Logic and the Limits of Mathematics

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Mathematicians speak a different language than us, the mere mortals. The German philosopher and mathematician Leibniz once envisioned that mathematicians can model everything in the world using the language of mathematics and consequently calculating the models can resolve every debate and open question.

Alas, only half of Leibniz's dream came true, while the other half proved to be an absolute impossibility. Between the late 19th century and the beginning of the 20th century, philosophers and mathematicians invented a language called *First-Order Logic* that can formalize and express everything we speak and mathematicians speak. The most wonderful part of this language is that it seems to be of a finitary system. That is, from a collection of basic and allegedly self-evident axioms, we can deduce, following a mechanistic algorithm, everything we care about, be it the truth about mathematics or the secret of how to get into Harvard.

Yet the work of Cantor, Gödel, Turing, and Tarski, along with many other logicians showed that this language, *First-Order Logic*, on which most of modern mathematics is built, has its limits. There are things that cannot be expressed. There are infinities that cannot be counted. There are truths that cannot be deduced. There are problems that cannot be computed.

"The meaning of the world is the separation of wish and fact", said the great logician Kurt Gödel.

Course Objectives

This course will cover the following topics:

1. Sets, Relations, and Functions

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2. The Language and Syntax of First Order Logic
3. The Uncountability of Real Numbers
4. Turing Machines and the Undecidability of the Entscheidungsproblem
5. Gödel's Incompleteness Theorems
6. Philosophical Discussions on the Incompleteness and Undecidability Results

You will gain the following skills:

1. How to understand and prove a mathematical statements using precise and rigorous natural languages
2. How to deal with difficult problems with care, patience, and courage
3. A solid foundation for future studies of college-level mathematics

Course Evaluation

Homework (80%)

Submit your solutions by 11:59 pm, Tuesday to pengyi0606@icloud.com

You can learn mathematics only by doing mathematics. Thus, doing a problem set weekly is mandatory. It provides you with a chance to get familiar with the materials covered in lectures. You are more than welcome to discuss problem sets with others, but solutions must be independently written.

You are expected to submit the problem set electronically by 11:59 pm, every Tuesday. You should send a email titled "you name + homework x" (where x is the cardinal number of that problem set), containing a PDF or JPG format document of your solutions. You can either handwrite the solutions or use LaTeX.

There should be at least 12 weeks of instructions. Thus, there should be at least 10 weeks with problem sets (i.e. no homework for the last week and the week with the midterm exam). The problem set of the lowest score will not count toward your final grades.

Exam (20%)

There will be one take-home midterm exam but no final exam. Details about the exam is to be announced later.

Textbook

We will primarily follow the open-source textbook collaboratively developed by The Open Logic Project. You can download it for free from https://builds.openlogicproject.org/. You are required to have a copy (either printed or electronic) and you should bring it to school every Wednesday. Should you decide to print a copy (which is highly discouraged), please note that we will at most use Chapters 1-6, 14-18, 22, 23, 26, 31-37, 70-72 and you should only print these chapters. You may also find the following textbooks helpful:

1. Velleman, *How to Prove It: A Structured Approach*
2. Hamilton, *Logic for Mathematicians*
3. Russell, *The Limits of Logic*, unpublished course notes: https://jefelino.github.io/limits-of-logic.pdf
4. Enderton, *A Mathematical Introduction to Logic*
5. Chiswell & Hodges, *Mathematical Logic*
6. Smullyan, *Gödel's Incompleteness Theorems*
7. Logic in Action Open Course Project: https://logicinaction.org/docs/lia.pdf

I strongly advise against watching or reading any Internet content about Gödel's incompleteness theorems in Chinese.

Office Hours

Immediately before and after class, or by appointment. Please do not ask anything related to mathematics via Wechat. I am always happy to schedule a Zoom meeting with you.

Schedule

This is a tentative schedule for the first four sessions.

1. Sets and Proof Methods
    - Section 1.1-1.4 and Chapter 70 of the textbook
2. Relations and Functions
    - Section 1.5, 2.1-2.4 and Chapter 3 of the textbook
3. The Size of Sets
    - Section 4.1-4.10 of the textbook
4. Review
    - No new readings