I borrowed materials from http://www.math.toronto.edu/~jmracek/Tutorial%2010.pdf
and Tapan Mitra's notes on mathematical economics.

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Implicit Function Thm (Module 6, Prop. 22, but with "x" & "p" switched; i.e., instead of expressing x as an implicit function of P, I'm expressing P (which I denote as "y") as a function of x)

Let $F : \overset{\text{open}}{U} \subset \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ be continuously differentiable.
Consider a pt $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^k$ s.t.

① $F(x_0, y_0) = 0$.

② $D_y F(x,y)\big|_{(x_0,y_0)}$ is invertible (i.e., $\det\left[D_y f(x_0, y_0)\right] \neq 0$).

Then $\exists$ a nbh $B_\varepsilon(x_0)$ around x_0 and a C^1 function
$h : B_\varepsilon(x_0) \subset \mathbb{R}^n \to \mathbb{R}^k$ s.t. $\forall\, x \in B_\varepsilon(x_0)$

① $F(x, h(x)) = 0$

② $\underbrace{Dh(x)}_{k \times n} = -\left[\underbrace{D_y F(x,y)}_{k \times k}\right]^{-1} \left[\underbrace{D_x F(x,y)}_{k \times n}\right]$

IFT gives sufficient cond. for which we can <u>locally</u> express the surface $F(x,y) = 0$ by $h(x) = y$ for some C^1 func h.

$\boxed{\text{Ex. 1}}$ Suppose $F(x,y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 and $F(0,0) = 0$.
What conditions on the partials of F, F_1 & F_2, will guarantee that $F(F(x,y), y) = 0$ has an implicit function $h(x) = y$ near $(0,0)$?

Define $z := F(x,y) \in \mathbb{R} \implies F(z,y)$ is C^1 and $F(z=0, y=0) = 0$.

By IFT, we need $D_y F(z,y)\big|_{(0,0)} \neq 0$

$$\implies D_y F(\underbrace{F(x,y)}_{z}, y)\big|_{(0,0)}$$

$$= \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial y}\bigg|_{(0,0)} \quad \text{by chain rule}$$

$$= \underbrace{\frac{\partial F}{\partial x}}_{F_1} \underbrace{\frac{\partial F}{\partial y}}_{F_2} + \underbrace{\frac{\partial F}{\partial y}}_{F_2} \Big|_{(0,0)}$$

$$= F_1(F_2 + 1) \Big|_{(0,0)}$$

$$\neq 0$$

$$\Rightarrow F_1 \neq 0 \text{ and } F_2 \neq -1$$

$\boxed{\text{Ex. 2}}$ Suppose a firm uses 2 inputs x_1 & x_2 w/ input costs w_1 & w_2 to produce a product that's sold at unit price $p > 0$.

The profit function is $f(x_1, x_2, w_1, w_2) = p x_1^a x_2^b - w_1 x_1 - w_2 x_2$ where $a+b<1$, $a,b>0$.

The FOC of firm's PMP:

$$f_1 \equiv \frac{\partial f}{\partial x_1} = pa x_1^{a-1} x_2^b - w_1 = 0$$

$$f_2 \equiv \frac{\partial f}{\partial x_2} = pb x_1^a x_2^{b-1} - w_2 = 0$$

How does optimal x_1^* change w.r.t. w_1 & w_2? (i.e., find $\frac{\partial x_1^*}{\partial w_1}$ & $\frac{\partial x_1^*}{\partial w_2}$)
You can assume the firm always uses positive amount of x_1 & x_2.
and FOC fully charaterizes the optimal solution.

The idea is to treat the FOC as our F func and express (x_1, x_2) as an implicit function of the "parameters," (w_1, w_2) in some nbh of (x_1^*, x_2^*, w_1, w_2).

Need to check ① $F(x_1, x_2, w_1, w_2) = [f_1 \ \ f_2]$ is C^1 (polynomials) ✓

② $F(x_1^*, x_2^*, w_1, w_2) = 0$ ✓

③ $D_{(x_1,x_2)} F(x_1^*, x_2^*, w_1, w_2)$ invertible

$$D_{(x_1,x_2)} F(x_1^*, x_2^*, w_1, w_2) = \begin{bmatrix} pa(a-1)x_1^{a-2}x_2^b & pab x_1^{a-1} x_2^{b-1} \\ pab x_1^{a-1} x_2^{b-1} & pb(b-1) x_1^a x_2^{b-2} \end{bmatrix} \Bigg|_{x_1^*, x_2^*} \equiv J$$

$$|J| = \underbrace{p^2 ab x_1^{*\,2(a-1)} x_2^{*\,2(b-1)}}_{>0} \underbrace{[1-a-b]}_{>0} > 0 \quad ✓$$

By IFT, at (x_1^*, x_2^*, w_1, w_2), there's locally a function $g: \mathbb{R}^2_{++} \to \mathbb{R}^2_{++}$
$g(w_1, w_2) = (x_1\ x_2)$ s.t. $F(g(w_1,w_2), w_1, w_2) = 0$ (which means $(x_1\ x_2)$ is optimal)

$$Dg(w_1,w_2) = \begin{bmatrix} \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\ \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2} \end{bmatrix} = -[D_x F]^{-1} D_w F$$

$$= -\frac{1}{|J|} \begin{bmatrix} pb(b-1)x_1^a x_2^{b-2} & -pabx_1^{a-1}x_2^{b-1} \\ -pabx_1^{a-1}x_2^{b-1} & pa(a-1)x_1^{a-2}x_2^b \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \frac{\partial x_1}{\partial w_1} = \frac{pb(b-1)x_1^a x_2^{b-1}}{|J|} < 0$$

$$\frac{\partial x_1}{\partial w_2} = \frac{-pabx_1^{a-1}x_2^{b-1}}{|J|} < 0 \quad (x_1 \ \&\ x_2 \text{ are complements})$$

Envelope Thm (Module 6, Prop. 25)

Let $f: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ and $g_i: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ $\forall\ i \in \{1, \dots, m\}$. f & g_i are C^1. Let $x \in \mathbb{R}^n$ (choice variable) & $p \in \mathbb{R}^k$ (parameter), $y_1, \dots, y_m \in \mathbb{R}$.

Consider: $V(p) = \max_x f(x,p)$ s.t. $g_i(x,p) \leq y_i\ \forall\ i$. ☆

Suppose for some open $A \times B \subset \mathbb{R}^n \times \mathbb{R}^k$, the solution to ☆ (x^*, p) is the graph of some C^1 function $h: B \to A$. (So $(x^*, p) = (h(p), p)$)

Let $I(h(p))$ denote the set of binding constraints at the solution $x^* = h(p)$, and suppose KKT cond. hold at $x^* = h(p)$ $\forall\ p \in B$ w/ λ_j denoting the associated multipliers for each $j \in I(h(p))$.

Then $$\frac{\partial V(p)}{\partial p_i} = \frac{\partial f(h(p),p)}{\partial p_i} - \sum_{j \in I(h(p))} \lambda_j \frac{\partial g_j(h(p),p)}{\partial p_i}$$

The Envelope thm tells us how the optimal value of an objective function changes (e.g., indirect utility) when we change the parameters (e.g., price & wealth) by simply differentiating the associated Lagrangian w.r.t. the param and evaluate at the solution.

Q: why do we focus on $I(h(P))$ only? CS!

$\boxed{\text{Ex. 3}}$ Back to Ex. 2, we have $F: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, where input $x^* = (x_1^*, x_2^*)$ is the choice var and input price $P = (w_1, w_2)$ is the param.

We've shown IFT applies, so $\exists$ C^1 func h s.t. $x^* = h(P)$ for some open nbh $A \times B \subset \mathbb{R}^2 \times \mathbb{R}^2$.

Since there's no constraint, the rest of the assumptions in the envelope thm are vacuously satisfied.

Hence envelope thm applies,

$$\text{and } \frac{\partial V(w_1, w_2)}{\partial w_1} = \frac{\partial f(\overbrace{h_1(w_1,w_2)}^{x_1^*}, \overbrace{h_2(w_1,w_2)}^{x_2^*}, w_1, w_2)}{\partial w_1}$$

$$= \underbrace{f_1}_{=0} \frac{\partial h_1}{\partial w_1} + \underbrace{f_2}_{=0} \frac{\partial h_2}{\partial w_1} + f_3$$

by FOC when evaluated at x^*

$$= f_3$$

$$= -x_1^*$$

(you can solve x_1^* in terms of w_1 & w_2 from the FOC, which involves lots of algebra ...)

So when price of input 1 increases by \$1, firm's profit decreases by $\$x_1^*$.

Note: Ex. 3 is an example of Remark 26 (where the optimization prob. is not constrained)