Advanced Topics in Astrodynamics and Trajectory Design

Circular Restricted Three Body Problem

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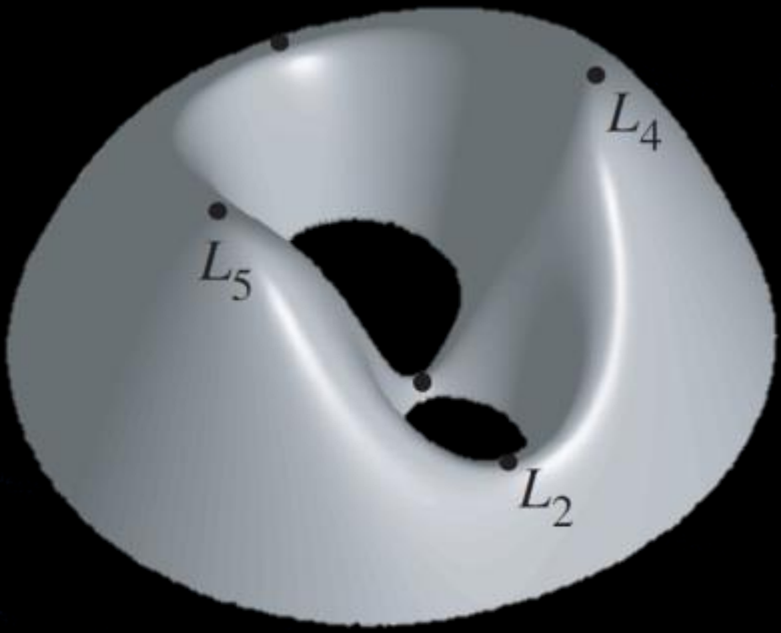
MSc in Astronautics and Space Engineering



Bibliography

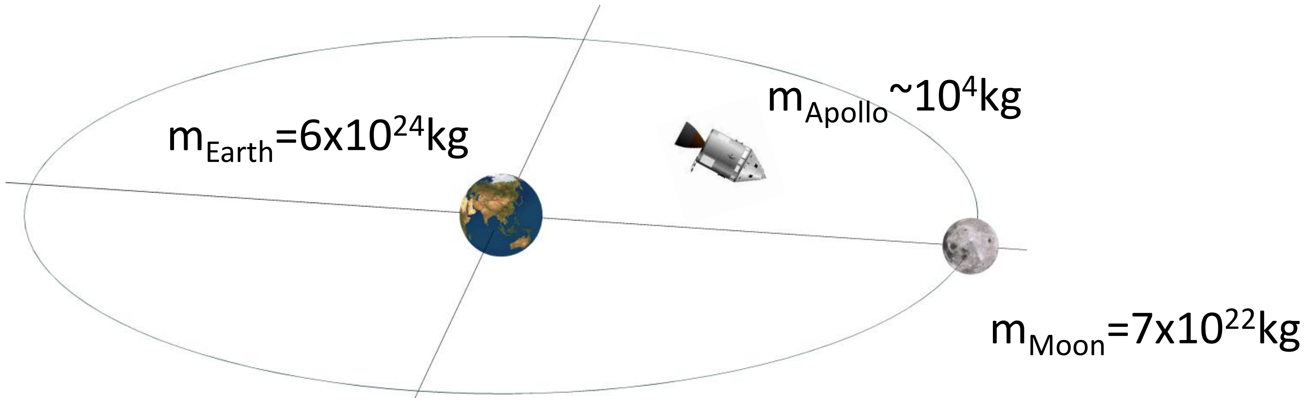
Some good books ordered by increasing depth and breath :

- *Orbital Mechanics,* V.A Chobotov, AIAA Education Series
- *Analytical Mechanics of Space Systems*, H.Shcaub, J.L.Junkins, AIAA Education Series
- *An Introduction to the Mathematics and Methods of Astrodynamics*, R.H.Battin, AIAA Educationan Series
- *Fundamentals of Astrodynamics and Applications*, D.A.Vallado, Space Technology Library

Circular Restricted Three Body Problem (CR3BP)

- "**Three-body**" refers to a problem that considers the motion of 3 bodies. Generally, 2 celestial bodies + spacecraft; Earth+Moon+S/C, Sun+Jupiter+asteroid.
- "**Restricted**" refers to the consideration that two of these bodies have masses that far outweigh the mass of the third.
- "**Circular**" refers to the fact that these two large bodies move in circular orbits around their common centre of mass.

CR3BP: General Description



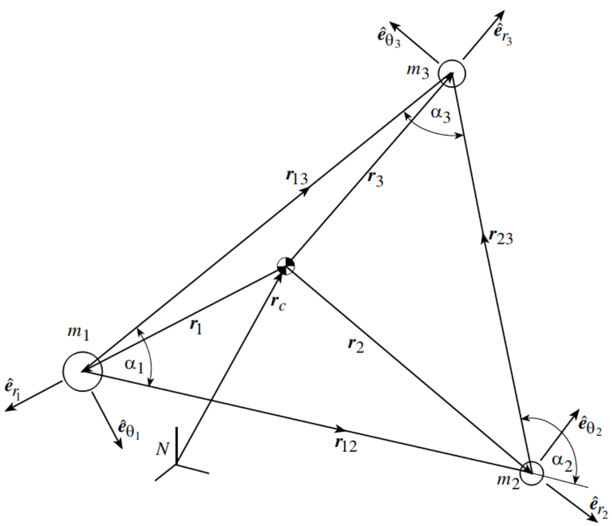


Figure 9.1: Illustration of Three-Body Problem

Courtesy (Shcaub & Junkins, 2002)

$$M\ddot{\mathbf{r}}_C = \mathbf{F}_{external} \approx 0$$

$$M = m_1 + m_2 + m_3$$

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \text{ where } i = 1,2,3$$

$$m_1\ddot{\mathbf{r}}_1 = G\sum_{j=1}^{3}\frac{m_1 m_j}{r_{1j}^3}\mathbf{r}_{1j}$$

$$m_2\ddot{\mathbf{r}}_2 = G\sum_{j=1}^{3}\frac{m_2 m_j}{r_{2j}^3}\mathbf{r}_{2j}$$

$$m_3\ddot{\mathbf{r}}_3 = G\sum_{j=1}^{3}\frac{m_3 m_j}{r_{3j}^3}\mathbf{r}_{3j}$$

CR3BP: General Description

- "**Restricted**" refers to the consideration that two of these bodies have masses that far outweigh the mass of the third.
  - ✓ **$m_3 \sim 0$**
- "**Circular**" refers to the fact that these two large bodies move in circular orbits around their common centre of mass.
  - ✓ **Constant angular velocity vector *ω***

$$n = \frac{2\pi}{\tau} = \sqrt{\frac{GM}{a^3}} \longrightarrow \omega = \sqrt{\frac{G(m_1 + m_2)}{r_{12}{}^3}}$$

CR3BP: Synodic or rotating reference frame

- A reference frame *F* centred at the barycentre and rotating with an angular velocity $\boldsymbol{\omega} = \omega \hat{\mathbf{k}}$

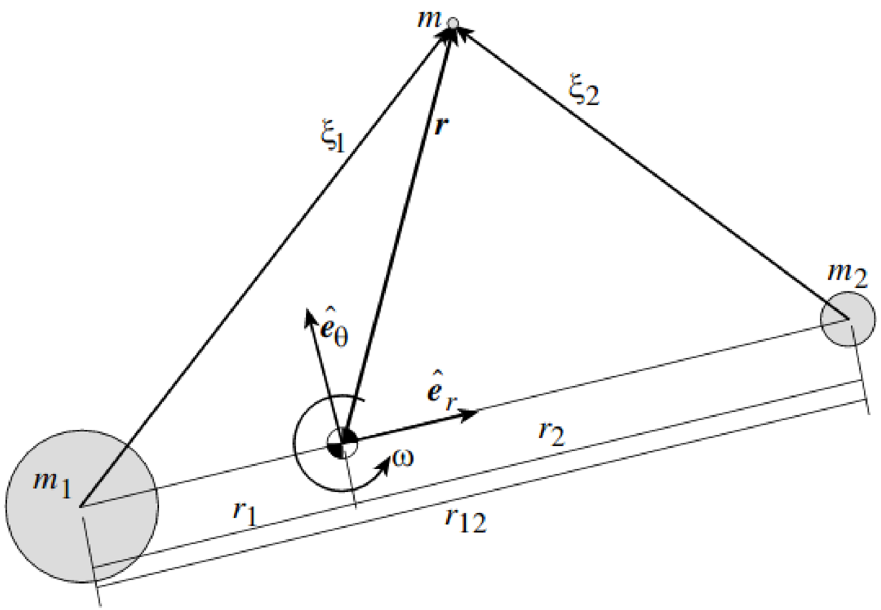
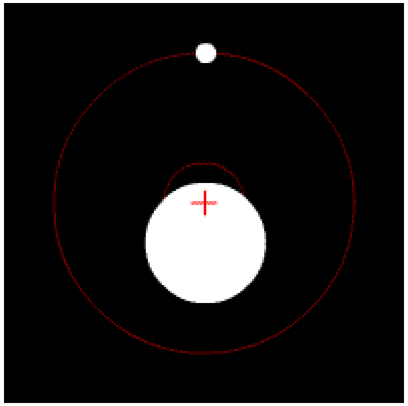


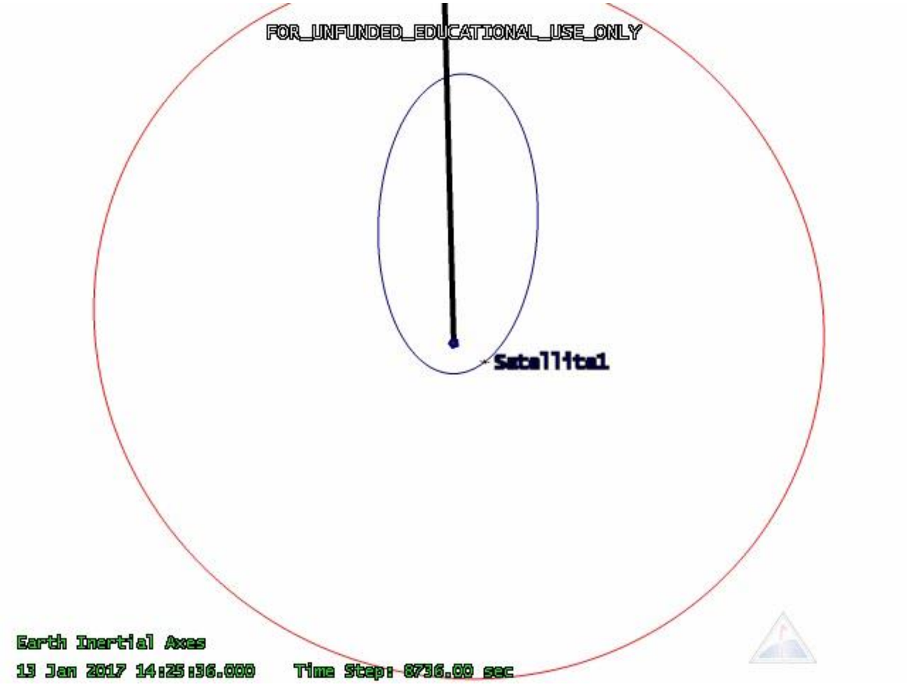
Figure 9.8: Illustration of Circular Restricted Three-Body Problem

Courtesy (Shcaub & Junkins, 2002)

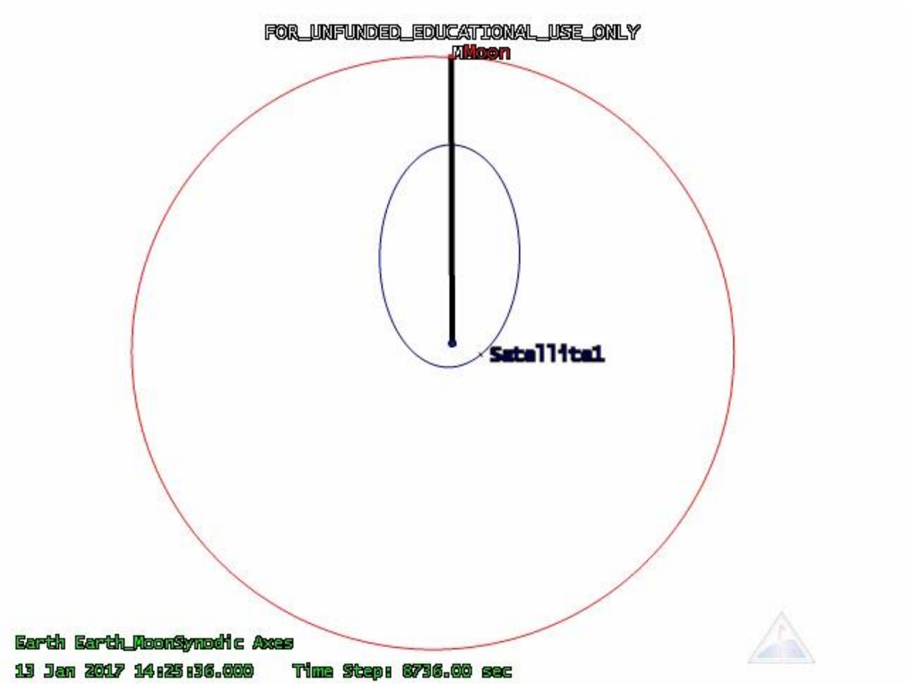


CR3BP: Synodic or rotating reference frame

Inertial orbit in inertial RF

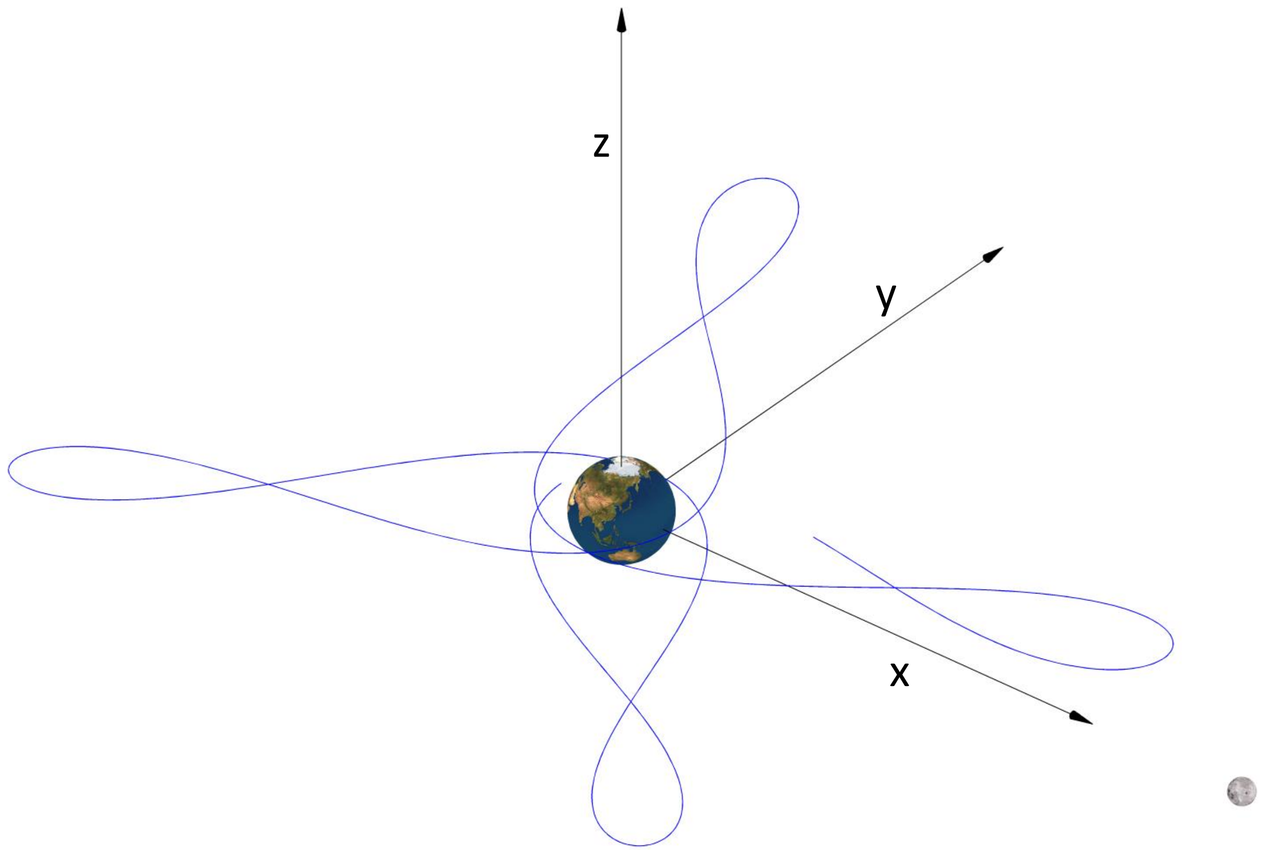


Inertial orbit in non-inertial RF



CR3BP: Synodic or rotating reference frame

Non-inertial orbit in non-inertial RF



Exercise 1

DemoEx1_CR3BP_Session4.m

1. Open file "*DemoEx1_CR3BP_Session4.m*"
2. Run each section in order one time.

1. Change the orbit by choosing a different apoapsis and periapis distance.

1. Attempt the following example:

   r_p=0.4u; r_a=0.86u
2. Attempt the following example:

   r_p=0.2u; r_a=0.7615u

What is happening in these strange orbits?

CR3BP: Equations of motion – Vector form

- Express ***r*** in $F:\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z\}$

$$\omega = \sqrt{\frac{G(m_1+m_2)}{r_{12}{}^3}}$$

$$\mathbf{r} = r_x\hat{\mathbf{e}}_r + r_y\hat{\mathbf{e}}_\theta + r_z\hat{\mathbf{e}}_z$$

$$\ddot{\mathbf{r}} + \underbrace{2\boldsymbol{\omega}\times\dot{\mathbf{r}}}_{\textit{Coriolis}} + \underbrace{\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}}_{\textit{Centripetal}} = \frac{\mathbf{F}_1}{m} + \frac{\mathbf{F}_2}{m}$$

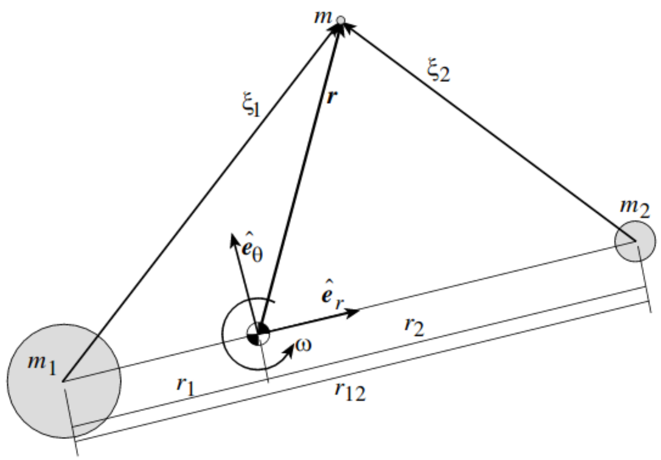


Figure 9.8: Illustration of Circular Restricted Three-Body Problem

$$\mathbf{F}_1 = -\frac{GmM_1}{\xi_1^3}\boldsymbol{\xi}_1$$

$$\mathbf{F}_2 = -\frac{GmM_2}{\xi_2^3}\boldsymbol{\xi}_2$$

$$\xi_i = \sqrt{(r_x - r_i)^2 + r_y^2 + r_z^2}$$

CR3BP: Equations of motion – scalar form

$$\left.\begin{aligned}
\ddot{r}_x - 2\omega\dot{r}_y - \omega^2 r_x &= -G\left(\frac{m_1}{\xi_1^3}\left(r_x - r_1\right) + \frac{m_2}{\xi_2^3}\left(r_x - r_2\right)\right) \\
\ddot{r}_y + 2\omega\dot{r}_x - \omega^2 r_y &= -G\left(\frac{m_1}{\xi_1^3} + \frac{m_2}{\xi_2^3}\right) r_y \\
\ddot{r}_z &= -G\left(\frac{m_1}{\xi_1^3} + \frac{m_2}{\xi_2^3}\right) r_z
\end{aligned}\right\}$$

$$\boxed{\mathbf{a} = -\frac{Gm}{r^2}\left(\frac{\mathbf{r}}{r}\right)}$$

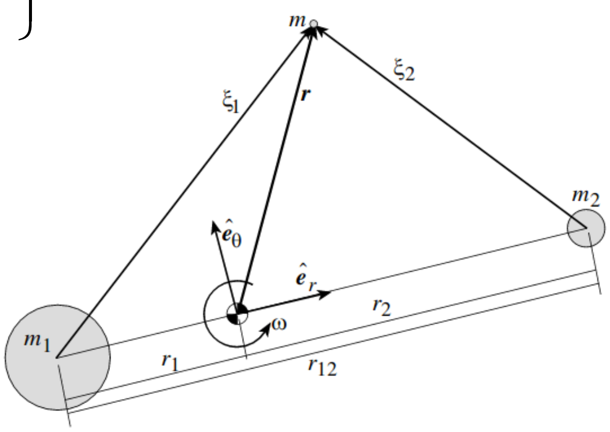


Figure 9.8: Illustration of Circular Restricted Three-Body Problem

CR3BP: Equations of motion – Potential function

$$U\left(r_x, r_y, r_z\right) = \frac{\omega^2}{2}\left(r_x{}^2 + r_y{}^2\right) + \frac{Gm_1}{\xi_1} + \frac{Gm_2}{\xi_2}$$

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} = \frac{\partial U}{\partial \mathbf{r}}$$

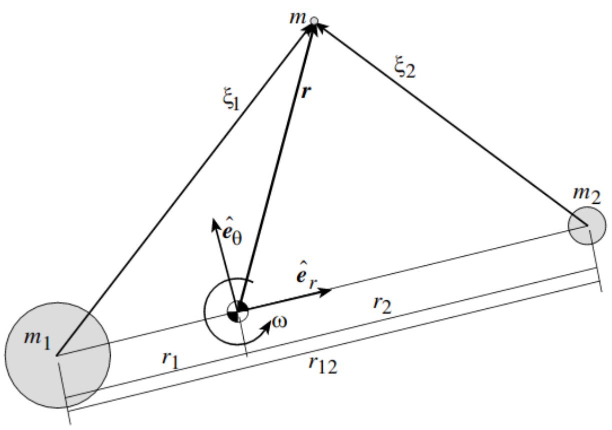


Figure 9.8: Illustration of Circular Restricted Three-Body Problem

CR3BP: Jacobi Integral

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} = \frac{\partial U}{\partial \mathbf{r}}$$

$$\left(\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}\right) \cdot \dot{\mathbf{r}} = \frac{\partial U}{\partial \mathbf{r}} \cdot \dot{\mathbf{r}}$$

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{\partial U}{\partial \mathbf{r}} \cdot \frac{d\mathbf{r}}{dt}$$

$$\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{dU}{dt}$$

$$\frac{1}{2}\frac{d}{dt}\left(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right) = \frac{dU}{dt}$$

$$v^2 = 2U - C$$

$$C = 2U - v^2$$

- Jacobi integral/constant = C

CR3BP: Jacobi Integral

$$C = \omega^2 \left(r_x{}^2 + r_y{}^2 \right) + \frac{2Gm_1}{\xi_1} + \frac{2Gm_2}{\xi_2} - v^2$$

- Basically, C is the classical energy integral expressed in rotating reference frame.
- Jacobi integral has two important uses:
  - ✓Verifying accuracy of numerical integration.
  - ✓Indicating regions of feasible motion.

CR3BP: Non-dimensional nomenclature

- The problem can be made non-dimensional by choosing the following units:
  - ✓ m_1+m_2 as the unit of mass
  - ✓ r_{12} as the unit of length
  - ✓ P/(2π) as the unit of time

$$\mu = \frac{m_2}{m_1 + m_2}$$

$$x_2 - x_1 = 1$$

$$x_1 = -\mu$$

$$x_2 = 1 - \mu$$

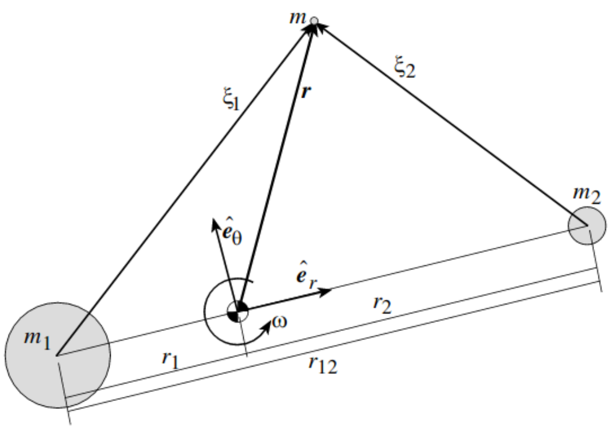


Figure 9.8: Illustration of Circular Restricted Three-Body Problem

CR3BP: Non-dimensional nomenclature

System	μ	L	V	T
Sun-Jupiter	9.537×10^{-4}	7.784×10^{8}	13.102	3.733×10^{8}
Sun-(Earth+Moon)	3.036×10^{-6}	1.496×10^{8}	29.784	3.147×10^{7}
Earth-Moon	1.215×10^{-2}	3.850×10^{5}	1.025	2.361×10^{6}
Mars-Phobos	1.667×10^{-8}	9.380×10^{3}	2.144	2.749×10^{4}
Jupiter-Io	4.704×10^{-5}	4.218×10^{5}	17.390	1.524×10^{5}
Jupiter-Europa	2.528×10^{-5}	6.711×10^{5}	13.780	3.060×10^{5}
Jupiter-Ganymede	7.804×10^{-5}	1.070×10^{6}	10.909	6.165×10^{5}
Jupiter-Callisto	5.667×10^{-5}	1.883×10^{6}	8.226	1.438×10^{6}
Saturn-Mimas	6.723×10^{-8}	1.856×10^{5}	14.367	8.117×10^{4}
Saturn-Titan	2.366×10^{-4}	1.222×10^{6}	5.588	1.374×10^{6}
Neptune-Triton	2.089×10^{-4}	3.548×10^{5}	4.402	5.064×10^{5}
Pluto-Charon	1.097×10^{-1}	1.941×10^{4}	0.222	5.503×10^{5}

TABLE 2.2.1. **Table of m_1-m_2 systems in the solar system.** Source: The first three are the values used in Koon, Lo, Marsden, and Ross [2000, 2001b]. The others are from the Jet Propulsion Laboratory's solar system dynamics website: `http://ssd.jpl.nasa.gov/`.

CR3BP: Non-dimensional dynamics

$$\left.\begin{aligned}
\ddot{x}-2\dot{y}-x &= -\frac{1-\mu}{\rho_1^3}\left(x-x_1\right)-\frac{-\mu}{\rho_2^3}\left(x-x_2\right)\\
\ddot{y}+2\dot{x}-y &= -\left(\frac{1-\mu}{\rho_1^3}+\frac{-\mu}{\rho_2^3}\right)y\\
\ddot{z} &= -\left(\frac{1-\mu}{\rho_1^3}+\frac{-\mu}{\rho_2^3}\right)z
\end{aligned}\right\}$$

$$\rho_i=\sqrt{\left(x-x_i\right)^2+y^2+z^2}$$

$$U\left(x,y,z\right)=\frac{1}{2}\left(x^2+y^2\right)+\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}$$

$$v^2=x^2+y^2+\frac{2\left(1-\mu\right)}{\rho_1}+\frac{2\mu}{\rho_2}-C$$

CR3BP: Equilibrium Positions

- Let say we have a trajectory with a state vector such as:

$$(\mathbf{x}, \dot{\mathbf{x}}) = (x, y, z, 0, 0, 0)$$

$$\ddot{x} = 2\dot{y} + x - \frac{1-\mu}{\rho_1^3}\left(x - x_1\right) - \frac{\mu}{\rho_2^3}\left(x - x_2\right)$$

$$\ddot{y} = -2\dot{x} + y - \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right)y$$

$$\ddot{z} = -\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right)z$$

CR3BP: Equilibrium Positions

- Let say we have a trajectory with a state vector such as:

$$(\mathbf{x}, \dot{\mathbf{x}}) = (x, y, z, 0, 0, 0)$$

$$\ddot{x} = 0$$

$$\ddot{y} = 0$$

$$\ddot{z} = 0$$

- We then have an equilibrium point. (or stationary point).

CR3BP: Equilibrium Positions

- Let's look for: $(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = (x, y, z, 0, 0, 0, 0, 0, 0)$

$$\ddot{x} = 2\dot{y} + x - \frac{1-\mu}{\rho_1^3}(x - x_1) - \frac{\mu}{\rho_2^3}(x - x_2)$$

$$\ddot{y} = -2\dot{x} + y - \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) y$$

$$\ddot{z} = -\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) z \longrightarrow \ddot{z} = 0 \rightarrow z = 0$$

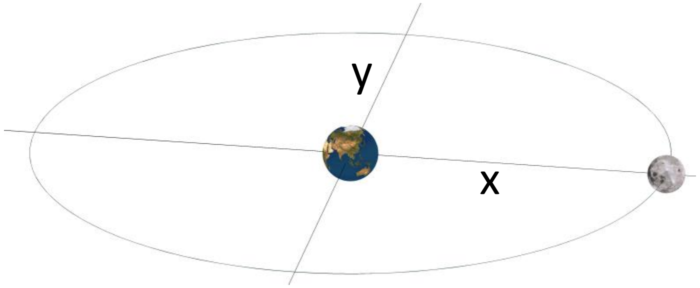
- All stationary points must lie in the orbital plane of m_1 & m_2

CR3BP: Equilibrium Positions

- Let's look for: $(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = (x, y, 0, 0, 0, 0, 0, 0, 0)$

$$\ddot{x} = 2\dot{y} + x - \frac{1-\mu}{\rho_1^3}(x - x_1) - \frac{\mu}{\rho_2^3}(x - x_2)$$

$$\ddot{y} = -2\dot{x} + y - \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) y$$



$$y - \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) y = 0$$

$y = 0$ *Collinear points*

$\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) = 1;\ \rho_1 = \rho_2$ *Equilateral points*

CR3BP: Equilibrium Positions (Collinear Solutions)

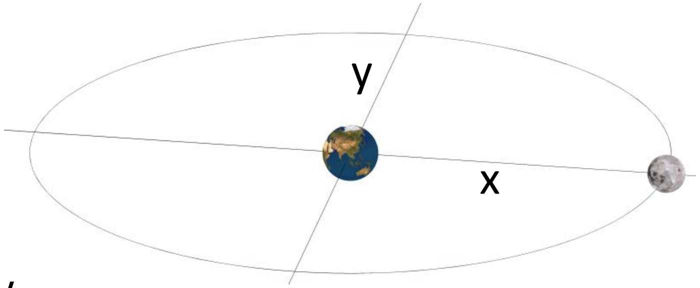
- Let's look for: $(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = (x, 0, 0, 0, 0, 0, 0, 0, 0)$

$$\rho_1 = \sqrt{(x - x_1)^2} \qquad x_1 = -\mu$$

$$\rho_2 = \sqrt{(x - x_2)^2} \qquad x_2 = 1 - \mu$$

$$\ddot{x} = 2\dot{y} + x - \frac{1-\mu}{\rho_1^3}(x - x_1) - \frac{\mu}{\rho_2^3}(x - x_2)$$

$$x - \frac{1-\mu}{|x - x_1|^3}(x - x_1) - \frac{\mu}{|x - x_2|^3}(x - x_2) = 0$$



L3: $x - x_1 < 0 \;\&\; x - x_2 < 0$

$$x + \frac{1-\mu}{(x - x_1)^2} + \frac{\mu}{(x - x_2)^2} = 0$$

L1: $x - x_1 > 0 \;\&\; x - x_2 < 0$

$$x - \frac{1-\mu}{(x - x_1)^2} + \frac{\mu}{(x - x_2)^2} = 0$$

L2: $x - x_1 > 0 \;\&\; x - x_2 > 0$

$$x - \frac{1-\mu}{(x - x_1)^2} - \frac{\mu}{(x - x_2)^2} = 0$$

Exercise 2A

Part A in:

DemoEx2_CR3BP_Session4.m

- For the Earth-Moon system, find the three collinear equilibrium points.

Hint: Use *fzero* function (i.e. Matlab) to solve the equations below.

L3: $x - x_1 < 0 \;\;\&\;\; x - x_2 < 0$ $$x + \frac{1-\mu}{(x-x_1)^2} + \frac{\mu}{(x-x_2)^2} = 0$$

L1: $x - x_1 > 0 \;\;\&\;\; x - x_2 < 0$ $$x - \frac{1-\mu}{(x-x_1)^2} + \frac{\mu}{(x-x_2)^2} = 0$$

L2: $x - x_1 > 0 \;\;\&\;\; x - x_2 > 0$ $$x - \frac{1-\mu}{(x-x_1)^2} - \frac{\mu}{(x-x_2)^2} = 0$$

Constants:

m_{Earth}=5.9737x10^{24}kg

m_{Moon}=7.3476x10^{22}kg

Exercise 2B

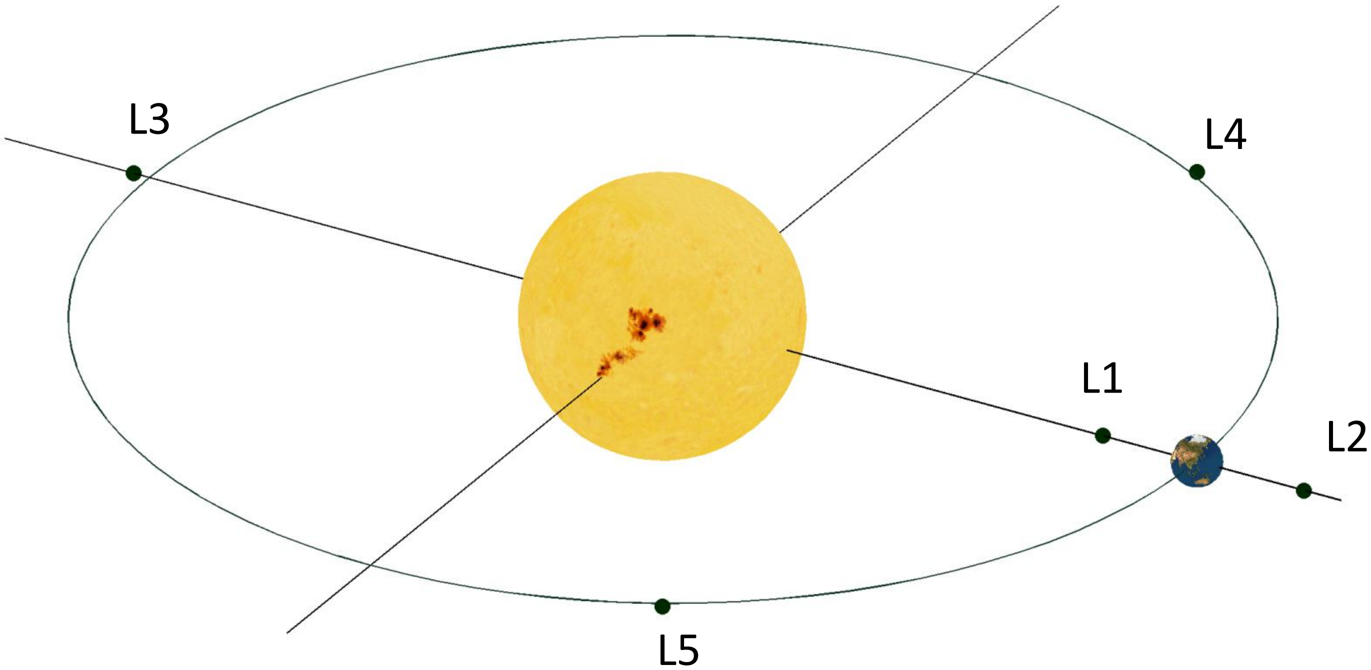
Part B in:

DemoEx2_CR3BP_Session4.m

For each equilibrium solution, propagate its initial condition and plot the results, both in rotating reference frame and inertial reference frame.

- Can you explain what you see?

CR3BP: Five Fixed Points (Lagrange or Libration Points)



CR3BP: Zero velocity curves

$$\left.\begin{aligned}\ddot{x}-2\dot{y}-x&=-\frac{1-\mu}{\rho_1^3}\left(x-x_1\right)-\frac{-\mu}{\rho_2^3}\left(x-x_2\right)\\ \ddot{y}+2\dot{x}-y&=-\left(\frac{1-\mu}{\rho_1^3}+\frac{-\mu}{\rho_2^3}\right)y\\ \ddot{z}&=-\left(\frac{1-\mu}{\rho_1^3}+\frac{-\mu}{\rho_2^3}\right)z\end{aligned}\right\}$$

$$\rho_i=\sqrt{\left(x-x_i\right)^2+y^2+z^2}$$

$$U\left(x,y,z\right)=\frac{1}{2}\left(x^2+y^2\right)+\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}$$

$$v^2=x^2+y^2+\frac{2\left(1-\mu\right)}{\rho_1}+\frac{2\mu}{\rho_2}-C$$

CR3BP: Zero velocity curves

$$v(x,y,z)^2 = \underbrace{x^2 + y^2 + \frac{2(1-\mu)}{\rho(x,y,z)_1} + \frac{2\mu}{\rho(x,y,z)_2}}_{\textit{Potential Energy}} - C$$

- If C>2U then $v^2 < 0 \longrightarrow v \equiv \text{Does not exist}$

Exercise 3

DemoEx3_CR3BP_Session4.m

- Compute the zero velocity curves for a trajectory with Jacobi Constant $C=L_1$.

CR3BP: Zero velocity curves at C_1=L_1

1. Compute (x,y,z) for the L1 point in the Earth-Moon System.
2. Create a grid in 3D and analyse the values of the following equation:

$$v^2 = x^2 + y^2 + \frac{2(1-\mu)}{\rho(x,y,z)_1} + \frac{2\mu}{\rho(x,y,z)_2} - C_1$$

3. Plot the isosurface for $v^2=0$

CR3BP: Zero velocity curves

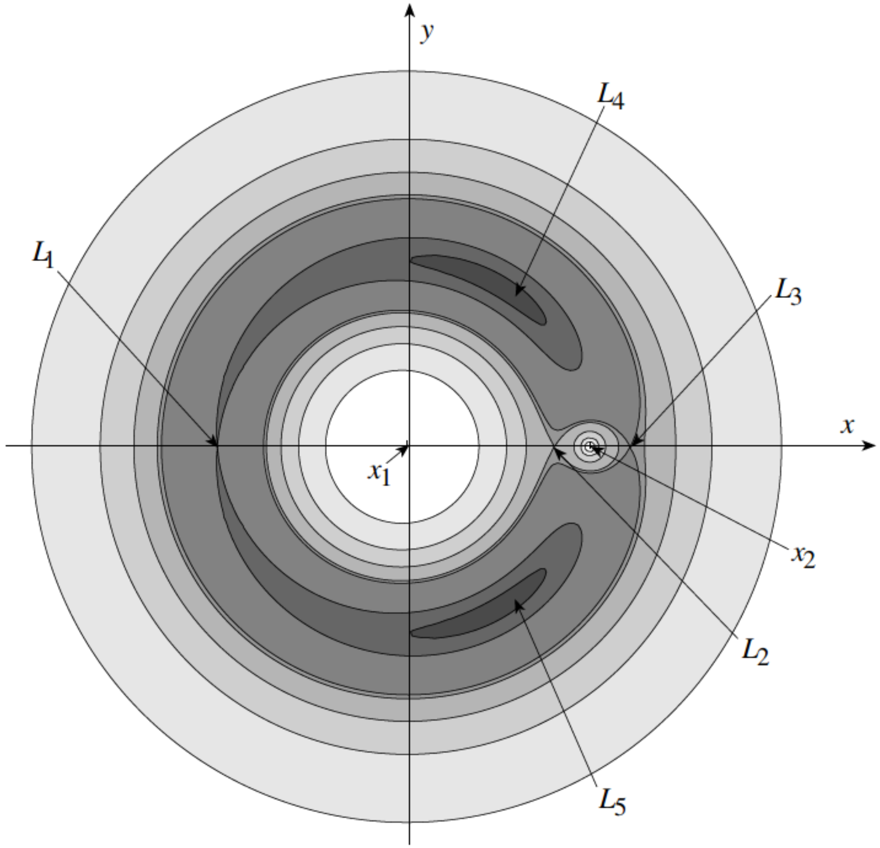


Figure 9.9: Zero Relative Velocity Surface Contours of the Earth-Moon System in the $x - y$ Plane

Courtesy (Shcaub & Junkins, 2002)

CR3BP: Regimes of Motion

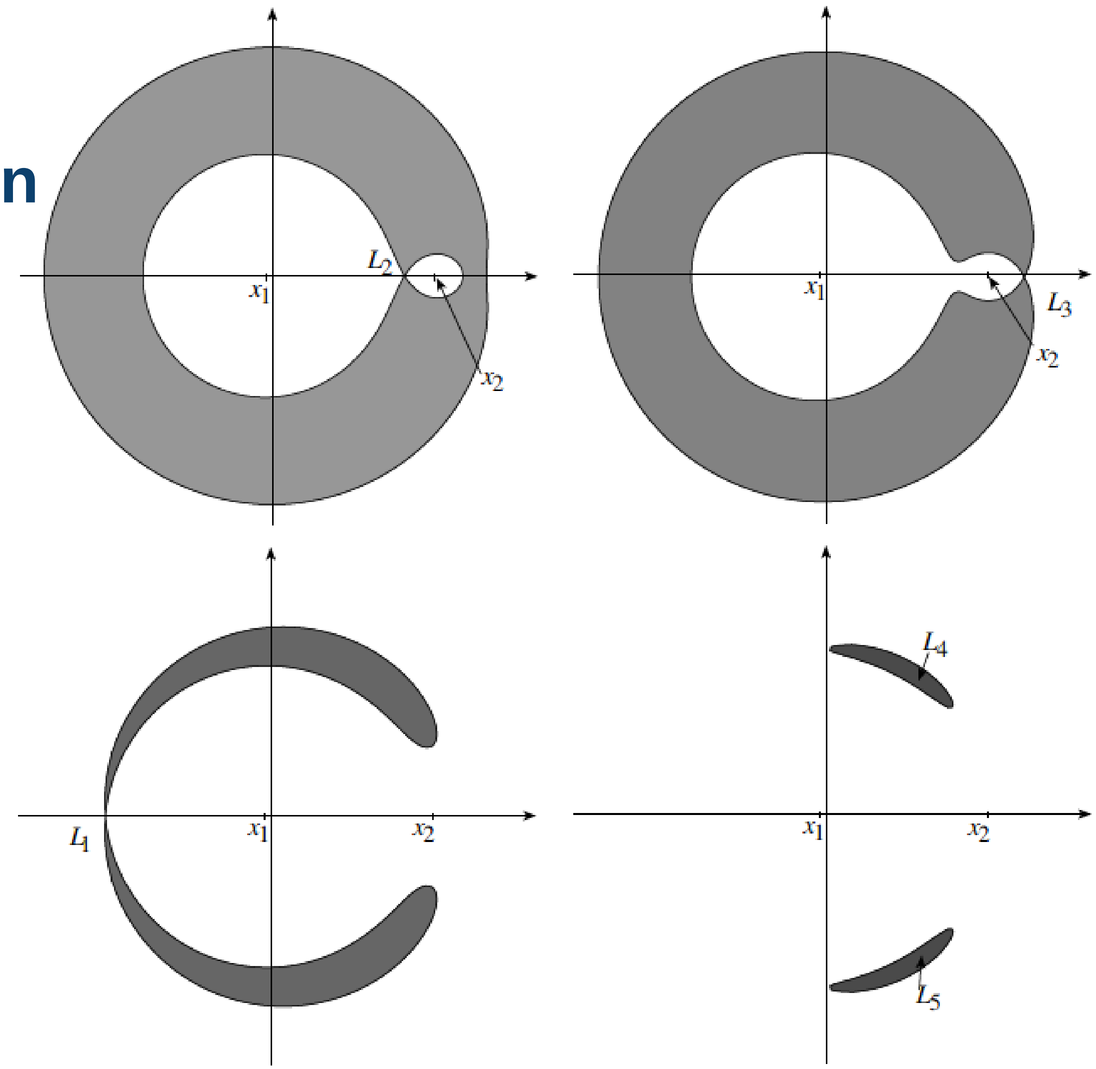


Figure 9.10: Critical Zero Relative Velocity Surface Contours of the Earth-Moon System Touching the Lagrange Stationary Points

Courtesy (Shcaub & Junkins, 2002)

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Advanced Topics in Astrodynamics and Trajectory Design

Libration Point Orbits (LPOs)

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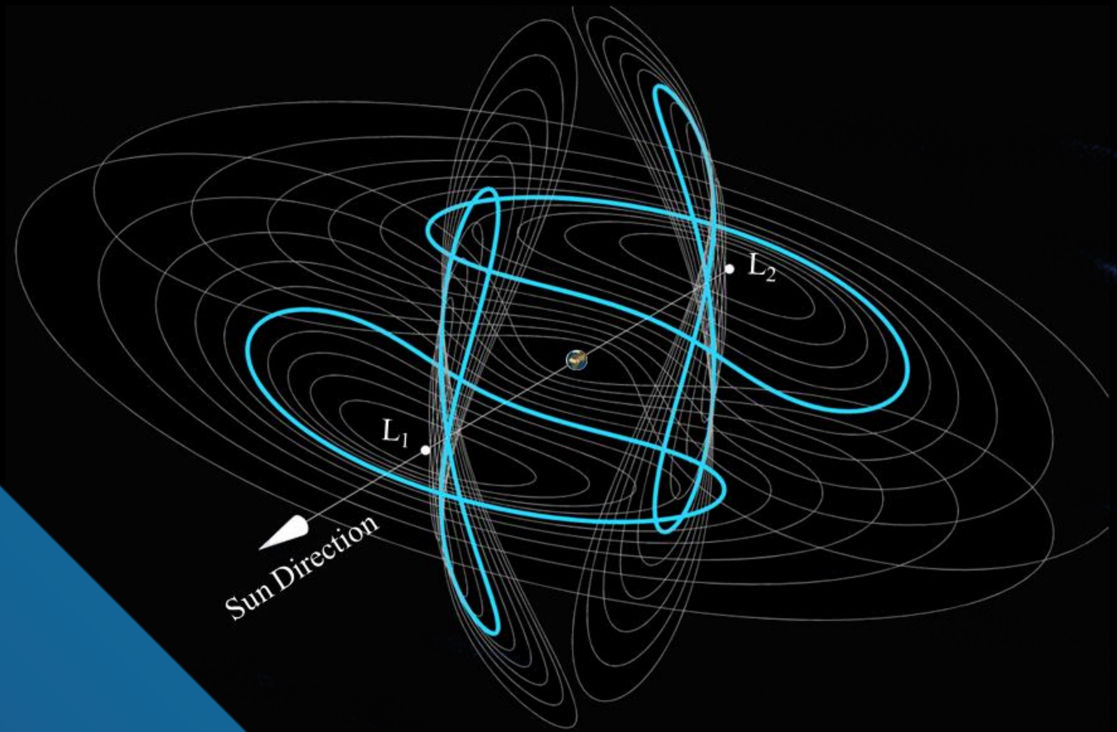
MSc in Astronautics and Space Engineering



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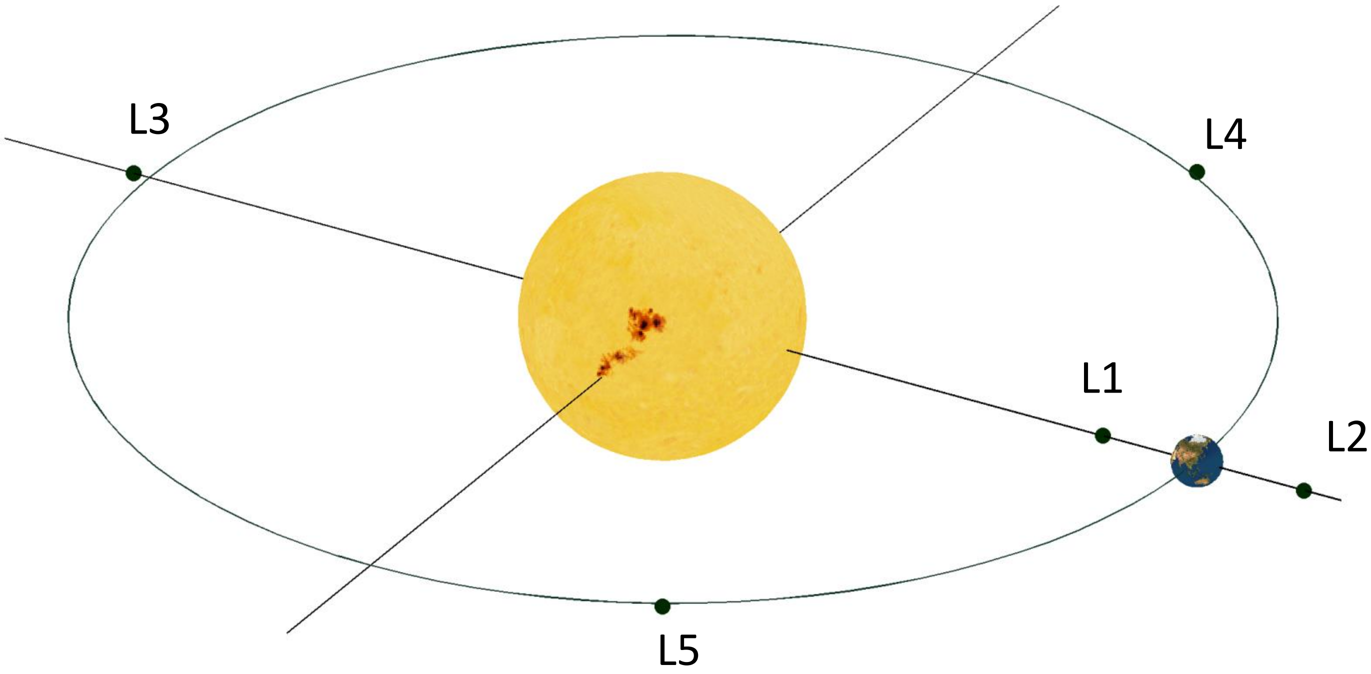
CR3BP: Five Fixed Points (Lagrange or Libration Points)

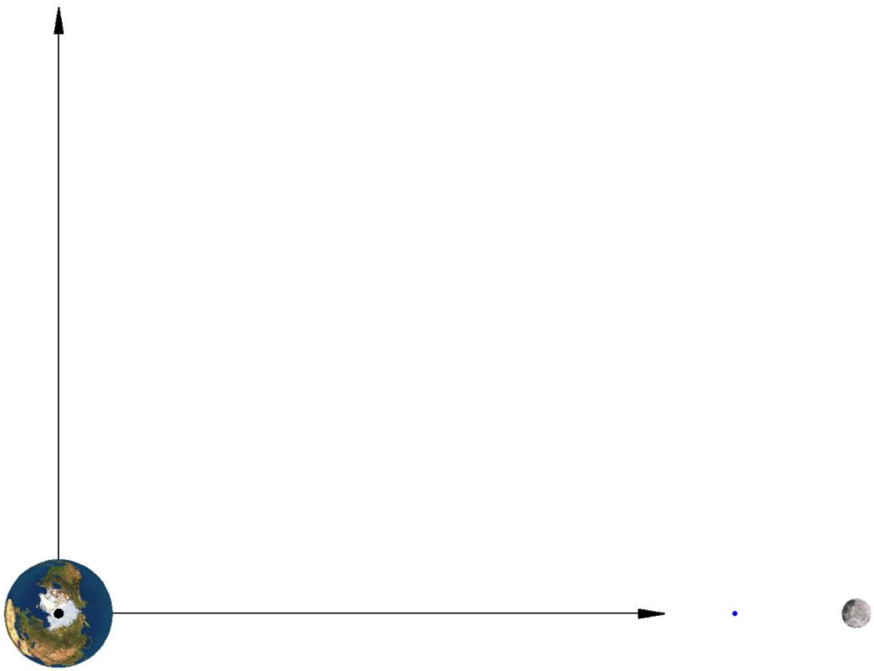


Motion of a spacecraft in an equilibrium point

- What happens when we place a spacecraft at the L1 equilibrium point?

Motion of a spacecraft in an equilibrium point

- What happens when we place a spacecraft at the L1 equilibrium point?

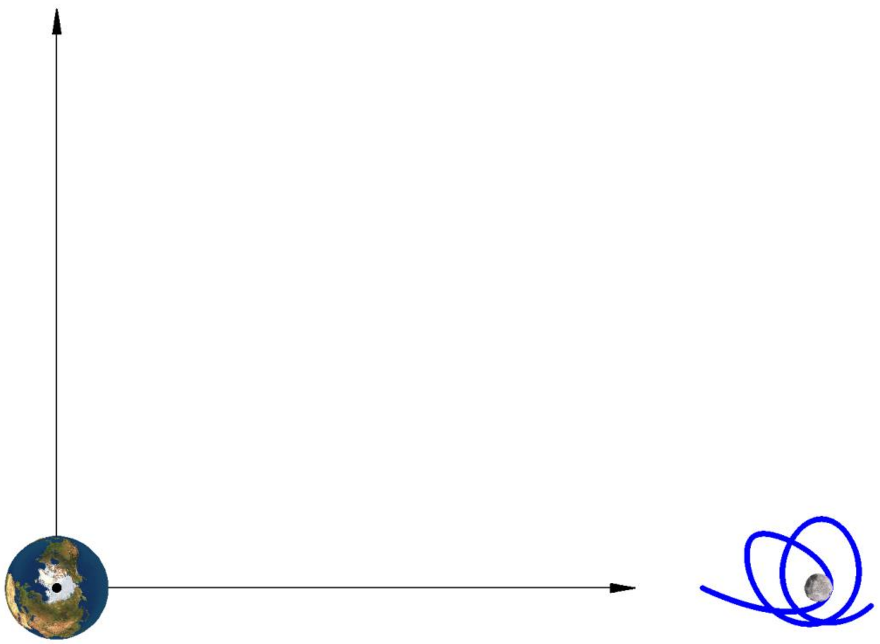


Motion of a spacecraft in an equilibrium point

- What happens when we place a spacecraft at the L1 equilibrium point, but with 1 cm/s error in its velocity?

Motion of a spacecraft in an equilibrium point

- What happens when we place a spacecraft at the L1 equilibrium point, but with 1 cm/s error in its velocity?

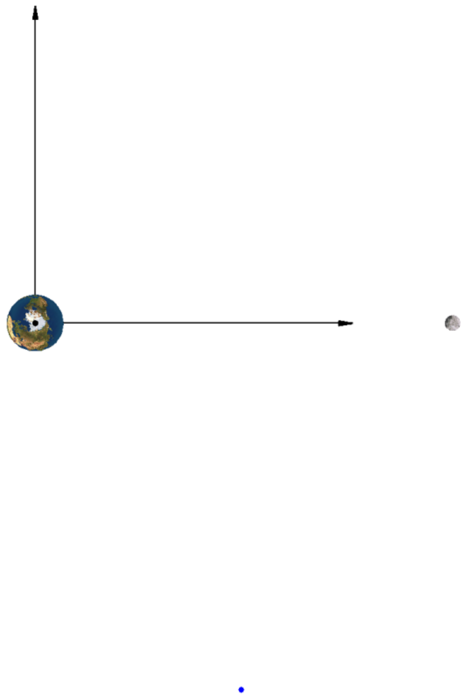


Motion of a spacecraft in an equilibrium point

- What happens when we place a spacecraft at the L5 equilibrium point?

Motion of a spacecraft in an equilibrium point

- What happens when we place a spacecraft at the L5 equilibrium point?

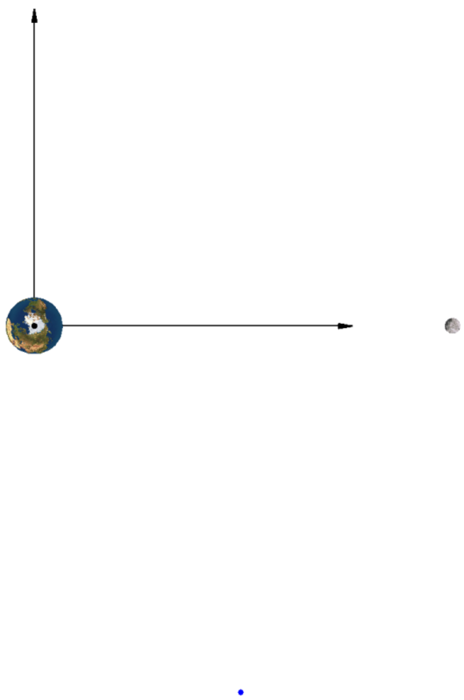


Motion of a spacecraft in an equilibrium point

- What happens when we place a spacecraft at the L5 equilibrium point, but with 1 cm/s error in its velocity?

Motion of a spacecraft in an equilibrium point

- What happens when we place a spacecraft at the L5 equilibrium point, but with 1 cm/s error in its velocity?

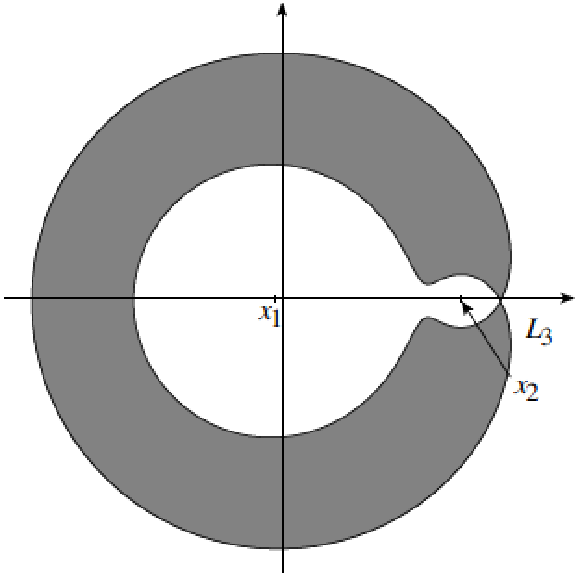


Motion of a spacecraft in an equilibrium point

- Would you say then that L5 is more stable than L1 point?
- Is L1 stable or unstable?
- Is L5 stable?

Linear Stability Analysis near fixed points

- Fixed points or Equilibrium points have an important “organizing role” of a system phase space.



- In particular, if we linearize the system in a neighbourhood of the fixed points:

$$\delta\dot{\mathbf{y}} = \mathbf{M}\delta\mathbf{y}$$

- The eigenvalues of the matrix M determine the stability of the fixed point

Linear Stability Analysis near fixed points

- Consider our dynamical model: $\ddot{\mathbf{r}} = f\left(\mathbf{r}, \dot{\mathbf{r}}\right)$
- Now, let us consider the motion near an equilibrium point $\mathbf{r}_0$: $\ddot{\mathbf{r}}(\mathbf{r}_0, \mathbf{0}) = \mathbf{0}$

$$\ddot{\mathbf{r}}(\mathbf{r}_0 + \delta\mathbf{r}, \mathbf{0} + \delta\mathbf{v}) = \mathbf{0} + \mathbf{M}\begin{pmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \end{pmatrix}$$

$$\mathbf{M} = \left(\begin{array}{ccc|ccc} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial \dot{y}} & \frac{\partial \dot{x}}{\partial \dot{z}} \\ \frac{\partial \dot{y}}{\partial x} & \cdots & & & & \\ \vdots & & & & & \\ \hline \vdots & & & & \ddots & \vdots \\ \frac{\partial \ddot{z}}{\partial x} & \frac{\partial \ddot{z}}{\partial y} & \frac{\partial \ddot{z}}{\partial z} & \frac{\partial \ddot{z}}{\partial \dot{x}} & \frac{\partial \ddot{z}}{\partial \dot{y}} & \frac{\partial \ddot{z}}{\partial \dot{z}} \end{array}\right)$$

Linear Stability Analysis near fixed points

- Consider or dynamical model: $\ddot{\mathbf{r}} = f\left(\mathbf{r}, \dot{\mathbf{r}}\right)$
- Now, let us consider the motion near an equilibrium point $\mathbf{r}_0$: $\ddot{\mathbf{r}}(\mathbf{r}_0, \mathbf{0}) = \mathbf{0}$

$$\ddot{\mathbf{r}}(\mathbf{r}_0 + \delta\mathbf{r}, \mathbf{0} + \delta\mathbf{v}) = \mathbf{0} + \mathbf{M}\begin{pmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \end{pmatrix}$$

$$\mathbf{M} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial y} & \frac{\partial \ddot{x}}{\partial z} & 0 & 2 & 0 \\ \frac{\partial \ddot{y}}{\partial x} & \frac{\partial \ddot{y}}{\partial y} & \frac{\partial \ddot{y}}{\partial z} & -2 & 0 & 0 \\ \frac{\partial \ddot{z}}{\partial x} & \frac{\partial \ddot{z}}{\partial y} & \frac{\partial \ddot{z}}{\partial z} & 0 & 0 & 0 \end{array}\right)$$

Linear Stability Analysis near fixed points

$$\left.\begin{array}{c}
\dot{x} = v_x \\
\dot{y} = v_y \\
\dot{z} = v_z \\
\hline
\ddot{x} = x + 2\dot{y} - \dfrac{1-\mu}{\rho_1^3}\left(x - x_1\right) - \dfrac{\mu}{\rho_2^3}\left(x - x_2\right) \\
\ddot{y} = y - 2\dot{x} - \left(\dfrac{1-\mu}{\rho_1^3} + \dfrac{\mu}{\rho_2^3}\right) y \\
\ddot{z} = -\left(\dfrac{1-\mu}{\rho_1^3} + \dfrac{\mu}{\rho_2^3}\right) z
\end{array}\right\}$$

$$\rho_i = \sqrt{\left(x - x_i\right)^2 + y^2 + z^2}$$

Linear Stability Analysis near fixed points

$$x_1 = -\mu$$
$$x_2 = 1 - \mu$$

$$\rho_i = \sqrt{\left(x - x_i\right)^2 + y^2 + z^2}$$

$$\mathbf{M} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline & & & 0 & 2 & 0 \\ & \mathbf{Q} & & -2 & 0 & 0 \\ & & & 0 & 0 & 0 \end{array}\right) \qquad \mathbf{Q} = \begin{pmatrix} \dfrac{\partial \ddot{x}}{\partial x} & \dfrac{\partial \ddot{x}}{\partial y} & \dfrac{\partial \ddot{x}}{\partial z} \\ \dfrac{\partial \ddot{y}}{\partial x} & \dfrac{\partial \ddot{y}}{\partial y} & \dfrac{\partial \ddot{y}}{\partial z} \\ \dfrac{\partial \ddot{z}}{\partial x} & \dfrac{\partial \ddot{z}}{\partial y} & \dfrac{\partial \ddot{z}}{\partial z} \end{pmatrix}$$

$$\frac{\partial \ddot{x}}{\partial x} = 1 - \frac{1-\mu}{\rho_1^3}\left(1 - \frac{3}{\rho_1^2}\left(x - x_1\right)^2\right) - \frac{\mu}{\rho_2^3}\left(1 - \frac{3}{\rho_2^2}\left(x - x_2\right)^2\right)$$

$$\frac{\partial \ddot{y}}{\partial y} = 1 + \frac{1-\mu}{\rho_1^3}\left(\frac{3y^2}{\rho_1^2} - 1\right) + \frac{\mu}{\rho_2^3}\left(\frac{3y^2}{\rho_2^2} - 1\right)$$

$$\frac{\partial \ddot{x}}{\partial y} = \frac{\partial \ddot{y}}{\partial x} = 3\left(\frac{1-\mu}{\rho_1^5}\left(x - x_1\right) + \frac{\mu}{\rho_2^5}\left(x - x_2\right)\right) y$$

$$\frac{\partial \ddot{y}}{\partial z} = \frac{\partial \ddot{z}}{\partial y} = 3\frac{1-\mu}{\rho_1^5} yz + 3\frac{\mu}{\rho_2^5} yz$$

$$\frac{\partial \ddot{x}}{\partial z} = \frac{\partial \ddot{z}}{\partial x} = 3\frac{1-\mu}{\rho_1^5}\left(x - x_1\right) z + 3\frac{\mu}{\rho_2^5}\left(x - x_2\right) z$$

$$\frac{\partial \ddot{z}}{\partial z} = \frac{1-\mu}{\rho_1^3}\left(\frac{3z^2}{\rho_1^2} - 1\right) + \frac{\mu}{\rho_2^3}\left(\frac{3z^2}{\rho_2^2} - 1\right)$$

Linear Stability Analysis near fixed points

$$\begin{pmatrix} \delta\dot{\mathbf{r}} \\ \delta\dot{\mathbf{v}} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \end{pmatrix}$$

- Recall, an eigenvector of a matrix **M** is any vector that when multiplied to **M**, it only gets scaled.

$$\mathbf{Md} = \lambda\mathbf{d} \qquad \det\left|\mathbf{M} - \lambda\mathbf{I}\right| = 0$$

- A linear ordinary differential equation such as this can then be solved as:

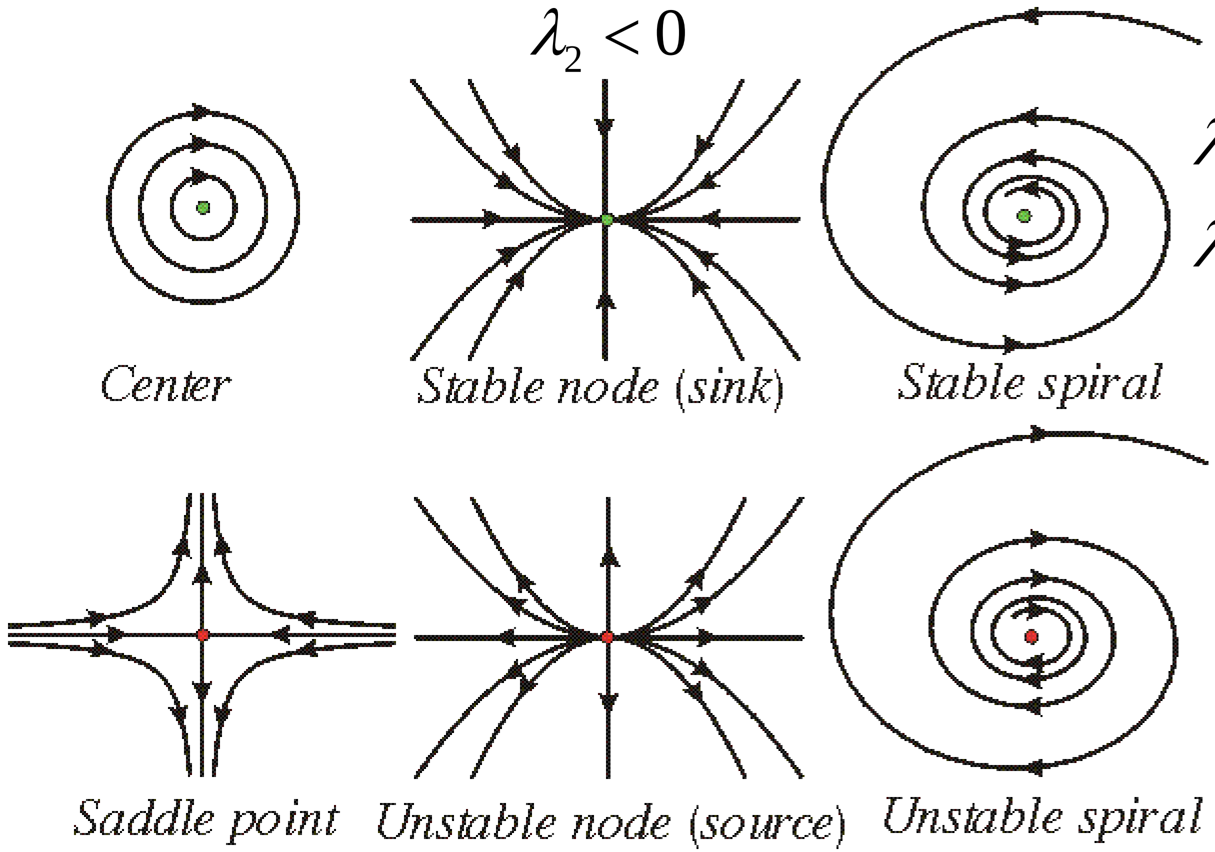
$$\delta\mathbf{x} = \sum_i c_i \mathbf{d}_i e^{\lambda_i t}$$

- Thus, only if λ are pure imaginary numbers the displacement will just rotate about the associated equilibrium point.

Classification of Fixed Points

$$\lambda_1 < 0$$
$$\lambda_2 < 0$$

$$\lambda_1 = +i\omega$$
$$\lambda_2 = -i\omega$$



$$\lambda_1 = -\alpha + i\beta$$
$$\lambda_2 = -\alpha - i\beta$$

$$\lambda_1 < 0$$
$$\lambda_2 > 0$$

$$\lambda_1 = \alpha + i\beta$$
$$\lambda_2 = \alpha - i\beta$$

$$\lambda_1 > 0$$
$$\lambda_2 > 0$$

Periodic Orbits near L1 and L2

- Let us prove the existence of periodic motion near the L1/L2 points.

Periodic Orbits near L1 and L2

$$(\mathbf{x}_{Li}, \dot{\mathbf{x}}_{Li}) = (x_{Li}, 0, 0, 0, 0, 0)$$

$$x_1 = -\mu$$
$$x_2 = 1 - \mu$$

$$\rho_i = \sqrt{\left(x - x_i\right)^2}$$

$$\Omega = +\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}$$

$$\mathbf{M} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1+2\Omega & 0 & 0 & 0 & 2 & 0 \\ 0 & 1-\Omega & 0 & -2 & 0 & 0 \\ 0 & 0 & -\Omega & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \\ \hline \ddot{x}' \\ \ddot{y}' \\ \ddot{z}' \end{pmatrix} = \mathbf{M} \begin{pmatrix} x' \\ y' \\ z' \\ \hline \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix}$$

$$\ddot{x}' - 2\dot{y}' - \left(1 + 2\Omega\right)x' = 0$$
$$\ddot{y}' + 2\dot{x}' + \left(\Omega + 1\right)y' = 0$$
$$\ddot{z}' + \Omega z' = 0$$

Periodic Orbits near L1 and L2

- Let us prove the existence of periodic motion near the L1/L2 points

$$\ddot{x}' - 2\dot{y}' - (1 + 2\Omega)x' = 0$$
$$\ddot{y}' + 2\dot{x}' + (\Omega + 1)y' = 0$$
$$\ddot{z}' + \Omega z' = 0$$

- In the linearized approximation, the out-of-plane component has no influence on the in-plane motion.

$$\ddot{z}' + \Omega z' = 0 \longrightarrow z' = A_z \sin\left(\sqrt{\Omega}t + \varphi\right)$$

- While in the X-Y plane, arbitrarily chosen initial conditions will give rise to unbounded motion. However, periodic motion is also possible if starting conditions are chosen suitably.

Periodic Orbits near L1 and L2

- Let us prove the existence of periodic motion near the L1/L2 points

$$\ddot{x}' - 2\dot{y}' - (1 + 2\Omega)\,x' = 0$$
$$\ddot{y}' + 2\dot{x}' + (\Omega + 1)\,y' = 0$$

$$x' = -A_x \cos(\omega_p t + \phi)$$
$$y' = kA_x \sin(\omega_p t + \phi)$$

$$k = \frac{\omega_p^2 + 1 + 2\Omega}{2\omega_p}$$

$$\omega_p^2 = \frac{2 - \Omega + \sqrt{9\Omega^2 - 8\Omega}}{2}$$

Periodic Orbits near L1 and L2

$$\ddot{x}' - 2\dot{y}' - \left(1 + 2\Omega\right) x' = 0$$

$$\ddot{y}' + 2\dot{x}' + \left(\Omega + 1\right) y' = 0$$

$$x' = -A_x \cos\left(\omega_p t + \phi\right) \qquad \dot{x}' = A_x \omega_p \sin\left(\omega_p t + \phi\right) \qquad \ddot{x}' = A_x \omega_p^2 \cos\left(\omega_p t + \phi\right)$$

$$y' = k A_x \sin\left(\omega_p t + \phi\right) \qquad \dot{y}' = k \omega_p A_x \cos\left(\omega_p t + \phi\right) \qquad \ddot{y}' = -k \omega_p^2 A_x \sin\left(\omega_p t + \phi\right)$$

$$A_x \omega_p^2 \cos\left(\omega_p t + \phi\right) = \left(2k\omega_p - 2\Omega - 1\right) A_x \cos\left(\omega_p t + \phi\right)$$

$$\omega_p^2 = 2k\omega_p - 2\Omega - 1$$

$$\boxed{k = \frac{\omega_p^2 + 1 + 2\Omega}{2\omega_p}}$$

Periodic Orbits near L1 and L2

$$\ddot{x}' - 2\dot{y}' - \left(1 + 2\Omega\right) x' = 0$$

$$\ddot{y}' + 2\dot{x}' + \left(\Omega + 1\right) y' = 0$$

$$x' = -A_x \cos\left(\omega_p t + \phi\right) \qquad \dot{x}' = A_x \omega_p \sin\left(\omega_p t + \phi\right) \qquad \ddot{x}' = A_x \omega_p^2 \cos\left(\omega_p t + \phi\right)$$

$$y' = kA_x \sin\left(\omega_p t + \phi\right) \qquad \dot{y}' = k\omega_p A_x \cos\left(\omega_p t + \phi\right) \qquad \ddot{y}' = -k\omega_p^2 A_x \sin\left(\omega_p t + \phi\right)$$

$$-k\omega_p^2 A_x \sin\left(\omega_p t + \phi\right) = -\left(2\omega_p + \left(\Omega + 1\right) k\right) A_x \sin\left(\omega_p t + \phi\right)$$

$$-k\omega_p^2 = -2\omega_p - \left(\Omega + 1\right) k \qquad k = \frac{\omega_p^2 + 1 + 2\Omega}{2\omega_p}$$

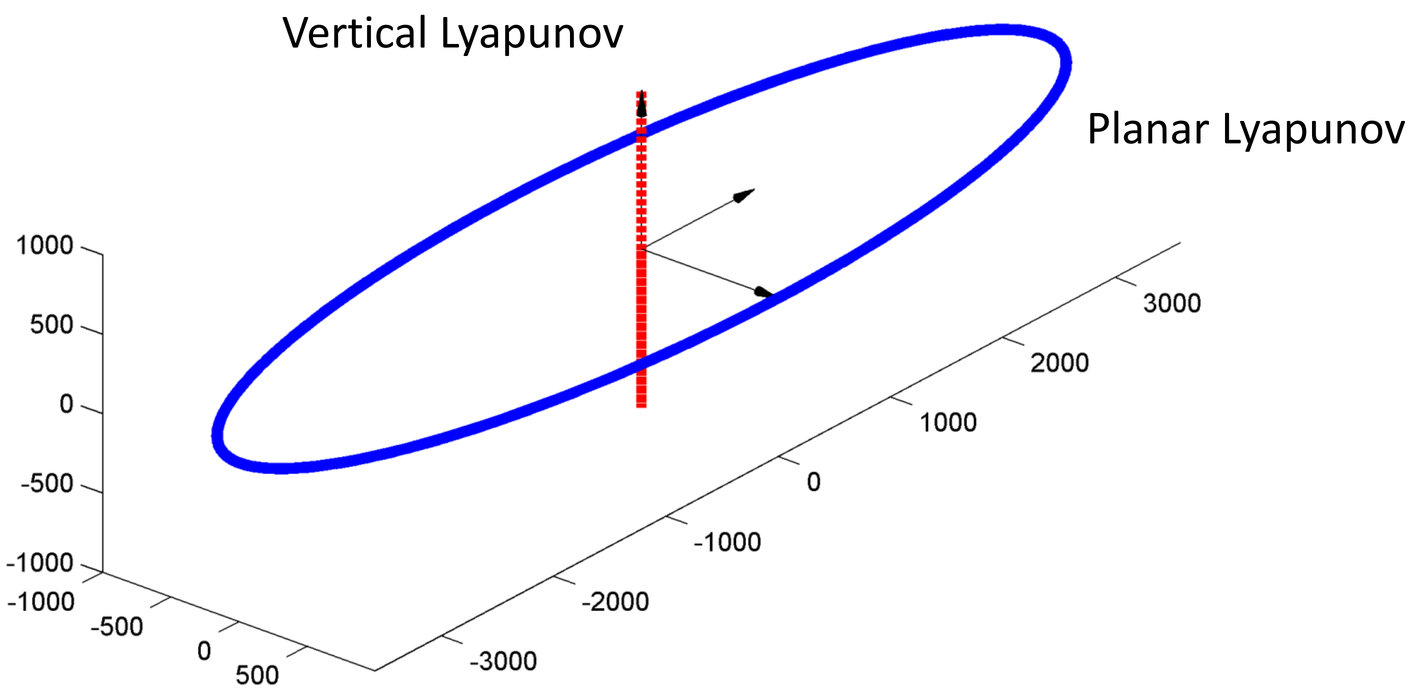
$$\boxed{\omega_p^2 = \frac{2 - \Omega + \sqrt{9\Omega^2 - 8\Omega}}{2}}$$

Periodic Orbits near L1 and L2

$$x' = -A_x \cos\left(\omega_p t + \phi\right)$$

$$y' = kA_x \sin\left(\omega_p t + \phi\right)$$

$$z' = A_z \sin\left(\sqrt{\Omega} t + \varphi\right)$$

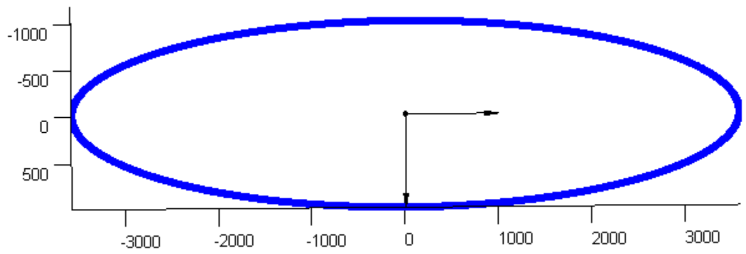


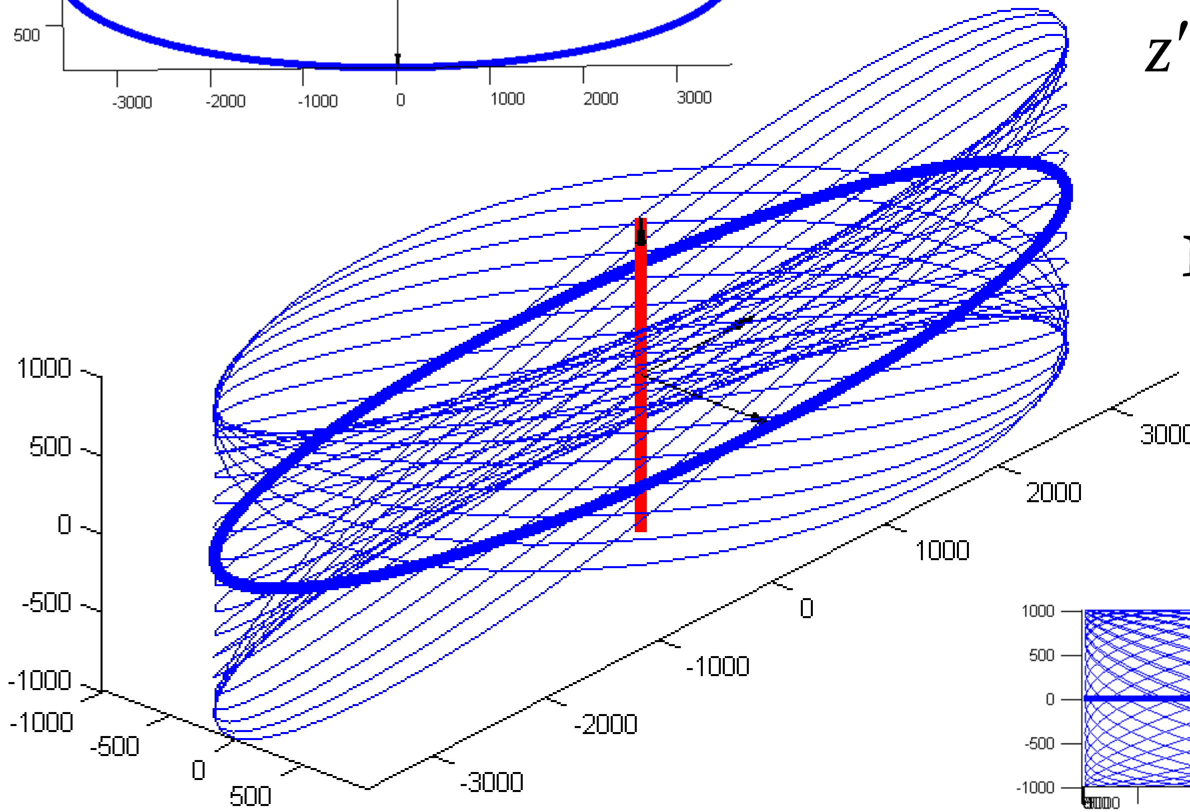
Periodic Orbits near L1 and L2

$$x' = -A_x \cos\left(\omega_p t + \phi\right)$$

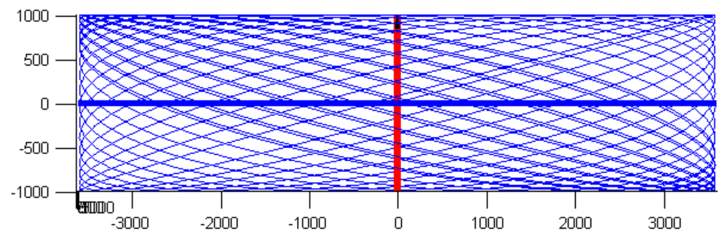
$$y' = kA_x \sin\left(\omega_p t + \phi\right)$$

$$z' = A_z \sin\left(\sqrt{\Omega} t + \varphi\right)$$





Lissajous Orbits



Exercise 2

Compute a planar Lyapunov Orbit for the Earth-Moon L1 point with an A_x of 1000km.

Periodic Orbits near L1 and L2

- Let us start computing a Lyapunov orbit by propagating the initial conditions as given by equations:

$$x' = -A_x \cos\left(\omega_p t + \phi\right) \qquad \dot{x}' = A_x \omega_p \sin\left(\omega_p t + \phi\right)$$

$$y' = kA_x \sin\left(\omega_p t + \phi\right) \qquad \dot{y}' = k\omega_p A_x \cos\left(\omega_p t + \phi\right)$$

$$\mathbf{x}_0 = \begin{pmatrix} -A_x & 0 & 0 & 0 & k\omega_p A_x & 0 \end{pmatrix}$$

$$t = 0$$

$$\phi = 0$$

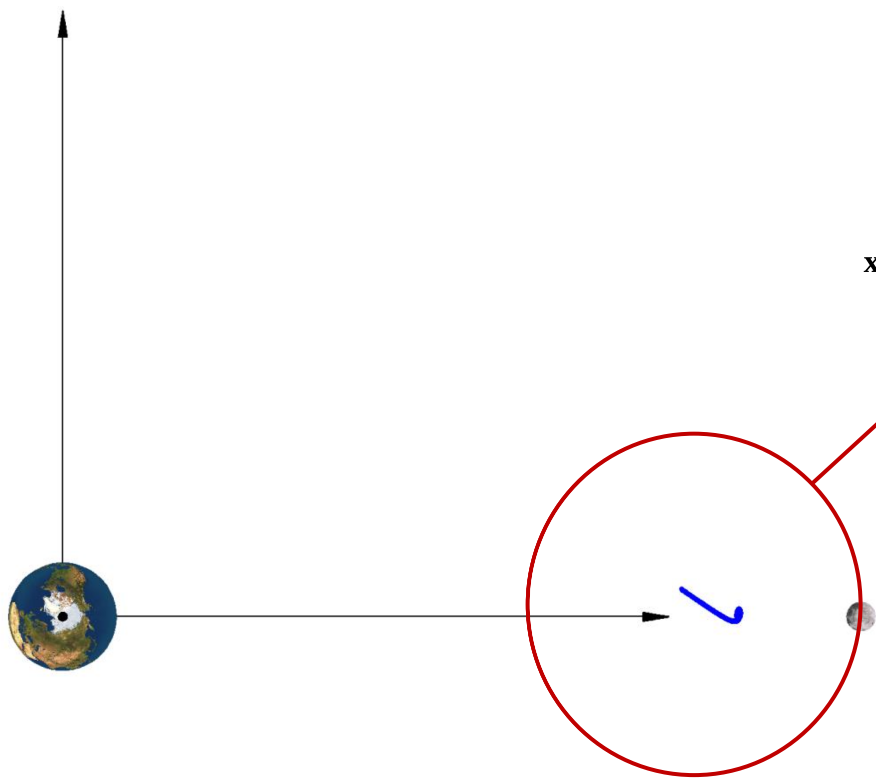
$$\omega_p = \sqrt{\frac{2 - \Omega + \sqrt{9\Omega^2 - 8\Omega}}{2}}$$

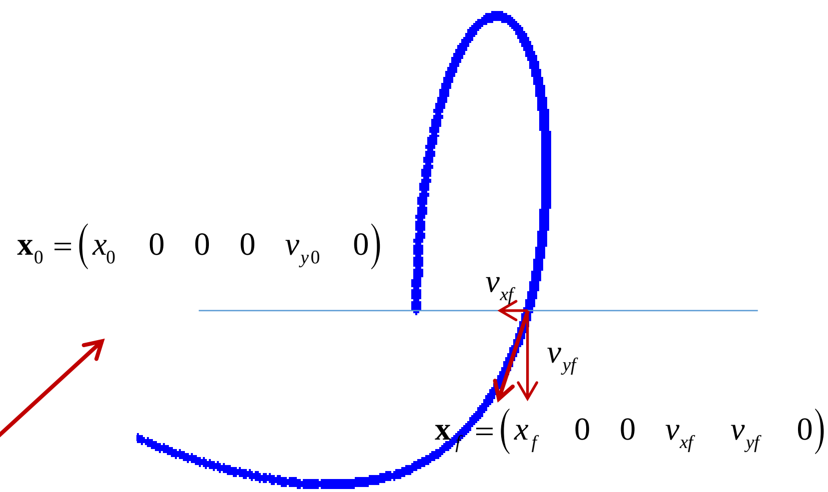
$$k = \frac{\omega_p^2 + 1 + 2\Omega}{2\omega_p}$$

$$\Omega = +\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}$$

Periodic Orbits near L1 and L2

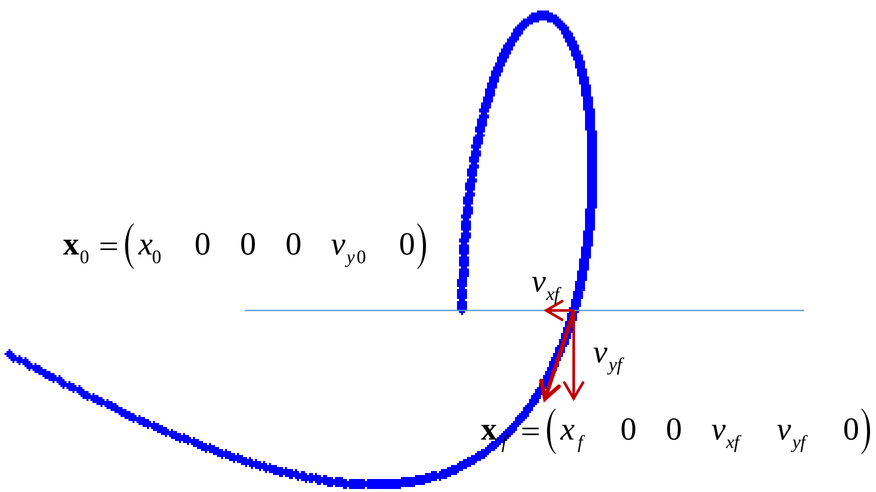
- Let us start computing a Lyapunov orbit by propagating the initial conditions.





- Due to the non-linearities, which were not considered in the orbit approximation

Periodic Orbits near L1 and L2



Transition Matrix:

$$\delta \mathbf{x} = \mathbf{\Phi}\left(\frac{T}{2}, 0\right)\delta \mathbf{x}_0$$

$$\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{M}(t)\mathbf{\Phi}(t, t_0)$$

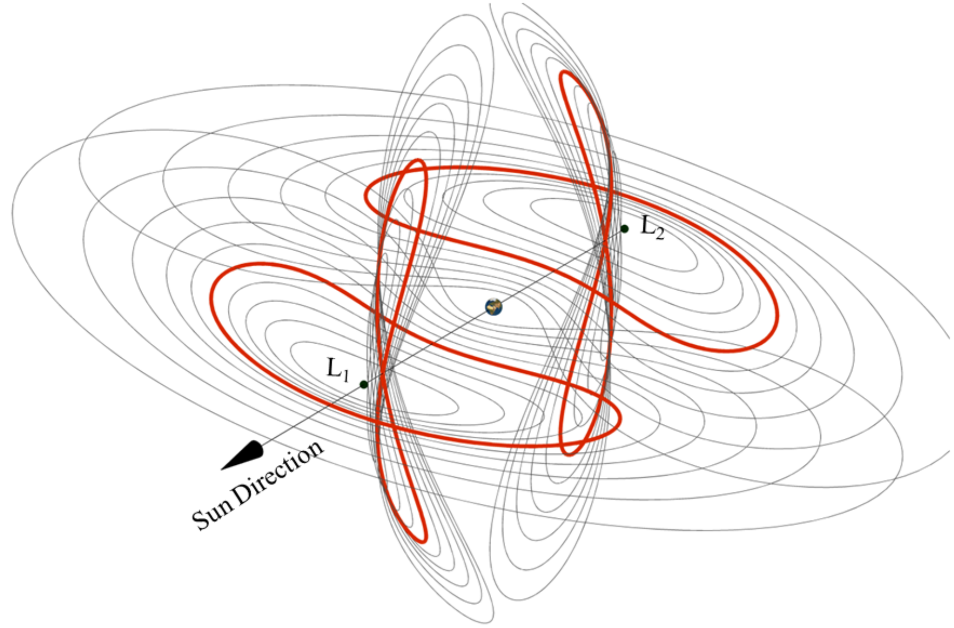
$$\mathbf{\Phi}(t_0, t_0) = \mathbf{I}$$

$$\begin{pmatrix} \Delta x_f \\ \Delta y_f \\ \Delta z_f \\ \Delta v_{xf} \\ \Delta v_{yf} \\ \Delta v_{zf} \end{pmatrix} = \mathbf{\Phi}_{6\times 6}\left(\frac{T}{2}, 0\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \Delta v_{y0} \\ 0 \end{pmatrix}$$

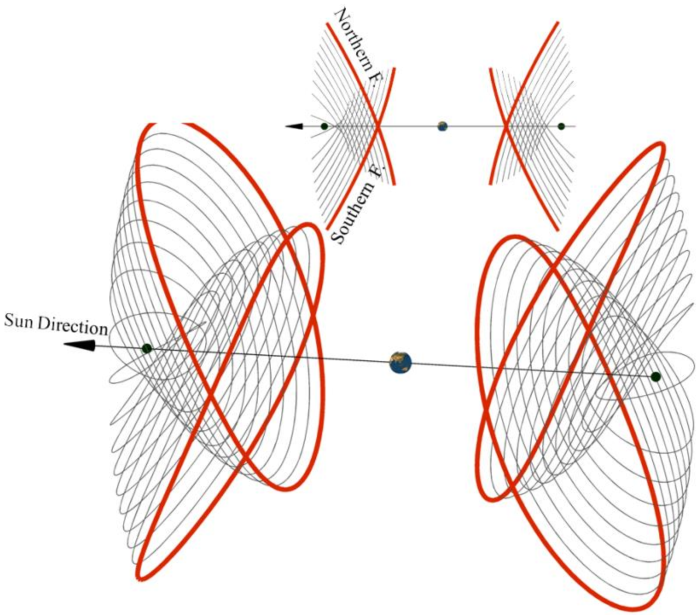
$$\Delta v_{xf} = \Phi_{(4\ \ 5)}\Delta v_{y0} \longrightarrow \Phi_{(4\ \ 5)}\Delta v_{y0} = -v_{xf} \longrightarrow \Phi_{(4\ \ 5)}\Delta v_{y0} = \frac{-v_{xf}}{\Phi_{(4\ \ 5)}}$$

Periodic Orbits near L1 and L2

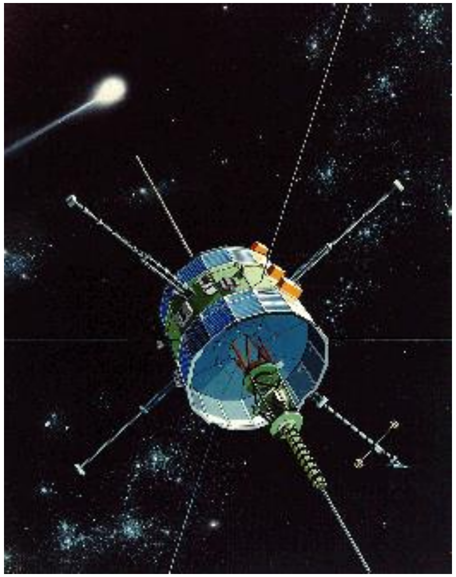
Lyapunov Orbits (Planar & Vertical)

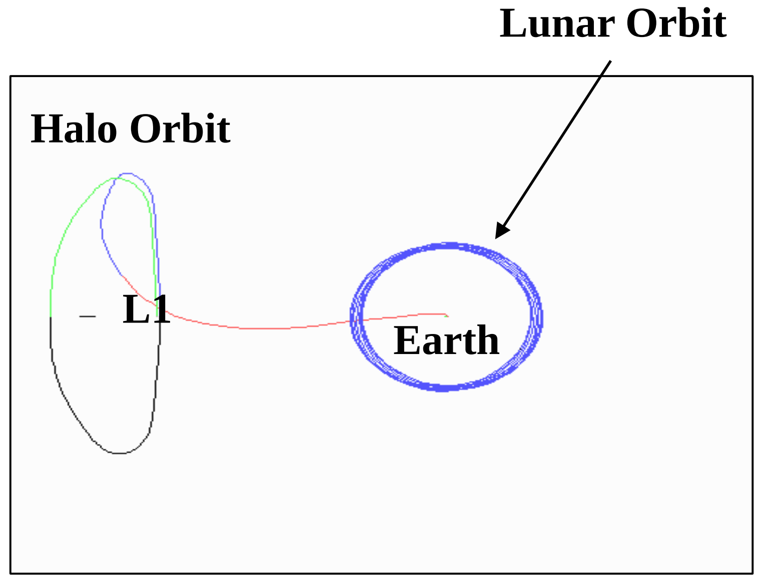


Halo Orbits



ISEE-3 / ICE

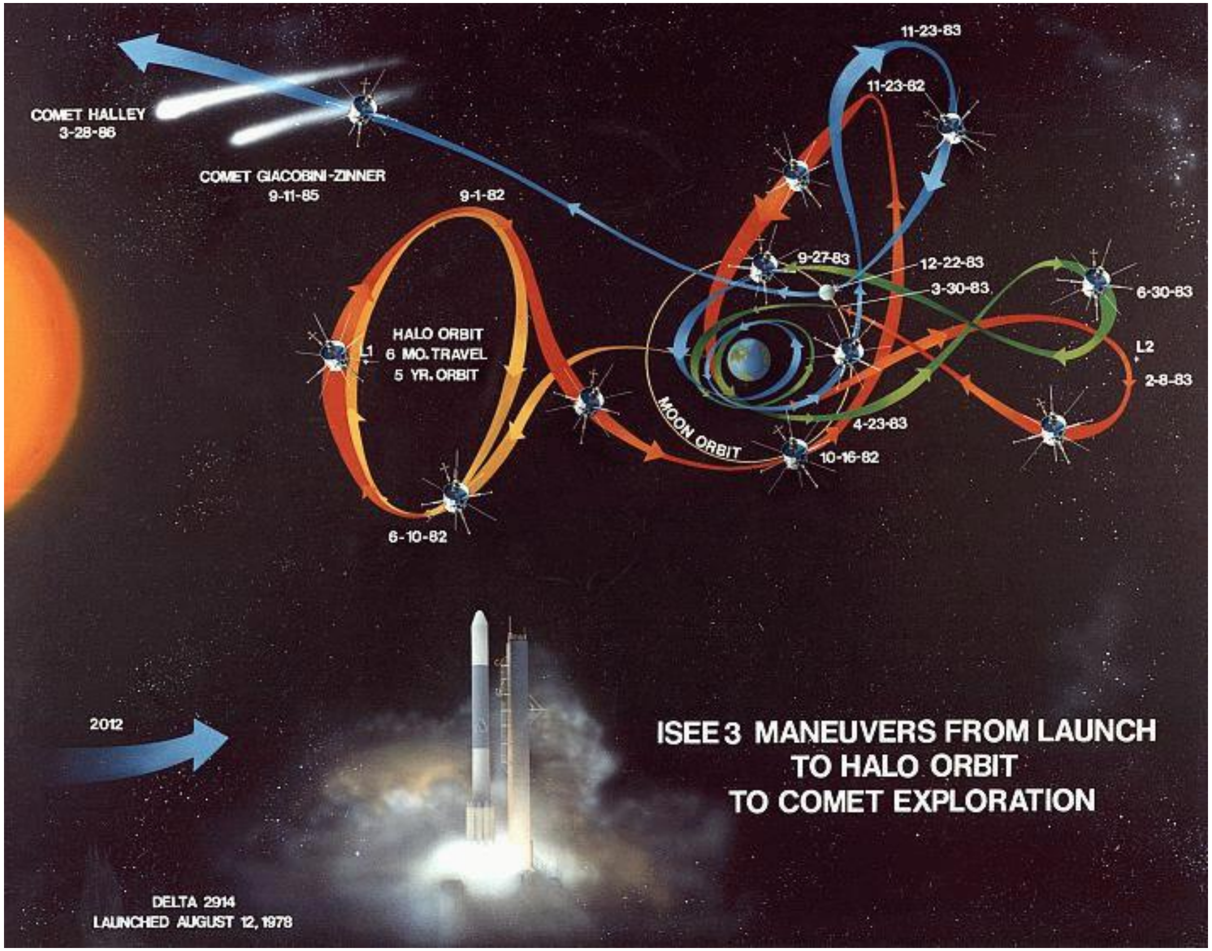




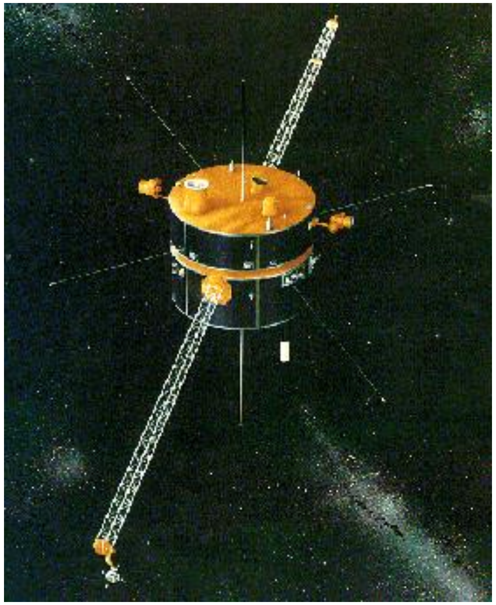
Solar-Rotating Coordinates, Ecliptic Plane Projection

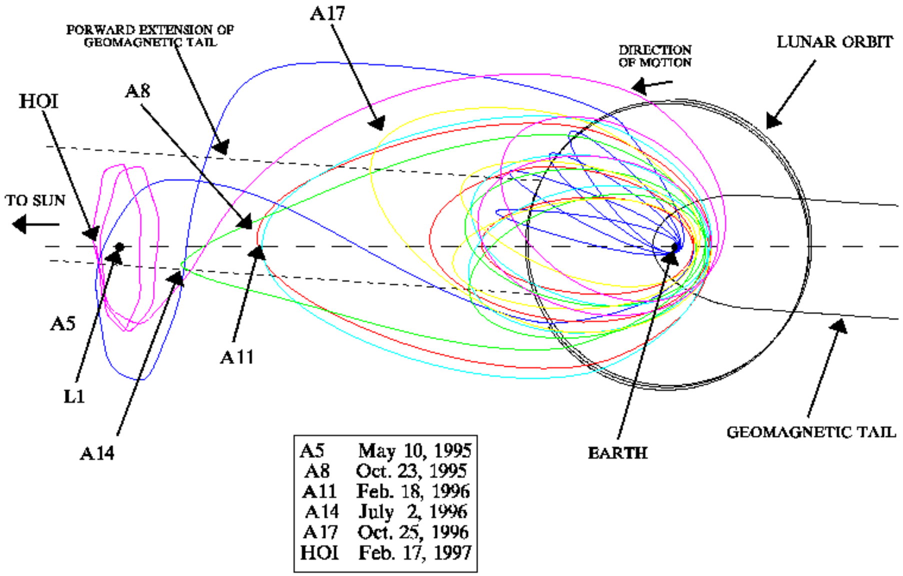
Mission:	**Investigate Solar-Terrestrial relationships, Solar Wind, Magnetosphere, and Cosmic Rays**
Launch:	**Sept., 1978, Comet Encounter Sept., 1985**
Lissajous Orbit:	**L1 Libration Halo Orbit, Ax=~175,000km, Ay = 660,000km, Az~ 120,000km, Class I**
Spacecraft:	**Mass=480Kg, Spin stabilized,**
Notable:	**First Ever Libration Orbiter, First Ever Comet Encounter**

ISEE-3 / ICE


COMET HALLEY
3-28-86
COMET GIACOBINI-ZINNER
9-11-85
9-1-82
11-23-83
11-23-82
9-27-83
12-22-83
3-30-83
6-30-83
HALO ORBIT
6 MO. TRAVEL
5 YR. ORBIT
L1
L2
2-8-83
MOON ORBIT
4-23-83
10-16-82
6-10-82
2012
ISEE 3 MANEUVERS FROM LAUNCH
TO HALO ORBIT
TO COMET EXPLORATION
DELTA 2914
LAUNCHED AUGUST 12, 1978

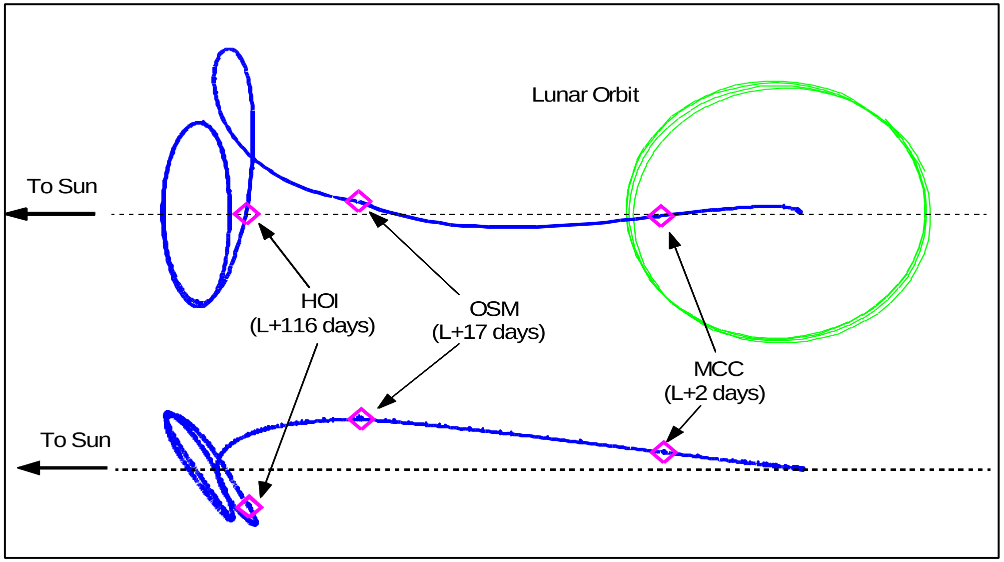
WIND





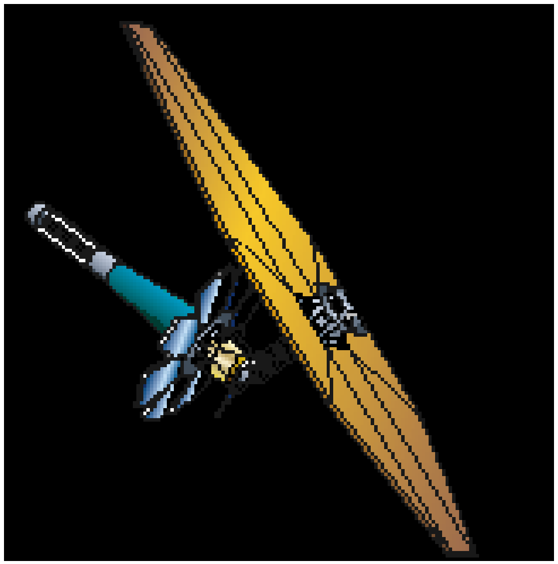
Mission:	**Investigate Solar-Terrestrial relationships, Solar Wind, Magnetosphere**
Launch:	**Nov., 1994, Multiple Lunar Gravity Assist**
Lissajous Orbit:	**Originally an L1 Lissajous Constrained Orbit, Ax~10,000km, Ay ~ 350,000km, Az~ 250,000km, Class I**
Spacecraft:	**Mass=1254Kg, Spin stabilized,**
Notable:	**First Ever Multiple Gravity Assist towards L1**

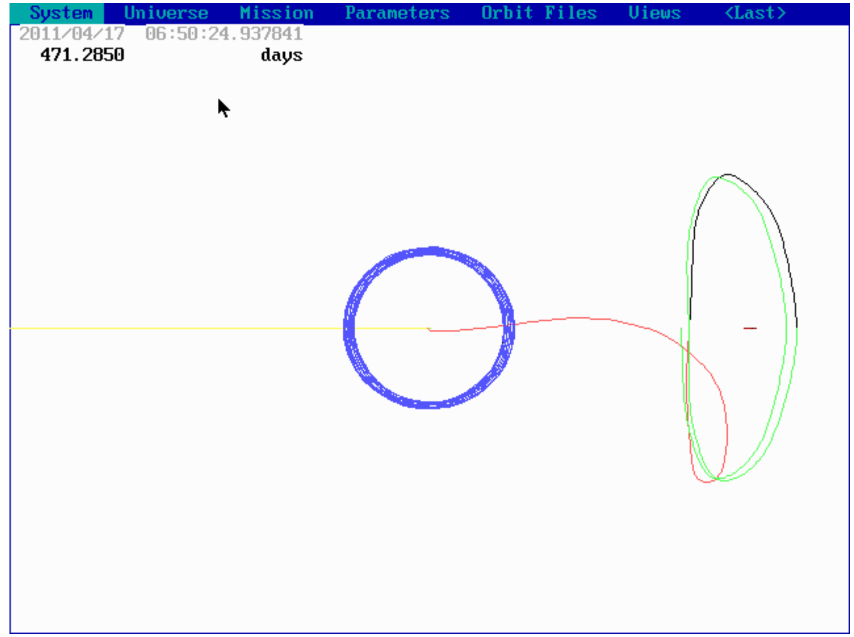
ACE



Mission:	**Investigate low-energy solar and high-energy galactic particles and Solar Wind**
Launch:	**August, 1997, Direct Transfer**
Lissajous Orbit: Class I	**L1 Lissajous Constrained Orbit, Ay~ <264,000, Ax~tbd km, Az~157,000km,**
Spacecraft:	**Mass=757Kg, Three Axis Stabilized, 'Solar' Pointing**
Notable:	**First Constrained Transfer Orbit**

James Webb Space Telescope





Mission: JWST Is Part of Origins Program. Designed to Be the Successor to the Hubble Space Telescope. NGST Observations in the Infrared Part of the Spectrum.

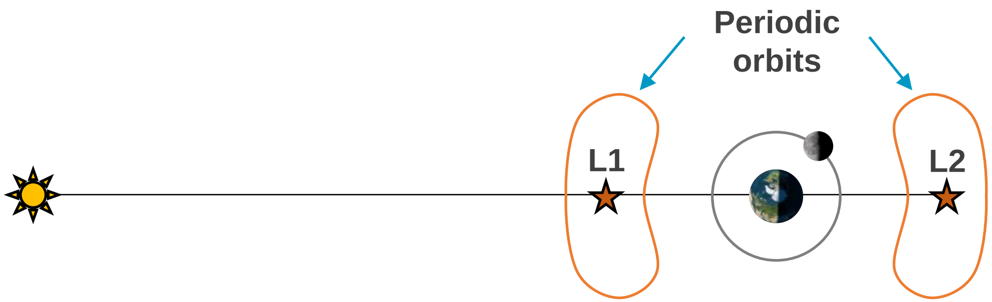
Launch: ~2011, Direct Transfer

Lissajous Orbit: L2 large lissajous, Ay~ 294,000km, Ax~800,000km, Az~ 131,000km, Class I or II

Spacecraft: Mass~6000kg, Three Axis Stabilized, 'Star' Pointing

Notable: Observations in the Infrared Part of the Spectrum. Important That the Telescope Be Kept at Low Temperatures, ~3^0K. Large Solar Shade/Solar Sail

Periodic Orbits near L1 and L2



- International Sun Earth Explorer 3 (1978)
- SOHO (1995)
- Wind (1994)
- Advanced Composition Explorer (1997)
- Genesis (2001)
- Deep Space Climate Observatory (2015)
- LISA Pathfinder (2015)

Previous and planned missions to L1 and L2

- Wilkinson Microwave Anisotropy Probe (2001)
- Herschel (2009)
- Planck (2009)
- Gaia (2013)
- James Webb Space Telescope (2019)
- Euclid (2020)

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