CS 4733 Class Notes: Forward Kinematics

Reference: Chapter 3, Robot Modeling and Control by Spong, Hutchinson and Vidyasgar, Wiley, 2006.

1 Establishing Frames Between Links of a Robot

- A robot is a series of $links$ and $joints$, which creates a $kinematic\ chain$. Each link connnects 2 adjacent joints, and each joint connects 2 adjacent links (see figure 1.)
- We need to set up a coordinate $frame$ for every joint of the robot. Once we do this, we can establish a set of transformations that will take us from one joint frame to the next.
- If we combine all these transformations from frame 0 to frame n, we can define the entire robot transformation matrix T_n^0.
- All joints, without exception, are represented by a Z axis. If we have a revolute (rotary motion) joint, we rotate about Z. If we have a prismatic joint (a linear sliding joint), we translate along Z. Notation: joint k connects link $k-1$ and link k, and it rotates θ_k about the Z_{k-1} axis. When joint k is actuated, link k moves.
- A robot with n joints will have $n+1$ links, since each joint connects 2 links. We number the joints from 1 to n, and we number the links from 0 to n starting at the base. We can think of link 0 as the fixed base of the robot that never moves.
- With the i^{th} joint we describe a joint variable q_i, which is an angular rotation if a revolute joint or a linear displacement if a prismatic joint.
- Each link has a coordinate frame attached to it. Frame $o_i x_i y_i z_i$ is attached to link i. This means that whatever robot motions occur, the coordinates of every point on link i are constant when expressed in the i^{th} coordinate frame.
- When joint i is actuated, the link i and its entire frame experience a resulting motion. Figure 1 shows a more general way to analyze the relationship between links and joints. Figure 2 shows some frames attached to a 3-link manupulator.

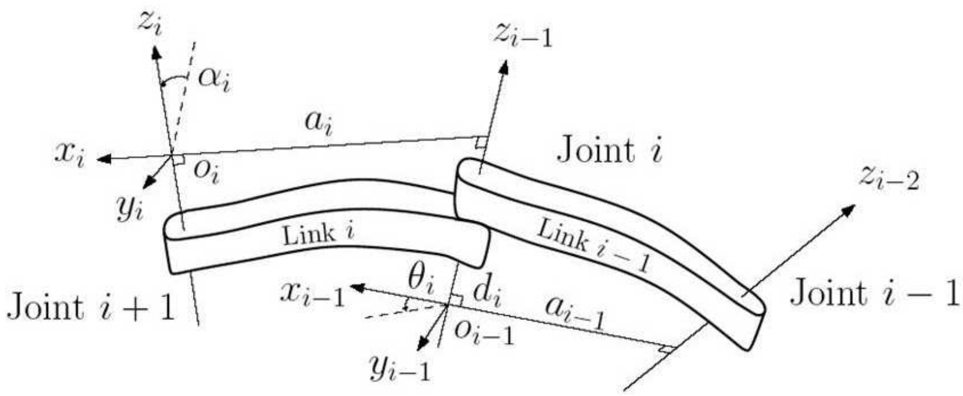


Figure 1: Relationship of joints and links on a robot mechanism

1.1 Creating Transform T_i^{i-1} from Frame $i-1$ to Frame i

We need to specify 4 parameters that will allow us to completely describe the transformation from one frame of the robot to the next. These parameters are called the $Denavit-Hartenberg$ parameters. When we describe a robot using this notation, we refer to it as $D-H$ notation.

1. Rotate about the Z_{i-1} axis by an angle of θ_i. θ_i is called the *joint angle*.
2. Translate along Z_{i-1} by d_i. d_i is called the *link offset* distance.
3. Translate along X_{i-1} by a_i. This will bring the orgins of the two coordinate frames together. a_i is called the *link length*.
4. Rotate about the X_{i-1} axis by an angle α_i. This angle is called the *link twist* angle, and it will align the Z axes of the two frames.

This entire process can be summarized by chaining together the 4 transformations above into a single composite transformation:

$$T_i^{i-1} = Rot(Z,\theta_i)\,Trans(Z,d_i)\,Trans(X,a_i)\,Rot(X,\alpha_i) \tag{1}$$

$$= \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

We will see that this choice of parameters is not unique. However, this is a standard way of specifying the relation of each coordinate frame to the next one in a serial kinematic chain.

1.2 Setting up a Table of D-H parameters

From the discussion above, all we need to provide to solve the forward kinematics of a robot are 4 parameters: θ_i , d_i , a_i , α_i. If we fill in a table like the one below, we can $completely$ specify the robot's forward kinematic structure:

Joint	θ	d	a	α
1				
2				
3				
...	...	...	...	...
N				

Using this table, we can just plug into the transform matrix for each link of the robot, and multiply them together. Note that the *joint variable* is either θ for revolute joints or *d* for prismatic joints.

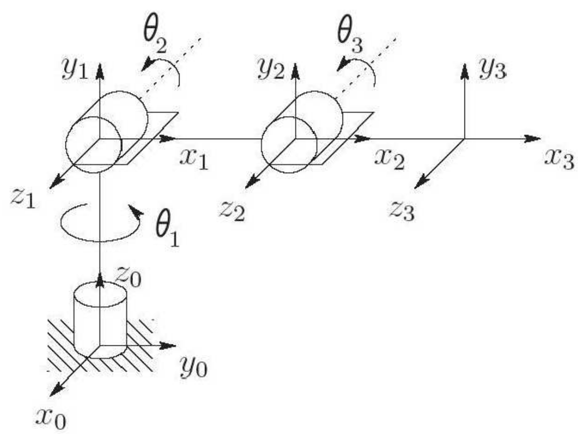


Figure 3.1: Coordinate frames attached to elbow manipulator.

Figure 2: D-H frames for 3 link elbow manipulator. Note: manipulator is pictured after rotation $(\theta_1, \theta_2, \theta_3) = (90, 0, 0)$

Figure 2 shows a 3-link elbow manipulator. Assume link lengths of a_2 and a_3 for links 2 and 3, and that the link 1 offset is d_1:

Joint	θ	d	a	α
1	θ_1	d_1	0	90
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0

$$A_1^0 = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;\ A_2^1 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;\ A_3^2 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$T_3^0 = A_1^0 A_2^1 A_3^2 = \begin{bmatrix} C_1C_2C_3 - C_1S_2S_3 & -C_1C_2S_3 - C_1S_2S_3 & S_1 & C_1C_2a_3 + C_1a_2 \\ S_1C_2C_3 - S_1S_2S_3 & -S_1C_2S_3 - S_1S_2C_3 & -C_1 & S_1C_2a_3 + S_1a_2 \\ S_2C_3 + C_2S_3 & -S_2S_3 + C_2C_3 & 0 & S_2a_3 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

If $(\theta_1, \theta_2, \theta_3) = (90, 0, 0)$, then substituting in T_3^0 we get the matrix:

$$T_3^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & a_2 + a_3 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

This shows the end effector location as (0,$a_2 + a_3$,d_1) as shown in the figure. Note also the directions of the axes in the final end-effector frame are equal to the column vectors of T_3^0.

2 Settting up a Frame Diagram

- In analyzing a robot mechanism, we often create a frame diagram that graphically shows the relationships between the DH-frames of the robot.
- Starting from a base coordinate frame, we align the Z axis with the first joint axis. Call this joint axis Z_{i-1}. Every joint axis in the mechanism will be a Z axis. The axis of the next joint in the chain Z_i is either parallel, intersecting or skew with Z_{i-1}.
  - if Z_i and Z_{i-1} are intersecting, then X_i is in the direction of the cross product of the 2 Z axes.
  - if Z_i and Z_{i-1} are parallel, then X_i is in the direction of common normal between the 2 parallel axes. Since there are many equal normals between the 2 parallel axes, we usually take as the X_i axis the normal through the origin O_{i-1} from the previous frame and establish origin O_i as the point of intersection of the normal with Z_i. Note that the link offset distance d_i will be zero in this case.
  - if Z_i and Z_{i-1} are skew, then X_i is in the direction of the common (unique) normal between the 2 axes.
  - Once we have a new X and Z axis, and an origin for the new frame we are done. The Y axis will simply be the cross product of Z and X.
- Using these rules we then move out the mechanism, a joint and a frame at a time, filling in the D-H parameters and setting up the manipulator transforms.
- We also usually designate a $zero - position$ frame diagram of the robot which is a graphical depiction of the frames when all joint variables are zero. The examples that follow will make this clear.
- Another analysis of the manipulator is to define its geometric workspace: the volume of space reachable by the endpoint of the manipulator. For the previous example of the elbow manipulator (fig. 2), the workspace is a sphere.

3 Example: Forward Kinematics, Cylindrical Manipulator

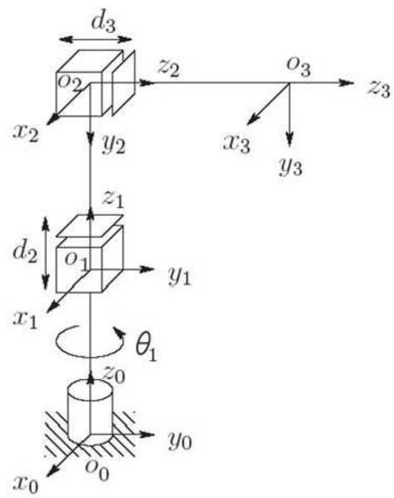


Figure 3.7: Three-link cylindrical manipulator.

Table 3.2: DH parameters for 3-link cylindrical manipulator.

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable

Figure 3: Three link Cylindrical Manipulator

Figure 5 shows a picture of this mechanism and its $frame\ diagram$. A frame diagram shows the robots confi guration for each link of the robot. The table of joint parameters is in fi gure 3. Substituting these values into the D-H frame transformation matrices we get:

$$A_1^0 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ A_2^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ A_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$T_3^0 = A_1^0 A_2^1 A_3^2 = \begin{bmatrix} C_1 & 0 & -S_1 & -S_1\, d_3 \\ S_1 & 0 & C_1 & C_1\, d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Note: d_1 could be zero, in which case the origin of link 1 would be the same as link 2's origin.

Example: Let $\theta_1 = 90$, $d_1 = 0$, $d_2 = 3$ and $d_3 = 5$. Then substituting into the transform matrix, we can see the manipulator endpoint (last column of matrix) is at $(-5, 0, 3)$.

4 Example: Forward Kinematics, Spherical Wrist

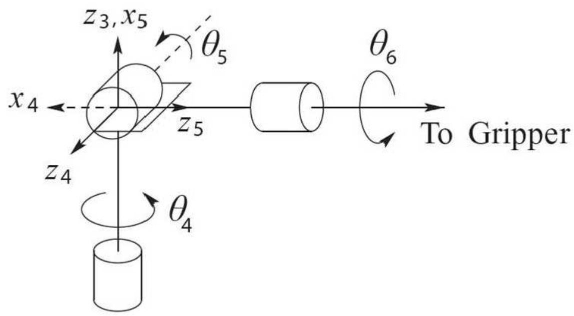


Figure 3.8: The spherical wrist frame assignment.

Table 3.3: DH parameters for spherical wrist.

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable

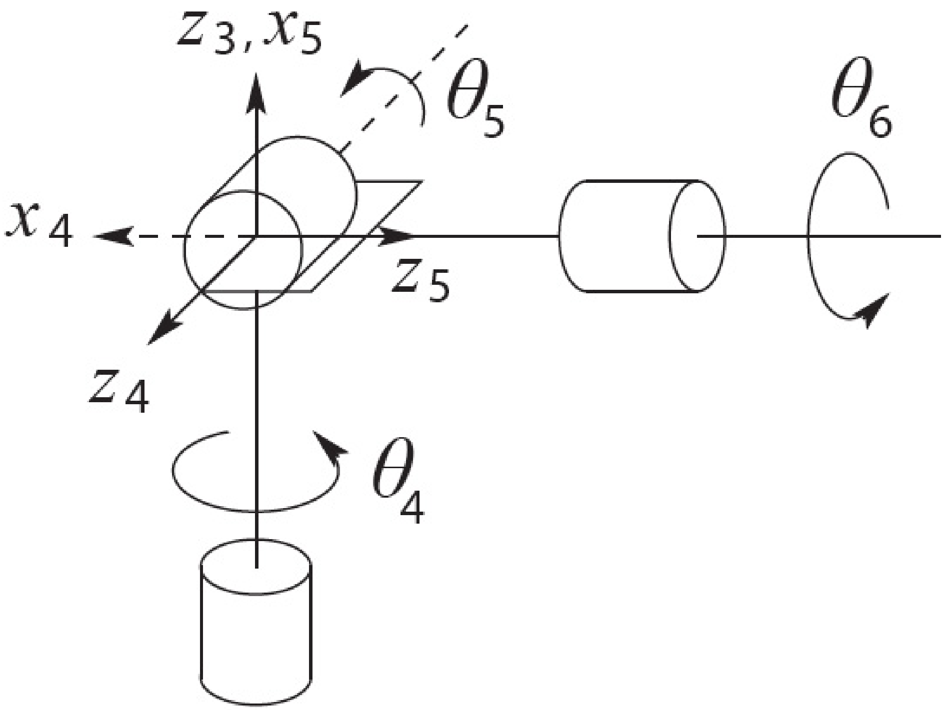
Figure 4: Spherical wrist frame assignments

Aa spherical wrist can be added to the cylindirical manipulator to orient the end-effector (gripper) in space. The first 2 angles effectively point the gripper in a spherical coordinate system, and the last angle is a roll angle that orients the gripper about the approach axis. The table of joint parameters is in figure 4. Substituting these values into the D-H frame transformation matrices we get:

$$A_4^3 = \begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \; A_5^4 = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \; A_6^5 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

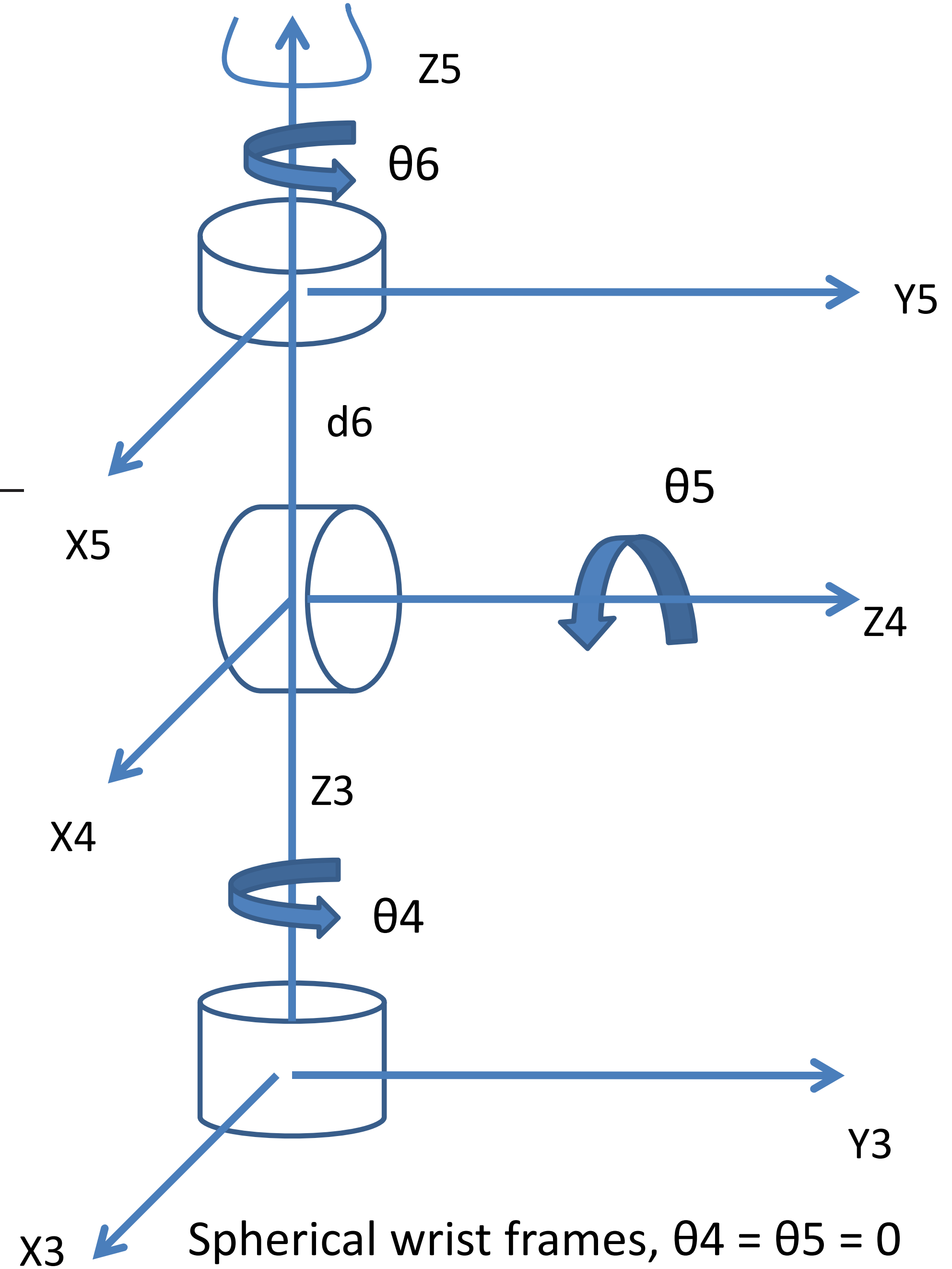
$$T_6^3 = A_4^3 A_5^4 A_6^5 = \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -C_4C_5S_6 - S_4C_6 & C_4S_5 & C_4S_5d_6 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & S_4S_5d_6 \\ -S_5C_6 & S_5S_6 & C_5 & C_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

You can append the 2 matrices $T_3^0\, T_6^3$ to get the final matrix ${}^0T_6$ of the cylindrical manipulator with the spherical wrist. This mechanism has 6 DOF and 6 joint variables: $\theta_1, d_2, d_3, \theta_4, \theta_5, \theta_6$.



Spherical wrist frames, θ4 = θ5 = - 90

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*



Spherical wrist frames, θ4 = θ5 = 0
zero position

5 Example: Forward Kinematics: 3 Link Manipulator B

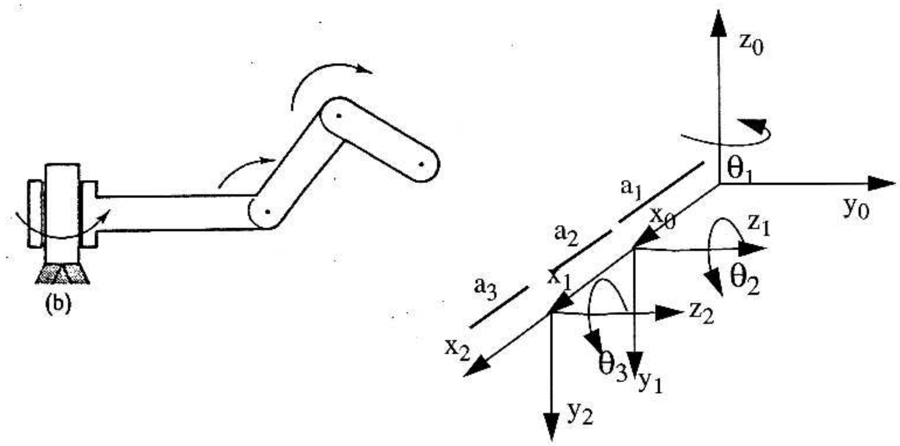


Figure 5: Manipulator B: Mechanism and frame diagram

Figure 5 shows a picture of this mechanism and its $frame\ diagram$. A frame diagram shows the robots confi guration for each link of the robot.

The table of joint parameters is as follows:

Joint	θ	d	a	α
1	θ_1	0	a_1	-90
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0

Substituting these values into the D-H frame transformation matrices we get (note: $C_{23} = Cos(\theta_2 + \theta_3)$, same for S_{23}.

$$A_1^0 = \begin{bmatrix} C_1 & 0 & -S_1 & a_1C_1 \\ S_1 & 0 & C_1 & a_1S_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;\ A_2^1 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;\ A_3^2 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$A_2^0 = \begin{bmatrix} C_1C_2 & -C_1S_2 & -S_1 & C_1C_2a_2 + C_1a_1 \\ S_1C_2 & -S_1S_2 & -C_1 & S_1C_2a_2 + S_1a_1 \\ -S_2 & -C_2 & 0 & -S_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;\ A_3^0 = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & -S_1 & C_1C_{23}a_3 + C_1C_2a_2 + C_1a_1 \\ S_1C_{23} & -S_1S_{23} & C_1 & S_1C_{23}a_3 + S_1C_2a_2 + S_1a_1 \\ -S_{23} & -C_{23} & 0 & -a_3S_{23} - S_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

If $\theta_1 = 90, \theta_2 = 90, \theta_3 = -90$, then

$$A_3^0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & a_1 + a_3 \\ 0 & -1 & 0 & -a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

6 Example: Forward Kinematics: 3 Link Manipulator D

Figure 3 shows a picture of this mechanism and its frame diagram.

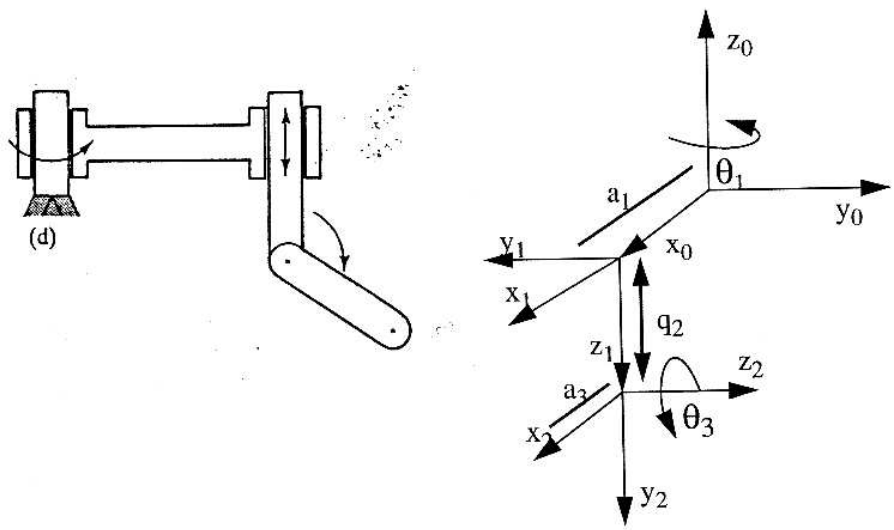


Figure 6: Manipulator D: Mechanism and frame diagram

The table of joint parameters is as follows:

Joint	θ	d	a	α
1	θ_1	0	a_1	180
2	0	q_2	0	90
3	θ_3	0	a_3	0

Substituting these values into the D-H frame transformation matrices we get:

$$A_1^0 = \begin{bmatrix} C_1 & S_1 & 0 & a_1C_1 \\ S_1 & -C_1 & 0 & a_1S_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \; A_2^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \; A_3^2 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$A_2^0 = \begin{bmatrix} C_1 & 0 & -S_1 & a_1C_1 \\ S_1 & 0 & C_1 & a_1S_1 \\ 0 & -1 & 0 & -q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \; A_3^0 = \begin{bmatrix} C_1C_3 & -C_1S_3 & -S_1 & C_1C_3a_3 + C_1a_1 \\ S_1C_3 & -S_1S_3 & C_1 & S_1C_3a_3 + S_1a_1 \\ -S_3 & -C_3 & 0 & -a_3S_3 - q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

if $\theta_1 = 90$ and $\theta_3 = 0$ and $q_2 = 5$:

$$A_3^0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & a_1 + a_3 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$