COMS 4733 Class Notes

1 Jacobians and Forces

To analyze the forces a manipulator applies, we are faced with the same problem as before: We wish to apply forces in Cartesian space with a manipulator, but we control these forces by applying torques to revolute joints in joint space.

However, we can relate these two by the idea of virtual work. The work done by the end-effector moving through Cartsian space must be the same as the work done by the joints rotating in joint space.

$$Work\ in\ Cartesian\ Space \equiv Work\ in\ Joint\ Space$$

$$\underbrace{F \cdot \delta x}_{Force\ times\ displacement\ in\ Cartesian\ Space=work} \equiv \underbrace{\tau \cdot \delta \Theta}_{Torque\ times\ joint\ displacement=work}$$

Work is a scalar quantity, and if we have a vector of forces, then we can write:

$$F^T \delta x = \tau^T \delta \Theta \qquad \begin{array}{l} \delta x = vector\ of\ Cartesian\ displacements \\ \delta \Theta = vector\ of\ joint\ displacements \end{array}$$

but: $\delta x = J \delta \Theta$ - our definition of the Jacobian (divide by δt to get velocity relationship)

- $F^T J \delta \Theta = \tau^T \delta \Theta$
- $F^T J = \tau^T$ since the above is true for all $\delta \Theta$.
- $J^T F = \tau$ by transposing both sides.
- $J^T F = \tau$

This says that the Jacobian transpose relates Cartesian forces at the end-effector with torques at the joints. This allows us to specify a force supplied by the end-effector and find out what the requisite joint torques are.

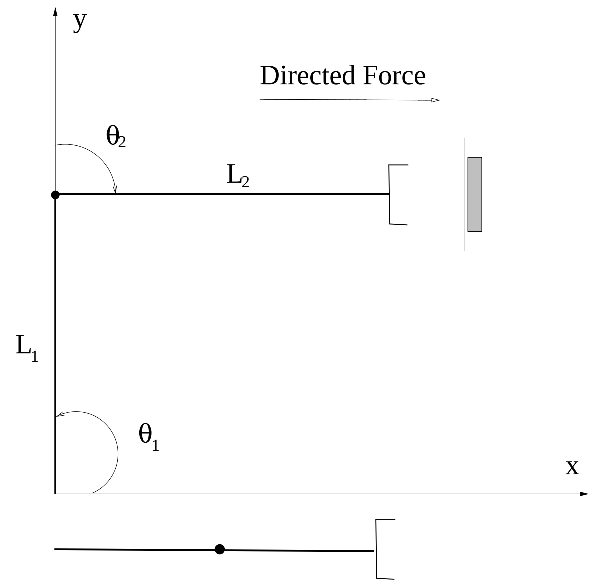

Figure 1: 2-Link Manipulator in configuration $\Theta_1 = 90$, $\Theta_2 = -90$

For the two Link planar RR Manipulator,

$$J = \begin{bmatrix} -S_1L_1 - L_2S_{12} & -L_2S_{12} \\ C_1L_1 + L_2C_{12} & L_2C_{12} \end{bmatrix}$$

$$J^T = \begin{bmatrix} -S_1L_1 - L_2S_{12} & C_1L_1 + L_2C_{12} \\ -L_2S_{12} & L_2C_{12} \end{bmatrix}$$

In figure 1, $\Theta_1 = 90$, $\Theta_2 = -90$ (Jacobian varies with <u>position</u>)

$$J = \begin{bmatrix} -L_1 & 0 \\ L_2 & L_2 \end{bmatrix}$$

$$J^T = \begin{bmatrix} -L_1 & L_2 \\ 0 & L_2 \end{bmatrix}$$

$$J^T F = \tau$$

$$\begin{bmatrix} -L_1 & L_2 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} F_X \\ F_Y \end{bmatrix} = \begin{bmatrix} -L_1F_X + L_2F_Y \\ L_2F_Y \end{bmatrix}$$

If $\Theta_1 = 90$, $\Theta_2 = -90$, to impose a force of N Newtons in the X directon

$$\begin{bmatrix} -L_1 & L_2 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} N \\ 0 \end{bmatrix} = \begin{bmatrix} -L_1N \\ 0 \end{bmatrix} \quad \begin{array}{l} \textit{you need apply} \\ \textit{a torque only} \\ \underline{\textit{to Joint 1}} \end{array}$$

Joint 2 cannot impart force in the X direction in this configuration.

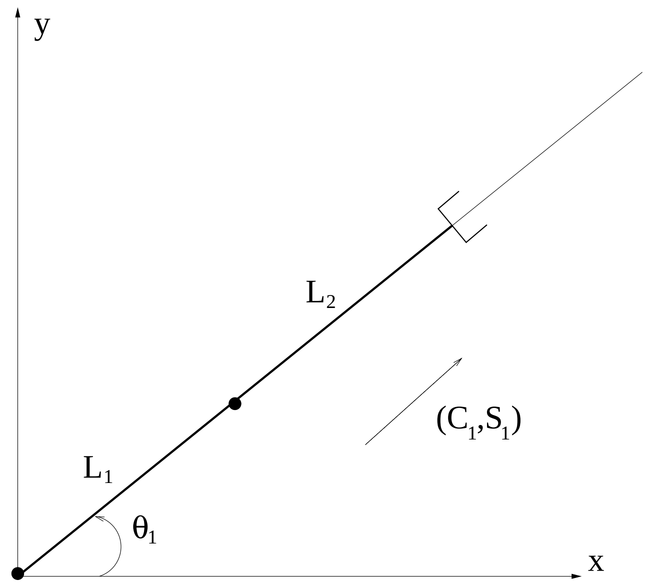

Figure 2: 2-Link Manipulator in configuration $\Theta_2 = 0$

Now, assume the configuration is such that $\Theta_2 = 0$ in the 2 Link Manipulator (see figure 2) - the arm is fully extended:

$$J^T = \begin{bmatrix} -L_1S_1 - L_2S_{12} & C_1L_1 + C_{12}L_2 \\ -L_2S_{12} & L_2C_{12} \end{bmatrix}$$

if $\Theta_2 = 0$,

$$J^T = \begin{bmatrix} -L_1S_1 - L_2S_1 & C_1L_1 + C_1L_2 \\ -L_2S_1 & L_2C_1 \end{bmatrix}$$

$$= \begin{bmatrix} -S_1(L_1 + L_2) & (L_1 + L_2)C_1 \\ -L_2S_1 & L_2C_1 \end{bmatrix}$$

$$\underline{J^TF = \tau}$$

Now solve $J^TF = 0$ -Find Cartesian Forces that cause no equivalent joint forces. <u>Null Space of J^T</u>.

$$\begin{bmatrix} -S_1(L_1 + L_2) & (L_1 + L_2)C_1 \\ -L_2S_1 & L_2C_1 \end{bmatrix} \begin{bmatrix} F_X \\ F_Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$F_X(-S_1(L_1 + L_2)) + F_Y(C_1(L_1 + L_2)) = 0$$

$$F_X(-L_2S_1) + F_Y(L_2C_1) \ = \ 0$$

The solution to these equations is that ratio of $\frac{F_X}{F_Y} = \frac{C_1}{S_1}$ This means that if forces are applied along vector $(C_1, \ S_1)$, <u>NO</u> joint forces are induced. The <u>structure itself absorbs the forces</u>