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6170 Section #1
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Plan for today:

- Review basics of logic & limit proofs.
- Q & A

Logic

something that connects multiple expressions in a statement

- Quantifier: an operator that specifies to what extent a statement applies over its domain.

a sentence that could be true / false.

set of values that may be substituted in place of the variables

We focus on $\forall$ (universal quantifier) & $\exists$ (existential quantifier)

E.g. Some birds sing all the time.

$\exists$ birds s.t. $\forall$ time, sing.

All birds sing sometime.

$\forall$ birds, $\exists$ time s.t. they sing.

Notice how changing the order of the quantifiers change the meaning of the sentence.

- Negation

E.g. Some birds sing all the time.
$\exists$ birds s.t. $\forall$ time, sing.

$\forall$ birds, $\exists$ time s.t. they don't sing.
(All birds don't sing sometime.)

E.g. Def. of $x_n \to x$: $\forall \varepsilon > 0, \exists N$ s.t. $\forall n > N$, $|x_n - x| < \varepsilon$

Negation : $\exists \varepsilon > 0$ s.t. $\forall N$, $\exists n > N$ s.t. $|x_n - x| \geq \varepsilon$

$\boxed{\text{E.g.}}$ Def of a set A being open: $\forall x \in A, \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq A$.

Negation: $\exists x \in A$ s.t. $\forall \varepsilon > 0$, $B_\varepsilon(x) \nsubseteq A$. (or $\exists y \in B_\varepsilon(x)$ s.t. $y \in A^c$)

- Other conditional statement

  Let $P(x)$ denote a statement with variable x. Let $Q(x)$ be defined similarly.

  Consider the statement: $\forall x \in D,\ P(x) \Rightarrow Q(x)$

  ↳ <u>Contrapositive</u>

  $\forall x \in D,\ \sim Q(x) \Rightarrow \sim P(x)$

  ↳ <u>Converse</u>

  $\forall x \in D,\ Q(x) \Rightarrow P(x)$

  ↳ <u>Inverse</u>

  $\forall x \in D,\ \sim P(x) \Rightarrow \sim Q(x)$

<u>Thm</u>: $(P(x) \Rightarrow Q(x)) \Leftrightarrow (\sim Q(x) \Rightarrow \sim P(x))$

Proof is by using truth table. (From math review camp(2020))

A	B	$A \to B$	$\sim B$	$\sim A$	$\sim B \to \sim A$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$\boxed{\text{E.g.}}$ If $x_n \to x \in \mathbb{R}$, then x_n is Cauchy.

Proved this directly yesterday.

Now let's try to prove it by contraposition.

WTS: If x_n if not Cauchy, then $x_n \not\to x$.

Start w/ the negation:

$\exists\, \varepsilon > 0$ s.t. $\forall\, N$, $\exists\, m, n > N$ s.t. $|x_m - x_n| \geq \varepsilon$.

$\varepsilon \leq |x_m - x_n| \leq |x_m - x| + |x_n - x|$

WLOG, assume $|x_n - x| \geq |x_m - x|$

$2|x_n - x| \geq \varepsilon$

$|x_n - x| \geq \frac{1}{2}\varepsilon$

Take $\varepsilon' = \frac{1}{2}\varepsilon > 0$. Then $\forall\, N$, $\exists\, n > N$ s.t. $|x_n - x| \geq \varepsilon'$.

$\Rightarrow x_n \not\to x$.

Limits

1 Prove $\lim_{n\to\infty} \frac{1}{1+n} = 0$

By defintion, $\forall\, \varepsilon > 0$, we want to find N s.t. $n > N \Rightarrow \left|\frac{1}{1+n} - 0\right| < \varepsilon$

For arbitrary ε, $\frac{1}{1+n} < \varepsilon \Rightarrow 1+n > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon} - 1$

So for any given ε, as long as we pick $n > \frac{1}{\varepsilon} - 1 \equiv N$, we have what we want.

2 Prove: If $a_n \to a$, $b_n \to b$. Then $a_n b_n \to ab$.

We again want to show that $\forall\, \varepsilon > 0$, $\exists\, N$ s.t. $|a_n b_n - ab| < \varepsilon$ 1

We know $\forall\, \varepsilon' > 0$, $\exists\, N_1$ s.t. $n > N_1 \Rightarrow |a_n - a| < \varepsilon'$ 2

N_2 $\quad n > N_2 \Rightarrow |b_n - b| < \varepsilon'$ 3

Take $N = \max\{N_1, N_2\}$

Relate 1 to 2 & 3 ?

$$|a_n b_n - ab| = |(a_n b_n - a_n b) + (a_n b - ab)| \leq |a_n b_n - a_n b| + |a_n b - ab|$$
$$= |a_n|\underbrace{|b_n - b|}_{<\varepsilon'} + |b|\underbrace{|a_n - a|}_{<\varepsilon'}$$

We're getting close! What else do we know about convergent seq? They're bounded.

$\exists M > 0$ s.t. $|a_n| < M$ $\forall n$. We can choose this M big enough so that $|b| < M$ as well.

Then we have $|a_n b_n - ab| \leq |a_n||b_n - b| + |b||a_n - a|$

$< M \cdot 2\varepsilon'$

we want this to be ε.
So set $\varepsilon' = \dfrac{\varepsilon}{2M}$

We now have $\forall \varepsilon > 0$, (take $\varepsilon' = \frac{\varepsilon}{2M}$)

$\exists N$ s.t. $n > N$ $(\Rightarrow |a_n - a| < \varepsilon', |b_n - b| < \varepsilon')$

$\Rightarrow |a_n b_n - ab| < (M \cdot 2\varepsilon' =)\ \varepsilon$

Additional question from Canvas Discussion

Module 1, Prop 27: Every seq. x_n has a monotonic subseq.

Proof:

For each n, define $S_n = \{x_n, x_{n+1}, \cdots\}$

Case 1: Suppose $\exists n \in \mathbb{N}$ s.t. S_n has no max.

Claim: Then $\forall k > n$, S_k doesn't have a max. (hw2)

We employ an "algorithm" to construct a nondecreasing seq.

Let $x_{n_0} = x_n$.

Choose x_{n_1} from $S_{(n_0+1)} = \{x_{n+1}, x_{n+2}, \cdots\}$ s.t. $x_{n_1} \geq x_{n_0}$.

We've guaranteed such x_{n_1} exists. Why? If no such x_{n_1} exists, then x_{n_0} is the largest element in $S_n = \{x_n\} \cup S_{n+1}$, but we assumed no max..

Choose x_{n_2} from $S_{(n_1+1)}$ s.t. $x_{n_2} \geq x_{n_1}$. Again such x_{n_2} exists; o.w. x_{n_1} is the largest element in S_{n_1}, where $n_1 > n$, violating the claim.

We can continue doing this for any x_{n_k} ...

E.g., We have the seq $\overset{x_0}{1}\ \overset{x_1}{\tfrac{1}{2}}\ \overset{x_2}{6}\ \overset{x_3}{0}\ \overset{x_4}{3}\ \overset{x_5}{7}\ \overset{x_6}{9}\ \overset{\cdots}{\cdots}$ and $S_0 = \{\overset{x_0}{1}, \overset{x_1}{\tfrac{1}{2}}, \overset{x_2}{6}, \overset{x_3}{0}, \overset{x_4}{3}, \overset{x_5}{7}, \overset{x_6}{9}, \overset{\cdots}{\cdots}\}$ has no max. Then in this case we start w/ $n=0$. $x_{n_0} = x_n = x_0 = 1$.

$S_{(n_0+1)} = S_1 = \{\overset{x_1}{\tfrac{1}{2}}, \overset{x_2}{6}, \overset{x_3}{0}, \overset{x_4}{3}, \overset{x_5}{7}, \overset{x_6}{9} \overset{\cdots}{\cdots}\}$. According to the algorithm, we pick $x_{n_1} \in S_1$ s.t. $x_{n_1} \geq x_{n_0} = 1$. We can pick $x_{n_1} = x_2 = 6$. Now $S_{(n_1+1)} = S_3 = \{\overset{x_3}{0}, \overset{x_4}{3}, \overset{x_5}{7}, \overset{x_6}{9} \overset{\cdots}{\cdots}\}$ and we can next pick $x_{n_2} = x_5 = 7 \geq x_{n_1}$, and $S_{(n_2+1)} = S_6 = \{\overset{x_6}{9}, \overset{\cdots}{\cdots}\}$

And then we keep doing this infinitely many times, we'll have a nondecreasing subsequence $x_{n_k} = \{\underset{x_{n_0}}{1}, \underset{x_{n_1}}{6}, \underset{x_{n_2}}{7}, \underset{\cdots}{\cdots}\}$

<u>Case 2</u>: $\forall n$, S_n has a max

We can construct a <u>nonincreasing</u> subseq from x_n recursively.

$n_0 = \min\{n \in \mathbb{N} \mid x_n = \max S_0\}$

Recursively pick indices $\cdots$

$n_{(k+1)}$

$= \min\{n \in \mathbb{N} \mid x_n = \max S_{(n_k+1)}\}$

you can have multiple elements w/ the same max value. Pick the one w/ the smallest index!

why not the largest index? It may not exist!

$\max\{n \in \mathbb{N} \mid x_n = \max\{2, 2, 2, \cdots\}\} = \infty$

Why is x_{n_k} nonincreasing? $S_{(n_k+1)} \subset S_{n_k} \Rightarrow \max S_{(n_k+1)} \leq \max S_{n_k}$

e.g., $x_n = \frac{1}{1+n}$, $S_0 = \{\underset{x_0}{1}, \underset{x_1}{\tfrac{1}{2}}, \underset{x_2}{\tfrac{1}{3}}, \cdots\}$, $x_0 = \max S_0$,

$0 = n_0 = \min\{n \in \mathbb{N} \mid x_n = \max S_0\}$