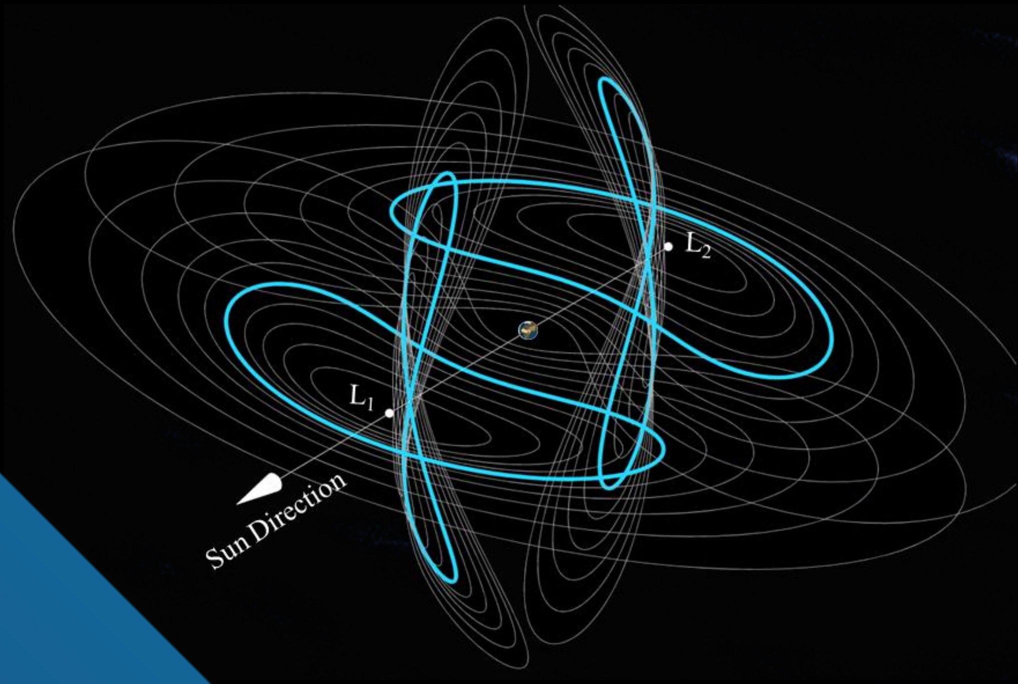
Mathematics and Programming for Astrodynamics & Trajectory Design

Joan Pau Sánchez

MSc in Astronautics and Space Engineering



Course Objectives

- This module is intended to provide you with an overview of state-of-the-art in applied mathematics and techniques for astrodynamics and trajectory design. The objective is to provide a fundamental understanding of the trending topics in mission analysis. The focus of the subject is not so much in deep understanding, but in providing breadth of knowledge and experiences.

Intended Learning Outcomes

On completion of this module you should :

- Be able to write reliable code to solve realistic mission analysis scenarios.
- Be able to apply a range of applied mathematical techniques to solve trajectory design problems and be able to independently expand on appropriate tools and know-how when necessary.
- Identify most common non-Keplerian orbit types and explain their applications.
- Reflect on the current challenges in trajectory design for both Earth observation

Course Overview

Session 1	Seminar	Lambert Arc Intro
	Workshop	Minimum Energy Transfer Algorithm and F & G Coefficients
Session 2	Seminar	Lambert Arc Session 2
	Workshop	Build your own Lambert-solver
Session 3	Seminar	Pork Chop Plots
	Workshop	Full Earth-Mars Transfer Analysis
Session 4	Seminar	Low Thrust Trajectories
	Workshop	Sub-optimal control law transfers
Session 5	Seminar	CR3BP & LPOs
	Workshop	

Bibliography

Some good books ordered by increasing depth and breath :

- *Orbital Mechanics,* V.A Chobotov, AIAA Education Series
- *Analytical Mechanics of Space Systems*, H.Shcaub, J.L.Junkins, AIAA Education Series
- *An Introduction to the Mathematics and Methods of Astrodynamics*, R.H.Battin, AIAA Educationan Series
- *Fundamentals of Astrodynamics and Applications*, D.A.Vallado, Space Technology Library

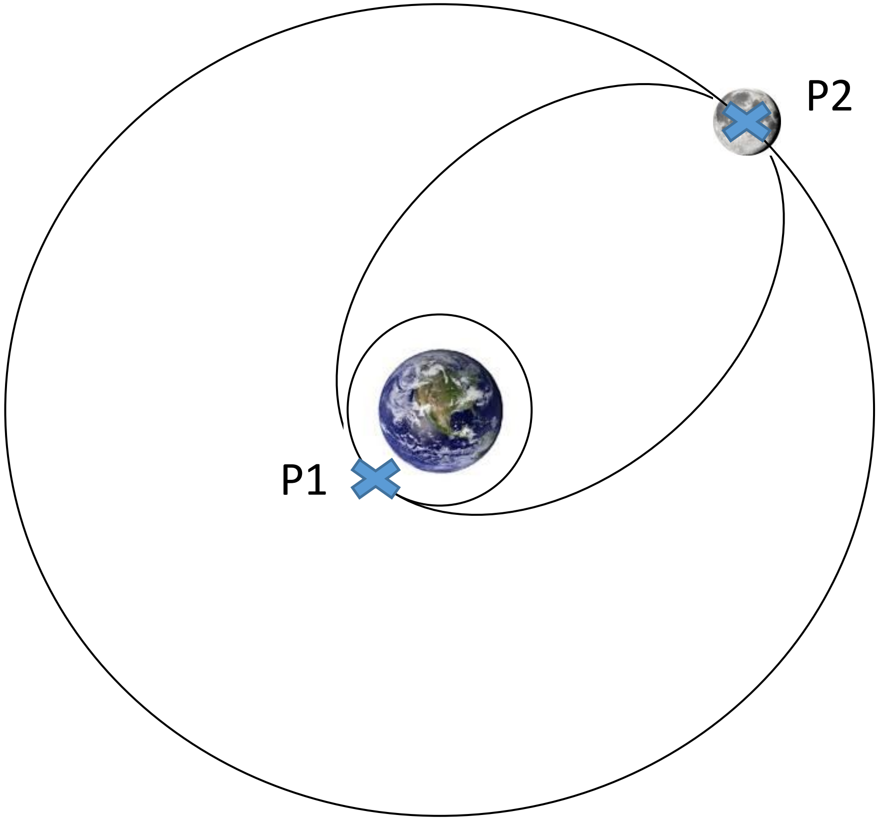
Session 1 - Content

A Lambert Arc DIY

- Review Hohmann
- Introduction of the Two-body Orbital Boundary-value Problem.
  (a.k.a. Lambert Arc)
- The Minimum Energy Orbit
- Lagrange Coefficients Solution to Orbital Motion
- A general algorithm to solve the Lambert Arc

Two-body Orbital Boundary-value Problem

- Two-body Orbital = Keplerian Motion



Hohmann transfer

Vis-viva Equation

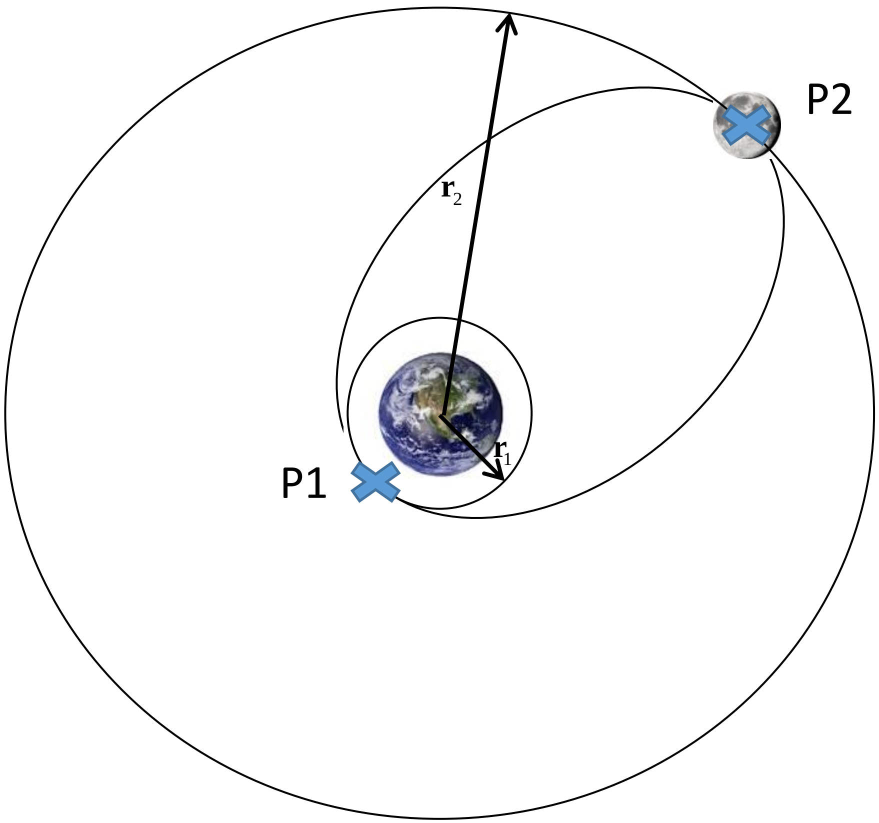
$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

Vis-viva solved for *v*

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Two-body Orbital Boundary-value Problem

- Two-body Orbital = Keplerian Motion



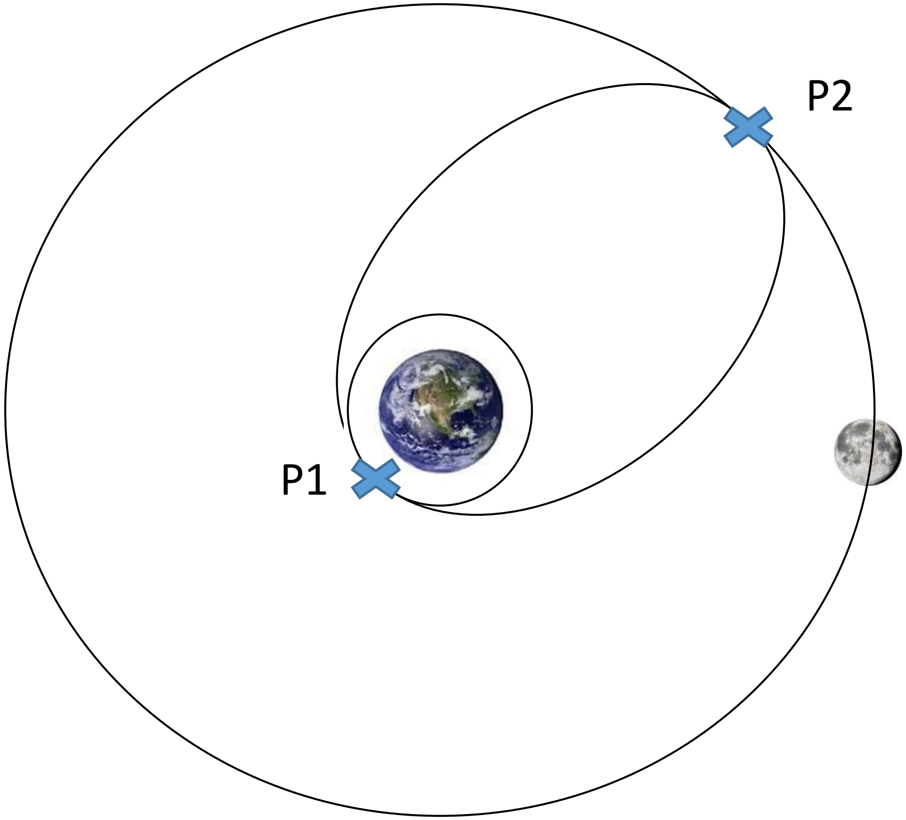
Hohmann transfer

$$\left.\begin{array}{l} v_1 = \sqrt{\dfrac{\mu_E}{r_1}} \\[2ex] v_2 = \sqrt{\mu_E\left(\dfrac{2}{r_1} - \dfrac{2}{r_1 + r_2}\right)} \end{array}\right\} \Delta v_s = v_2 - v_1$$

$$\left.\begin{array}{l} v_3 = \sqrt{\mu_E\left(\dfrac{2}{r_2} - \dfrac{2}{r_1 + r_2}\right)} \\[2ex] v_4 = \sqrt{\dfrac{\mu_E}{r_2}} \end{array}\right\} \Delta v_f = v_4 - v_3$$

Two-body Orbital Boundary-value Problem

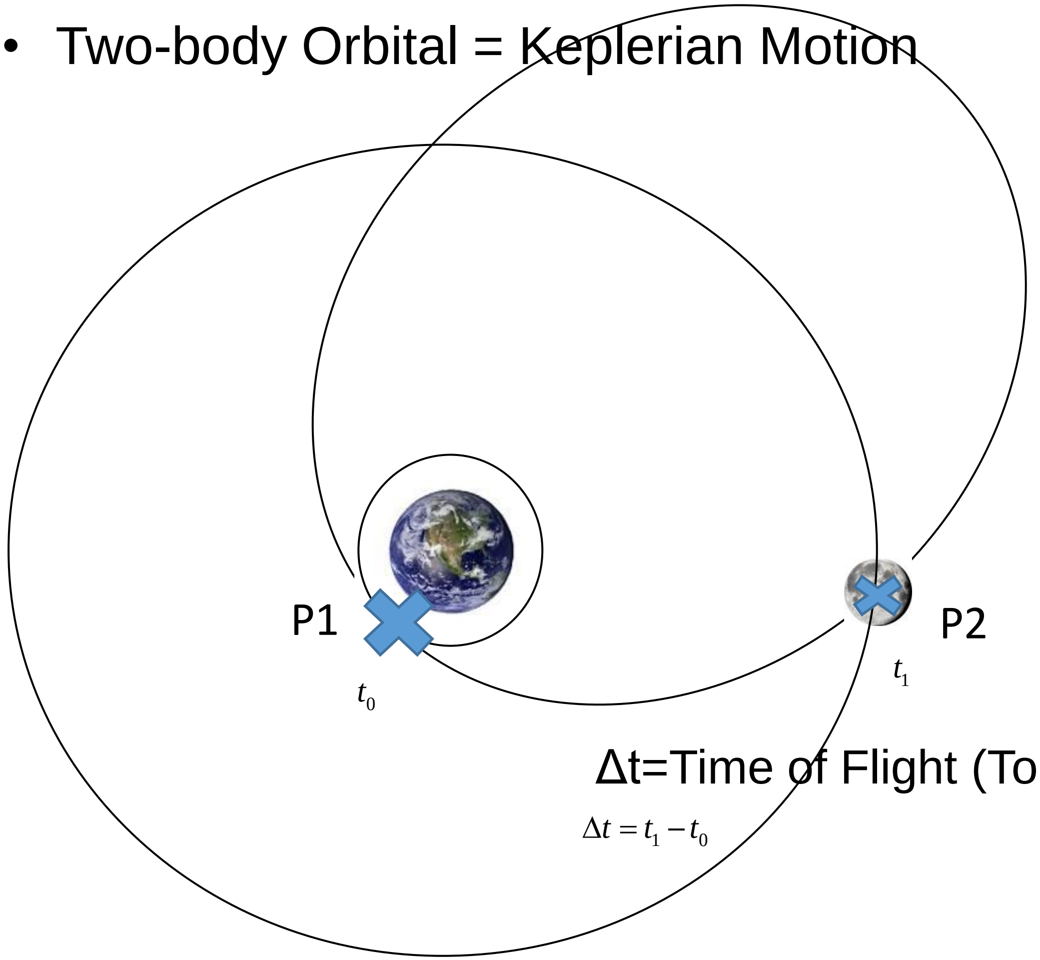
- Two-body Orbital = Keplerian Motion



What if the Moon is not there?

Two-body Orbital Boundary-value Problem

- Two-body Orbital = Keplerian Motion

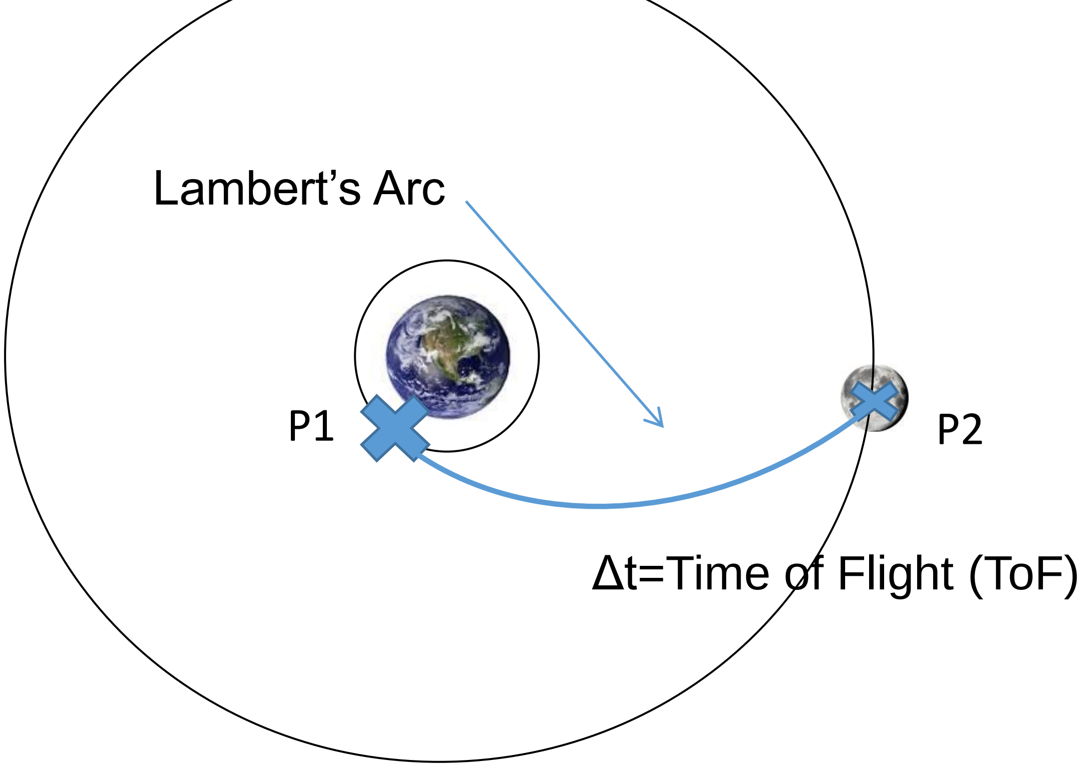


Δt=Time of Flight (ToF)

$\Delta t = t_1 - t_0$

Two-body Orbital Boundary-value Problem

- Two position vectors & time = Lambert's Problem
  - ✓ Solved not only for trajectory design but also for orbital determination.



Lambert’s Problem: Historical Perspective

- 1743: Leonhard Euler defines the problem and finds an infinite series solution for a parabolic case.
- 1761-1771: Johann H. Lambert geometric solutions.
- 1857: Gauss’s solution provides geometrical insight.
- Universal variables:
  - ✓Lancaster & Blanch (1969), Gooding (1988,1990), Izzo (2015). Bate (1971), Vallado (1997), Luo (2011), Thomson (1929), Arora (2013). Battin-Vaughan (1984), Loechler (1988), Shen (2004), MacLellan (2005).
- Semi-major axis:
  - ✓ Thorne (1995,2014), Prussing (2000), Chen (2013), Wailliez (2014).
- Semi-latus rectum (*p*-iteration):
  - ✓ Herrick-Liu (1959), Boltz (1984), Bate (1971).
- Eccentricity vector:
  - ✓ Avanzini (2008), He (2010), Zhang (2010, 2011), Wen (2014).

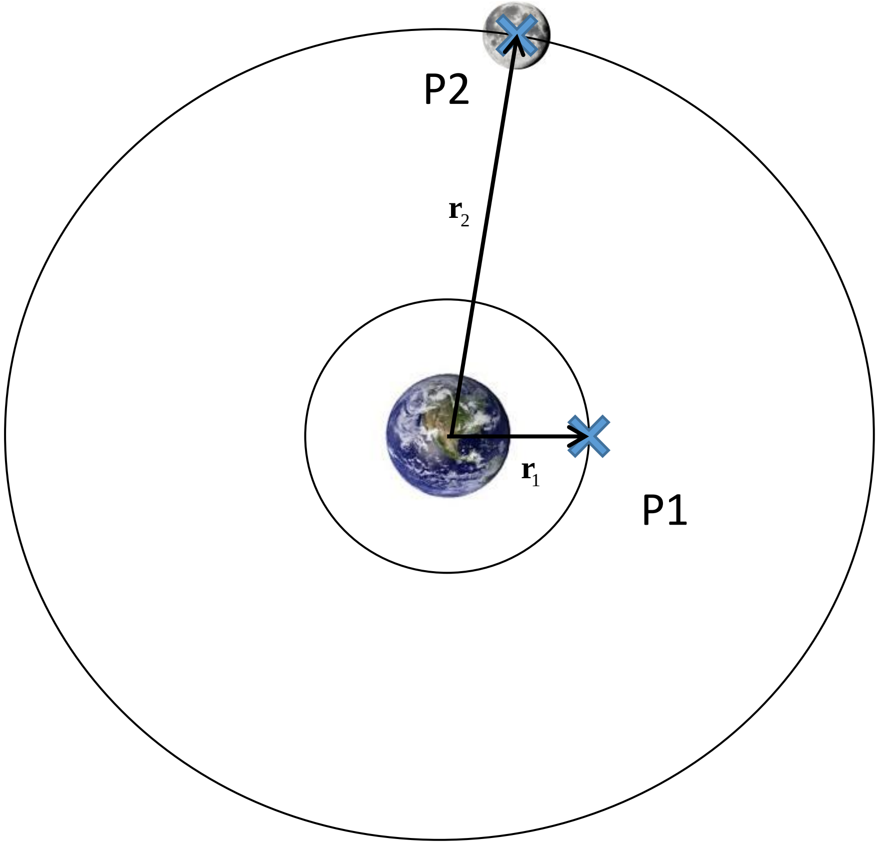
Lambert's Problem

- We will attempt to solve it using a general approach that may be used to solve other targeting problems in higher fidelity dynamics (i.e. *n*-body problem or with other perturbations).
  - ✓ *Pros*: The techniques used are applicable to many other problems and you may come across them in future occasions.
  - ✓ *Cons*: The final algorithm is not very robust and its convergence tends to fail in many occasions.
- Algorithm as described in: *Analytical Mechanics of Space Systems*, H.Shcaub, J.L.Junkins, AIAA Education Series.

Lambert's Problem – Be patient!

- 1. Minimum Energy Transfer
    - ✓ MinETransfer $\left(\mathbf{r}_1, \mathbf{r}_2 \Rightarrow a_{\min}, e_{\min}, \Delta t_{\min}\right)$
    - ✓ MinETransfer $\left(\mathbf{r}_1, \mathbf{r}_2, t_m \Rightarrow \dot{\mathbf{r}}_1\right)$
- 2. Simple propagation solution - F & G Solutions for elliptical orbits:
    - ✓*FGKepler* $\left(\mathbf{r}_0, \dot{\mathbf{r}}_0, \Delta\theta \Rightarrow \mathbf{r}_f\right)$
    - ✓*FGKepler_dt* $\left(\mathbf{r}_0, \dot{\mathbf{r}}_0, \Delta t \Rightarrow \mathbf{r}_f\right)$
- *3. State Transition Matrix for Two-body-problem*
    - ✓*STM_Lambert*: $\left(\mathbf{r}_0, \dot{\mathbf{r}}_0, \Delta t \Rightarrow \frac{\partial \mathbf{r}_f}{\partial \dot{\mathbf{r}}_0}\right)$
- *4. Implement a Differential Corrector*
- *5. Implement a Continuation Method*

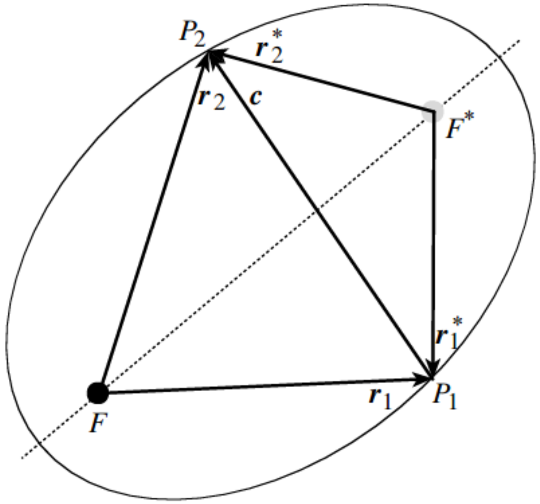
Minimum Energy Transfer



- Connect P1 and P2.

Minimum Energy Transfer

(Schaub & Junkins, 2002)



Which of the following is correct?

A) $r_1 + r_1^* = a$

$r_2 + r_2^* = a$

B) $r_1 + r_1^* = c$

$r_2 + r_2^* = c$

C) $r_1 + r_1^* = 2a$

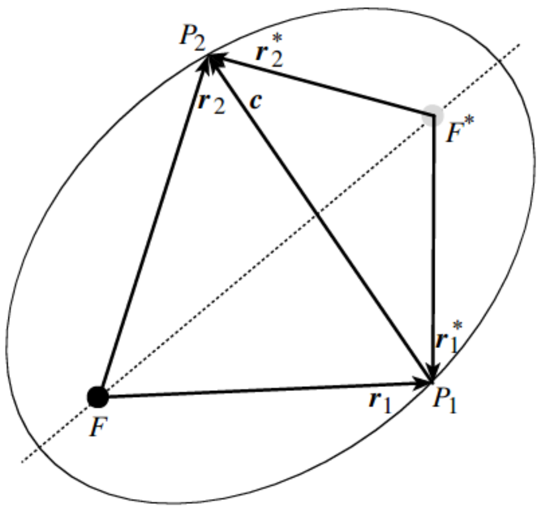
$r_2 + r_2^* = 2a$

D) $r_1 + r_1^* = r_2 + r_2^*$

1. Download and install Socrative Student App or connect into website.
2. Connect into Room: GPQK5UNSK
3. Wait for question to appear, put your name in and answer.

Minimum Energy Transfer

(Schaub & Junkins, 2002)



- Connect P1 and P2.

$$r_1 + r_1^* = 2a$$
$$r_2 + r_2^* = 2a$$

$$\underbrace{r_1 + r_2}_{\textbf{Fixed}} + r_1^* + r_2^* = 4a$$

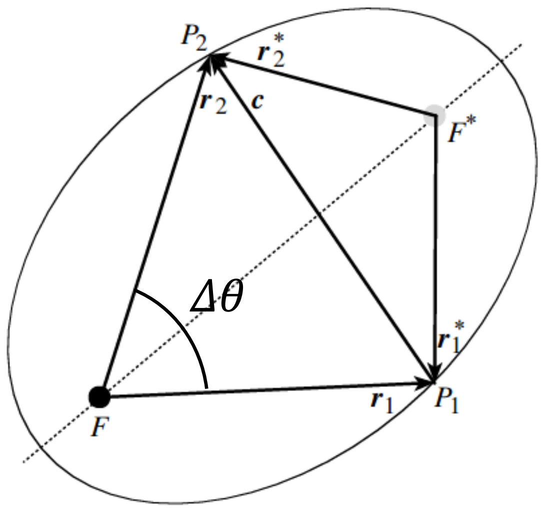
- 3 unknown parameters: r_1^*, r_2^*, a
- Note the following inequality constraint: $c \leq r_1^* + r_2^*$

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon = -\frac{\mu}{2a}$$

- Hence, the smaller *a* the smaller the energy ε.

Minimum Energy Transfer

(Schaub & Junkins, 2002)



- Hence, minimizing ε:

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon = -\frac{\mu}{2a}$$

$$\underbrace{r_1 + r_2}_{\textbf{Fixed}} + r_1^* + r_2^* = 4a$$

$$r_1^* + r_2^* = c$$

$$a_{\min} = \frac{1}{4}\left(r_1 + r_2 + c\right)$$

$$c = \left|\mathbf{r}_2 - \mathbf{r}_1\right| = \sqrt{r_2^2 + r_2^1 - 2r_1r_2\cos\left(\Delta\theta\right)}$$

Minimum Energy Transfer: Algorithm MinETransfer

Algorithm *MinETransfer* : $\left(\mathbf{r}_1, \mathbf{r}_2 \Rightarrow a_{\min}, e_{\min}, \Delta t_{\min}\right)$

Reference Books for all the algorithms I will present:

- *Orbital Mechanics,* V.A Chobotov, AIAA Education Series
- *Analytical Mechanics of Space Systems*, H.Shcaub, J.L.Junkins, AIAA Education Series
- *An Introduction to the Mathematics and Methods of Astrodynamics*, R.H.Battin, AIAA Educationan Series
- *Fundamentals of Astrodynamics and Applications*, D.A.Vallado, Space Technology Library

Minimum Energy Transfer: Algorithm MinETransfer

Algorithm *MinETransfer* : $\left(\mathbf{r}_1, \mathbf{r}_2, t_m \Rightarrow a_{\min}, e_{\min}, \Delta t_{\min}\right)$

$$c = \left|\mathbf{r}_2 - \mathbf{r}_1\right|$$

$$r_1 = \left|\mathbf{r}_1\right| \qquad r_2 = \left|\mathbf{r}_2\right|$$

$$a_{\min} = \frac{1}{4}\left(r_1 + r_2 + c\right)$$

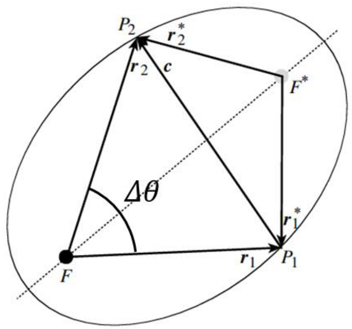
$$\cos\Delta\theta = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}$$

$$p_{\min} = \frac{r_1 r_2}{c}\left(1 - \cos\Delta\theta\right)$$

$$e_{\min} = \sqrt{1 - \frac{p_{\min}}{a_{\min}}}$$

$$\beta_e = 2\sin^{-1}\left(\sqrt{\frac{2a_{\min} - c}{2a_{\min}}}\right)$$

$$\Delta t_{\min} = \sqrt{\frac{a_{\min}^3}{\mu}}\left[\pi - t_m \cdot \left(\beta_e - \sin\beta_e\right)\right]$$



Where $t_m(+1)$ refers at Δt for the short path transfers and $t_m(-1)$ for the long path transfer.

Minimum Energy Transfer

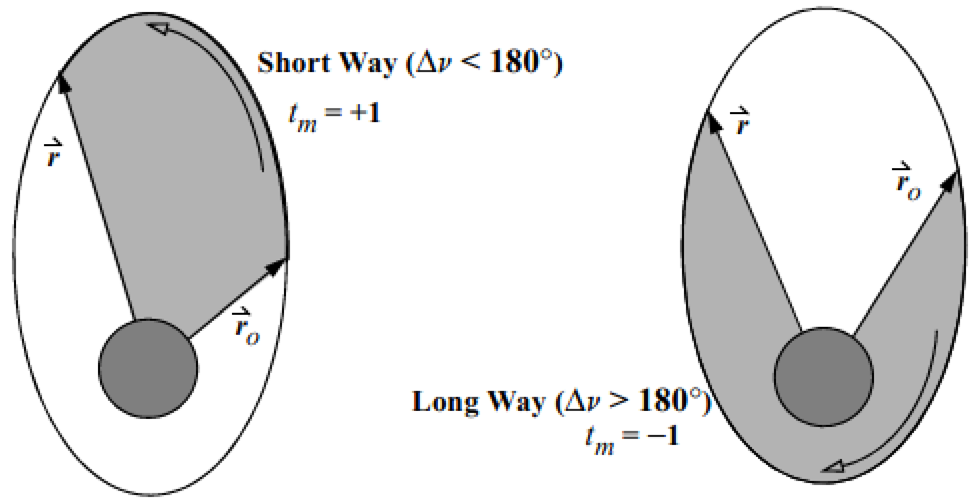


Figure 7-8. **Transfer Methods, t_m, for the Lambert Problem.** Traveling between the two specified points can take the long way or the short way. For the long way, the change in true anomaly exceeds 180°.

*Courtesy (*Vallado, 2013)

Exercise 1

ExoMars Trace Gas Orbiter departed Earth on 14/03/2016, and arrived at Mars on 15/10/2016. What is the minimum energy orbit that would link the position of Earth and Mars those two days? What is the time of flight of the minimum energy orbit? Was this the trajectory followed by ExoMars TGO?

Hints:

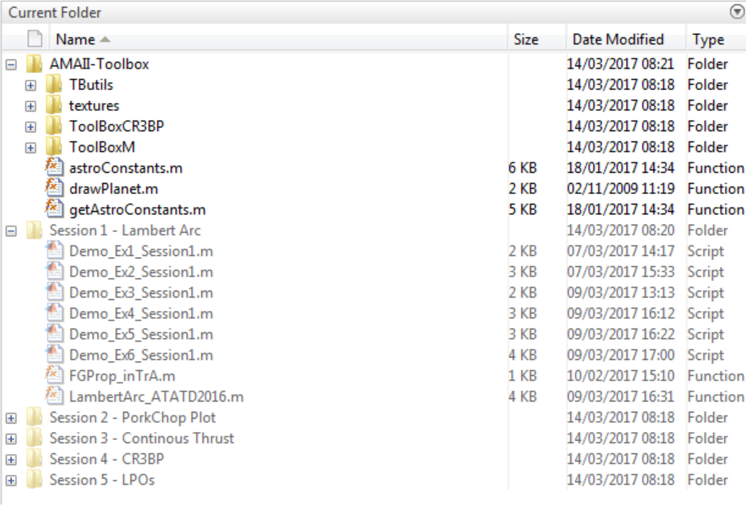
Use functions *EphSS_car* to compute the position of Earth and Mars at those dates (index Earth 3, index Mars 4).

Result Guide: $\Delta t_{\min}^{ShortPath} = 231.8 \text{ days}$

Session 1 MATLAB Guide

The following bullet points provide some basic hints on how completing the tasks of this course:

- Complete all the exercises by creating one script for each task. This way you will be able to revise easily your calculations, as well as comparing with the exercise solutions.
- Create a sensible working directory.
- Note that AMAII-Toolbox is in the path of the working environment.



Current Folder

Name	Size	Date Modified	Type
AMAII-Toolbox		14/03/2017 08:21	Folder
TButils		14/03/2017 08:18	Folder
textures		14/03/2017 08:18	Folder
ToolBoxCR3BP		14/03/2017 08:18	Folder
ToolBoxM		14/03/2017 08:18	Folder
astroConstants.m	6 KB	18/01/2017 14:34	Function
drawPlanet.m	2 KB	02/11/2009 11:19	Function
getAstroConstants.m	5 KB	18/01/2017 14:34	Function
Session 1 - Lambert Arc		14/03/2017 08:20	Folder
Demo_Ex1_Session1.m	2 KB	07/03/2017 14:17	Script
Demo_Ex2_Session1.m	3 KB	07/03/2017 15:33	Script
Demo_Ex3_Session1.m	2 KB	09/03/2017 13:13	Script
Demo_Ex4_Session1.m	3 KB	09/03/2017 16:12	Script
Demo_Ex5_Session1.m	3 KB	09/03/2017 16:22	Script
Demo_Ex6_Session1.m	4 KB	09/03/2017 17:00	Script
FGProp_inTrA.m	1 KB	10/02/2017 15:10	Function
LambertArc_ATATD2016.m	4 KB	09/03/2017 16:31	Function
Session 2 - PorkChop Plot		14/03/2017 08:18	Folder
Session 3 - Continous Thrust		14/03/2017 08:18	Folder
Session 4 - CR3BP		14/03/2017 08:18	Folder
Session 5 - LPOs		14/03/2017 08:18	Folder

FOLLOW VERY CLOSELY THE WORK GUIDE IN CANVAS

Download Session 1 MATLAB Guide (PART A) from BB

Exercise 1

ExoMars Trace Gas Orbiter departed Earth on 14/03/2016, and arrived at Mars on 15/10/2016. What is the minimum energy orbit that would link the position of Earth and Mars those two days? What is the time of flight of the minimum energy orbit? Was this the trajectory followed by ExoMars TGO?

Hints:

Use functions *EphSS_car* to compute the position of Earth and Mars at those dates (index Earth 3, index Mars 4).

When completed, connect to Socrative Room GPQK5UNSK and indicate it so in the relevant question

Result Guide: $\Delta t_{\min}^{ShortPath} = 231.8 \text{ days}$

Initial Value Problem and Position and Velocity as a Function of Time

Initial Value Problem:

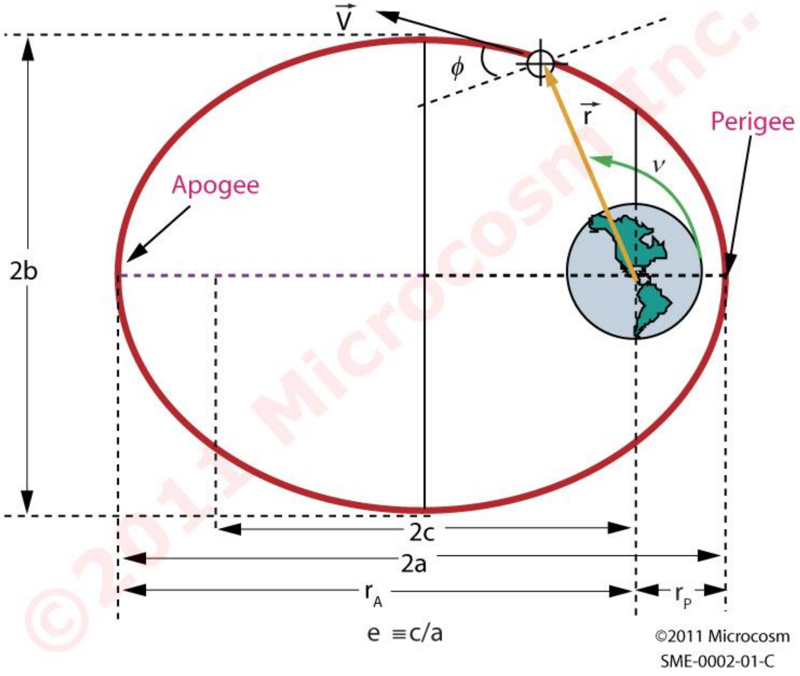
We may typically know the position and velocity vectors of a spacecraft at a given time (e.g. $\boldsymbol{r}_1$ and $\boldsymbol{v}_1$) and need to know the spacecraft position and velocity some time later (e.g. $\boldsymbol{r}_2$ and $\boldsymbol{v}_2$).

Position and Velocity Problem:

$$\boxed{\left(\mathbf{r}(t), \mathbf{v}(t)\right)}$$

Kepler's Equation

$$E - e\sin E = nt \qquad nt = M$$



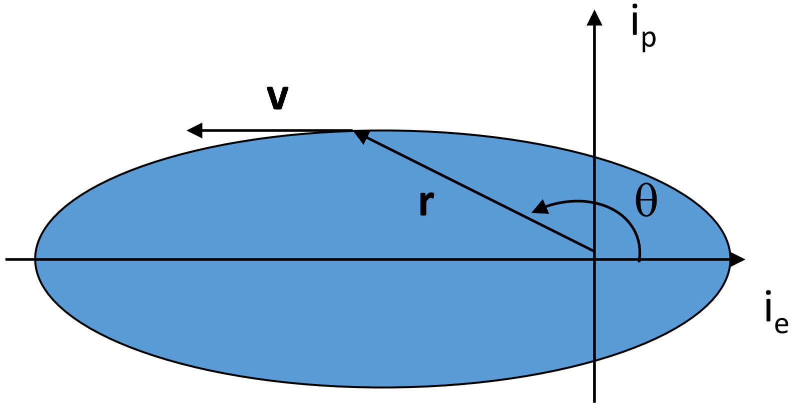
Initial Value Problem:
Lagrange Coefficients Solution to Orbital Motion

The general problem is to find the position and velocity vectors $\boldsymbol{r_2}$ and $\boldsymbol{v_2}$ at a given time $t_1+\Delta t$ once the position and velocity vectors $\boldsymbol{r_1}$ and $\boldsymbol{v_1}$ at t_1 are known.

Note that any position vector $\boldsymbol{r}$ and the velocity vector $\boldsymbol{v}$ at a given true anomaly θ can be expressed in terms of orbital plane coordinates as follows:

$$\mathbf{r} = r\cos\theta\mathbf{i}_e + r\sin\theta\mathbf{i}_p$$

$$\mathbf{v} = -\frac{\mu}{h}\sin\theta\mathbf{i}_e + \frac{\mu}{h}(e+\cos\theta)\mathbf{i}_p$$

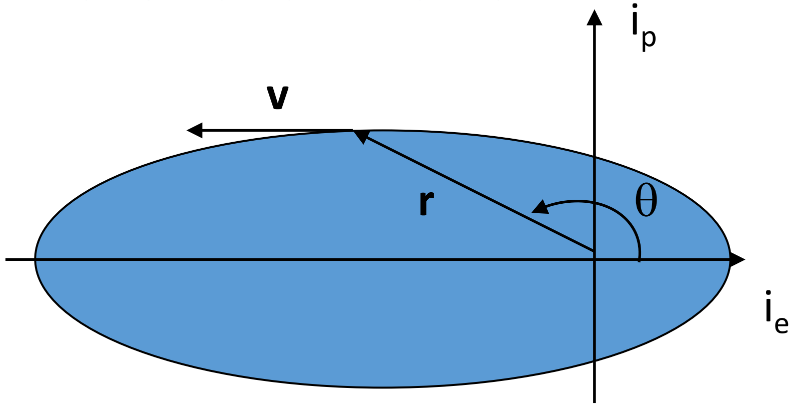


Where h is the angular momentum

Lagrange Coefficients Solution to Orbital Motion

$$\mathbf{r} = r\cos\theta\,\mathbf{i}_e + r\sin\theta\,\mathbf{i}_p$$

$$\mathbf{v} = -\frac{\mu}{h}\sin\theta\,\mathbf{i}_e + \frac{\mu}{h}(e+\cos\theta)\,\mathbf{i}_p$$



$$r = \frac{h^2}{\mu(1+e\cos\theta)}$$

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(r\cos\theta\,\mathbf{i}_e + r\sin\theta\,\mathbf{i}_p\right)$$

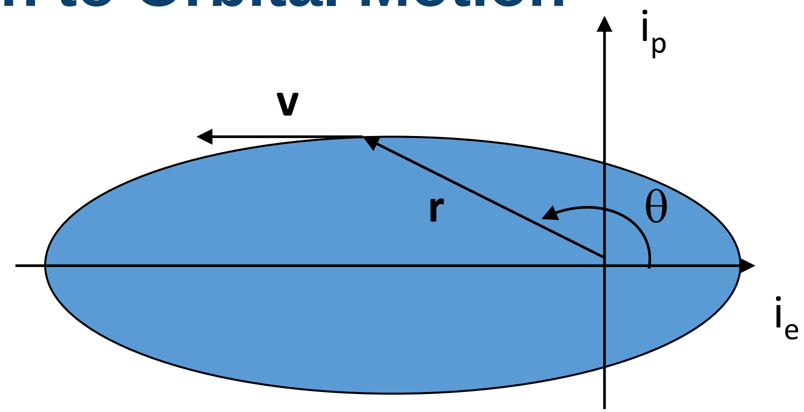
$$h = r^2\dot{\theta}$$

$$h^2 = \mu p$$

Lagrange Coefficients Solution to Orbital Motion

$$\mathbf{r} = r\cos\theta\,\mathbf{i}_e + r\sin\theta\,\mathbf{i}_p$$

$$\mathbf{v} = -\frac{\mu}{h}\sin\theta\,\mathbf{i}_e + \frac{\mu}{h}(e+\cos\theta)\,\mathbf{i}_p$$



$$\mathbf{r}_1 = r_1\cos\theta_1\,\mathbf{i}_e + r_1\sin\theta_1\,\mathbf{i}_p$$

$$\mathbf{v}_1 = -\frac{\mu}{h}\sin\theta_1\,\mathbf{i}_e + \frac{\mu}{h}(e+\cos\theta_1)\,\mathbf{i}_p$$

Lagrange Coefficients Solution to Orbital Motion

These equations are valid at the initial point $\boldsymbol{r_1}$ and $\boldsymbol{v_1}$. We can now invert the relationship and obtain the coordinate unit vectors as a function of the initial position and velocity vectors:

$$\mathbf{i}_e = \frac{\mu}{h^2}(e+\cos\theta_1)\mathbf{r}_1 - \frac{r_1}{h}\sin\theta_1\mathbf{v}_1$$

$$\mathbf{i}_p = \frac{\mu}{h^2}\sin\theta_1\mathbf{r}_1 + \frac{r_1}{h}\cos\theta_1\mathbf{v}_1$$

Now substituting the above relations into the general form equation from slide 25, we can obtain:

$$\mathbf{r}_2 = F\mathbf{r}_1 + G\mathbf{v}_1$$

$$\mathbf{v}_2 = \dot{F}\mathbf{r}_1 + \dot{G}\mathbf{v}_1$$

Where F, G, Fdot and Gdot are known as Lagrange coefficients.

Lagrange Coefficients Solution to Orbital Motion

$$\mathbf{i}_e = \frac{\mu}{h^2}(e + \cos\theta_1)\mathbf{r}_1 - \frac{r_1}{h}\sin\theta_1\mathbf{v}_1$$

$$\mathbf{i}_p = \frac{\mu}{h^2}\sin\theta_1\mathbf{r}_1 + \frac{r_1}{h}\cos\theta_1\mathbf{v}_1$$

$$\mathbf{r}_2 = r_2\cos\theta_2\mathbf{i}_e + r_2\sin\theta_2\mathbf{i}_p$$

$$\mathbf{v}_2 = -\frac{\mu}{h}\sin\theta_2\mathbf{i}_e + \frac{\mu}{h}(e + \cos\theta_2)\mathbf{i}_p$$

$$\mathbf{r}_2 = r_2\cos\theta_2\left(\frac{\mu}{h^2}(e + \cos\theta_1)\mathbf{r}_1 - \frac{r_1}{h}\sin\theta_1\mathbf{v}_1\right) + r_2\sin\theta_2\left(\frac{\mu}{h^2}\sin\theta_1\mathbf{r}_1 + \frac{r_1}{h}\cos\theta_1\mathbf{v}_1\right)$$

$$\mathbf{v}_2 = -\frac{\mu}{h}\sin\theta_2\left(\frac{\mu}{h^2}(e + \cos\theta_1)\mathbf{r}_1 - \frac{r_1}{h}\sin\theta_1\mathbf{v}_1\right) + \frac{\mu}{h}(e + \cos\theta_2)\left(\frac{\mu}{h^2}\sin\theta_1\mathbf{r}_1 + \frac{r_1}{h}\cos\theta_1\mathbf{v}_1\right)$$

Finally, arrange $\mathbf{r}_1$ and $\mathbf{v}_1$ terms together to obtain the Lagrange coefficients.

Lagrange Coefficients Solution to Orbital Motion

Let us consider the true anomaly difference $\Delta\theta = \theta_2 - \theta_1$, and define;

$$e\cos\theta_1 = \frac{p}{r_1} - 1 \qquad e\sin\theta_1 = \frac{\sqrt{p}}{r_1}\sigma_1 \text{ where } \sigma_1 = \frac{\mathbf{r}_1 \cdot \mathbf{v}_1}{\sqrt{\mu}}$$

$$r(\Delta\theta) = \frac{p r_1}{r_1 + (p - r_1)\cos\Delta\theta - \sqrt{p}\sigma_1 \sin\Delta\theta}$$

Then the Lagrange coefficients can be written in the following way:

$$F = 1 - \frac{r}{p}(1 - \cos\Delta\theta); \quad G = \frac{r r_1}{\sqrt{\mu p}}\sin\Delta\theta$$

$$\dot{F} = \frac{\sqrt{\mu}}{r_1 p}\left[\sigma_1(1 - \cos\Delta\theta) - \sqrt{p}\sin\Delta\theta\right]; \quad \dot{G} = 1 - \frac{r_1}{p}(1 - \cos\Delta\theta)$$

Minimum Energy Transfer

Statement: Find the $\boldsymbol{v}_1$ for minimum energy orbit.

Recall, $\mathbf{r}_2 = F\mathbf{r}_1 + G\mathbf{v}_1$

$$\mathbf{v}_1 = \frac{1}{G}\left(\mathbf{r}_2 - F\mathbf{r}_1\right)$$

$$F = 1 - \frac{r_2}{p}(1 - \cos\Delta\theta); \quad G = \frac{r_2 r_1}{\sqrt{\mu p}}\sin\Delta\theta$$

MinETransfer $\left(\mathbf{r}_1, \mathbf{r}_2 \Rightarrow a_{\min}, e_{\min}\right)$

Minimum Energy Transfer: Algorithm MinETransfer

Algorithm *MinETransfer* : $\left(\mathbf{r}_1, \mathbf{r}_2, t_m \Rightarrow \dot{\mathbf{r}}_1\right)$

$$c = \left|\mathbf{r}_2 - \mathbf{r}_1\right|$$

$$\cos\Delta\theta = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}$$

$$\sin\Delta\theta = t_m \sqrt{1-\cos^2\Delta\theta}$$

$$p_{\min} = \frac{r_1 r_2}{c}\left(1-\cos\Delta\theta\right)$$

$$F = 1 - \frac{r_2}{p_{\min}}(1-\cos\Delta\theta)$$

$$G = \frac{r_2 r_1}{\sqrt{\mu\, p_{\min}}}\sin\Delta\theta$$

$$\dot{\mathbf{r}}_1 = \frac{1}{G}\left(\mathbf{r}_2 - F\mathbf{r}_1\right)$$

F & G Solutions for elliptical orbits.

Algorithm *FGKepler_trA*: $\left(\mathbf{r}_0, \dot{\mathbf{r}}_0, \Delta\theta \Rightarrow \mathbf{r}_f\right)$

$$\mathbf{h} = \mathbf{r}_0 \times \dot{\mathbf{r}}_0$$

$$p = \frac{h^2}{\mu}$$

$$\sigma_0 = \frac{\mathbf{r}_0 \cdot \dot{\mathbf{r}}_0}{\sqrt{\mu}}$$

$$r_f = \frac{p r_0}{r_0 + (p - r_0)\cos\Delta\theta - \sqrt{p}\,\sigma_0 \sin\Delta\theta}$$

$$F = 1 - \frac{r_f}{p}(1 - \cos\Delta\theta)$$

$$G = \frac{r_f r_0}{\sqrt{\mu p}} \sin\Delta\theta$$

$$\mathbf{r}_f = F\mathbf{r}_0 + G\dot{\mathbf{r}}_0$$

Download Session 1 MATLAB Guide (PART A) from BB

Exercise 2

ExoMars Trace Gas Orbiter departed Earth on 14/03/2016, and arrived at Mars on 15/10/2016.

Program the F and G solutions to the two body problem. Verify the answer by comparing it to a numerical integration of the differential equations of motion:

$$\boxed{\ddot{\mathbf{r}} + \frac{\mu_E}{r^3}\mathbf{r} = 0}$$

Using the F and G techniques, plot the orbit of Mars, Earth and the minimum energy transfer for ExoMars TGO.

When completed, connect to Socrative Room GPQK5UNSK and indicate it so in the relevant question

Result Guide: $\dot{r}_1 = \begin{bmatrix} -7.72 & -31.04 & -2.26 \end{bmatrix}$ km/s

Mathematics and Programming for Astrodynamics & Trajectory Design

Cranfield Aerospace

End of day 1