Advanced Topics in Astrodynamics and Trajectory Design

Continuous Thrust Transfers

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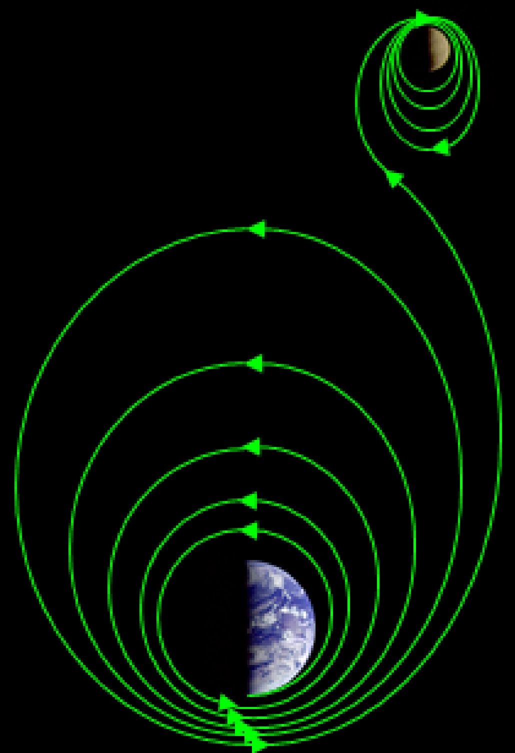
MSc in Astronautics and Space Engineering



www.cranfield.ac.uk

Bibliography

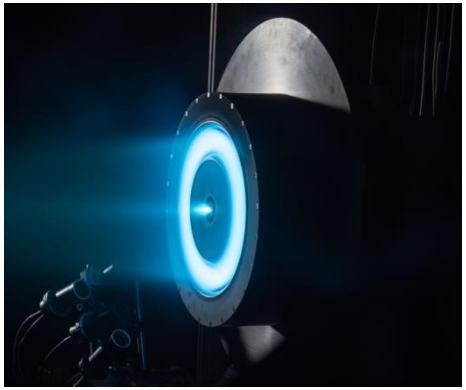
Some good books ordered by increasing depth and breath :

- *Orbital Mechanics,* V.A Chobotov, AIAA Education Series
- *Analytical Mechanics of Space Systems*, H.Shcaub, J.L.Junkins, AIAA Education Series
- *An Introduction to the Mathematics and Methods of Astrodynamics*, R.H.Battin, AIAA Educationan Series
- *Fundamentals of Astrodynamics and Applications*, D.A.Vallado, Space Technology Library
- **Biesbroek, Robin. *Lunar and Interplanetary Trajectories*. Springer International Publishing, 2016.**

Continuous Thrust Transfers

- Non-Impulsive Δv
- Problem Formulation
- Gravity Losses
- Very low thrust hence just a perturbation
- Method of variation of parameters

Non-Impulsive Manoeuvres



- All space manoeuvres require actually finite burns.
  - ✓ Specially! If we model a electric propulsion system.

- Compute how long would the burn be to achieve escape velocity from a circular 250 km altitude parking orbit if the spacecraft was to use

- Isp=320 s
- Thrust 425 N

apogee motor such as the one in the picture. Assume a spacecraft with a wet mass of 2000 kg.

Socrative: Connect into Room: **GPQK5UNSK**

Compute how long would the burn be to achieve escape velocity from a circular 250 km altitude parking orbit if the spacecraft was to use apogee motor such as the one in the picture. Assume a spacecraft with a wet mass of 2000 kg.

Constants:

R_E=6378 km – Radius of the Earth

μ_E= 3.986×10^5 km^3/s^2 - The gravitational parameter of the Earth

A) 16 sec

B) 4.2 h

C) 2.6 h

D) <1 sec

- Isp=320 s
- Thrust 425 N

Low Thrust Engines

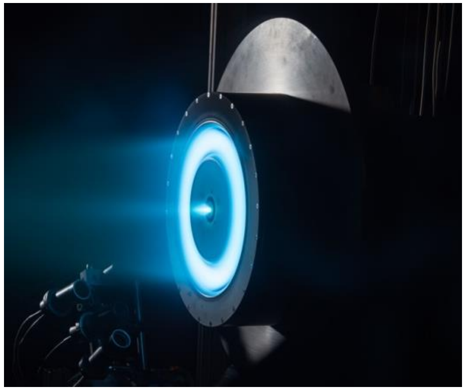
- The example above was for a typical “high” thrust engine.
- Low thrust engines have thrust typically < 1N. However, they are the most efficient engines available.

Table extracted from Biesbroek, R. (2016). Lunar and interplanetary trajectories.

Table 1.3 Overview of engine types and typical values for the specific impulse

Engine type	Typical specific impulse Isp [s]	Applications
Cold-gas	50	Spacecraft attitude control
Mono-propellant	220	Attitude control or spacecraft main engine for all orbit transfers
Solid	250	Launchers and rocket stages
Bi-propellant	320	Larger attitude control or spacecraft main engine for all orbit transfers. More effective than mono-propellant, but more complex too
Cryogenic	450	Launchers (cryogenic engines can only be used until a few days after launch)
Hall-effect (HET)	1650	Deep-space maneuvers/spiraling out to the Moon
Ion	3000	Deep-space maneuvers

Non-Impulsive Manoeuvres



- All space manoeuvres require actually finite burns.
  - ✓ Specially! If we model a electric propulsion system.

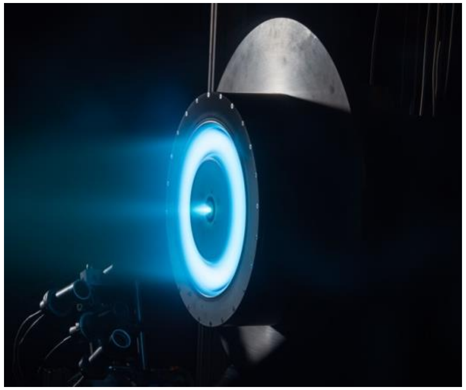
Equation of Motion

$$\ddot{\mathbf{r}} = -\frac{\mu_E}{r^3}\mathbf{r} + \mathbf{a}$$

→ No closed form solution

- Low thrust transfers generally require some very complex analysis (e.g., optimal control theories).
- However, some approximations can be done that provide a working solution (e.g. sub-optimal control laws, shape-based methods).

Electric Propulsion Fundamentals

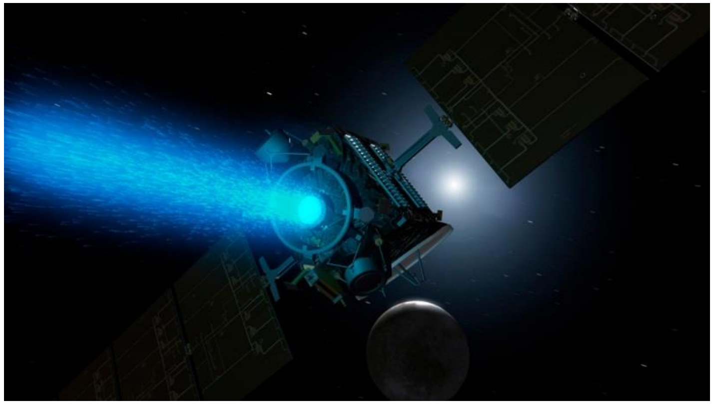


- The propellant is converted into charged particles or plasma, and these are accelerated using electrostatic or electromagnetic forces.
- The propellant is ejected at exhaust velocities about 10 times greater than those of conventional chemical rockets.
- However, the propellant mass-flow is extremely small, and thus the thrust is also very small.

$$T = \dot{m} v_e$$

- Consequently, the Electric Propulsion system must operate nearly continuously so that the low acceleration produces noticeable changes in the trajectory.

Electric Propulsion Fundamentals



$$T = \dot{m} v_e$$

$$v_e = g_0 \cdot I_{sp}$$

$$P_{out} = \frac{1}{2} \dot{m} v_e^2$$

Thruster efficiency $\eta = \frac{P_{out}}{P_{in}}$

$$\eta = \frac{P_{out}}{P_{in}} = \frac{\frac{1}{2} \dot{m} v_e^2}{P_{in}} = \frac{T v_e}{2 P_{in}}$$

$$T = \frac{2 \eta P_{in}}{g_0 I_{sp}}$$

Low Thrust Transfer: Problem Formulation

- Let $\mathbf{X_0}$ be an initial state vector such as (x_0, y_0, z_0, v_{x0}, v_{y0},v_{z0}), and ***u***(t) the force vector provided by our propulsion system.
- ***u***(t) is generally referred as *control inputs*.
- Our spacecraft is subject to 2-body motion, thus:

Equation of Motion

$$\ddot{\mathbf{r}} = -\frac{\mu_E}{r^3}\mathbf{r} + \mathbf{a}$$

$$\mathbf{a} = \frac{\mathbf{u}}{m}$$

Low Thrust Transfer: Problem Formulation

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- Our spacecraft is subject to 2-body motion, thus:

$$\ddot{x} = -\frac{\mu_E}{r^3} x + a_x$$

$$\ddot{y} = -\frac{\mu_E}{r^3} y + a_y$$

$$\ddot{z} = -\frac{\mu_E}{r^3} z + a_z$$

Low Thrust Transfer: Problem Formulation

- Let $\mathbf{X_0}$ be an initial state vector such as (x_0, y_0, z_0, v_{x0}, v_{y0},v_{z0}), and $\boldsymbol{u}$(t) the force vector provided by our propulsion system.
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$$\frac{dz}{dt} = v_z$$

$$\frac{dv_x}{dt} = -\frac{\mu_E}{r^3} x + a_x$$

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$$\frac{dv_z}{dt} = -\frac{\mu_E}{r^3} z + a_z$$

$$\mathbf{a} = \frac{\mathbf{u}}{m} \qquad \dot{m} = \frac{T}{v_e}$$

$$\frac{dm}{dt} = \frac{T}{v_e} \Longrightarrow (x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0}, m_0)$$

Low Thrust Transfer: Problem Formulation

- Let $\mathbf{X_0}$ be an initial state vector such as (x_0, y_0, z_0, v_{x0}, v_{y0},v_{z0}), and $\boldsymbol{u}$(t) the force vector provided by our propulsion system.
- $\boldsymbol{u}$(t) is generally referred as *control inputs*.
- Our spacecraft is subject to 2-body motion, thus:

$$\boxed{\ddot{\mathbf{r}} = -\frac{\mu_E}{r^3}\mathbf{r} + \frac{\mathbf{u}(t)}{m}}$$

- Solving the optimal control problem requires us to find the *control history* $\boldsymbol{u}$(t) generally subject to a series of *constraints* and *boundary conditions.*
- This is out of the scope of this course (but some notions are provided).

Trajectory Optimization

- The standard form of an optimization problem is:

$$\begin{aligned} \underset{x}{\text{minimize}} \;& f(x) \\ \text{subject to} \;& g_i(x) \le 0, \; i = 1, \ldots, m \\ & h_j(x) = 0, \; j = 1, \ldots, p \end{aligned}$$

✓The **objective function** $f(x) : \mathbb{R}^n \to \mathbb{R}$

✓Subject to **inequality** and **equality constraints.**

- **Nonlinear programming** (NLP) is generally referred to the process of solving an **optimization problem**.

Optimal Control Problem for a continuous thrust transfer

- NLP provides a finite set of variables ***x*** that minimize a given objective function.
- Instead, an optimal control problem involves finding the continuous function ***u***(t) that minimizes the objective function.
- Since solving an infinite set of variables becomes rather cumbersome, the general approach is to **transcribe** or convert the *infinite-dimensional optimal control* problem into a *finite-dimensional NLP*.

Optimal Control Problem for a continuous thrust transfer

- NLP provides a finite set of variables ***x*** that minimize an objective func.
- An optimal control problem involves finding the continuous function ***u***(t).
- **Transcribe** the *infinite-dimensional optimal control* problem into a *finite-dimensional NLP*.

Gradient-based methods

$$f = c(x) \qquad c(x^*) = 0$$

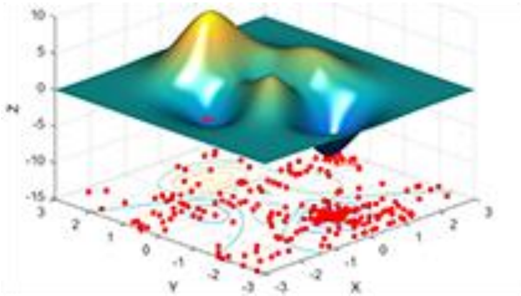
$$c(\bar{x}) = c(x) + \frac{dc}{dx}(\bar{x} - x)$$

$$c(\bar{x}) = c(x^*) = 0$$

$$0 = c(x) + \frac{dc}{dx}(\bar{x} - x)$$

$$\bar{x} = x - \left[\frac{dc}{dx}\right]^{-1} c(x)$$

Heuristic methods



A **heuristic** technique is any **approach** to problem solving that employs a practical method not guaranteed to be optimal (e.g. genetic algorithms, Particle Swarm Optimization, etc)

Low Thrust Transfer: Sub-optimal Control Laws.

Lagrange's Planetary Equations

Gaussian form

$$\frac{da}{dt}=\frac{2a^2v}{\mu}a_t$$

$$\frac{de}{dt}=\frac{1}{v}\left[2\left(e+\cos f\right)a_t-\frac{r}{a}\sin fa_n\right]$$

$$\frac{di}{dt}=\frac{r\cos\theta}{h}a_h$$

$$\frac{d\Omega}{dt}=\frac{r\sin\theta}{h\sin i}a_h$$

$$\frac{d\omega}{dt}=\frac{1}{ev}\left[2\sin fa_t+\left(2e+\frac{r}{a}\cos f\right)a_n\right]-\frac{r\sin\theta\cos i}{h\sin i}a_h$$

$$\frac{dM}{dt}=n-\frac{b}{eav}\left[2\left(1+\frac{e^2r}{p}\right)\sin fa_t+\frac{r}{a}\cos fa_n\right]$$

$f \equiv$ true anomaly

$\vartheta \equiv$ the argument of latitude (i.e., $\omega+f$)

The acceleration vector (a_t,a_n,a_h) is expressed in a Cartesian rotating frame (i_t,i_n,i_h), where i_t is the direction along the orbit velocity vector, i_h is parallel to the orbital momentum vector, thus the out-of-plane direction, and i_n completes the right-hand coordinates system

Low Thrust Transfer: Sub-optimal Control Laws.

What then would be the *best* control law to achieve a escape trajectory?

$$\frac{da}{dt}=\frac{2a^2v}{\mu}a_t$$

$$\frac{de}{dt}=\frac{1}{v}\left[2\left(e+\cos f\right)a_t-\frac{r}{a}\sin fa_n\right]$$

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Low Thrust Transfer: Simplified Control Scenarios.

- Let $\mathbf{X_0}$ be an initial state vector such as (x_0, y_0, z_0, v_{x0}, v_{y0},v_{z0} , m_0).

$$\frac{dx}{dt} = v_x$$

$$\frac{dy}{dt} = v_y$$

$$\frac{dz}{dt} = v_z$$

$$\frac{dv_x}{dt} = -\frac{\mu_E}{r^3} x + \frac{T_{\max}}{m} \frac{v_x}{|v|}$$

$$\frac{dv_y}{dt} = -\frac{\mu_E}{r^3} y + \frac{T_{\max}}{m} \frac{v_y}{|v|}$$

$$\frac{dv_z}{dt} = -\frac{\mu_E}{r^3} z + \frac{T_{\max}}{m} \frac{v_z}{|v|}$$

$$\frac{dm}{dt} = \frac{T}{v_e}$$

- Propagate, using Matlab, the escape trajectory described in the previous exercise using a constant thrust in tangential direction.

Low Thrust Trajectory - Exercise 2

Propagate, using Matlab ODE solvers, the escape trajectory described in the previous exercise using a constant thrust in tangential direction.

Initial Orbit: Circular 250 km altitude parking orbit.

Apogee motor specs: Thrust 425 N, Isp 320s.

Spacecraft wet mass : 2000 kg.

Low Thrust Transfer: Simplified Control Scenarios.

$$\frac{dx}{dt} = v_x$$

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$$\frac{dv_x}{dt} = -\frac{\mu_E}{r^3}x + \frac{T_{\max}}{m}\frac{v_x}{|v|}$$

$$\frac{dv_y}{dt} = -\frac{\mu_E}{r^3}y + \frac{T_{\max}}{m}\frac{v_y}{|v|}$$

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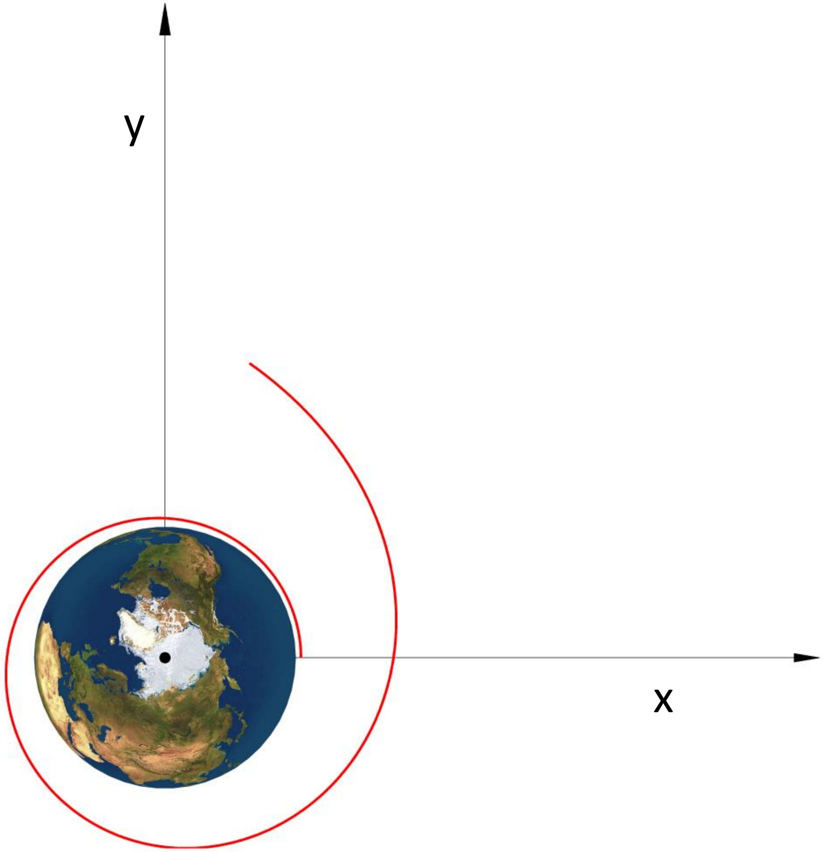
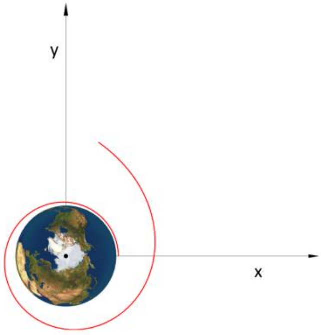


Low Thrust Transfer: Simplified Control Scenarios.



$$\frac{dx}{dt}=v_x$$

$$\frac{dy}{dt}=v_y$$

$$\frac{dz}{dt}=v_z$$

$$\frac{dv_x}{dt}=-\frac{\mu_E}{r^3}x+\frac{T_{\max}}{m}\frac{v_x}{|v|}$$

$$\frac{dv_y}{dt}=-\frac{\mu_E}{r^3}y+\frac{T_{\max}}{m}\frac{v_y}{|v|}$$

$$\frac{dv_z}{dt}=-\frac{\mu_E}{r^3}z+\frac{T_{\max}}{m}\frac{v_z}{|v|}$$

$$\frac{dm}{dt}=\frac{T}{v_e}$$

- Is the final state vector in a escape trajectory?

```
%----------------------------------------------------------
% Is the final state vector in a escape trajectory?
%[T,X,TE,YE,IE]=ode45(F,[0,tf],SV0,options);
SV_final=X(end,:); % final state vector

% Final distance to Earth
rf=sqrt(SV_final(1)^2+SV_final(2)^2+SV_final(3)^2);
% Escape velocity at the final distance
vesc=sqrt(2*muEarth/rf);
% Our final velocity
vf=sqrt(SV_final(4)^2+SV_final(5)^2+SV_final(6)^2);
% is vf>vesc
vf>vesc
% no
%----------------------------------------------------------
```

Low Thrust Trajectory - Exercise 2B

Ensure the spacecraft thrust until reaching escape condition.

Initial Orbit: Circular 250 km altitude parking orbit.

Apogee motor specs: Thrust 425 N, Isp 320s.

Spacecraft wet mass : 2000 kg.

Notes: Use MATLAB's ODE event feature to stop the propagation when it reaches escape velocity. For information on how to do this go to:

https://uk.mathworks.com/help/matlab/math/ode-event-location.html

- This is the suggested event function:

```
function [value,isterminal,direction]=event_Escape(t,x)
%---------------------------------------------------------------------------
muEarth = getAstroConstants('Earth','mu');
%---------------------------------------------------------------------------
% Current velocity
v=sqrt(x(4)^2+x(5)^2+x(6)^2);
% Escape velocity at altitude
vesc=sqrt(2*muEarth/sqrt(x(1)^2+x(2)^2+x(3)^2));

value = v-vesc; % when s/c reaches scape velocity value will be zero
isterminal = 1;
direction = 0;
end
```

Low Thrust Transfer: Simplified Control Scenarios.

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$$\frac{dy}{dt} = v_y$$

$$\frac{dz}{dt} = v_z$$

$$\frac{dv_x}{dt} = -\frac{\mu_E}{r^3}x + \frac{T_{\max}}{m}\frac{v_x}{|v|}$$

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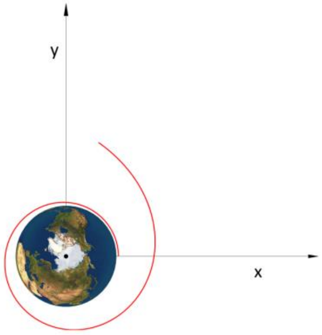
$$\frac{dv_z}{dt} = -\frac{\mu_E}{r^3}z + \frac{T_{\max}}{m}\frac{v_z}{|v|}$$

$$\frac{dm}{dt} = \frac{T}{v_e}$$



- How much is the gravity loss generated if we plan to thrust until escape condition with out apogee motor?

$$\Delta v_{GravityLoss} \approx 1.5 \text{ km/s}$$

Earth-Moon Smart-1 Trajectory

Smart-1 spacecraft (367 kg) is launched into a 250 km circular altitude orbit. Smart-1 propulsion system uses a Hall-effect Plasma Thruster with Isp of 3,000s and thruster efficiency of η=0.7. The input power is of 1190 W. What would it be the total DV to reach Lunar altitude? And what the propellant mass used?