Logic and the Limits of Mathematics: Part 1

Yi Peng

February 10, 2023

Outline

1. Course Logistics
2. Perspectives from Philosophy as well as History of Mathematics
   - Philosophy and Artificial Intelligence
   - Foundations of Mathematics: Some Histories
3. Sets, Relations, and Functions
   - Sets

Section 1

Course Logistics

Assignments

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- You can learn mathematics only by doing mathematics. Thus, doing a problem set weekly is mandatory. It provides you with a chance to get familiar with the materials covered in lectures. You are more than welcome to discuss problem sets with others, but solutions must be independently written.

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- You are expected to submit the problem set electronically by 11:59 pm, every Tuesday. You should send a email titled "you name + homework x" (where x is the cardinal number of that problem set), containing a PDF or JPG format document of your solutions. You can either handwrite the solutions or use LaTeX.

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- The problem set of the lowest score will not count toward your final grades.

Exam

There will be one take-home midterm exam but no final exam.
Details about the exam is to be announced later.

Textbook

Open Logic Project Builds ×

builds.openlogicproject.org

Gmail 翻译 YouTube

Open Logic Project Builds

This site contains PDFs built from the **source LaTeX files** of the most recent version of the **Open Logic Text**.

Complete PDFs

We have PDFs of the complete text in the Open Logic `master` branch, arranged in a somewhat sensible manner, including editorial comments. It's not intended as a textbook, but it shows what's there.

- **Open Logic Text, Complete Clean Version**: one big PDF of all the material, without extra markup.
- **Open Logic Text, Complete Debug Version**: one big PDF with additional markup to identify source files and OLT-specific commands.

Textbook

BRING A COPY WHEN YOU ATTEND THE LECTURES (either printed or electronic). Should you decide to print a copy (which is highly discouraged), please note that we will at most use Chapters 1-6, 14-18, 22, 23, 26, 31-37, 70-72 and you should only print these chapters.

Office Hours

Immediately before and after class, or by appointment.
Please do NOT ask anything related to mathematics via Wechat.
I am always happy to schedule a Zoom meeting with you.

Schedule for Part 1

1. Sets and Proof Methods
   - Section 1.1-1.4 and Chapter 70 of the textbook
2. Relations and Functions
   - Section 1.5, 2.1-2.4 and Chapter 3 of the textbook
3. The Size of Sets
   - Section 4.1-4.10 of the textbook
4. Review
   - No new readings

About the Instructor

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- always prefers to be contacted by email

Wechat Group

内容...

Survey

Please raise you hand if and only if you:

- are currently taking or have taken calculus

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- are doing some mathematics or physics 竞赛/科研
- are currently taking or have taken any philosophy course

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- are currently taking or have taken any logic course (except this course)
- are a mathematical genius

Section 2

Perspectives from Philosophy as well as History of Mathematics

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- Philosophy and Artificial Intelligence
- Foundations of Mathematics: Some Histories

3 Sets, Relations, and Functions
- Sets

Are minds machines?

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The Axiomatic Method in Mathematics

内容...

Section 3

Sets, Relations, and Functions

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Definition of Sets

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- There is one and only one set containing no elements. It is called the empty set and denoted "$\emptyset$".
- In normal mathematics, we consider every subject of our investigation as a set.

Extensionality

Definition (1.1 Extensionality)

Let A and B be two sets. $A = B$ if and only if every element of A is also an element of B, and vice versa.

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Example

$\{a, a, b\} = \{a, b\} = \{b, a\}$

Subsets

Definition (1.5 Subset)

Let A and B be two sets. $A \subseteq B$ if and only if for all $x \in A$, $x \in B$. If $A \subseteq B$ and $A \neq B$, we write $A \subsetneq B$, where we say A is a proper subset of B.

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Example

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- $\{a\} \subsetneq \{a, b\}$

Definition (Proposition (1.8))

We can rewrite Definition 1.1 as $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Exercise

Exercise 1

Let A, B, C be three sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

Power Sets

Definition (1.10 Power Set)

The set consisting of all subsets of a set A is called the power set of A, written $\wp(A)$ or $\mathcal{P}(A)$. i.e.

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}$$

Example

List all elements of $\mathcal{P}(\{a, b, c\})$.

Unions and Intersections

Definition (1.15 Union)

The union of two sets A and B, written $A \cup B$, is the set of all things which are elements of A, or B, or both. i.e.

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Exercise

Example

$\{a, b, c\} \cap \{0, 1\} = \emptyset$
$\{a, b, c\} \cup \{0, 1\} = \{a, b, c, 0, 1\}$

Exercise 2

Let A, B, C be three sets. Show that

1. $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$
2. $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$

Logical Connectives

Negation

We denote that not p as $\neg p$. Now, clearly whether $\neg p$ is true depends on the truth value of p. Intuitively, we know that $\neg p$ is true *if and only if* p is false.

We may use a *Truth Table* to describe possible situations in which $\neg p$ takes different truth values.

p	¬	p
1	0	1
0	1	0

To prove a statement taking form of $\neg\phi$

Format 1

Suppose (toward a contradition) ϕ.

[The proof of the contradiction goes here.]

Then there is a contradiction. Therefore, $(\neg\phi)$.

Format 2: *reductio ad absurdum*

Suppose (toward a contradition) $\neg\phi$.

[The proof of the contradiction goes here.]

Then there is a contradiction. Therefore, ϕ.

Logical Connectives

Conjunction

For a compound statement that p and q, we denote it as $(p \wedge q)$. Now, clearly whether $(p \wedge q)$ is true depends on the truth values of its components. Intuitively, we know that $(p \wedge q)$ is true *if and only if* p is true and q is true.

We may use a *Truth Table* to describe possible situations in which $(p \wedge q)$ takes different truth values.

p	q	p	∧	q
1	1	1	1	1
1	0	1	0	0
0	1	0	0	1
0	0	0	0	0

To prove a statement taking form of $(\phi \wedge \psi)$

Format

First prove that ϕ.

[The proof goes here.]

Then prove that ψ.

[The proof goes here.]

Therefore, $(\phi \wedge \psi)$.

- **To use a statement taking form of $(\phi \wedge \psi)$**

Treat this statements separately as two distinct atomic statements: ϕ and ψ.

Logical Connectives

Disjunction

For a compound statement that p or q, we denote it as $(p \vee q)$. Now, clearly whether $(p \vee q)$ is true depends on the truth values of its components. Intuitively, we know that $(p \vee q)$ is false *if and only if* p is false and q is false.

We may use a *Truth Table* to describe possible situations in which $(p \vee q)$ takes different truth values.

p	q	p	∨	q
1	1	1	1	1
1	0	1	1	0
0	1	0	1	1
0	0	0	0	0

To prove a statement taking form of $(\phi \vee \psi)$

Format 1

Prove that ϕ.

[The proof goes here.]

Therefore, $(\phi \vee \psi)$.

Format 2

Prove that ψ.

[The proof goes here.]

Therefore, $(\phi \vee \psi)$.

To use a statement taking form of $(\phi \vee \psi)$

Format 1

There are two cases in which $(\phi \vee \psi)$ holds.
Case 1: ϕ is true.

[The proof goes here.]

Case 2: ψ is true.

[The proof goes here.]

We covered all possibilities. Therefore, the goal is true.

To use a statement taking form of $(\phi \vee \psi)$

Format 2

Suppose ϕ and $\neg\psi$.

[The proof goes here.]

Format 3

Suppose $\neg\phi$ and ψ.

[The proof goes here.]

Logical Connectives

Conditional (Material Implication)

For a compound statement that if p then q, we denote it as $(p \to q)$. Now, in daily use, "if...then..." can express abundantly different meanings. However, in the remaining of this course, for simplicity we just **define** that $(p \to q)$ is false *if and only if* p is true and q is false.

We may use a *Truth Table* to describe possible situations in which $(p \to q)$ takes different truth values.

p	q	p	→	q
1	1	1	1	1
1	0	1	0	0
0	1	0	1	1
0	0	0	1	0

To prove a statement taking form of $(\phi \to \psi)$

Format 1

Suppose ϕ.

[The proof of ψ goes here.]

Therefore, $(\phi \to \psi)$.

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Suppose ϕ.

[The proof of ψ goes here.]

Therefore, $(\phi \to \psi)$.

Format 2 (Contrapositive/Contraposition)

Suppose $\neg\psi$.

[The proof of $\neg\phi$ goes here.]

Therefore, $(\phi \to \psi)$.

To use a statement taking form of $(\phi \to \psi)$

Format 1: *Modus Ponens*

Given $(\phi \to \psi)$ and ϕ, we have ψ.

To use a statement taking form of $(\phi \to \psi)$

Format 1: *Modus Ponens*

Given $(\phi \to \psi)$ and ϕ, we have ψ.

Format 2: *Modus Tollens*

Given $(\phi \to \psi)$ and $\neg\psi$, we have $\neg\phi$.

Logical Connectives

Biconditional

For a compound statement that p if and only if q, we denote it as $(p \leftrightarrow q)$. Now, clearly whether $(p \leftrightarrow q)$ is true depends on the truth values of its components. Intuitively, we know that $(p \leftrightarrow q)$ is true *if and only if* $(p \to q)$ is true and $(q \to p)$ is true. In other words, $(p \leftrightarrow q)$ is false *if and only if* p and q take different truth values.

We may use a *Truth Table* to describe possible situations in which $(p \leftrightarrow q)$ takes different truth values.

p	q	p	↔	q
1	1	1	1	1
1	0	1	0	0
0	1	0	0	1
0	0	0	1	0

To prove a statement taking form of $(\phi \leftrightarrow \psi)$

Format

First, we prove $\phi \rightarrow \psi$

[The proof goes here.]

Then, we prove $\psi \rightarrow \phi$

[The proof goes here.]

Therefore, $(\phi \leftrightarrow \psi)$.

To prove a statement taking form of $(\phi \leftrightarrow \psi)$

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First, we prove $\phi \rightarrow \psi$

[The proof goes here.]

Then, we prove $\psi \rightarrow \phi$

[The proof goes here.]

Therefore, $(\phi \leftrightarrow \psi)$.

- **To use a statement taking form of $(\phi \leftrightarrow \psi)$.**
  Now, we can see that $(\phi \leftrightarrow \psi)$ is equivalent to that $(\phi \rightarrow \psi)$ and $(\psi \rightarrow \phi)$. Thus, we can treat it as to separate statements: $(\phi \rightarrow \psi)$ and $(\psi \rightarrow \phi)$.

A General Strategy for Writing Proofs

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- Step 4: Apply the strategy we introduced before on logical connectives.
- Step 5 (not required): Praying to the God of Mathematics.
- Step 6: Write your proof in plain English. If you have to use mathematical or logical symbols, try to explain in plain English what you mean by these symbols as much as possible.

Exercise

Definition (1.22 difference)

The set difference $A \backslash B$ (or $A - B$) is the set of all elements of A which are not also elements of B. i.e.

$$A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Exercise 3

1. We know there is a set containing no elements, i.e. the empty set $\emptyset$. Show that $\emptyset$ is unique. i.e. show that there is no other set than $\emptyset$ containing no elements. Hint: proof by contradiction.
2. Show that $A - B = B - A$ if and only if $A = B$. Hint: contraposition.

Arbitrary Unions and Arbitrary Intersections

Definition (1.19 Arbitrary Union)

If A is a set of sets, then $\bigcup A$ is the set of elements of elements of A:

$$\bigcup A = \{x \mid x \text{ belongs to an element of } A\}, \text{ i.e.,}$$
$$= \{x \mid \text{there is a } B \in A \text{ such that } x \in B\}$$

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Definition (1.20 Arbitrary Intersection)

If A is a set of sets, then $\bigcap A$ is the set of objects which all elements of A have in common:

$$\begin{aligned}\bigcap A &= \{x \mid x \text{ belongs to every element of } A\}, \text{ i.e.,}\\ &= \{x \mid \text{for all } B \in A, \text{we have } x \in B\}\end{aligned}$$

Exercise

Example

Suppose $A = \{\{a, b\}, \{a, d, e\}, \{a, d\}\}$. Then $\bigcup A = \{a, b, d, e\}$ and $\bigcap A = \{a\}$.

Exercise

Example

Suppose $A = \{\{a, b\}, \{a, d, e\}, \{a, d\}\}$. Then $\bigcup A = \{a, b, d, e\}$ and $\bigcap A = \{a\}$.

Exercise 4

1. $\bigcup\{A, B\} = A \cup B$
2. $\bigcap\{A, B\} = A \cap B$