Yiqi Liu
6170 Section #7
10/22/21

Overview of materials for the second midterm:

Module 4 - Correspondences

- closed / convex / compact-valued; local boundedness; graph
- Continuity: uhc & lhc.
- singleton-valued
  ↳ + locally bounded give us
  lhc <=> association w/ a cont func <=> uhc. (HW#6)

- Berge's theorem of the maximum

  $f: X \times \Theta \to \mathbb{R}$ cont, X closed, $\phi: \Theta \rightrightarrows X$ is cont., non-empty-valued, and locally bounded.

  Then we have $\sigma(\theta) \equiv \arg\max_{z \in \phi(\theta)} f(z,\theta)$ is nonempty $\forall \theta$, uhc and locally bounded. $f^*(\theta) \equiv \sup\{f(z,\theta) : z \in \phi(\theta)\}$ is cont.

  - If in addition, f is quasi-concave in X and ϕ is convex-valued, then σ is convex-valued; if f is strictly quasi-concave, then σ is singleton-valued (hence associated w/ a cont. func)

Module 5 - Differentiation

- Single-variable:
  $f: X \to \mathbb{R}$, $X \subseteq \mathbb{R}$ & open, is differentiable at $x \in X$ if $\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ exists.
  - Differentiability => continuity
  - MVT: $f: [a,b] \to \mathbb{R}$ cont & diff. on (a,b). Then $\exists\, c \in (a,b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b-a}$.
  - Taylor polynomials & Taylor's thm

- Linear algebra & differentiation in higher dimensions

Module 4, Ex 2 : Prove if $\Phi : X \rightrightarrows Y$ is uhc, then $\Phi(x)$ is closed $\forall\, x \in X$.

Some of you tried "if $\Phi(x)$ not closed for some x, then $\forall\, x_n \to x$ and some $y_n \to y$, $y \notin \Phi(x)$, violating uhc." Note that here $y_n \in \Phi(x_n)\ \forall\, n$, not $\Phi(x)$. But for the contradiction to be valid, you need a convergent seq $y_n \in \Phi(x)$ s.t. $y_n \to y \notin \Phi(x)$.

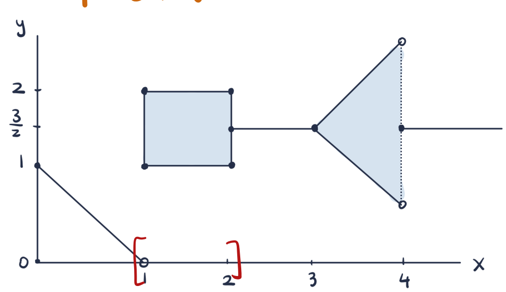
Since Φ is uhc, $G \equiv \{(x,y) \in X \times Y : y \in \Phi(x)\}$ is closed.
For any $x \in X$, $\{x\} \times Y$ is closed. Why? Y is closed by definition, and $\forall$ convergent seq $(x_n, y_n) \in \{x\} \times Y$, $x_n = x\ \forall\, n$ (const. seq) so clearly the limit pt $(x,y) \in \{x\} \times Y$.

Now, consider $(\{x\} \times Y) \cap G = \{(x,y) \in \{x\} \times Y : y \in \Phi(x)\}$

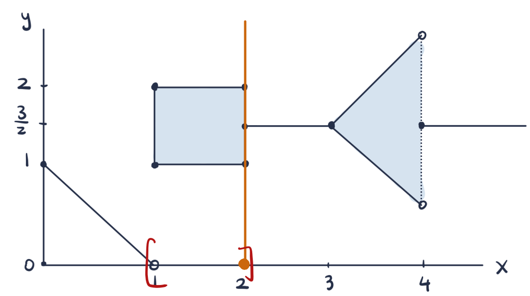
$= \{x\} \times \Phi(x)$ is closed

$\Rightarrow \Phi(x)$ is closed.

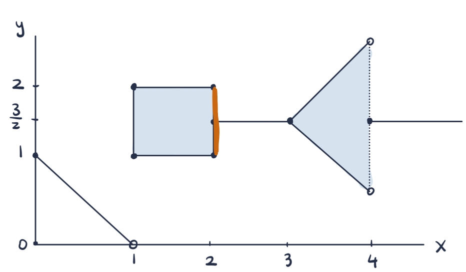
$\Phi : [1,2] \to \mathbb{R}_+$ is uhc.



$G = \{(x,y) \in X \times Y : y \in \Phi(x)\}$



Take $x = 2$. $\{2\} \times Y$

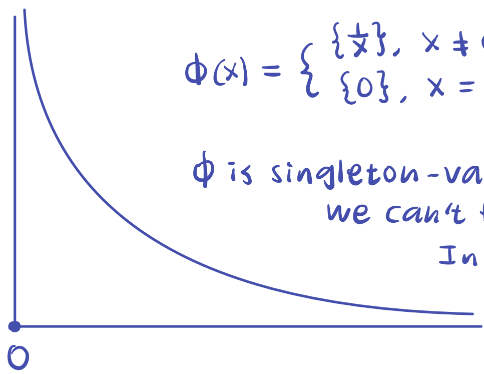


$(\{2\} \times Y) \cap G = \{2\} \times [1,2]$

Module 4, Ex. 5: A singleton-valued correspondence $\phi: X \rightrightarrows Y$ that's locally bounded is uhc iff lhc iff associated w/ a cont. func.

Note that Ex 4 tells us: If ϕ is singleton-valued,
then lhc $\Leftrightarrow$ associated w/ a cont. func.
$\Downarrow$
uhc

Why uhc $\not\Rightarrow$ associ. w/ cont. f here?



$$\phi(x) = \begin{cases} \{\frac{1}{x}\}, & x \neq 0 \\ \{0\}, & x = 0 \end{cases} \quad \text{associated w/ } f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

ϕ is singleton-valued and uhc. But note that $\forall x_n \to 0$, we can't find $y_n \in \phi(x_n)$ s.t. y_n converges. In fact, the only choice of y_n is $\frac{1}{x_n}$ and it diverges. So the def of uhc at $x=0$ holds vacuously true here.

But we can solve this issue by imposing local boundedness of ϕ.

Lemma: $y_n \to y$ iff $\forall$ subseq y_{n_k}, $\exists$ convergent sub-subseq $y_{n_{k_l}} \to y$.

Take any $x_n \to x \in X$. WTS $f(x_n) \to f(x)$.

Since ϕ is locally bounded, $\exists$ bounded set B and $\varepsilon > 0$ s.t. $\|x_n - x\| < \varepsilon \Rightarrow \phi(x_n) \subset B \Rightarrow f(x_n) \subset B$ (since $\phi(x_n)$ contains $f(x_n)$ only). For this ε, by def of seq. convergence, $\exists N \in \mathbb{N}$ s.t. $n > N$ $\Rightarrow \|x_n - x\| < \varepsilon \Rightarrow f(x_n) \subset B$. This implies all but finitely many elements of $f(x_n)$ are in B $\Rightarrow f(x_n)$ is a bounded seq.

Any subseq $f(x_{n_k})$ is also bounded. By B-W, $\exists$ convergent sub-subseq $f(x_{n_{k_l}}) \to y_l \in Y$. Notice y_l can be different for different l. But ϕ uhc. implies $y_l \in \phi(x) \Rightarrow y_l = f(x)$ $\forall l$.

Now we can apply the lemma and conclude $f(x_n) \to f(x)$.

• Differentiation in higher dimensions

$\boxed{\text{DEF}}$ If E is an open subset of $\mathbb{R}^m$ and $f: E \to \mathbb{R}^n$. For $x \in E$, if $\exists$ transformation A from $\mathbb{R}^m$ to $\mathbb{R}^n$ s.t.

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - A \cdot h \|}{\| h \|} = 0$$

Then f is differentiable at x at A is the derivative of f at x.

E.g., consider $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x + y^2, x^3 + 5y)^T$

At $(1,1)^T$, take $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. Show $A = f'(1,1)$.

WTS $$\lim_{h_1, h_2 \to 0} \frac{\left\| f(1+h_1, 1+h_2) - f(1,1) - \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\|}{\sqrt{h_1^2 + h_2^2}} = 0$$

$$\lim_{h_1, h_2 \to 0} \frac{\left\| f(1+h_1, 1+h_2) - f(1,1) - \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right\|}{\sqrt{h_1^2 + h_2^2}}$$

$$= \lim_{h_1, h_2 \to 0} \frac{\left\| (1+h_1 + (1+h_2)^2, (1+h_1)^3 + 5(1+h_2))^T - (2,6)^T - (h_1 + 2h_2, 3h_1 + 5h_2)^T \right\|}{\sqrt{h_1^2 + h_2^2}}$$

$$= \lim_{h_1, h_2 \to 0} \frac{\| (h_2^2, 3h_1^2 + h_1^3)^T \|}{\sqrt{h_1^2 + h_2^2}}$$

$$= \lim_{h_1, h_2 \to 0} \frac{\sqrt{h_2^4 + (3h_1^2 + h_1^3)^2}}{\sqrt{h_1^2 + h_2^2}}$$

$$= 0$$

$\boxed{\text{Taylor's Thm}}$ Suppose $f \in C^k[a,b]$ and $f^{(k+1)}$ exists on (a,b).

Then for any $c, d \in (a,b)$, $\exists\, p$ between c & d s.t.

$$f(c) = \underbrace{\sum_{n=0}^{k} \frac{f^{(n)}(d)}{n!}(c-d)^n}_{T_k(c)} + \frac{f^{(k+1)}(p)}{(k+1)!}(c-d)^{k+1}$$

Proof: $T_k(x) \equiv \sum_{n=0}^{k} \frac{f^{(n)}(d)}{n!}(x-d)^n$

$$M \equiv \frac{f(c) - T_k(c)}{(c-d)^{k+1}}$$

By construction, $f(c) = T_k(c) + M(c-d)^{k+1} = T_k(c) + f(c) - T_k(c)$

So we WTS: $M = \frac{f^{(k+1)}(p)}{(k+1)!}$

Define $g(x) = f(x) - T_k(x) - M(x-d)^{k+1}$.

Note ① $T_k^{(n)}(d) = \frac{d^n}{dx^n}\left[f(d) + f'(d)(x-d) + \frac{1}{2}f''(d)(x-d)^2 + \cdots \right.$
$\left. \cdots + \frac{1}{k!}f^{(k)}(d)(x-d)^k \right] \Big|_{x=d}$

$n=0$: $f(d) + f'(d)(x-d) + \frac{1}{2}f''(d)(x-d)^2 + \cdots \Big|_{x=d} = f(d)$

$n=1$: $f'(d) + f''(d)(x-d) + \cdots + \frac{1}{(k-1)!}f^{(k)}(d)(x-d)^{k-1} \Big|_{x=d} = f'(d)$

$n=2$: $f''(d) + \cdots + \frac{1}{(k-2)!}f^{(k)}(d)(x-d)^{k-2} \Big|_{x=d} = f''(d)$

Keep doing this you'll find $T_k^{(n)}(x)\big|_{x=d} = T_k^{(n)}(d) = f^{(n)}(d)$

for $n = 0, 1, \cdots, k$.

$$\Rightarrow g^{(n)}(d) = \underbrace{f^{(n)}(x) - T_k^{(n)}(x)}_{=0} - M \underbrace{\frac{(k+1)!}{(k+1-n)!}(x-d)^{(k+1-n)}}_{=0 \text{ at } x=d} \Bigg|_{x=d}$$

(I double checked the algebra and it should be correct (:

$g^{(n)}(d) = 0$ for $n = 0, \cdots, k.$

Also, $g(c) = 0$ by construction.

MVT: $f : [a,b] \to \mathbb{R}$ cont & diff. on (a,b). Then $\exists\ c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Since $g \in C^k[a,b]$, by MVT, $\exists\ P_1$ between c & d s.t.

$$g'(P_1) = \frac{g(c)-g(d)}{c-d} = 0 \quad \text{since } g(c) = 0,\ g(d) = g^{(0)}(d) = 0$$

$\exists\ P_2$ between P_1 & d s.t.

$$g''(P_2) = \frac{g'(P_1)-g'(d)}{P_1 - d} = 0 \quad \text{since } g'(P_1) = 0 \ \&\ g'(d) = 0$$

$$\vdots$$

$$g^{(k+1)}(P_{k+1}) = \frac{g^k(P_k) - g^k(d)}{P_k - d} = 0$$

$$\text{But } g^{(k+1)}(x) = f^{(k+1)}(x) - \underbrace{T_k^{(k+1)}(x)}_{=0} - M(k+1)!$$

$=0$ since $T_k^{(k)}(x) = f^{(k)}(d)$, a constant. and if we take $T_k^{(k+1)}(x)$, It'll be 0.

$$\Rightarrow g^{(k+1)}(P_{k+1}) = 0 = f^{(k+1)}(P_{k+1}) - M(k+1)!$$

$$\Rightarrow M = \frac{f^{(k+1)}(P_{k+1})}{(k+1)!}.$$

$$\Rightarrow g(c) = 0 = f(c) - T_k(c) - \frac{f^{(k+1)}(P_{k+1})}{(k+1)!}(c-d)^{k+1}$$