1 Inverse Kinematics

1. Forward Kinematics is a mapping from joint space Q to Cartesian space W:

   $$F(Q) = W$$

   This mapping is one to one - there is a unique Cartesian configuration for the robot for a given set of joint variables. Inverse Kinematics is a method to find the inverse mapping from W to Q:

   $$Q = F^{-1}(W)$$

2. The inverse kinematics problem has a wide range of applications in robotics. Most of our high level problem solving about the physical world is posed in Cartesian space. While we can reason about the physical world in Cartesian terms, the robot is actuated in joint space - that is what we ultimately can control. Once we solve a problem for its Cartesian space constraints, we need to map these constraints into the robot's joint space using inverse kinematics. For example, if we specify a straight line trajectory for a robot arm, we need to break that trajectory into a set of joint space values over time to get the robot to follow the line.

3. The inverse kinematics mapping is typically one to many. There are usually multiple sets of joint variables that will yield a particular Cartesian configuration. When solving the inverse problem, we often have to choose one solution from a number of valid solutions. There are also degenerate cases with an infinite number of solutions (called *singularities*).

4. Some solutions of the inverse mapping may not be physically realizable. This is due to manipulators having physical joint limits that prevent the mechanism from achieving certain joint configurations that may be solutions to the inverse kinematics problem (e.g. a joint may not have a full 360 degree motion).

5. There may not be a closed form solution to the inverse problem at all for some manipulators. However, most manipulators use a 3 DOF wrist that has intersecting axes. This allows us to separate the inverse problem into a 3 DOF problem for finding the endpoint of the wrist and a 3 DOF problem for finding the orientation of the wrist. This does in fact have a closed form solution.

6. Numerical methods can be used to find a solution to the inverse problem if a closed form solution does not exist.

7. A *redundant* robot is one that has extra DOF's (more than the space the robot works in requires). For example, a 7-DOF robot has an extra DOF if it is used in our normal 6-DOF Cartesian space. This can be useful for reaching around obstacles, and avoiding collisions with other objects in the workspace.

8. To solve inverse kinematics, we use a variety of methods: geometric, trigonometric and algebraic. There are certain forms that you can recognize and then use the appropriate method to solve for a joint variable.

9. Once you solve for a joint variable, you can think of the manipulator as a reduced DOF mechanism - with one less joint. Now solve this manipulator's inverse problem and keep doing this until all joints are solved for.

2 Cylindrical and Spherical Robot Inverse Kinematics

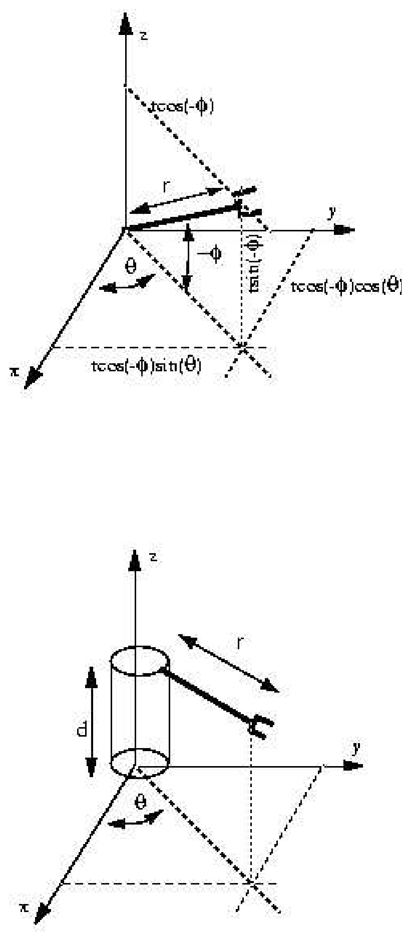

Figure 1: Top: 3-DOF Spherical Robot. This robot's final transform is found by $Rot(Z,\theta)\,Rot(Y,\phi)\,Trans(x,r)$ (note: these are not D-H transforms). Bottom: 3-DOF Cylindrical Robot. This robot's final transform is found by $Rot(Z,\theta)\,Trans(Z,d)\,Trans(x,r)$ (note: these are not D-H transforms)

1. **CYLINDRICAL:** Given the final T matrix of the cylindrical robot with $P = (P_x, P_y, P_z)$, find Θ, d, r for the cylindrical robot.

$$T_3^0 = \begin{bmatrix} Cos\Theta & -Sin\Theta & 0 & rCos\Theta \\ Sin\Theta & Cos\Theta & 0 & rSin\Theta \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. First, we look for positions in the T matrix that have a single variable isolated. P_z in the matrix T_3^0 (last column, third row) is only dependent on the variable d, so we can directly solve for d: $d = P_z$.

3. We notice that $P_x^2 + P_y^2 = r^2$. So $r = \pm\sqrt{P_x^2 + P_y^2}$.

4. To solve for Θ, we can take the ratio of P_y and P_x:

$$\frac{P_y}{P_x} = \frac{rSin\Theta}{rCos\Theta} = Tan\Theta \ , \ \ \Theta = atan2(P_y \, , \, P_x)$$

5. Note there are two solutions for r and Θ values to reach this position in space. If we take the postiive value of r, then $\Theta = atan2(P_y \, , \, P_x)$, and if we take the negative value of r then $\Theta = atan2(-P_y \, , \, -P_x)$

   The two solutions are equivalent to (r, Θ) and $(-r, \Theta + 180))$. However, the negative value of r may not be physically realizable in an actual robot - the arm may only extend radially forward. Also, consider $P = (0, 0, K)$ - can we solve for inverse kinematics in this robot configuration?

SPHERICAL robot has 3 variables: Radius r, Longitude angle Θ and latitude angle Φ. For the Spherical robot the final matrix is:

$$T_3^0 = \begin{bmatrix} Cos\Theta Cos\Phi & -Sin\Theta & Cos\Theta Sin\Phi & rCos\Phi Cos\Theta \\ Sin\Theta Cos\Phi & Cos\Theta & Sin\Theta Sin\Phi & rCos\Phi Sin\Theta \\ -Sin\Phi & 0 & Cos\Phi & -rSin\Phi \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1. r is the length of the the spherical arm radius and is equal to:

$$r = \pm\sqrt{P_X^2 + P_y^2 + P_z^2}$$

2. To solve for Φ, $P_z = rSin\Phi$, and $\Phi = Sin^{-1}(P_z / -r)$. Note r has 2 values from above.

3. As in the cylindrical robot, the ratio of P_x and P_y yields a solution for Θ:

$$\frac{P_y}{P_x} = \frac{rCos\Phi Sin\Theta}{rCos\Phi Cos\Theta} = Tan\Theta \ , \ \ \Theta = atan2(P_y \, , \, P_x) \, or \, \Theta = atan2(-P_y \, , \, -P_x)$$

4. As in cylindrical robot, there are multiple solutions. To get the second solution, find the point on the sphere directly opposite where the manipulator is (if at P_x, P_y, P_z choose $-P_x, -P_y, -P_z$). The second solution is the latituide and longitude of the new point with r's value negated. It also may not be physically realizable with the manipulator. Note singularity $P = (0, 0, K)$.

3 Adept Robot Inverse Kinematics

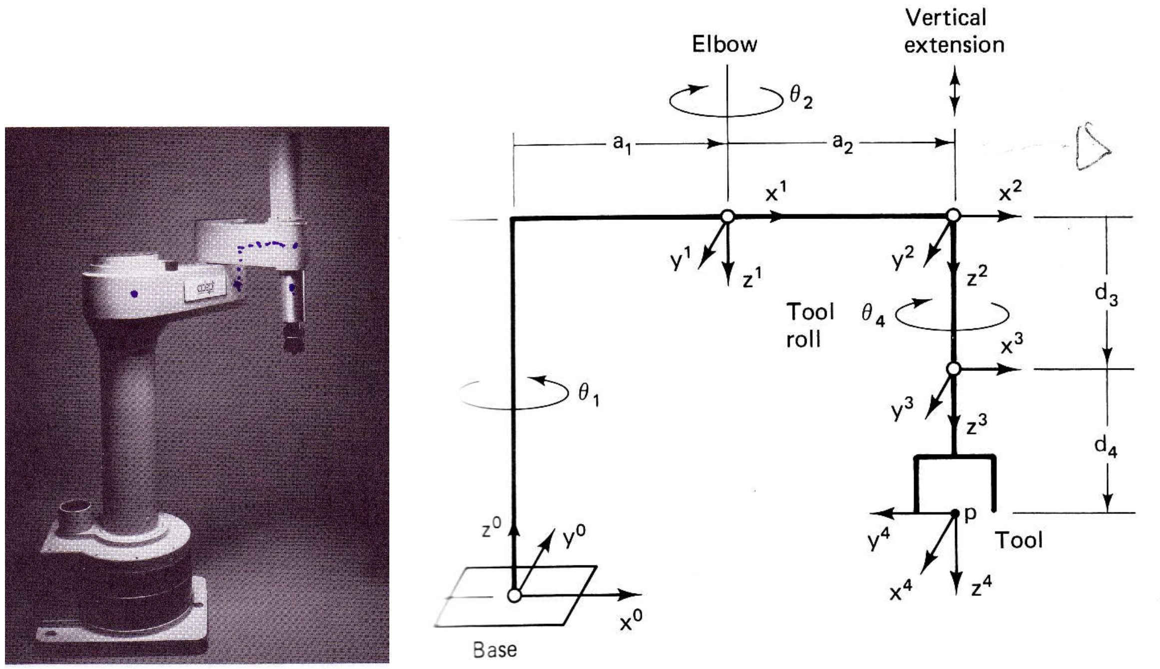

Figure 2: Adept 1 Scara Robot arm. This arm is in a R-R-P-R configuration. θ_1, θ_2 , θ_4 are the revolute joint angle variables and q_3 is the prismatic joint variable. The robot is pictured in the $Home$ position in the frame diagram using the values of the joint variables listed in the table below.

1. Given the final position of the robot P=(P_x, P_y, P_z). Find Θ_1, Θ_2, q_3, and Θ_4 for the scara robot. The final T matrix is given below:

$$T_4^0 = \begin{bmatrix} C_{1-2-4} & S_{1-2-4} & 0 & a_1C_1 + a_2C_{1-2} \\ S_{1-2-4} & -C_{1-2-4} & 0 & a_1S_1 + a_2S_{1-2} \\ 0 & 0 & -1 & d_1 - q_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. To find Θ_2: if we square and sum P_x and P_y, we can get an expression in Θ_2:

$$P_x^2 + P_y^2 = (a_1C_1 + a_2C_{1-2})^2 + (a_1S_1 + a_2S_{1-2})^2$$

$$P_x^2 + P_y^2 = a_1^2 + a_2^2 + 2a_1a_2C_1(C_1C_2 + S_1S_2) + 2a_1a_2S_1(S_1C_2 - S_2C_1)$$

$$P_x^2 + P_y^2 = a_1^2 + a_2^2 + 2a_1a_2C_1^2C_2 + 2a_1a_2S_1^2C_2$$

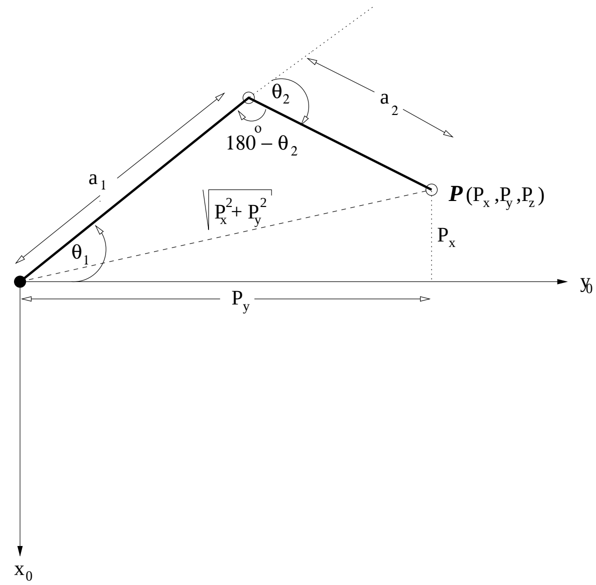

Figure 3: Solution to Θ_2 of Adept, as seen from above (along Z axis)

$$P_x^2 + P_y^2 = a_1^2 + a_2^2 + 2a_1a_2C_2$$

$$C_2 = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2} \; ; \; S_2 = \sqrt{1 - C_2^2}$$

$$\Theta_2 = \pm\, Cos^{-1}\left(\frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2}\right)$$

3. This is really just the derivation of the Law of Cosines which we can also use to find Θ_2 (see figure above):

$$a_1^2 + a_2^2 - 2a_1a_2Cos(180 - \Theta_2) = P_x^2 + P_y^2 \quad (Law\ of\ Cosines)$$

$$Cos(180 - \Theta_2) = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{-2a_1a_2}$$

$$-Cos(\Theta_2) = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{-2a_1a_2}$$

$$Cos(\Theta_2) = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2}$$

4. To solve for Θ_1, we solve for the following:

$$\begin{array}{ll} a_1 C_1 + a_2 C_{1-2} = P_x & Two\,equations\,in\,two \\ a_1 S_1 + a_2 S_{1-2} = P_y & unknowns\,(C_1, S_1) \\ & (\Theta_2\,known\,from\,above) \end{array}$$

$$a_1 C_1 + a_2 C_1 C_2 + a_2 S_1 S_2 = P_x \ , \ a_1 S_1 + a_2 S_1 C_2 - a_2 S_2 C_1 = P_y$$

$$(a_1 + a_2 C_2) C_1 + (a_2 S_2) S_1 = P_x \ , \ (-a_2 S_2) C_1 + (a_1 + a_2 C_2) S_1 = P_y$$

$$S_1 = \frac{a_2 S_2 P_x + (a_1 + a_2 C_2) P_y}{(a_2 S_2)^2 + (a_1 + a_2 C_2)^2}$$

$$C_1 = \frac{(a_1 + a_2 C_2) P_x - a_2 S_2 P_y}{(a_2 S_2)^2 + (a_1 + a_2 C_2)^2}$$

$$\Theta_1 = atan_2(a_2 S_2 P_x + (a_1 + a_2 C_2) P_y, (a_1 + a_2 C_2) P_x - a_2 S_2 P_y)$$

5. To solve for q_3:

$$P_z = d_1 - q_3 - d_4 \ ; \ q_3 = d_1 - d_4 - P_z$$

6. To solve for Θ_4: The final roll angle cannot be determined from the position vector $[P_x, P_y, P_z]$. If we are given the orientation matrix, then we can use the ratios of N_x, N_y to find Θ_4

$$Tan_{1-2-4} = \frac{S_{1-2-4}}{C_{1-2-4}} = \frac{N_y}{N_x}$$

$$\Theta_1 - \Theta_2 - \Theta_4 = atan_2(N_y, N_x)$$

$$\Theta_4 = -atan_2(N_y, N_x) + \Theta_1 - \Theta_2$$

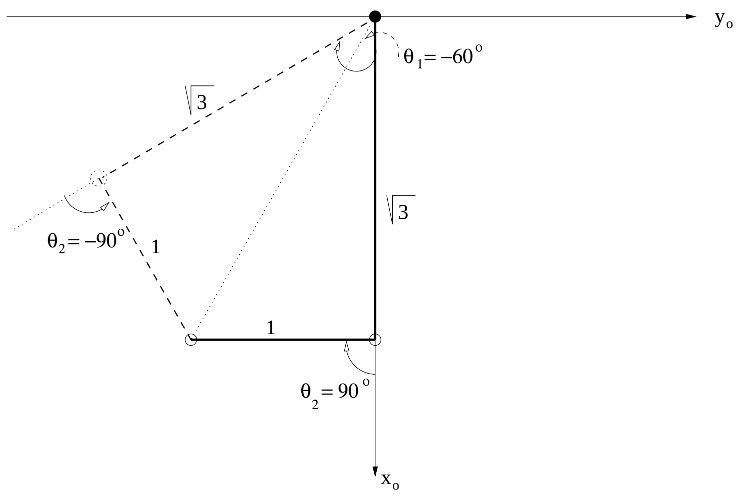

Figure 4: Example solution ignoring Θ_4 with 2 arm positions

7. Example Solution (no Θ_4) if $a_1 = \sqrt{3},\ a_2 = 1,\ d_1 = 5,\ d_4 = 2$ and P=$\sqrt{3}$,-1,1], solve for joint variables:

$$\Theta_2 = \pm Cos^{-1}\left(\frac{(P_x^2 + P_y^2 - a_1^2 - a_2^2)}{2a_1a_2}\right) = \pm Cos^{-1}(\frac{0}{2\sqrt{3}}) \ = \pm 90^\circ$$

$$\Theta_1 = atan_2(a_2S_2P_x + (a_1 + a_2C_2)P_y, (a_1 + a_2C_2)P_x - a_2S_2P_y)$$

$$if\ \Theta_2 = +90^\circ,\ \Theta_1 = atan_2(0, 4) = 0^\circ$$

$$if\ \Theta_2 = -90^\circ,\ \Theta_1 = atan_2(-\sqrt{3}, 1) = -60^\circ$$

$$q_3 = d_1 - d_4 - P_z = 5 - 2 - 1 = 2$$

Two Solutions:

Θ_1	Θ_2	q_3
0°	90°	2
-60°	-90°	2