# Lagrange points - Mission to L2

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## I. Mission to Lagrange point L2

The following report analyses what are Lagrange points and how the gravitational force acts on them. Then, it preceeds to calculate the Lagrange point between Earth and the Sun for a hypothetical mission.

### A. *Lagrange points destination analysis*

Lagrange points are zones in space where gravitational forces of a two body system produce enhanced equilibrium regions of attraction and repulsion. These regions are highly advisable for space missions since they reduce drastically the fuel consumption needed to remain in that position. In other words, at Lagrange points, the gravitational pull of two large masses precisely equals the centripetal force required for a small object to move with them [1].

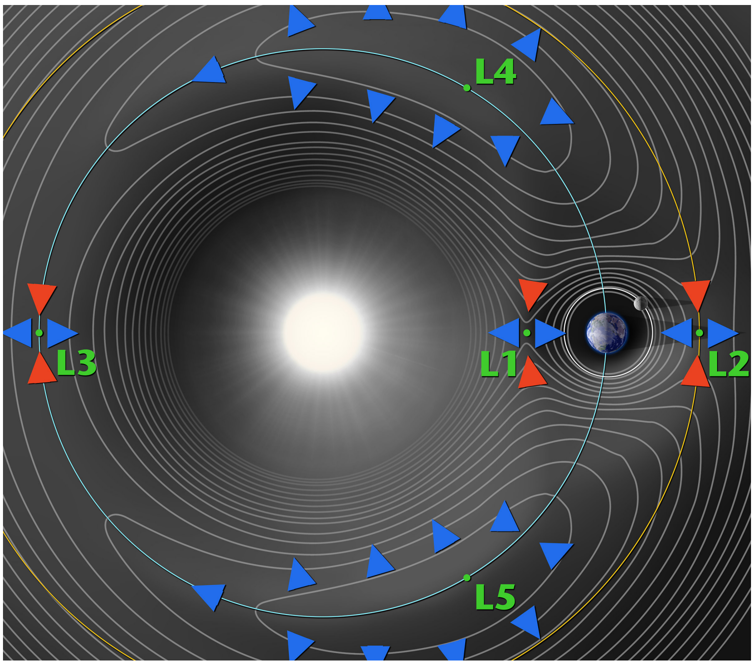


Fig. 1. Visual representation of Lagrange points (not drawn in scale). Source: [1].

Any system of two large objects in space, such as the Earth and our Sun, will produce 5 points of stability, where the gravitational pull of both objects cancel each other out to create a small area where stations or satellites could be kept at a relative halt, without the need for control thrusters or orbital correction. Although this resemble a three body problem, as the third body is much less massive it is called a restricted three-body problem.

As mentioned before, these points exists in any system of two larger masses and a third much smaller mass, i.e. a space probe between Sun-Earth Lagrange points system. Nevertheless, systems like this exists between any other objects, for instance, Earth-Moon and Sun-Jupiter system.

Taking as an example the Sun-Earth Lagrange points system [2]:

- **L1**: On a direct vector between Earth and the Sun lies the Lagrange L1 position. This is the point where the direct gravitational influence of the Earth and Sun cancels out.
  Lagrange point one is particularly, useful as it allows for a satellite placed there to not only remain equidistant for both the Sun and Earth but also provides a position for a consistent visual the sun and the illuminated surface on Earth.
  One example, of a satellite is the Solar and Heliospherical Observatory or *SOHO* satellite, which provided with Sun's gas currents below the Sun's visible surface and tracking changes in its magnetic fields.
- **L2**: On the same direction vector, but beyond Earth's orbit lies L2. The Earth gravity well maintains the position of L2 in spite of its higher orbit around the Sun, which increases the orbital period.
  On disadvantage at being at L2 point is that if there is a satellite communication between Earth and the satellite, every time there's a solar flare, which gives out radio waves that interfere with radio communications so typically it is advisable to point the communication disk to the Earth but also to the Sun behind it. So normally an orbit oscillating around L2 solves this issue, these orbits are called Halo orbits.
  The L2 point of the Earth-Sun system would be future home of the James Webb Space Telescope. L2 is ideal for astronomy because a spacecraft is close enough to readily communicate with Earth, can keep Sun, Earth and Moon behind the spacecraft for solar power and (with appropriate shielding) provides a clear view of deep space for the telescopes.
- **L3**: The third Lagrange point is located on the other side of the system, far beyond the Sun. This is the point where the Sun and Earth's gravitational pull align in the opposite direction to L2.
  An object place there would be eclipsed by the Sun and completely hidden from Earth. An applications of this point is to predict solar storms coming. Looking at the far side of the Sun is particularly useful as the Sun rotates and in the Earth would be able to foresee this event.
  Lagrange points 1, 2 and 3 are considered unstable equilibrium points meaning an object placed there will be in equilibrium as long as they are not influenced by other forces. For this reason, these positions will still need constant thrust control to correct any misalignment. A spacecraft must use frequent rocket firings to stay in so-called Halo orbits around the Lagrangian point.
- **L4 and L5**: Lagrange points L4 and L5 are located by charting equilateral triangles above and below the direct vector of the Sun to find the gravitational barycenter.
  L4 and L5 are extremely stable gravity wells that pull objects into themselves so many asteroids have been captured naturally by these wells. These regions would not require $\Delta V$ or orbital correction to maintain their position relative to the Earth and Sun.
  What makes this regions interesting is that the distance and travel times between Earth and those regions always remain the same.

Let's dive on how to represent graphically the Lagrangian problem. Kepler's laws only applies to two body problems so the addition of a third mass into the equation would not be applicable.

The problem this mission is facing is a three-body problem in which the satellite's motion is affected by the gravitational motion of both the Sun and Earth. While there is no solution to general three body problem the following steps shows a Lagrangian approach to the classical Newtonian mechanics and reshape it to eliminate the need of balancing forces and inertias. In Lagrangian mechanics, the solution is obtained by taking derivatives of the kinetic and potential energy functions.

Working out the equations of a two body problem, assuming the Sun and other and its planet in a circular orbit with the respective masses to be $m_p$ and the Sun's mass $m_s$, and the distance between their centers to be $R$ and $\mu$ represent [3] [4]:

$$M = m_s + m_p$$

Where:

$$m_p = \mu M \qquad \text{and} \qquad m_s = (1-\mu)M$$

The center of mass of a two-body system is called barycenter. The barycenter is located in the direction between the two bodies. With a distance $\mu R$ from the Sun and a distance $(1-\mu)R$ from the planet. Both the Sun and the planet resolve about the barycenter with an angular speed $\omega$, where

$$\omega^2 = \frac{GM}{R^3}$$

The period is related to the angular speed through the relation

$$T = \frac{2\pi}{\omega}$$

which leads to well-known expresion for Kepler's Third Law, which states that the square of the period is proportional to the cube of the distance:

$$T^2 = \frac{4\pi^2 R^3}{GM}$$

The above expressions are the result of the *two body problem*, however, the case of study considers a *three body problem*. Hence, the Lagrangian methodology is to be applied.

First, let's introduce the reference frame. Figure 2 shows the global three body reference frame. The Sun is represented in yellow and also the satellite and Earth are depicted with its respective distances from the Sun $r_s$ and $r_p$, respectively.

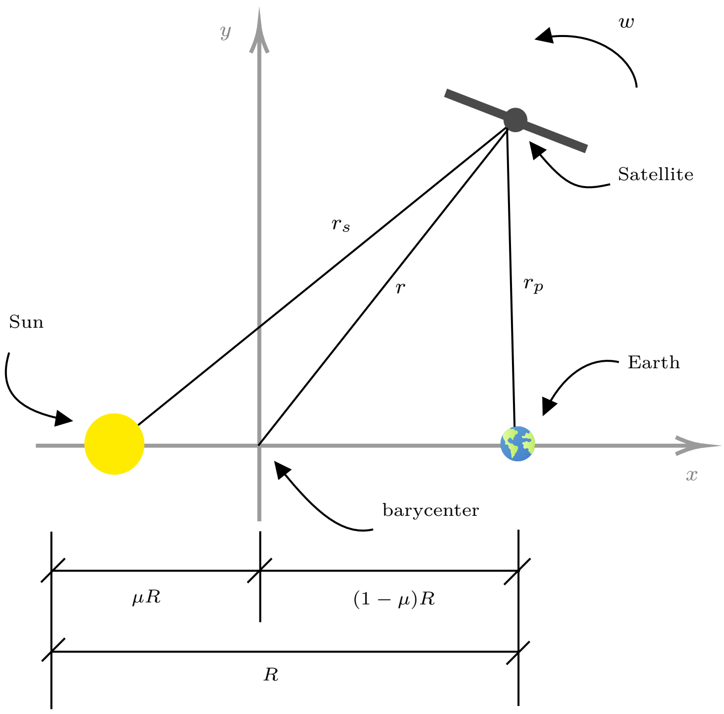


Fig. 2. Three body problem rotating reference frame. Source: Own.

The origin is located in the barycenter and on the $x$ axis between the Sun and Earth. What makes this reference frame highly interesting is the fact that it rotates with the angular velocity $w$. Besides, within this reference frame both the Sun and Earth is motionless and the problem is reduced to find the points where the planet is motionless too. Nonetheless, a rotating reference frame is not inertial.

As the reference frame is not inertial, acceleration terms are meant to be considered. According to Alembert's Principle [5], which is an extension of the virtual work from static to dynamical systems and separates the total forces and inertial forces (which are considered due to the movement of the non-inertial reference frame to maintain the validity of Newton's second Law):

$$F = ma \iff F - ma = 0 \tag{1}$$

D'Alembert's Principle is applicable to energy terms as well. In the restricted three body problem, the potential energy of the satellite has three terms:

$$U = \underbrace{-\frac{Gm_s m}{r_s}}_{\text{(A)}} \underbrace{-\frac{Gm_p}{r_p}}_{\text{(B)}} \underbrace{-\frac{1}{2}m\left(r\omega\right)^2}_{\text{(C)}} \tag{2}$$

where

1) Gravitational potential energy due to the Sun
2) Gravitational potential energy due to the Earth
3) Centrifugal potential energy due to the rotating frame[1].

and

- $m$ is the satellite's mass.
- $r$ is the distance between the satellite and the barycenter.
- $r_s$ is the distance between the satellite and the Sun.
- $r_p$ is the distance between the planet and the satellite.

Now, let's transform the above mentioned expression into an adimensional expression. To do so, the first thing is to substitute the expressions for $m_s$, $m_p$ and $\omega^2$ into the expression for $U$.

$$U = -\frac{Gm(1-\mu)}{r_s} - \frac{G\mu m M}{r_p} - \frac{GMm}{2R^3}r^2 \tag{3}$$

Rewriting the $r$ terms using non-dimensional variables,

$$r = \rho R, \quad r_s = \rho_s R, \quad r_p = \rho_p R \tag{4}$$

Rewriting the expression pf the potential energy $U$:

$$U = \frac{GMm}{R}\left[-\frac{1-\mu}{\rho_s} - \frac{\mu}{\rho_p} - \frac{1}{2}\rho^2\right] \tag{5}$$

Then, finding the stationary points of $U$ now reduces to finding the stationary points of non-dimensional within the brackets, which is represented by $u$.

$$u = -\frac{1-\mu}{\mu_s} - \frac{\mu}{\rho_p} - \frac{1}{2}\rho^2 \tag{6}$$

Briefly, this changed the $x$ and $y$ axis, see Figure

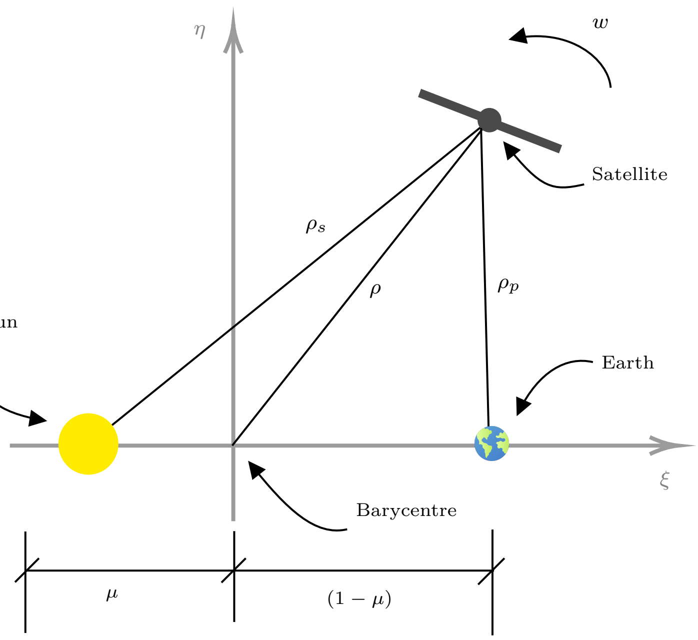


Fig. 3. Three body problem reference frame (adimensional). Source: Own.

[1]This third term would not appear in a potential energy expression written for an inertial frame.

Expressing $u$ in terms of $\xi$ and $\eta$ using Pythagoras theorem,

$$\rho^2 = \xi^2 + \eta^2 \tag{7}$$

$$\rho_s^2 = (\xi + \mu)^2 + \eta^2 \tag{8}$$

$$\rho_p^2 = [\xi - (1 - \mu)]^2 + \eta^2 \tag{9}$$

The resultant expression of the adimensional gravitational energy $u$ goes,

$$u = -\frac{1 - \mu}{\sqrt{(\xi + \mu)^2 + \eta^2}} - \frac{\mu}{\sqrt{[\xi - (1 - \mu)]^2 + \eta^2}} - \frac{1}{2}(\xi^2 + \eta^2) \tag{10}$$

The above expression is represented in the Figure below:

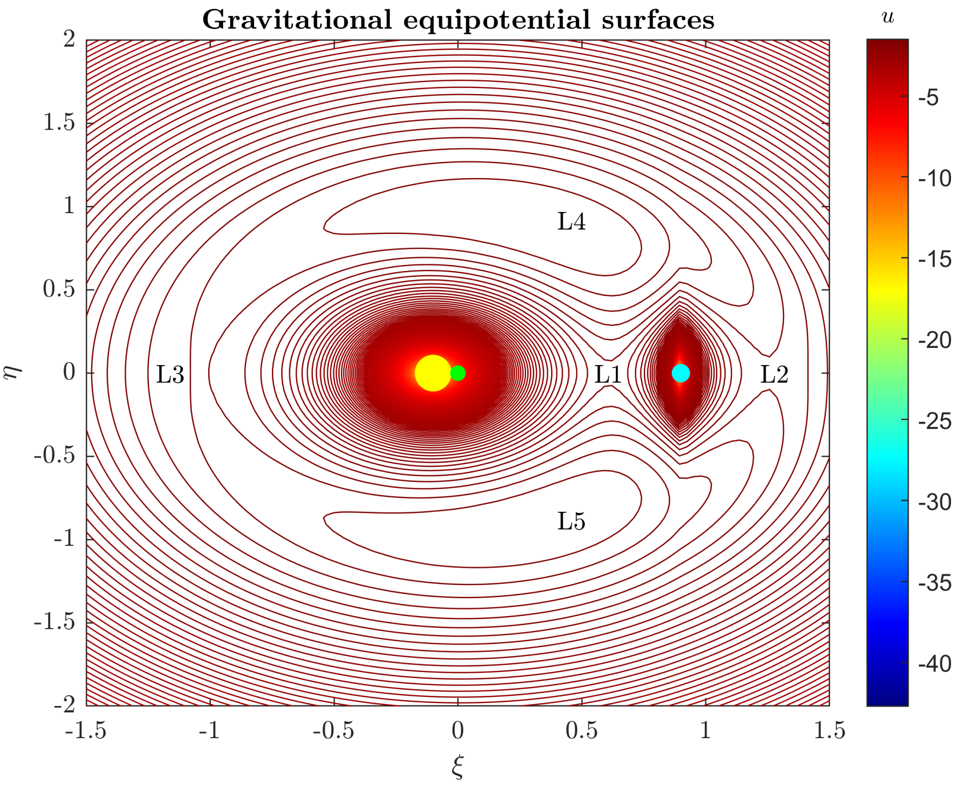


Fig. 4. Representation of gravitational equipotential surfaces of Sun-Earth system and Lagrange points in a rotating system coordinates. Source: Own.

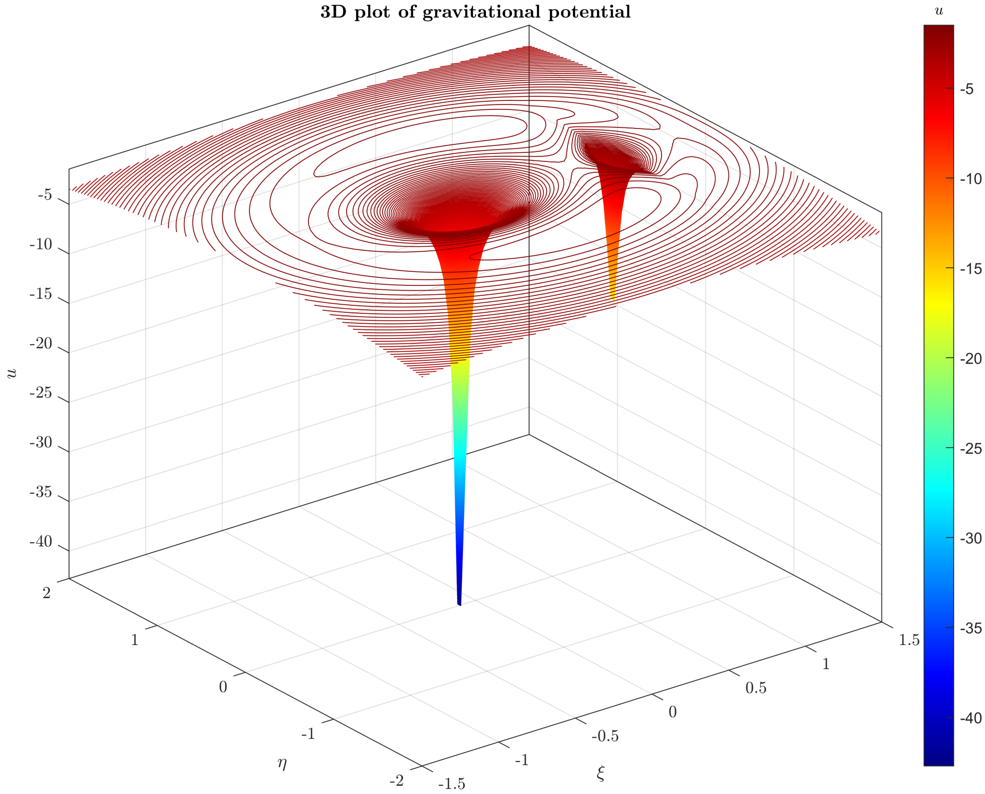


Fig. 5. 3D Representation of gravitational equipotential energy of Sun-Earth system. Source: Own.

The gravitational energy (10) is presented in 2D and 3D contour plots. In Figure 4 is represented the plot of $u$ as a function of $\varepsilon$ (abscissa) and $\eta$ (ordinate). The Sun and Earth are depicted in colour yellow and blue, respectively. Also, the barycenter is depicted which at the same time is the coordinates origin $(0, 0)$ of the reference frame. Additionally, the figure shows 5 regions in which potential gravity energy is almost null due to force equilibrium between the two bodies (Sun-Earth).

It is noticeable to mention that the plot atop is presented with a $\mu = 0.1$ since that is the value that allows the plot to illustrate all Lagrange points. Otherwise, it will place points L1 and L2 so close to Earth that it would not be able distinguish them clearly.

These contour lines represent a equipotential value of gravitational energy, that is, they represent equal spacing in the value of $u$.

Thereby, the coordinates of the Lagrange points (though they are spatial regions) can be extrapolated. The table here-under shows the location of the 5 Lagrange points.

| **Point** | $\varepsilon$ | $\eta$ |
|---|---|---|
| L1 | 0.609 | 0.000 |
| L2 | 1.260 | 0.000 |
| L3 | -1.042 | 0.000 |
| L4 | 0.400 | 0.866 |
| L5 | 0.400 | -0.866 |

TABLE I. Lagrange point locations

Finally, Figure 5 presents the a three-dimensional plot of the gravitational potential energy. This figure clearly presents the different stability Lagrange points in which the influence of $u$ is very small, whereas near the two great bodies there is a deep hole of gravitational potential energy. This explains that a satellite nearby those bodies will be highly influenced by the gravitational pull of the two massive bodies but if they stay at Lagrange points, those effects are much minor. Furthermore, recall that previously it was mentioned that L1, L2 and L3 were meta-stable points in the way that those points were unstable equilibrium points. Take a look at L1, the picture above clearly supports this phenomena as L1 is in between the two gravitational holes.

The L2 Lagrange point is selected to be the target destination, mainly due to its thermal stability, low radiation and absence of eclipses.

## References


[1] NASA. *What is a Lagrange Point?* 2020. URL: https://solarsystem.nasa.gov/resources/754/what-is-a-lagrange-point/.

[2] ESA. *What are Lagrange points?* 2020. URL: https://www.esa.int/Enabling_Support/Operations/What_are_Lagrange_points.

[3] Dr. Drang. *Lagrange points redux.* 2020. URL: https://leancrew.com/all-this/2016/08/lagrange-points-redux/.

[4] NASA. *The Lagrange points.* 2020. URL: https://map.gsfc.nasa.gov/ContentMedia/lagrange.pdf.

[5] Britannica. *D'Alembert's principle.* 2020. URL: https://www.britannica.com/science/dAlemberts-principle.


## II. Code

```
%% Equipotential surfaces and Langrange points plot

% Clear Command Window, clear workspace variables and close any MATLAB windows
clc;
clear;
close all;

% Reference frame parameters
mu = 0.1; % Ratio between Earth's mass and total mass (set 0.1 to zoom out)
xi = linspace(-1.5,1.5,100); % Adimentional 'x' variable
eta = linspace(-2,2,100); % Adimentional 'y' variable
[xi,ETA] = meshgrid(xi,eta); % Create 2D mesh

% Adimentional gravitational potential energy
f=@(xi,eta) -(1-mu)./(sqrt((xi+mu).^2+eta.^2)) - (mu)./(sqrt((xi-(1-mu)).^2)+eta.^2) ...
    - 0.5*(xi.^2+eta.^2);
Z=f(xi,ETA);

% Plot gratitaional equipotential energy
set(groot,'defaultAxesTickLabelInterpreter','latex');
set(groot,'defaulttextinterpreter','latex');
set(groot,'defaultLegendInterpreter','latex');

% 2D plot
figure(1);
hold on;
box on;
colormap(jet);
contour(xi,ETA,Z,700);
colorbar;
cbar=colorbar;
title(cbar,{'$u$'},'interpreter','latex');
title('\textbf{Gravitational equipotential surfaces}');
xlabel('$\xi$');
ylabel('$\eta$');
scatter(0,0,'filled','MarkerFaceColor',[0 1 0]);
scatter(0.9,0,50,'filled','MarkerFaceColor',[0 1 1]);
txt1 = {'L1'};
text(0.55,-0.02,txt1);
scatter(-0.1,0,200,'filled','MarkerFaceColor',[1 1 0]);
txt2 = {'L2'};
text(1.22,-0.02,txt2);
scatter(0,0,'filled','MarkerFaceColor',[1 1 0]);
txt1 = {'L3'};
text(-1.22,-0.02,txt1);
```

```
scatter(0,0,'filled','MarkerFaceColor',[0 1 0]);
txt4 = {'L4'};
text(0.4,0.9,txt4);
scatter(0,0,'filled','MarkerFaceColor',[0 1 0]);
txt4 = {'L5'};
text(0.4,-0.9,txt4);
scatter(0,0,'filled','MarkerFaceColor',[0 1 0]);
hold off;

% 3D plot
figure(2)
box on;
contour3(xi,ETA,Z,700);
colormap(jet);
colorbar;
cbar=colorbar;
title(cbar,{'$u$'},'interpreter','latex');
title('\textbf{3D plot of gravitational potential}');
xlabel('$\xi$');
ylabel('$\eta$');
zlabel('$u$');
hold off;
```