Mathematics and Programming for Astrodynamics & Trajectory Design

Pork-Chop Plots

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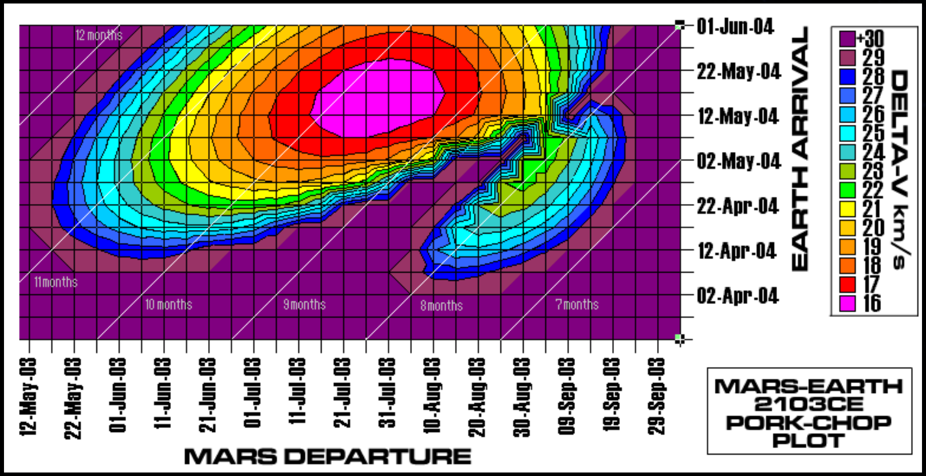


Bibliography

Some good books ordered by increasing depth and breath :

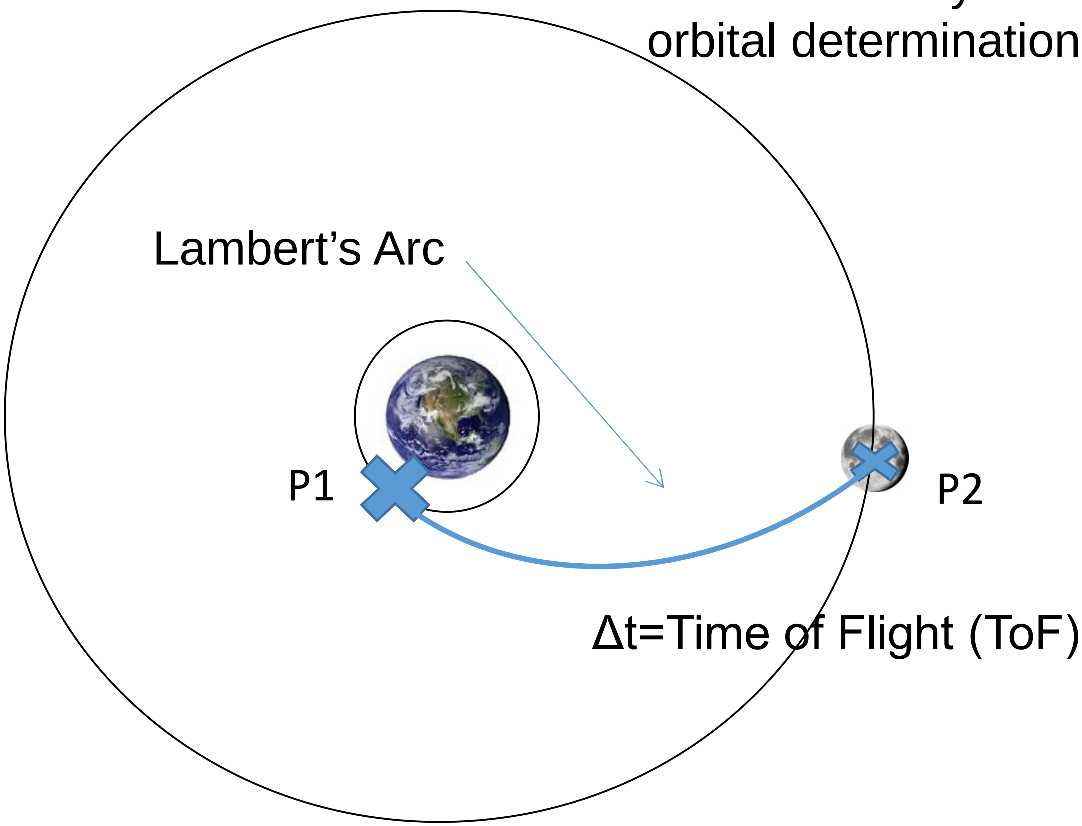
- *Orbital Mechanics,* V.A Chobotov, AIAA Education Series
- *Analytical Mechanics of Space Systems*, H.Shcaub, J.L.Junkins, AIAA Education Series
- *An Introduction to the Mathematics and Methods of Astrodynamics*, R.H.Battin, AIAA Educationan Series
- *Fundamentals of Astrodynamics and Applications*, D.A.Vallado, Space Technology Library
- Biesbroek, Robin. *Lunar and interplanetary trajectories*. Springer International Publishing, 2016.

Quick Summary Session 1

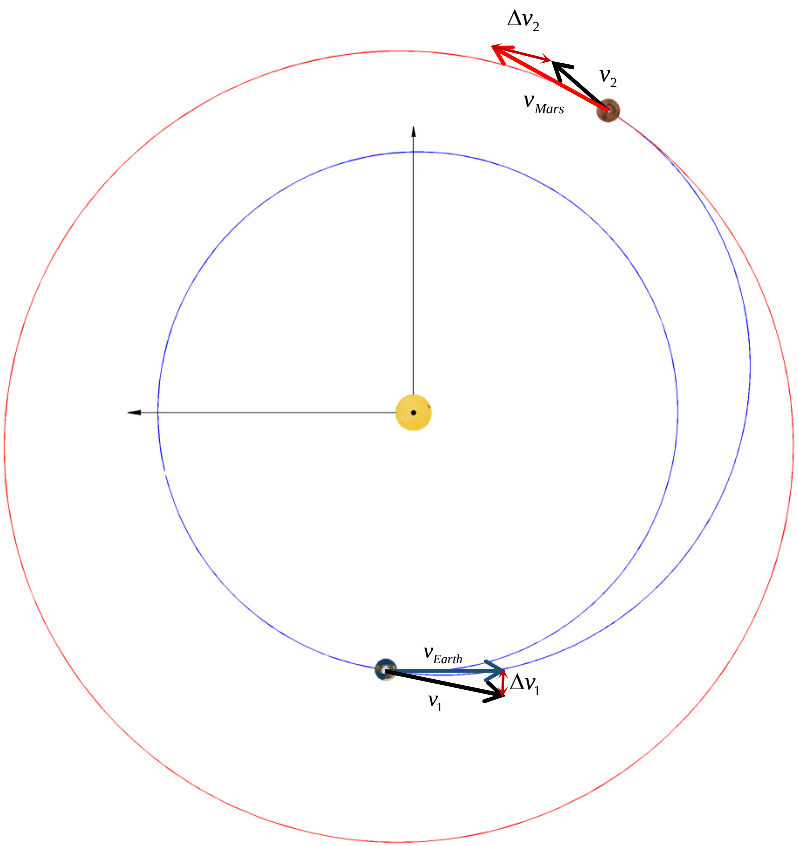
- Review Hohmann
- Introduction of the Two-body Orbital Boundary-value Problem.
  (a.k.a. Lambert Arc)
- The Minimum Energy Orbit
- Lagrange Coefficients Solution to Orbital Motion
- State Transition Matrix
- Differential Corrector
- Continuation Method
- A general algorithm to solve the Lambert Arc

Two-body Orbital Boundary-value Problem

- Two position vectors & time = Lambert's Problem
  - ✓ Solved not only for trajectory design but also for orbital determination.



Interplanetary Transfers



ExoMars TGO transfer to Mars:

- Departure: 14/03/2016
- Arrival: 15/10/2016

$$[v_1, v_2] = Lambert\left(r_1, r_2, ToF, t_m, \mu_{Sun}\right)$$

$$\Delta v_1 = \left|v_1 - v_{Earth}\right|$$

$$\Delta v_2 = \left|v_{Mars} - v_2\right|$$

$$\Delta v_1 = 3.75 \text{ km/s}$$

$$\Delta v_2 = 3.77 \text{ km/s}$$

$$\Delta v_{Total} = \Delta v_1 + \Delta v_2 = 7.5 \text{ km/s}$$

Session 2 - Content

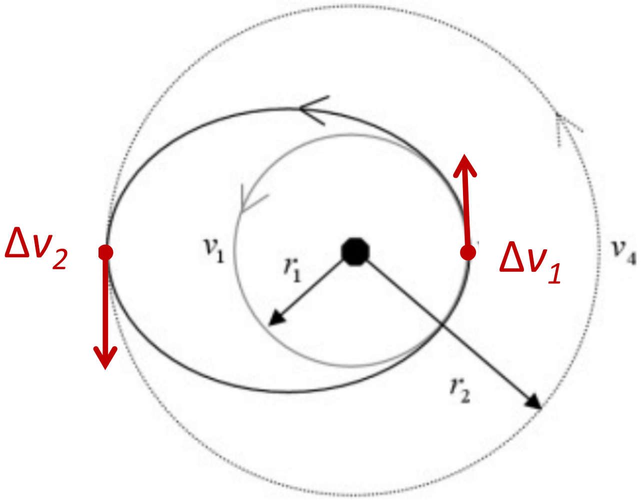
- Pork-chop plot and re-cap interplanetary transfers.
  - ✓Escape velocity
  - ✓C3 or specific kinetic energy
  - ✓Synodic period for a transfer opportunity
- Exercise: full mission analysis for a mission to Mars.

Recall AMA

What is the minimum DV for a Hohmann transfer Earth to Mars?

Recall AMA

What is the minimum DV for a Hohmann transfer Earth to Mars?



Δv_1 – Earth Departure

$$\Delta v_1 = \sqrt{\mu_{Sun}\left(\frac{2}{r_\oplus} - \frac{1}{a_{E2M}}\right)} - \sqrt{\frac{\mu_{Sun}}{r_\oplus}}$$

$$\Delta v_1 = \sqrt{\frac{\mu_{Sun}}{r_\oplus}}\left[\sqrt{\left(\frac{2r_M}{r_\oplus + r_M}\right)} - 1\right] = 2.945\,\text{km/s}$$

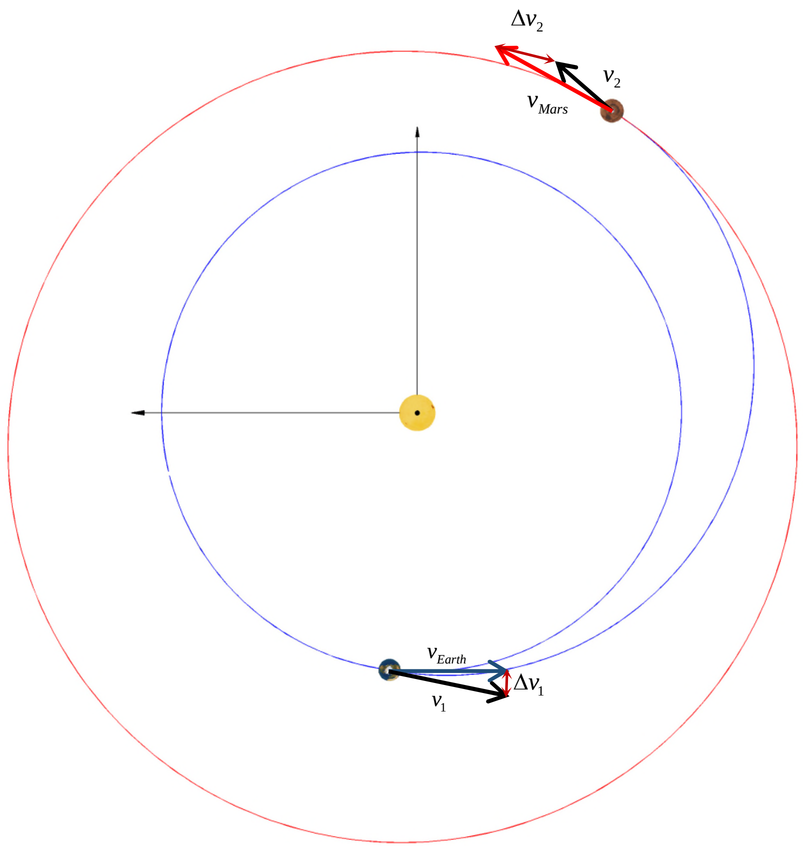
Δv_2 – Mars Arrival

$$\Delta v_2 = \sqrt{\frac{\mu_{Sun}}{r_M}} - \sqrt{\mu_{Sun}\left(\frac{2}{r_M} - \frac{1}{a_{E2M}}\right)}$$

$$\Delta v_2 = \sqrt{\frac{\mu_{Sun}}{r_M}}\left[1 - \sqrt{\left(\frac{2r_\oplus}{r_\oplus + r_M}\right)}\right] = 2.649\,\text{km/s}$$

$$\Delta v_{total} = \Delta v_1 + \Delta v_2 = 5.594\ \text{km/s}$$

Interplanetary Transfers



Then, why ExoMars TGO flew that particular trajectory? Was there any other better trajectory?

$$[v_1, v_2] = Lambert\left(r_1, r_2, ToF, t_m, \mu_{Sun}\right)$$

$$\Delta v_1 = \left| v_1 - v_{Earth} \right|$$

$$\Delta v_2 = \left| v_{Mars} - v_2 \right|$$

$$\Delta v_1 = 3.75 \text{ km/s}$$

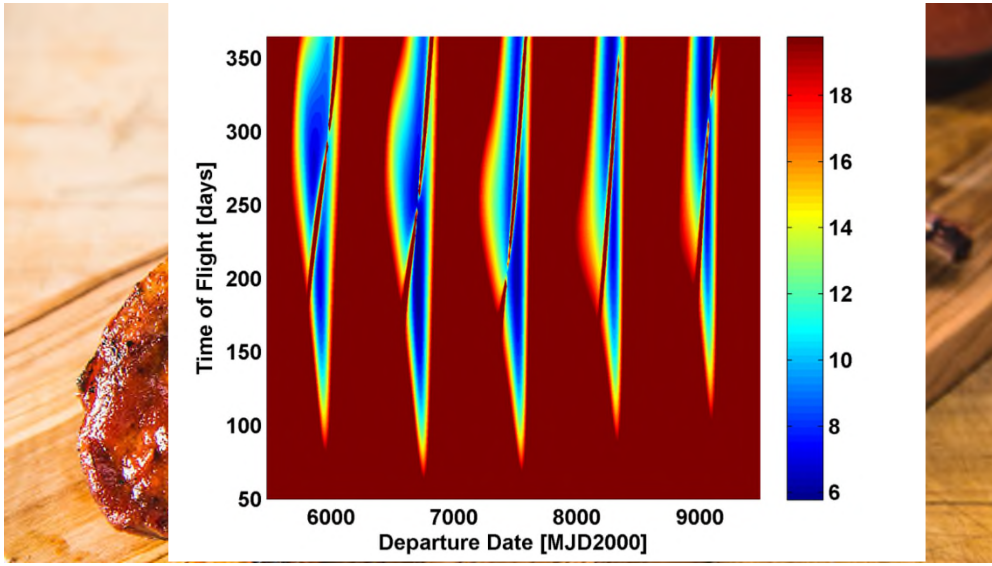
$$\Delta v_2 = 3.77 \text{ km/s}$$

$$\Delta v_{Total} = \Delta v_1 + \Delta v_2 = 7.5 \text{ km/s}$$

What the heck is a Pork-chop plot!

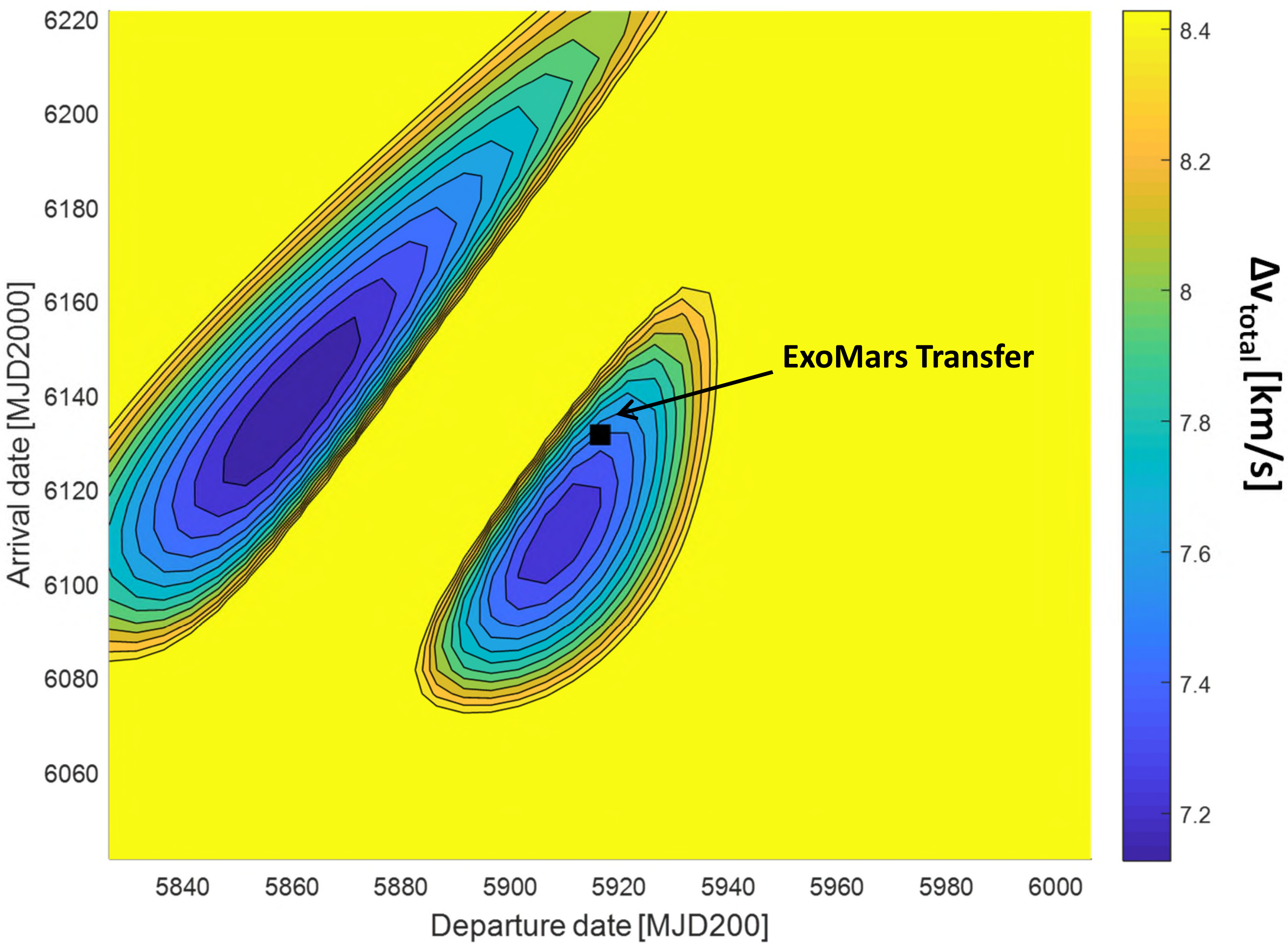
A Pork-chop is the first menu item on a trip to Mars*

- A space mission analyst is the person responsible of the trajectory design for a space mission.
- Mission analysis often use graphical methods to explore the best opportunities for launch.
- Lambert Arcs can be computed for a grid of departure and arrival dates, and a coloured plot can then easily indicate the best launch opportunities.
- Such a plot is well known as a pork-chop.

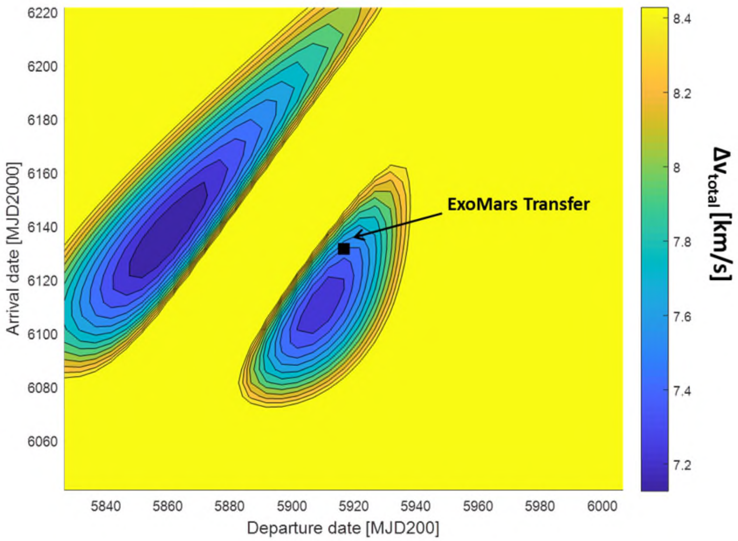


*https://mars.jpl.nasa.gov/spotlight/porkchopAll.html

Interplanetary Transfers



Interplanetary Transfers

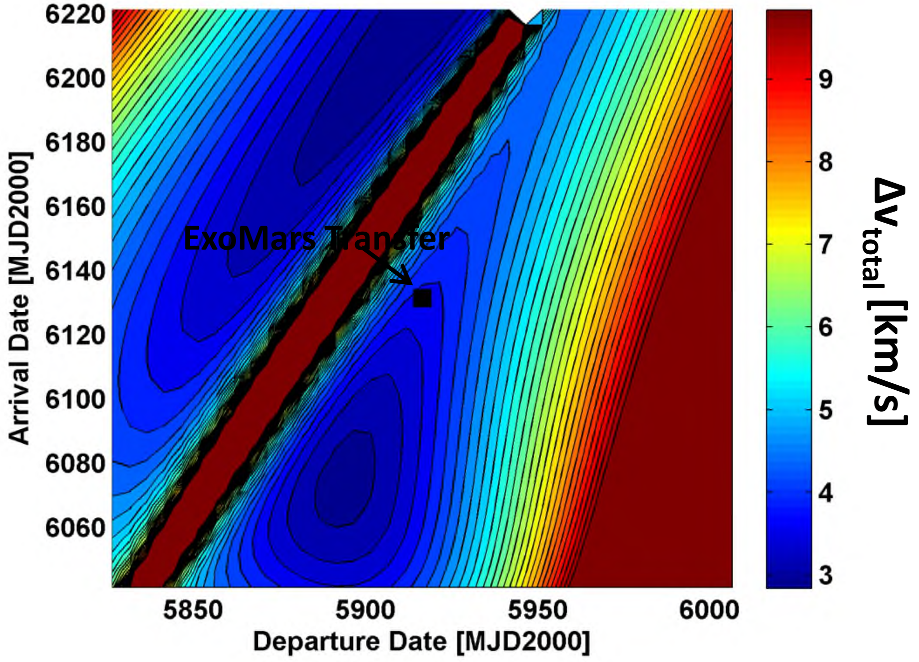


1. Why we don’t seem to have any launch opportunity as good as the Hohmann Transfer (i.e. 5.6 km/s)?

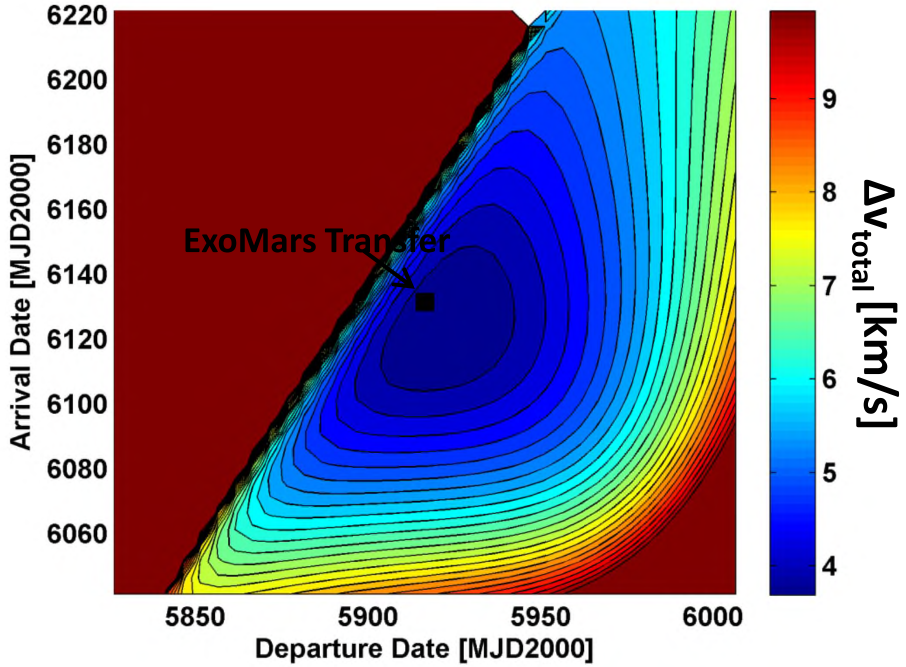
2. Why do you think TGO did not benefit from the 7.2 km/s opportunity in 5912 MJD2000? Or better yet, the 7.1 km/s opportunity in 5826 MJD2000?

Interplanetary Transfers

$$\Delta v = \left| v_1 - v_{Earth} \right|$$



$$\Delta v = \left| v_{Mars} - v_2 \right|$$



Interplanetary Transfers – Patched conics (Reminder)

- The patched conic method is a technique to estimate the Δv costs for interplanetary transfers and fly-bys.

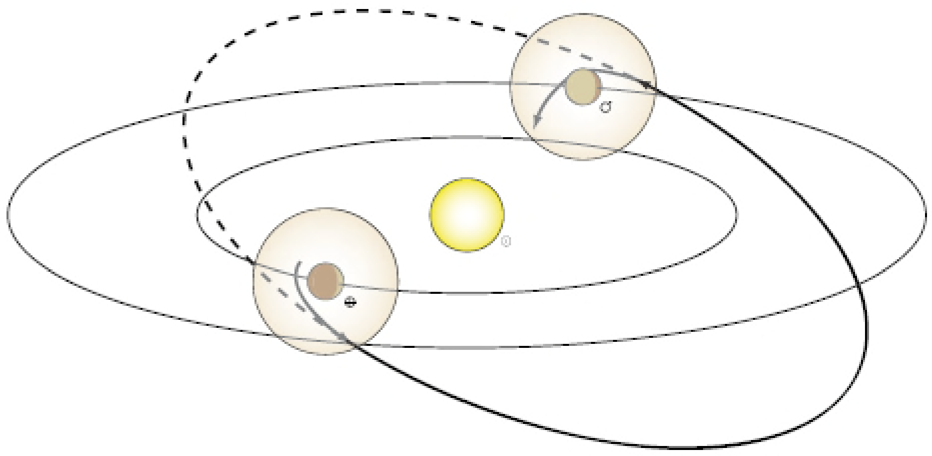
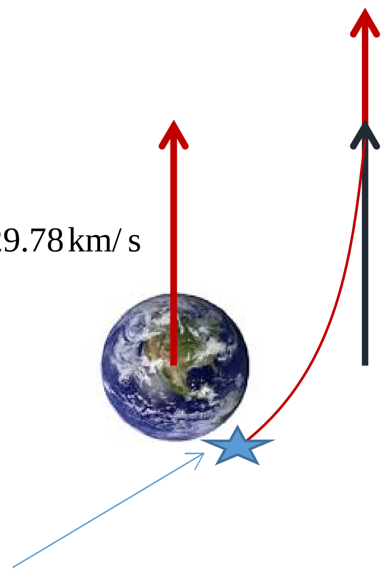


Figure 10.5: Approximating a Trajectory Among Multiple Bodies Through Spheres of Influences

1. Initial Parking orbit around Earth (e.g. circular 250 km)
2. Depart from Earth in a Hyperbolic escape trajectory.
3. Heliocentric cruise in an elliptic orbit.
4. Hyperbolic entry around Mars.
5. Capture manoeuvre during Hyperbolic transit at Mars.

Interplanetary Transfers – Earth Departure

$$v_{\oplus} = \sqrt{\frac{\mu_{Sun}}{r_{\oplus}}} = 29.78\,\mathrm{km/s}$$



$$v_{s/c} = v_{\oplus} + v_{\infty}^{+}$$

Hyperbolic escape velocity:

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon \quad \text{Vis viva}$$

$$\frac{v_{\infty}{}^{2}}{2} - \frac{\mu}{r_{\infty}} = \varepsilon \longrightarrow \varepsilon = \frac{v_{\infty}^{2}}{2}$$

$$\text{if } r_{\infty} = \infty \Rightarrow \frac{\mu}{r_{\infty}} = 0$$

Energy is conserved, so at periapsis:

$$\frac{v_{p}{}^{2}}{2} - \frac{\mu}{r_{p}} = \varepsilon \longrightarrow \frac{v_{p}{}^{2}}{2} = \frac{v_{\infty}^{2}}{2} + \frac{\mu}{r_{p}} \longrightarrow v_{p} = \sqrt{2\left(\frac{\mu_{\oplus}}{r_{p}} + \frac{v_{\infty}^{2}}{2}\right)}$$

Interplanetary Transfers – Earth Launch

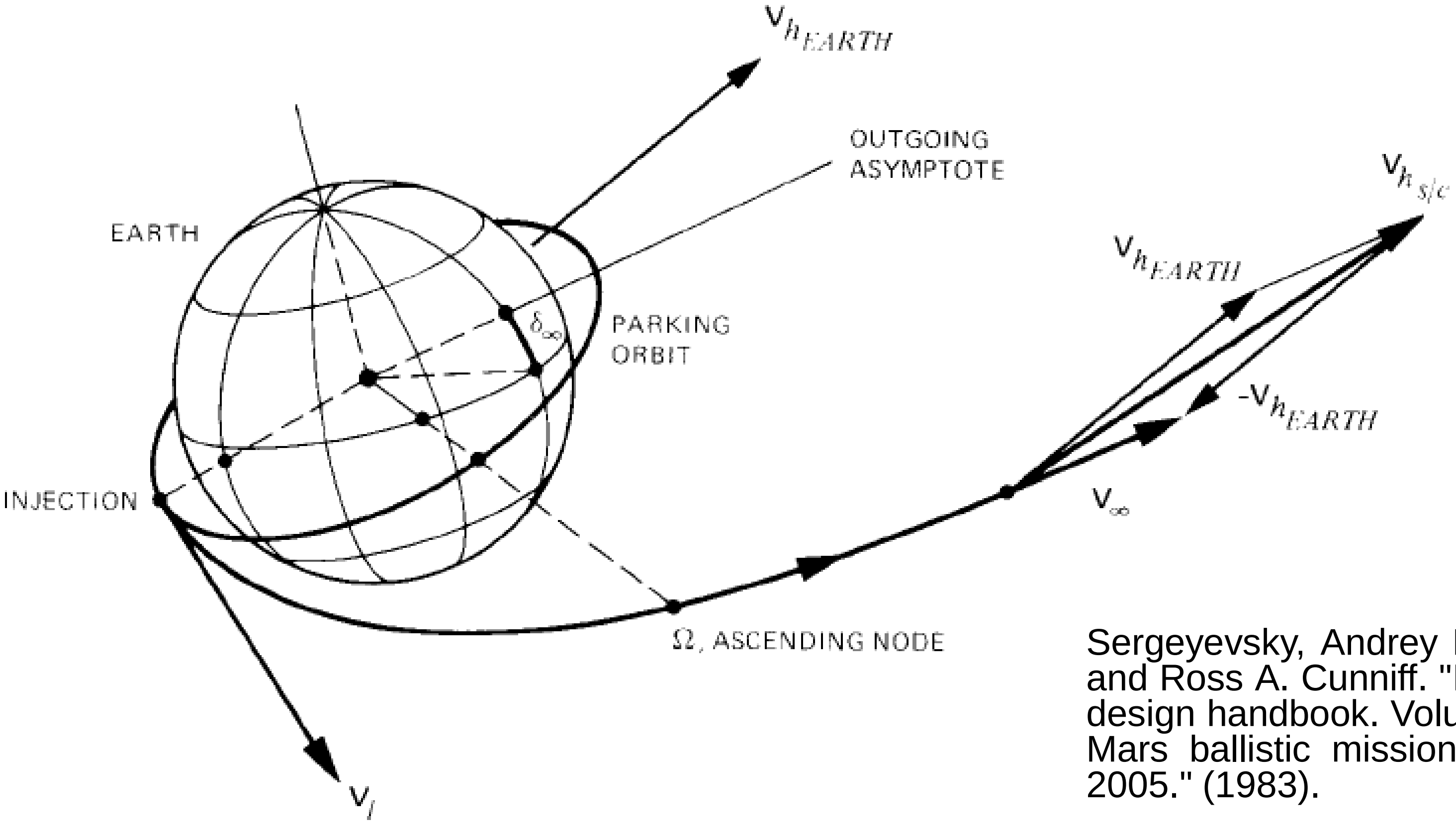


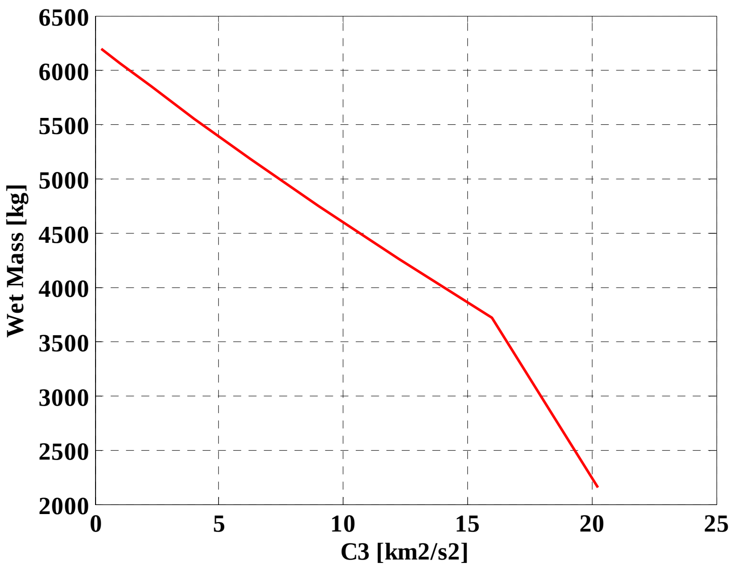
Fig. 2. Departure geometry and velocity vector diagram

Sergeyevsky, Andrey B., Gerald C. Snyder, and Ross A. Cunniff. "Interplanetary mission design handbook. Volume 1, part 2: Earth to Mars ballistic mission opportunities, 1990-2005." (1983).

Interplanetary Transfers – Earth Escape

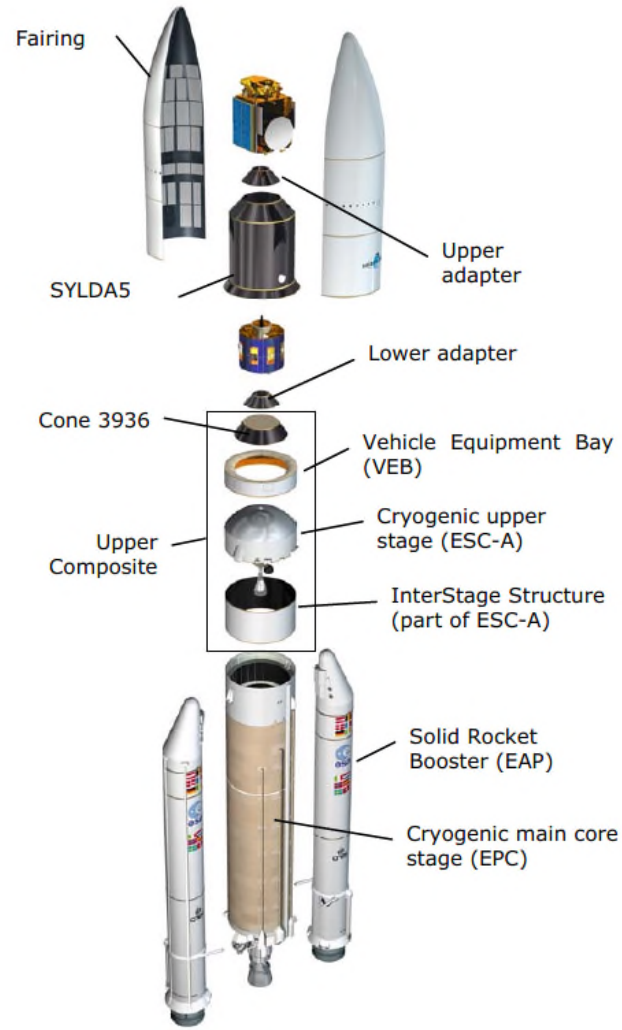
- The launch vehicle (LV), *generally*, inserts the spacecraft into escape trajectory.
- The LV performance is measured with the mass and escape C3.

$$C_3 = v_\infty^2 = 2\varepsilon$$



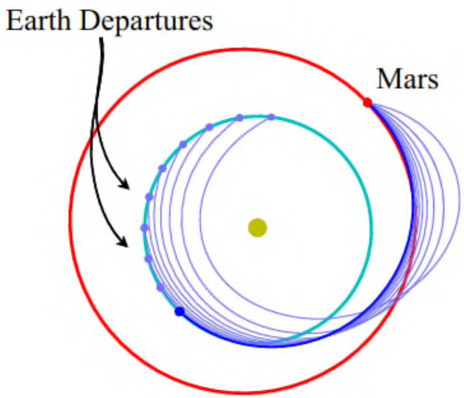
C3 [km2/s2]	Wet Mass [kg]
0	6191
1	6059
2.25	5844
4	5549
6.25	5180
9	4744
12.25	4251
16	3714
20.25	2155

Ariane 5 Escape Performance

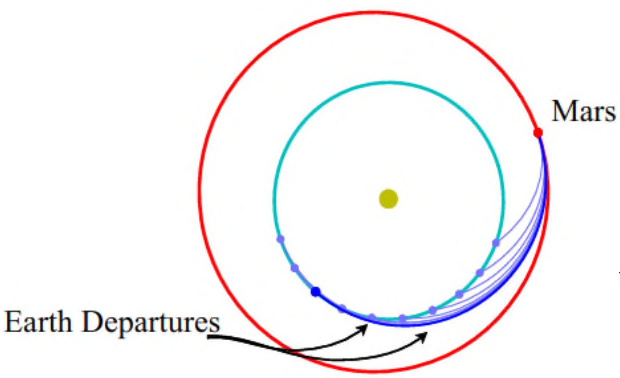


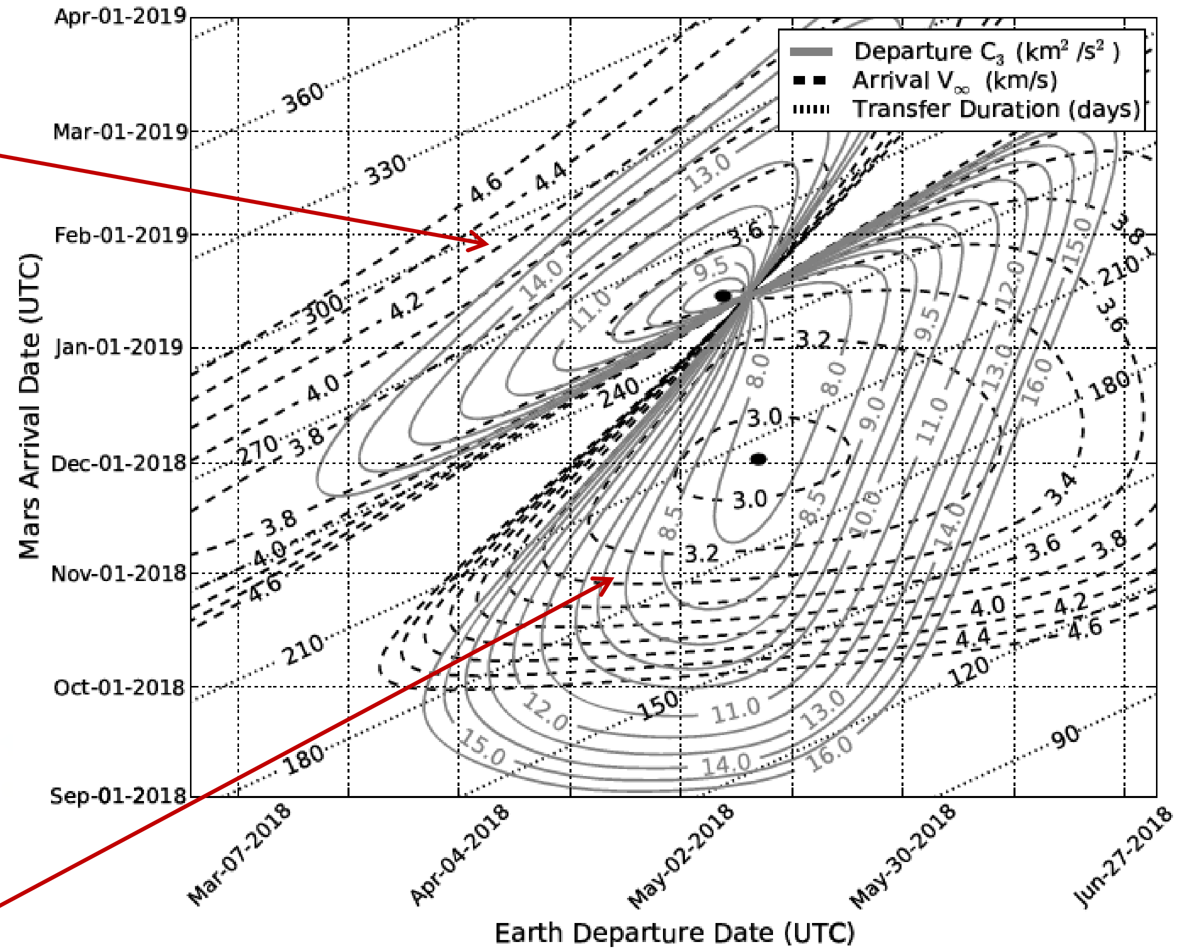
Interplanetary Transfers – Porkchop Plots

Type II - Long



Type I - Short



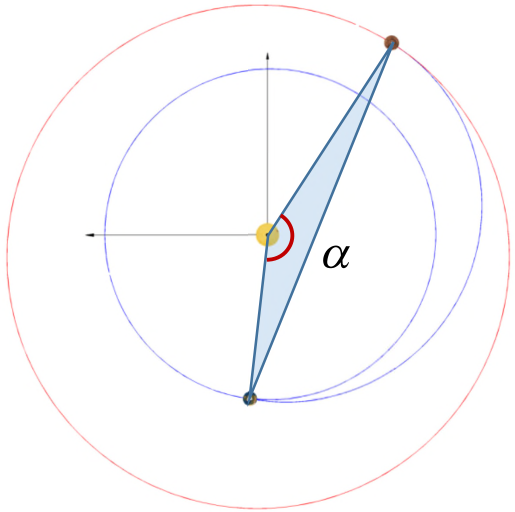


Source (Vallado, 2013)

Interplanetary Transfers – Synodic Period

- A synodic period T_s is defined as the time required for a particular phase angle between planets to repeat itself.

$$T_s = \frac{2\pi}{\left|n_2 - n_1\right|}$$



$$n_{Earth} = \sqrt{\frac{\mu_S}{a_E^3}} \approx 2\times10^{-7}\,\text{rad/s}$$

$$n_{Mars} = \sqrt{\frac{\mu_S}{a_M^3}} \approx 10^{-7}\,\text{rad/s}$$

$$T_s = \frac{2\pi}{\left|n_E - n_M\right|} = 2.14\ \text{years}$$

Interplanetary Transfers – Synodic Period

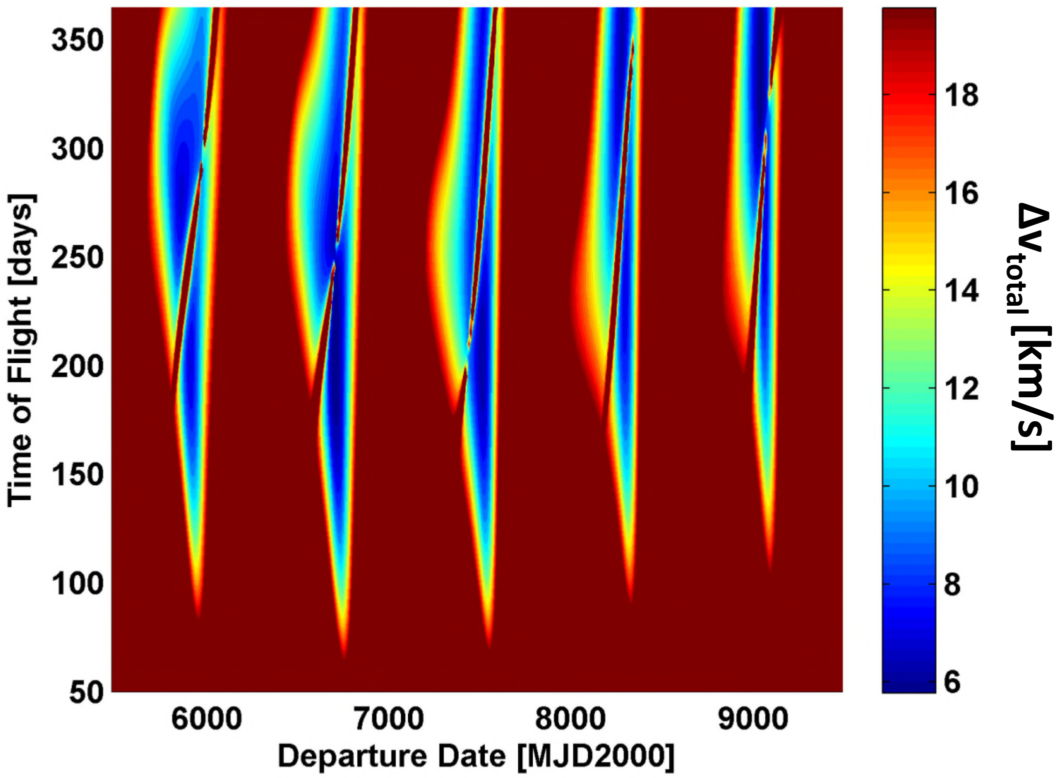
- A synodic period T_s is defined as the time required for a particular phase angle between planets to repeat itself.

$$T_s = \frac{2\pi}{|n_2 - n_1|}$$

Table 12.2: Synodic Periods between Earth and Other Planets

Planet	Heliocentric Ang. Rate [deg/year]	Revolution Period about the Sun [years]	Synodic Period [years]
Mercury ☿	1493.04	0.24	0.318
Venus ♀	584.60	0.62	1.600
Mars ♂	191.20	1.88	2.138
Jupiter ♃	30.30	11.88	1.093
Saturn ♄	12.18	29.57	1.036
Uranus ⛢	4.27	84.17	1.013
Neptune ♆	2.17	165.40	1.007
Pluto ♇	1.45	248.81	1.005

Interplanetary Transfers – Pork-chop Plot



Exercise 2

Create a script that computes the total Δv for all the possible transfers of the following grids of departure and arrival times.

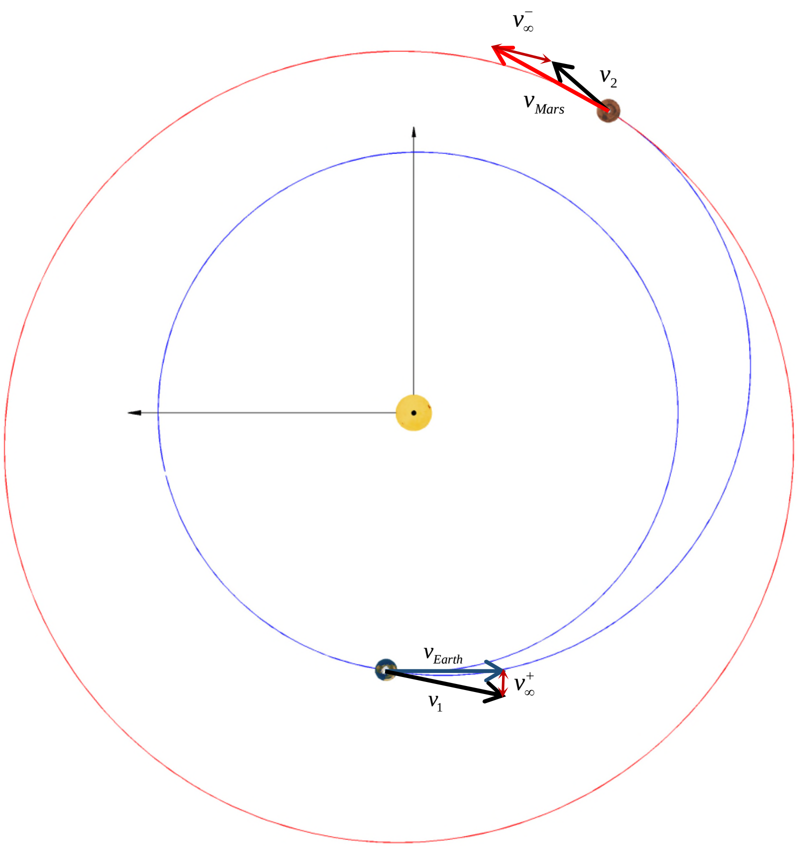
```
%-------------------------------------------------
% Earth to Mars from 14/3/16 to 15/10/2016
InTime=date2mjd2000([2016 03 14 0 0 0]);
FinTime=date2mjd2000([2016 10 15 0 0 0]);

% Generate Grid of conditions to analyse
Range=3*30; % days
DepartureGrid=InTime-Range:5:InTime+Range;
ArrivalGrid=FinTime-Range:5:FinTime+Range;
```

Plot these results as a colour map, where X and Y axis show departure and arrival times and colour denotes total Δv.

```
minDV=min(min(DV_solutions));
figure
hold on
contourf(DepartureGrid,ArrivalGrid,DV_solutions',[minDV:0.1:8.5])
plot(InTime,FinTime,'MarkerFaceColor',[0 0 0],'MarkerEdgeColor',[0 0 0],...
    'MarkerSize',10,...
    'Marker','square',...
    'LineStyle','none');
```

When completed, connect to Socrative Room GPQK5UNSK and indicate it so in the relevant question

Interplanetary Transfers - Exercise 3



Compute the actual total DV of ExoMars TGO transfer assuming a initial circular orbit at the Earth of 250 km altitude, and a final Mars operational orbit 400 km altitude circular orbit.

Interplanetary Transfers - Exercise 3

Compute the actual total DV of ExoMars TGO transfer assuming a initial circular orbit at the Earth of 250 km altitude, and a final Mars operational orbit 400 km altitude circular orbit.

1. Use algorithm LambertArc_ATATD: $\left(\mathbf{r}_{Earth}, \mathbf{r}_{Mars}, \Delta t, t_m, \mu_{sun} \Rightarrow \dot{\mathbf{r}}_{SC@E}, \dot{\mathbf{r}}_{SC@M}\right)$
2. Compute v_∞'s: $v_{\infty @E} = \left|\dot{\mathbf{r}}_{SC@E} - \dot{\mathbf{r}}_{Earth}\right|$ $\quad v_{\infty @M} = \left|\dot{\mathbf{r}}_{SC@M} - \dot{\mathbf{r}}_{Mars}\right|$
3. Compute velocities at parking orbit and periapis: $v_{p.o.} = \sqrt{\dfrac{\mu_\oplus}{r_p}}$

$$v_{rp} = \sqrt{2\left(\frac{\mu_\oplus}{r_p} + \frac{v_\infty^2}{2}\right)}$$

When completed, connect to Socrative Room GPQK5UNSK and indicate it so in the relevant question

Interplanetary Transfers - Exercise 3

Compute the actual total DV of ExoMars TGO transfer assuming a initial circular orbit at the Earth of 250 km altitude, and a final Mars operational orbit 400 km altitude circular orbit.

Solving the Lambert Arc

```
r1=rEarth;
r2=rMars;
tm=+1; % Short Way (+1);

[r1dot,r2dot]=LambertArc_ATATD2016(r1, r2, TOFETG_MJD2000, tm, muSun);

v_inf_Earth=norm(vEarth-r1dot)
v_inf_Mars=norm(vMars-r2dot)
```

```
v_inf_Earth =

       3.7416


v_inf_Mars =

       3.7695
```

Interplanetary Transfers - Exercise 3

Compute the actual total DV of ExoMars TGO transfer assuming a initial circular orbit at the Earth of 250 km altitude, and a final Mars operational orbit 400 km altitude circular orbit.

Computation of actual DVs using patched conic approximation

```
%--------------------------------------------------------------------------
% Extra Constants
muEarth=getAstroConstants('Earth','mu');
RE=getAstroConstants('Earth','Radius');

muMars=getAstroConstants('Mars','mu');
RM=getAstroConstants('Mars','Radius');
%--------------------------------------------------------------------------
% Earth Departure
RC=RE+250; % Parking Orbit [km]
Vc=sqrt(muEarth/RC); % velocity in parking orbit
Vp=sqrt(2*(muEarth/RC+v_inf_Earth^2/2)); % velocity periapsis in escape orbit

DVE1=Vp-Vc % Actual manoeuvre to be provided by the spacecraft propulsion
            % system

% Mars Arrival
RO=RM+400;
Vp=sqrt(2*(muMars/RO+v_inf_Mars^2/2)); % velocity periapsis in arrival hyperbolic
                                       % orbit
Vc=sqrt(muMars/RO); % velocity in operational orbit

DVM2=Vp-Vc % Actual manoeuvre that need to be performed by the spacecraft
            % in order to be inserted into a permanently capture trajectory
%--------------------------------------------------------------------------
% Transfer DV using a patched conic approximation.
DVTotal=DVE1+DVM2
```

```
DVE1 =

    3.8328


DVM2 =

    2.7055


DVTotal =

    6.5384
```

ATD2022-23 Progress Check

Please, connect to Socrative (Room: GPQK5UNSK) and select the last completed exercise.

Assessed Exercises

- Simple: Earth-Mars Proton Launch transfer
- Medium: Earth-Asteroid Rendezvous F-Class Mission
- Medium to high: Asteroid Mining Technology Demonstrator.

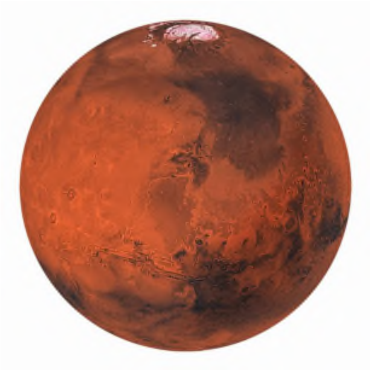
Deadline before 24th March at 13h UK Time.

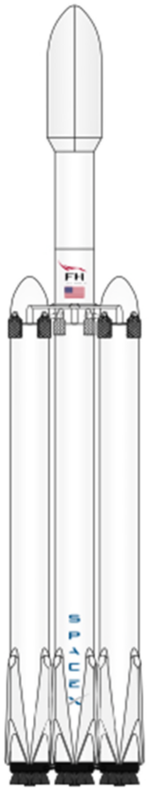
Deadline for part-timers 7th April at 13h UK Time.

The Turnitin Submission should be both the report and all the code.

Assessed Exercises 1

Your space agency is preparing a Mars remote sensing observation mission. The mission is intended to be launched between 1/01/2024 and 31/12/2025. The orbiter has a mass of 1250 kg and a propulsion system with an Isp of 310s. The targeted final orbit is a 400 km circular orbit. Perform a complete assessment of Earth-Mars direct insertion launch opportunities for a Falcon 9 Heavy launch with booster recovery (see below performance of this launcher). Your space agency programme manager is interested to know the extend of the launch window opportunity and your recommendation as baseline launch date.





Falcon Heavy

C3 [km2/s2]	Wet Mass [kg]
0	5750
5	5000
10	4250
15	3600
20	3000
25	2400
30	1900
40	1000

C3 Performance for a Direct Insertion escape trajectory for Falcon 9 Heavy with booster recovery

Assessed Exercises 2

A team of planetary scientist want to propose a exploration mission to a Potentially Hazardous Asteroid for the Fast-Class Mission Call from ESA's Science Directorate. The mission boundary conditions state that your spacecraft will be launched together with ESA M-Class Mission ARIEL and piggy-backed to the Sun-Earth L2 point. Your mission will be deployed at the L2 point between 1/1/2028 and 31/12/2028. The total wet mass of your mission cannot be larger than 850 kg and the main propulsion system for the spacecraft has an Isp of 310s. Find a good rendezvous opportunity that is suitable for their mission.

Hint: Departure from L2 is equivalent to a v_∞ =0 km/s.

If their payload weights 20 kg. Can you find any suitable asteroid to rendezvous with? What are the launch window opportunities? Assume that about 15% of the final dry mass is available for payload.

Assessed Exercises 3 – Asteroid Mining Tech Demo

Search for Asteroid Sample Return mission opportunities for commercial nanosatellite demonstrators for asteroid mining.

The mission should be launched not before 1/1/2030 and completed not later than 31/12/2035. Given the constraints on launch and Δv capability, **ideally**, the nanosat transfer opportunity should satisfy the following requirements:

- The hyperbolic excess velocity of the departure shall be v_∞<1.5 km/s.
- The hyperbolic excess velocity of the arrival shall be v_∞<1.5 km/s.
- The asteroid arrival rendezvous manoeuvre shall be < 500 m/s
- The departure rendezvous manoeuvre shall be < 500 m/s
- Asteroid operation phase should be between 2 to 6 month long.

Provide an analysis of transfer opportunities satisfying the above boundary conditions or challenge the requirements if they turn out to be impossible to satisfy.