Toda Lattice, Double Bracket

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1 Equivalence between ODE and Lie bracket notation

The central idea of analog sorting as described in the papers is using the Toda Lattice, which is described using the Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{1}^{n} p_k^2 + \sum_{1}^{n-1} exp(q_k - q_{k+1})$$

A system of equations are associated with this, defined as the Hamiltonian equations (can be derived using total differential)

$$\dot{p_k} = -\frac{\partial H}{\partial q_k}$$
$$\dot{q_k} = \frac{\partial H}{\partial p_k}.$$

Calculating this, we get

$$\dot{p_k} = exp(q_{k-1} - q_k) - exp(q_k - q_{k+1})$$
$$\dot{q_k} = p_k$$

where $e^{x_0 - x_1} = e^{x_n - x_{n+1}} = 0$

1.1 Toda Lattice - ODE form

This system of ODEs can be changed to the system described in (http://hrl.harvard.edu/analog/) using a change of variables.

$$x_k = -\frac{1}{2}p_k$$
$$y_k = \frac{1}{2}e^{(q_k - q_{k+1})/2}$$

In that case,

$$\begin{aligned}
\dot{x_k} &= -\frac{1}{2}\dot{p_k} \\
&= -\frac{1}{2}(exp(q_{k-1} - q_k) - exp(q_k - q_{k+1})) \\
&= -\frac{1}{2}(4y_{k-1}^2 + 4y_k^2) \\
&= 2y_k^2 - 2y_{k-1}^2
\end{aligned}$$

and

$$\begin{aligned}
\dot{y_k} &= \frac{1}{2}e^{(q_k - q_{k+1})/2}(\frac{\dot{q}_k - \dot{q}_{k+1}}{2}) \\
&= y_k \frac{p_k - \dot{p}_{k+1}}{2} \\
&= y_k(x_{k+1} - x_k)
\end{aligned}$$

Taking into account the boundary conditions $y_0 = y_n = 0$ and we get the desired system of ODEs.

1.2 Toda Lattice - Jacobi Matrix

The connection to the Double Bracket notation can be made through the Jacobi Matrix form of the Toda Lattice. The Jacobi Matrix is given by

$$H = \begin{bmatrix} x_1 & y_1 & 0 & \dots & 0 \\ y_1 & x_2 & y_2 & \dots & 0 \\ & & \ddots & & \\ & & y_{n-2} & x_{n-1} & y_{n-1} \\ 0 & \dots & & y_{n-1} & x_n \end{bmatrix}$$

This is also the form that H is in for analog sorting.

In order to get the double bracket form, we need a diagonal matrix $N = diag(n, n-1, \dots, 1)$, so we need

$$N = \begin{bmatrix} n & 0 & \dots & 0 \\ 0 & n-1 & & \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & 1 \end{bmatrix}$$

We then get

$$HN = \begin{bmatrix} nx_1 & (n-1)y_1 & \dots & 0 \\ ny_1 & (n-1)x_2 & \dots & 0 \\ \vdots & & \ddots & 0 \\ & & 2x_{n-1} & y_{n-1} \\ 0 & \dots & 2y_{n-1} & x_n \end{bmatrix}$$

$$NH = \begin{bmatrix} nx_1 & ny_1 & \dots & 0 \\ (n-1)y_1 & (n-1)x_2 & \dots & 0 \\ \vdots & & \ddots & 0 \\ & & 2x_{n-1} & 2y_{n-1} \\ 0 & \dots & y_{n-1} & x_n \end{bmatrix}$$

$$HN - NH = \begin{bmatrix} 0 & -y1 & \dots & & 0 \\ y_1 & 0 & -y_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & y_{n-2} & 0 & -y_{n-1} \\ 0 & & \dots & y_{n-1} & 0 \end{bmatrix}$$

$$H[H,N] = \begin{bmatrix} y_1^2 & -x_1y_1 & -y_1y_2 & 0 & \dots & 0 \\ x_2y_1 & -y_1^2 + y_2^2 & -x_2y_2 & 0 & \dots & \\ y_1y_2 & \ddots & & \ddots & & 0 \\ \vdots & & & & y_{n-1}^2 - y_{n-2}^2 & -x_{n-1}y_{n-1} \\ 0 & \dots & & & x_ny_{n-1} & -y_{n-1}^2 \end{bmatrix}$$

$$[H,N]H = \begin{bmatrix} y_1^2 & -x_2y_1 & -y_1y_2 & 0 & \dots & 0 \\ x_1y_1 & y_1^2 - y_2^2 & -x_3y_2 & 0 & \dots & \\ y_1y_2 & \ddots & & \ddots & & 0 \\ \vdots & & & & -y_{n-1}^2 + y_{n-2}^2 & -x_ny_{n-1} \\ 0 & \dots & & & x_{n-1}y_{n-1} & y_{n-1}^2 \end{bmatrix}$$

Therefore, we get

$$[H,[H,N]] = \begin{bmatrix} 2y_1^2 & y_1(x_2 - x_1) & 0 & & \dots & 0 \\ y_1(x_2 - x_1) & -2y_1^2 + 2y_2^2 & y_2(x_3 - x_2) & 0 & \dots & \\ 0 & \ddots & & \ddots & & 0 \\ \vdots & & & & 2y_{n-1}^2 - 2y_{n-2}^2 & y_{n-1}(x_n - x_{n-1}) \\ 0 & \dots & & & y_{n-1}(x_n - x_{n-1} & -2y_{n-1}^2 \end{bmatrix}$$

which is exactly $\dot{H}$

2 Related work

Below is a list of articles that are related to this topic:

- Bloch, Anthony M., and Alberto G. Rojo. "Sorting: The Gauss Thermostat, the Toda Lattice and Double Bracket Equations,." *Three Decades of Progress in Control Sciences* (2010): 35-48.

- Guseinov, Gusein Sh. “A Class of Complex Solutions to the Finite Toda Lattice.” *Mathematical and Computer Modelling* 57.5-6 (2013): 1190-202.
- Tomei, Carlos. “The Toda Lattice, Old and New.” *JGM Journal of Geometric Mechanics* 5.4 (2013): 511-30. Web.