks. In other words, if the new concept is a dependent relationship, students need to realize that it takes two sets to form a relationship and those sets and the pattern relating them need to be readily identified. Understanding the connections provides students with insights into the nature of the relationship between the two sets (second core idea); and this type of relationship must satisfy a special requirement in order to be considered a function (third core idea).

The core ideas of a function can also help teachers restore for students some of the connections and links that are lost in the presentation in textbooks, in which knowledge "has to be taken apart and ordered sequentially . . . into a large number of isolated bits of knowledge" (Eisenberg 1992, p. 168). According to Donovan and Bransford (2005), the key to restoring such connections is recognizing that "memory of factual knowledge is enhanced by conceptual knowledge, and conceptual knowledge is clarified as it is used to help organize constellations of important details" (p. 7). In this section I share how some mathematics teachers use language and the core ideas to introduce the concept of function to students. I also offer some adaptations of their ideas, inviting readers to take them as a starting point to explore how the core ideas can help students improve their sense of what to do when working with functions.

Introducing function *to prealgebra students*

A prealgebra teacher decided to use these ideas to introduce students to the concept of function. At the start of the activity, students were asked to find as many meanings as they could for the word *function,* using the Internet, dictionaries, and thesauruses, and talking to adults, such as their parents or teachers, to complete the assignment. The next day, students shared what the word meant to them. As they shared the different meanings of the word, the teacher would interject, when appropriate, how some meanings of the word implied action—that is, to function is to do something. To transition the discussion, the teacher showed them **figure 4** and explained how this concept map would help students see the relationships among the mathematical ideas about functions. Then the teacher showed the PowerPoint presentation on functions as a way to introduce and clarify the meaning of the three core ideas.

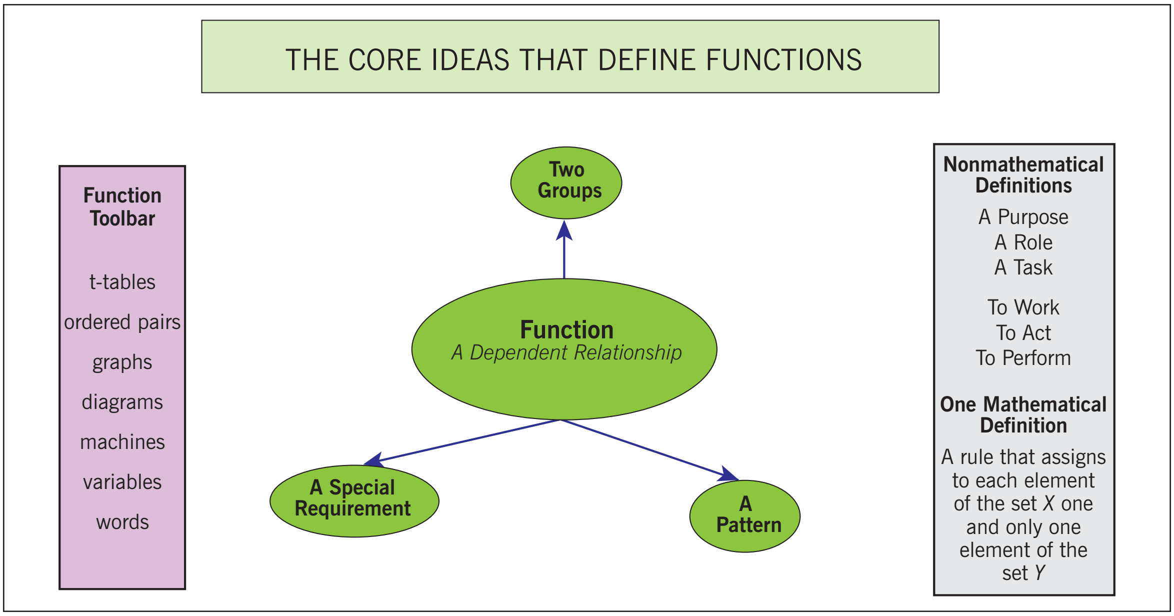

Fig. 4 Core ideas that define functions

The teacher then showed the class a function machine—just the machine. Students were told to look for a pattern between the numbers that were put into the machine and the numbers that came out. A 5 became an 11, a 6 became a 12, a 9 became a 15, an 8 would become a what? Almost immediately, a student said that the machine was adding 6, and the teacher asked the class to decide if this is in fact the case. When everyone agreed, the teacher distributed a copy of the concept map as shown in **figure 4**, led a whole-class discussion on how this example related to the core ideas, and eventually came up with **figure 5**. After the class worked through a few more machine examples, students were told to come up with their own function machines for the next class session. The next day, students were ready to share their machines with the class, and the teacher introduced the many ways in which one can represent a function. In reflecting about this experience, the teacher concluded that students were much more engaged and interested in learning about functions.

One can adapt this activity to provide students with various nonmathematical definitions (**fig. 6**) of the word *function* to facilitate a similar exchange of ideas on its everyday meanings. In either case, the point is to establish the link to students' prior knowledge by having them think about and discuss what the word *function* means in everyday language in order to begin their transition toward a better understanding of what a function is, looks like, and does, *mathematically*. In fact, terms such as *opposite, intercept, variable, slope,* and *limit,* in addition to those identified in *Principles and Standards for School Mathematics* (NCTM 2000, p. 63), would also work well in this activity. Schwartzman's mathematical dictionary (1994) and a regular English dictionary

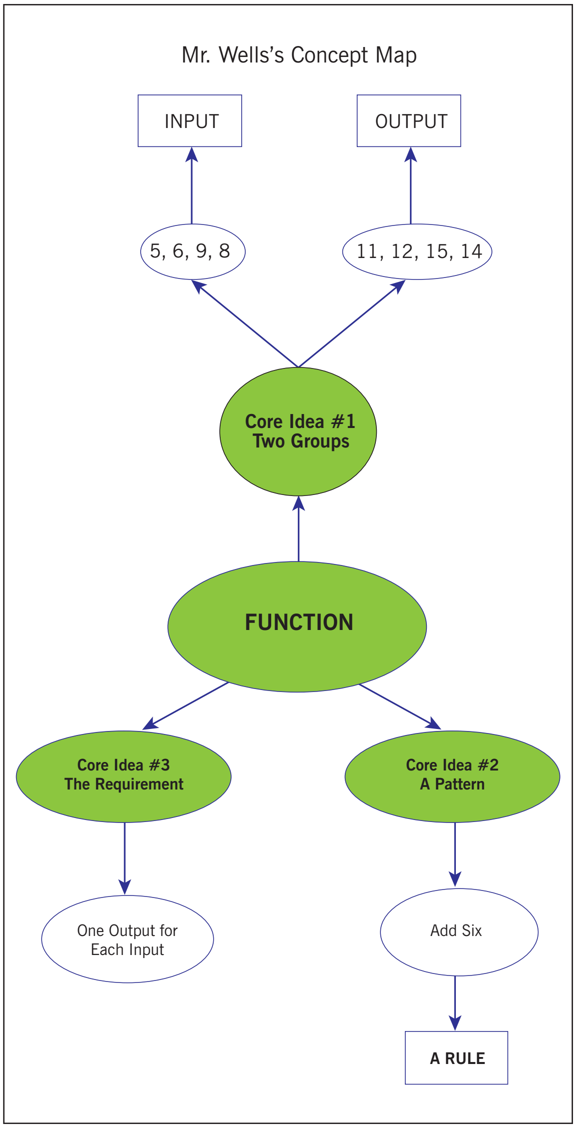

Fig. 5 Concept map used in a prealgebra class

If something functions, it works.	A purpose	A role
A formal social gathering	To perform	An action or activity proper to a person or thing
The purpose for which something is designed	A ceremonious public or social occasion or gathering	To work
To operate	To serve	To perform a function
The acts or operations expected of a person or thing	A job	A task
An action	A behavior	To operate in an expected or proper manner
An activity	To act	To do

Fig. 6 Nonmathematical definitions

are useful in preparing introductions of these concepts. Another adaptation of the activity is to make a class bulletin board of **figure 3**, in addition to the handout, to help students internalize a visual representation of the concept as it is developed over time.

Introducing function *to algebra students*

An algebra teacher introduced the concept of functions by leading a class discussion about the everyday meaning of the word and using the PowerPoint presentation to reinforce the core ideas. The class was shown a picture of a boy dropping a ball from a height of 48 inches. Students were told that after each bounce, the ball would reach half its previous height. The task for the students was to make a table of entries and a graph and to find an equation that would help them predict the height of the ball given the number of bounces the ball had completed. Before letting his students begin their work, the teacher facilitated a whole-class discussion to identify how the core ideas of a function applied to this problem. **Figure 7** reflects the results of the discussion. Afterward, groups were formed, and students began to work on the task. While walking around the room observing student work, the teacher was able to see the different ways, some right and some wrong, in which students answered the questions. After some time, some students were asked to present their work to illustrate that there are different ways to represent the same kind of information. This was also an opportunity to clarify some misconceptions and address some mistakes. Most of the discussion, however, revolved around trying to find the right function to represent the pattern—not an easy task, since it was an exponential function. Once the function was established, students used it to find the height for different numbers of bounces. The algebra teacher, like the prealgebra teacher, was encouraged by students' reactions and the results of using the core ideas to organize instruction and learning.

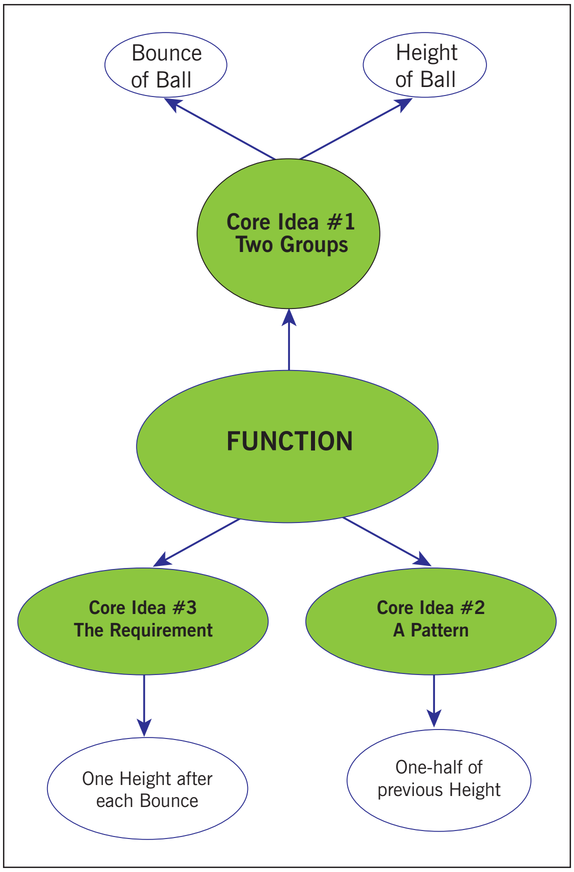

Fig. 7 Class concept map generated by a discussion in an algebra class

The challenge for the algebra teacher is to find a way to help students recognize that the same concepts and skills covered in the text are also being addressed through word problems, like the one about the bouncing ball. One adaptation, then, can be to make clearer in the concept map the relationship between the wording in the problem and the conventional mathematical notation used to express those same ideas. For instance, as students presented their graphs, the discussion could have included how the x-axis represented the number of bounces for the ball and that the y-axis represented the height of the ball. The concept map could thus be updated to show some of the more familiar mathematical notation, *e.g.,* x and y, next to the name of the two groups. Consequently, when the section on the Cartesian coordinate system begins, terms such as *abscissa, ordinate,* and *ordered pairs* can be linked to, and used to expand upon, the first core idea of a function. Also, through the use of a class bulletin board with the core ideas, the teacher can begin to form in students a collective understanding of the relationships among the details of the scenario, the concept of function, and the mathematics topic of the day.

TAKE 2: WHAT DOES $f(x)$ MEAN ANYWAY?

Technically, $f(x) = 2x + 30$ tells students all there is to know about the function. The notation is intended to convey to the student what this *thing* is *doing* to numbers from the domain. To help make this less abstract and more meaningful to students, ask students to figure out how much money they would spend at an amusement park if the entrance fee is \$30 and the price for each ride is \$2. Students will see that the amount of money spent will depend on the number of rides that they take—*two groups:* money spent and rides taken. Asking students to find out the amount of money spent for any particular number of rides taken will give them a better sense of what this function is doing. Also, guiding them to represent their thinking through the use of a t table or ordered pairs, e.g., (0, 30), (1, 32), (2, 34), and then graphing those points, will give students additional insights about *the pattern*: a line that rises from left to right or numbers that increase by two each time. A clearer picture begins to form regarding the domain and the range, as well as how the function meets the special requirement: the number of rides taken determines one and only one amount to spend.

Of particular importance is to link such insights to the conventional notation and technical language of functions. That is, the notation $f(x) = 2x + 30$ gives a name for this *thing* called a function, i.e., f; it explicitly identifies the variable for the domain of the function, x; it represents the range of the function, $f(x)$; and it shows what the function looks like through a set of operations represented symbolically by $2x + 30$. This set of operations also clarifies what the function is *doing* to establish the dependent relationship between the domain and the range. **Figure 8** shows how a concept map centered on the core ideas can clarify for students the meaning of $f(x) = 2x + 30$.

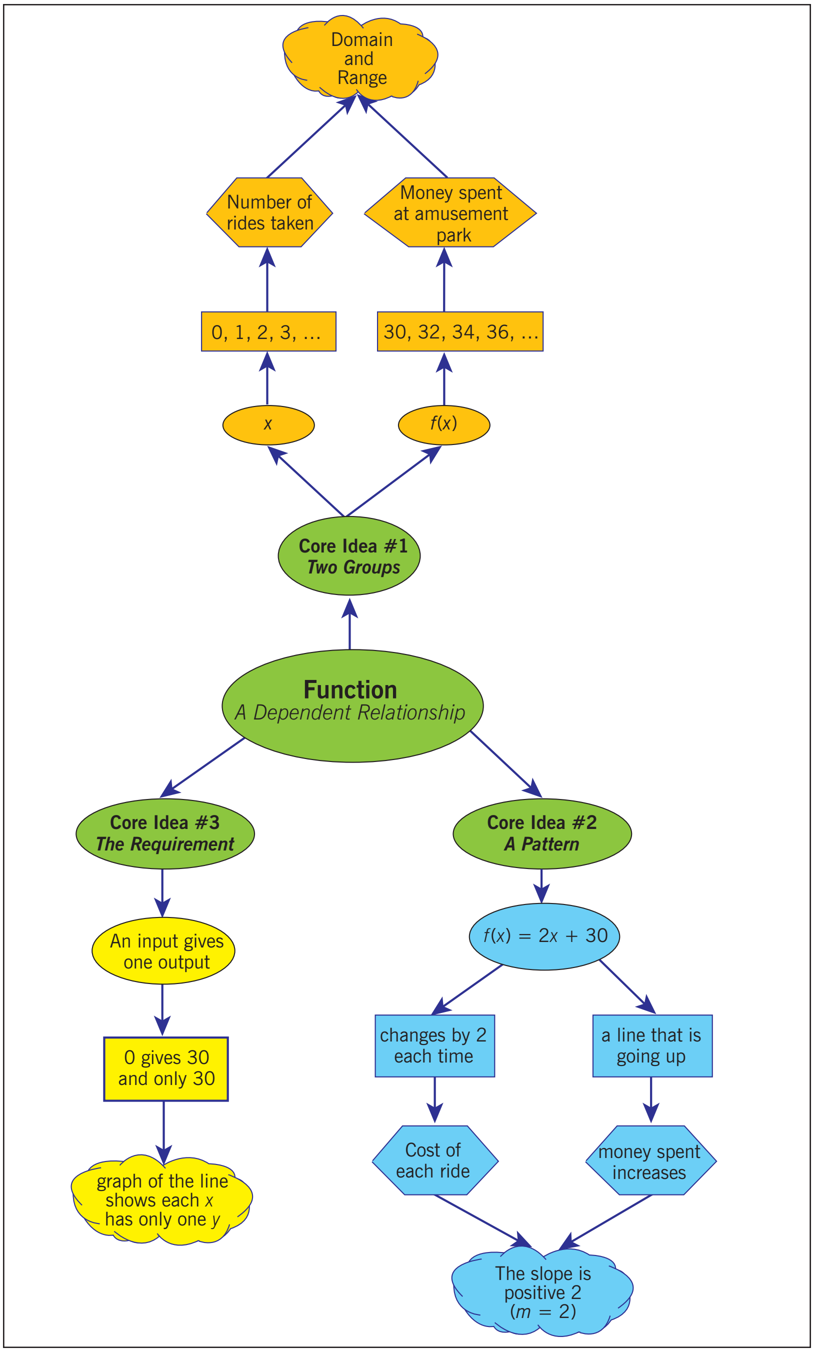

Fig. 8 How a concept map can clarify the meaning of $f(x) = 2x + 30$

CONCLUSION

By studying words and definitions through different sources, one can form a basic idea of what the concept requires and what it is meant to accomplish.

—J. Santillanes, algebra teacher

In many regards, the statement by Mr. Santillanes encapsulates the essence of this article because "mathematics requires careful reasoning about precisely defined objects and concepts" (Ball et al. 2006, par. 8), particularly when the same word, like the word *function,* can have many different and valid mathematical definitions. Accordingly, through the use of language one can sift through these multiple definitions to uncover the core ideas that define the concept of function. To be clear, one goal of this article is to illustrate how the use of everyday language can serve as an effective conduit to help students learn the very technical and powerful language of mathematics. The intent is to raise the level of sophistication in which students communicate mathematically as their understanding of and experience with a concept evolves. From this vantage point, language becomes a bridge, rather than a barrier, to student success with mathematics.

Another goal of this article is to posit that students develop a better sense for functions (Eisenberg 1992) when teachers begin with the core ideas. Using these ideas to organize lessons and to facilitate whole-class discussions can be the framework to meet such a goal because "when one understands Big ideas, mathematics is no longer seen as a set of disconnected concepts, skills, and facts. Rather, mathematics becomes a coherent set of ideas" (Charles 2005, p. 10). Also, "using concepts to organize information stored in memory allows for much more effective retrieval and application" (Donovan and Bransford 2005, p. 7), which can be of great help to students as they try to solve problems on their own. Therefore, from this vantage point, the core ideas can become anchors for both student and teacher as they engage in the teaching and learning of mathematics.

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