Toda Lattice, Role of N and y_k

Lusa Zhan

November 2016

1 Role of N

In order to understand the role of N fully, we need to consider *Theorem 1.5* in Helmke and Moore. To summarize for our purposes, the theorem states the following:

Given $H = [H, [H, N]]$ for each N, converges to an equilibrium as $t \to \infty$. If $N = (\mu_1, \ldots, \mu_n)$ where $\mu_1 > \cdots > \mu_n$, then the Hessian of $f_N(H) = \frac{1}{2}\|N - H\|^2$ is nonsingular and negative definite.

That means that the linearization at a critical point is

$$\dot{\xi}_{ij} = -(\lambda_{\pi(i)} - \lambda_{\pi(j)})(\mu_i - \mu_j)\xi_{ij}$$

where $\xi = [H_\infty, N]$. Since the Hessian is negative definite, ξ must be at the max, so H_∞ must be sorted.

2 Role of y_k

This section aims to explain the choice of initial values for the y_k's to start the algorithm.

First of all, as mentioned previously, $y_0 = y_n = 0$. To analyze the initial values of y_k, we consider the relation to $\dot{x}_k$. To recall,

$$\begin{aligned} \dot{x_k} &= 2y_k^2 - 2y_{k-1}^2 \\ \dot{y_k} &= y_k(x_{k+1} - x_k) \end{aligned}$$

If $y_k = 0$, then $\dot{x}_k$ would be 0, meaning $x_k(t) = x_k(0), \forall t$. This means that the values along the diagonal of H will remain constant and never get sorted.

In addition, the y_k values should be as small as possible (but still positive). Recall that the eigenvalues will be the values along the diagonal of H as $t \to \infty$. Consider the determinant $f_n = \det(H - \lambda I)$, where n is the size of the matrix. Since H is a tridiagonal matrix, this determinant can be formulated using a recurrence relation on the size of the matrix n.

$$
\begin{aligned}
f_n &= (x_n - \lambda) f_{n-1} - y_{n-1}^2 f_{n-2} \\
f_1 &= x_1 - \lambda
\end{aligned}
$$

We can use this to calculate the first few f_n and set them to 0 (what we would do to calculate the eigenvalues):

$$
\begin{aligned}
f_1 &= x_1 - \lambda = 0 \\
f_2 &= (x_2 - \lambda)(x_1 - \lambda) - y_1^2 = 0 \\
f_3 &= (x_3 - \lambda)(x_2 - \lambda)(x_1 - \lambda) - (x_3 - \lambda) y_1^2 - (x_1 - \lambda) y_2^2 = 0
\end{aligned}
$$

Intuitively, the eigenvalues will be closest to x_k if the y_k values are smaller.

3 Related work

Below is a list of articles that are related to this topic:

- Brockett, R.w. "Dynamical Systems That Sort Lists, Diagonalize Matrices, and Solve Linear Programming Problems." Linear Algebra and Its Applications 146 (1991): 79-91.

- Helmke, Uwe, and John B. Moore. "Double Bracket Isospectral Flows." Optimization and Dynamical Systems. London: Springer-Verlag, 1994. 43-80. Print.