COMPUTATIONAL AEROSPACE ENGINEERING

ASSIGNMENT 2

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1 Part 1 (basic)

1.1 Statement

The goal of this assignment is to develop a Matlab program able to solve any two dimensional heat conduction problem using bilinear quadrilateral elements.

To test the performance of the program, consider the heat conduction problem depicted in Figure 1. The coordinates are given in meters. The conductivity matrix is isotropic, with $\kappa = \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\kappa = 10\text{W}/°\text{C}$. The temperature $u = 0$ is prescribed along edges AB and AD. The heat fluxes $q \cdot n = 0$ and $q \cdot n = 20W/m$ are prescribed on edges BC and CD, respectively. A constant heat source $f = 4\,W/m^2$ is applied over the plate.

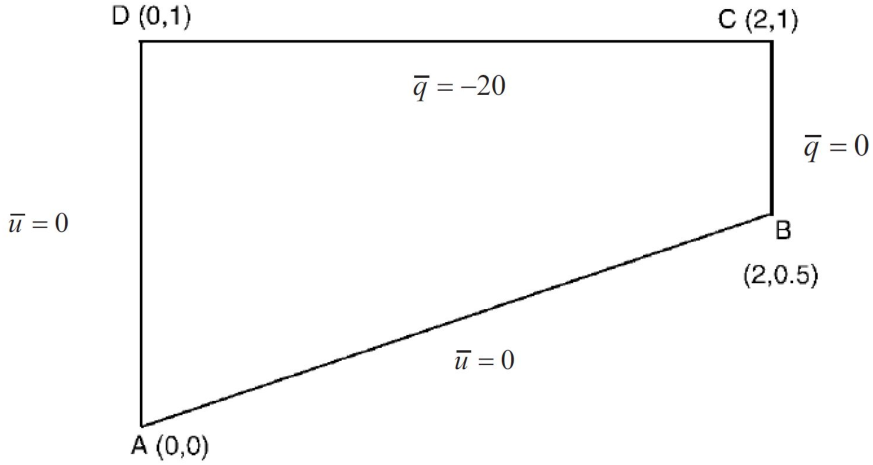


Figure 1 Structure 1

To assess convergence upon mesh refinement, launch 4 different analyses with increasing number of finite elements. In particular, use structured meshes with

- $n_{el} = 1$ element
- $n_{el} = 5 \times 5$ element
- $n_{el} = 10 \times 10 = 100$ elements
- $n_{el} = 30 \times 30 = 900$ elements

For these three cases, plot the distribution of temperature along the edge DC in the same graph.

1.1.1 Programming tasks

The requested MATLAB program has not been developed from scratch. Rather, it has been used the "template" code *mainHEATC.m*, which already incorporates preprocess and postprocess facilities (using the commercial software GID).

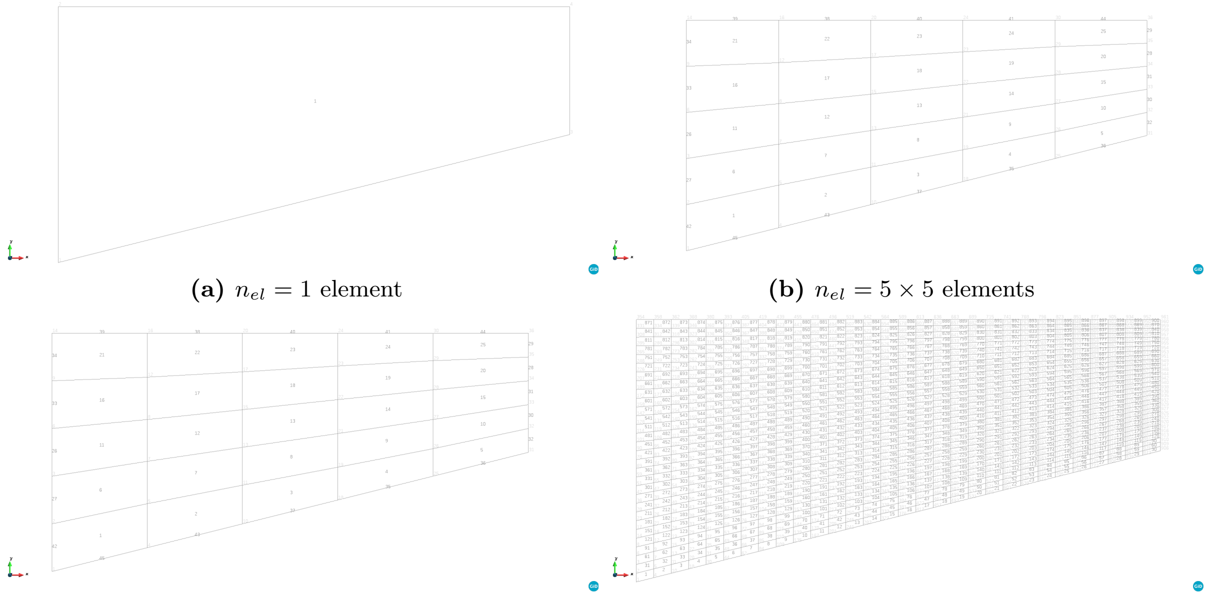
There are some parts missing which have been implemented in the attached code, mainly :

- The assembly of the conductance matrix, as well as the shape function routines of bilinear quadrilateral elements.
- The assembly of the vector of global source flux vector $(\boldsymbol{F}_s)$, the solution of the final system of equations, and the computation of the heat flux vector at each Gauss point.

1.1.2 Guidelines for pre-processing

Following the guidelines explained in this section, we shall proceed to solve the problem.

First, we begin to set the mesh of the structure. In order to asses a convergence further convergence analysis, different number of finite elements were set. The dimensions of the structure was given by the problem statement, thus, according to Figure 1, the different structures mesh were generated as shown in the following Figure 2.



(a) $n_{el} = 1$ element

(b) $n_{el} = 5 \times 5$ elements

(c) $n_{el} = 10 \times 10$ elements

(d) $n_{el} = 30 \times 30$ elements

Figure 2 Mesh element

Note that in generating the geometry, an external package was added to set the structure's material and surfaces. Each surface is associated with a number. This number is used in the MATLAB input data file to assign the corresponding conductivity. Furthermore, next step was to identify lines with different boundary conditions as well. This number is used to impose both Dirichlet and Neumann Boundary Conditions.

Subsequently, the following step was the generation of the **finite element mesh**. To do so, a finite quadrilateral type element was selected and using structured mesh. At the same time, the boundary mesh was also generated during this stage.

Finally, the final step of the pre-process was to export all the coordinates and structure's properties.

1.1.3 Guidelines for post-processing

Once the MATLAB code calculates the temperatures for each node, as well as the heat flux vector between them, it is time to view the results.

In order to draw the temperature ranges, the contour fill tool is employed for each mesh so as to see how the vales fluctuate throughout the surface. Cold colors such as blue will indicate the coldest areas while vivid colours such as red will indicate the maximum values of temperature.

Similarly, the 2D Heat flux analysis requires the Display Vectors tool and adopts the same color scale as the temperatures.

1.1.4 Temperature results

Once the temperatures are obtained for each mesh, they are plotted as follows:

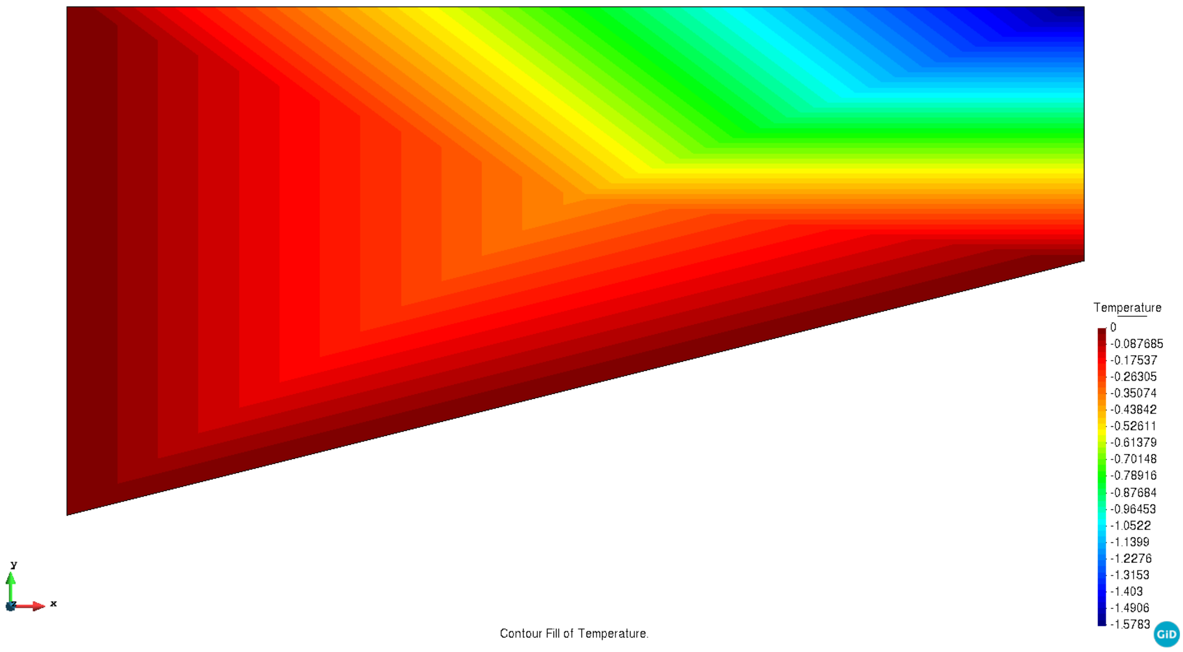


Figure 3 Temperature distribution for $n_{el} = 1$ element

The above image show the different temperature distribution considering a single element. Notice that the temperature in the boundary that were set as they were boundary conditions (Dirichlet) are have all $T = 0$ °C. Furthermore, due to the lack of mesh elements, the temperature distribution is not smooth.

The temperatures are presented smooth since the parameter d was passed to the post process MATLAB file. However, GID has built-in shape functions and it interpolates the result. Thereby, it shows a smooth transition inside the element.

The surface above has negative temperature. This is explained as the heat flux $q \cdot \vec{n} = 20$ W/m is positive towards the external part of the structure, therefore, the heat is flowing from the inside of the structure towards the external side.

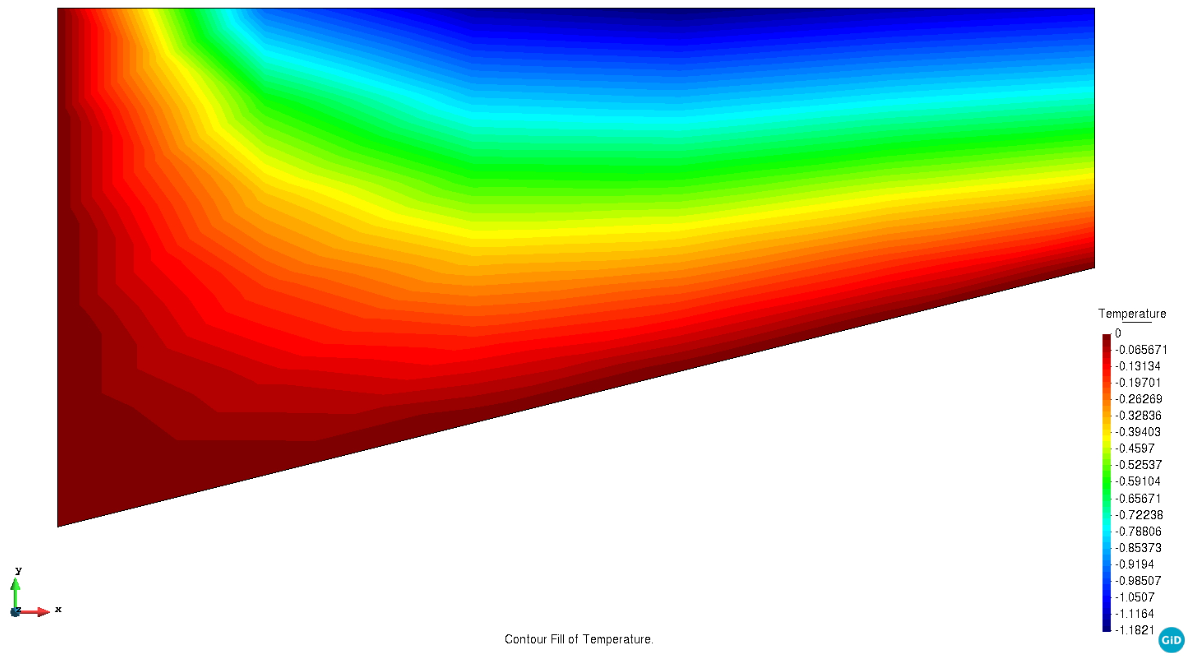


Figure 4 Temperature distribution for $n_{el} = 5 \times 5$ elements

Additionally, the temperature distribution for $n_{el} = 5 \times 5$ elements is plotted in the above Figure 4. This temperature distribution is far more smooth than the previous one. However, the southwest part of the mesh appears to be irregular despite the temperature is being a boundary condition. It is a transition phase so increasing the number of elements will increase the smoothness of the temperature distribution.

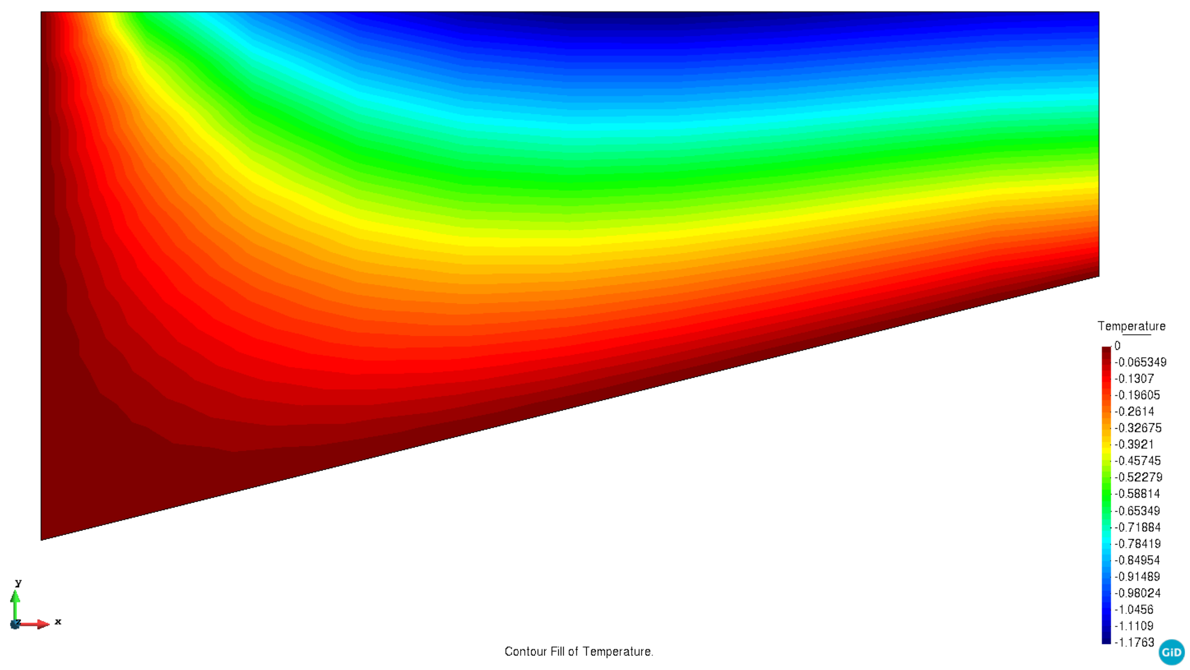


Figure 5 Temperature distribution for $n_{el} = 10 \times 10$ elements

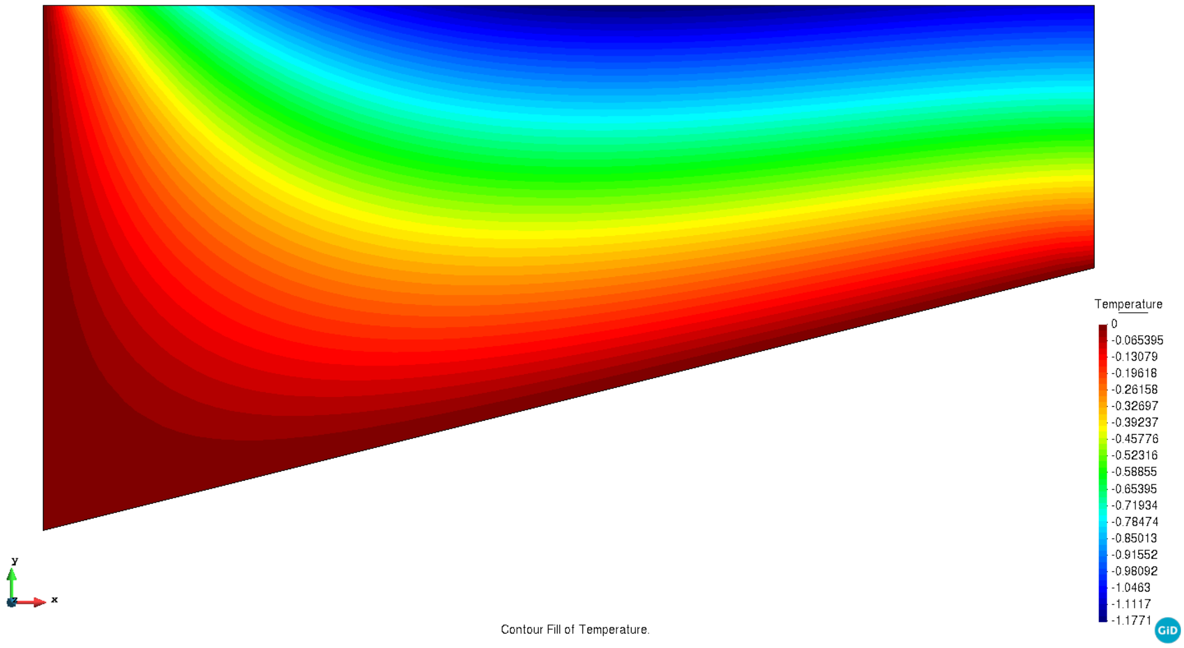
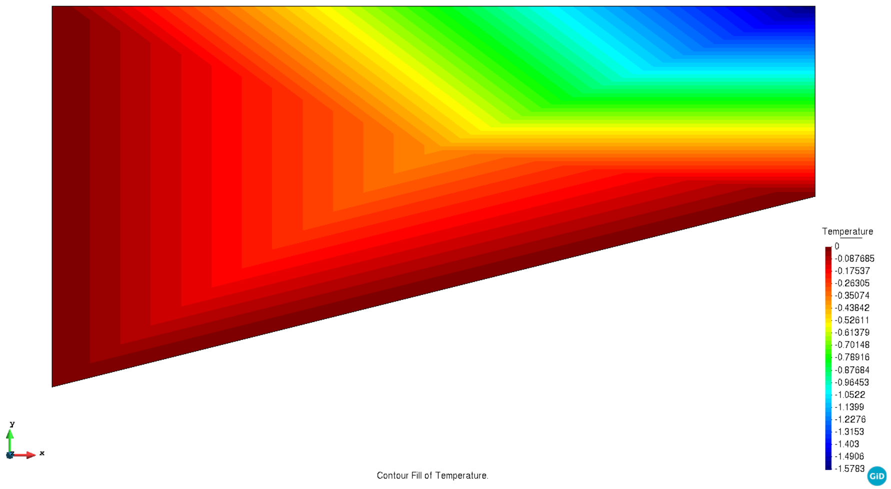


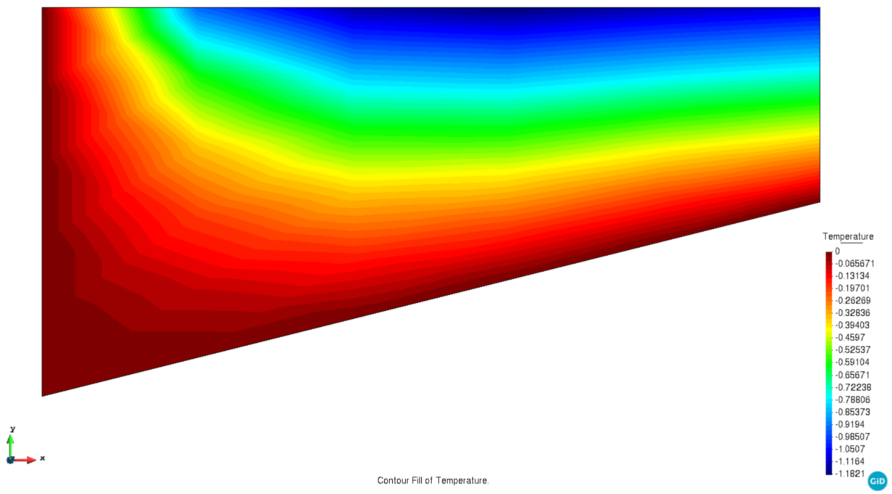
Figure 6 Temperature distribution for $n_{el} = 30 \times 30$ elements

For both the $n_{el} = 10 \times 10$ elements and $n_{el} = 30 \times 30$ elements, the results of the temperature distribution can be considered to have converged and smoothness is generalised. The variation between them is approximately 0.5%.

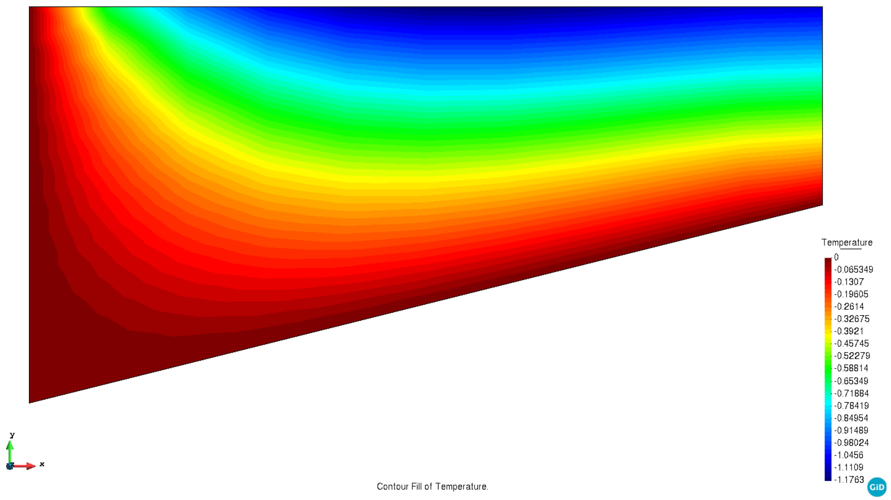
Finally, it can be seen that the result complies with both the Neumann and Dirichlet boundary conditions, maintaining the temperature on the left and bottom sides while the vectors can be seen at the Heat Flux analysis 13.



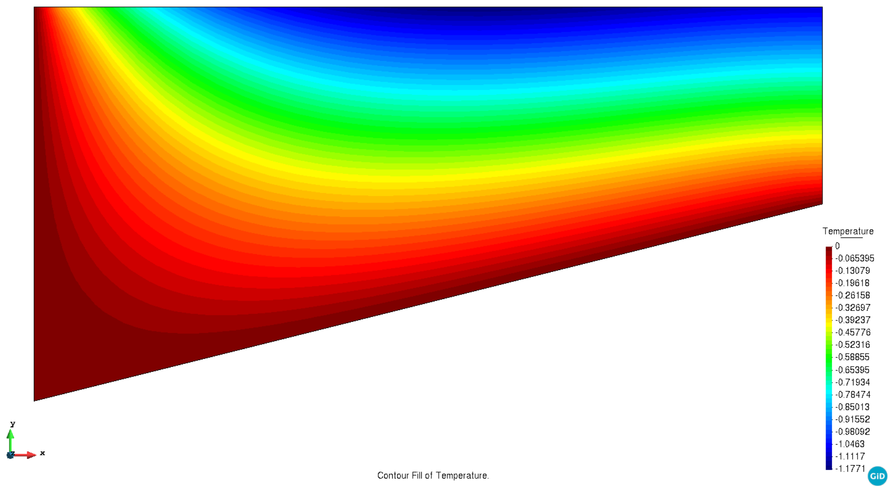
(a) $n_{el} = 1$ element



(b) $n_{el} = 5 \times 5$ elements



(c) $n_{el} = 10 \times 10$ elements



(d) $n_{el} = 30 \times 30$ elements

Figure 7 Temperature distribution for all elements

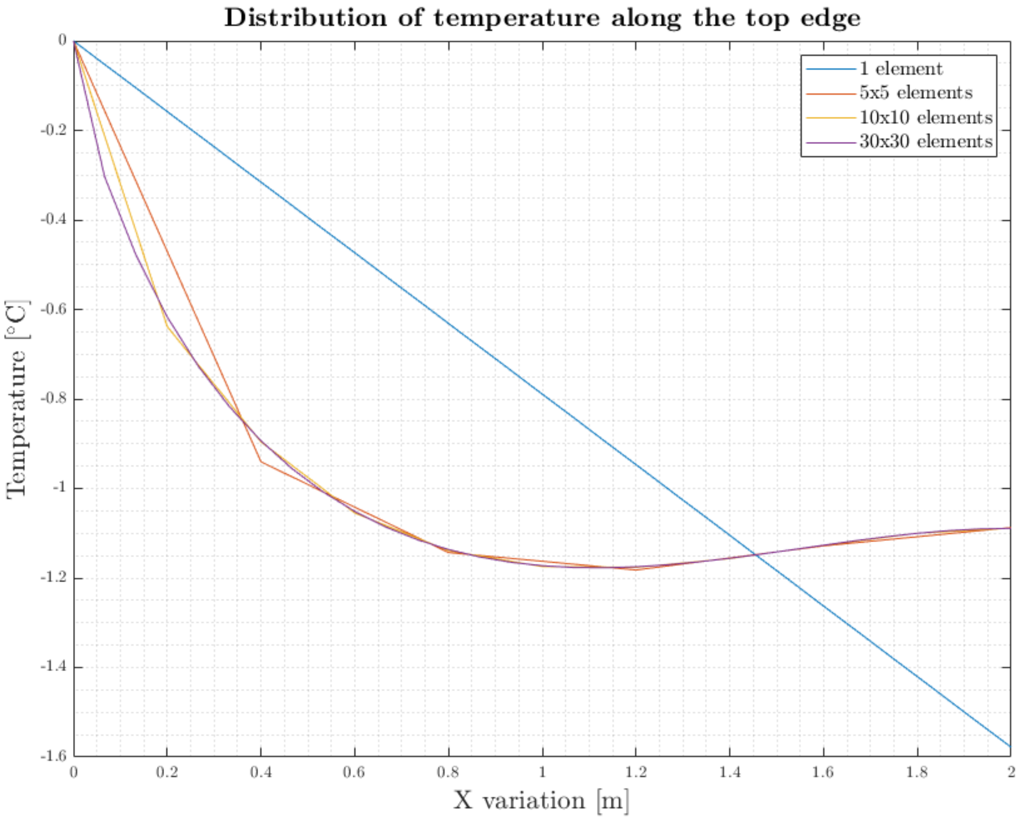


Figure 8 Distribution of temperature along the top edge

As it can be seen in Fig.8, the distribution of temperature along the top edge is highly dependant on the number of divisions. While 1 element does not represent the physical system being studied, he option of 25 elements could be considered quite precise for the low number of terms it involves and the subsequent code speed. However, analysing the convergence it can be seen that values over a 100 elements are a decent approximation while the best option, which comprises 900 elements, has already converged and no further change is noticeable.

1.1.5 Heat Flux results

Once the heat fluxes are obtained for each mesh, they are plotted as follows:

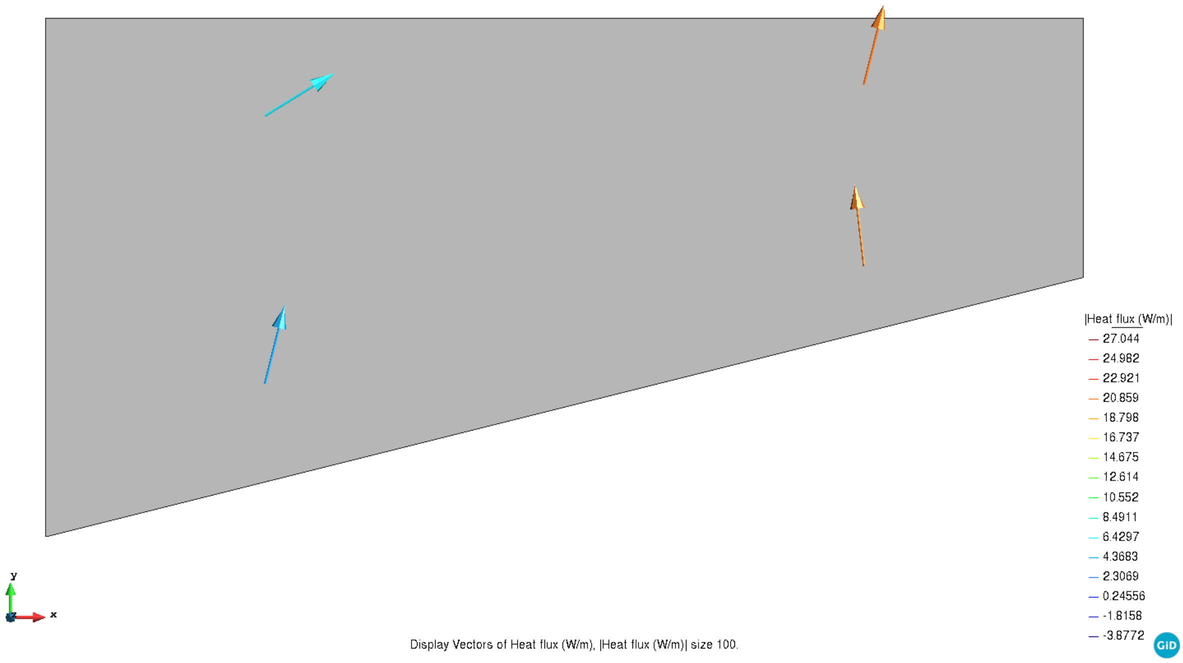


Figure 9 Heat Flux distribution for $n_{el} = 1$ element

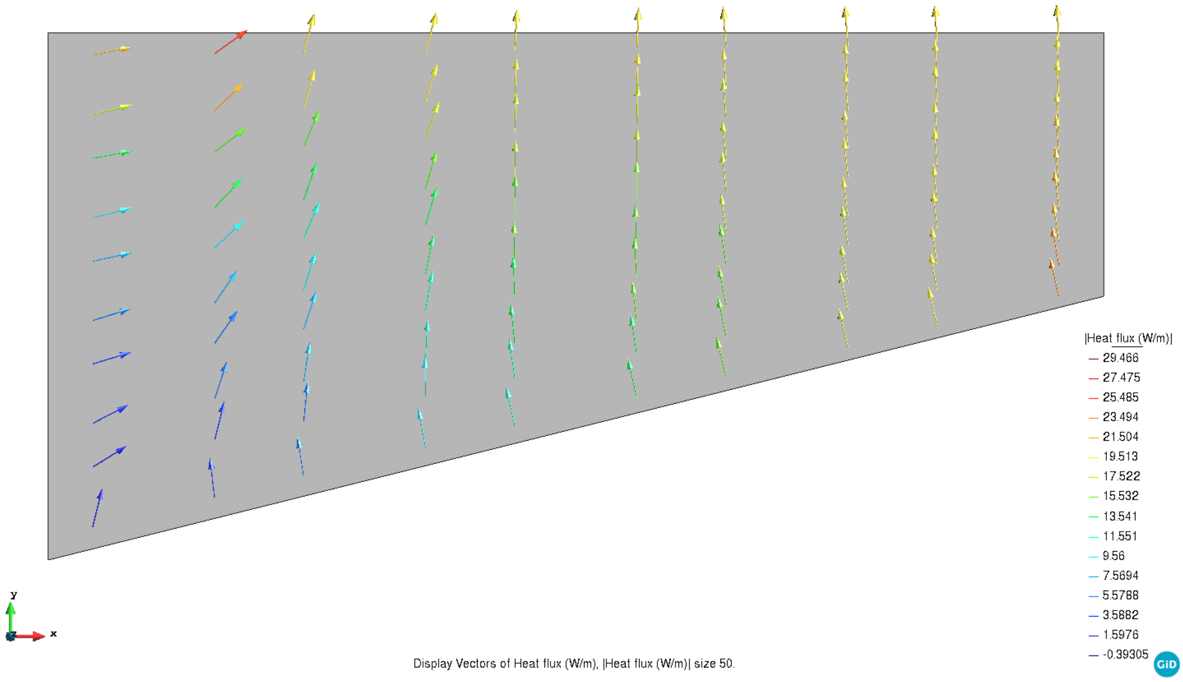


Figure 10 Heat Flux distribution for $n_{el} = 5 \times 5$ elements

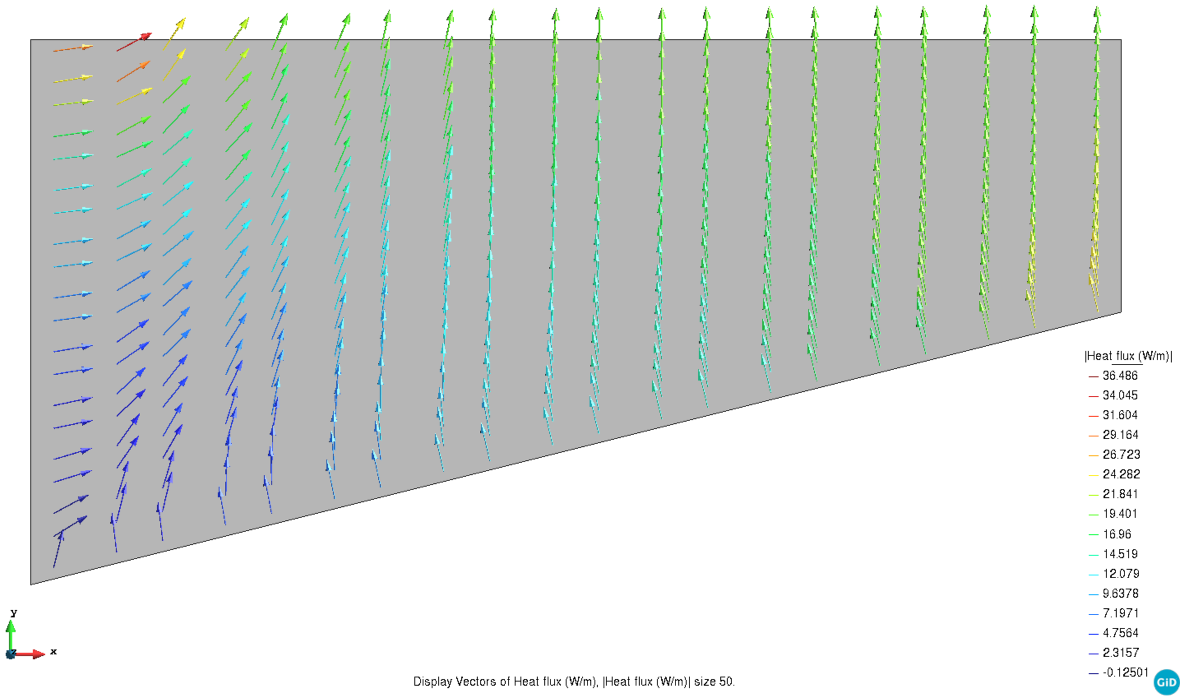


Figure 11 Heat Flux distribution for $n_{el} = 10 \times 10$ elements

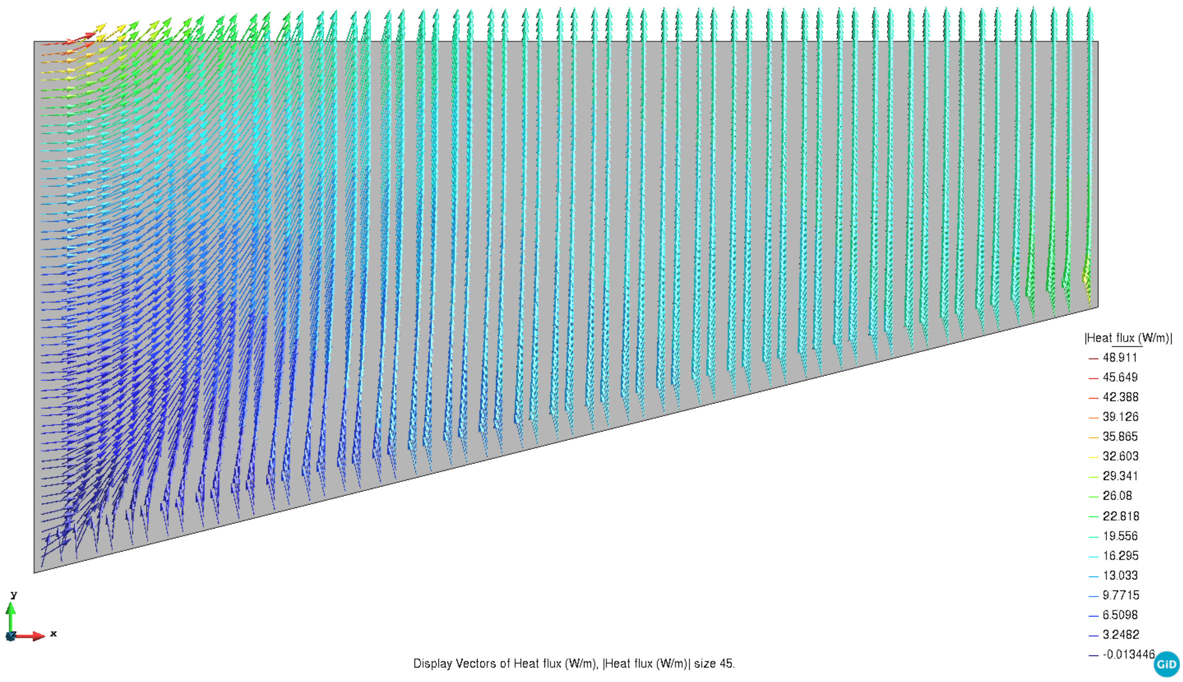
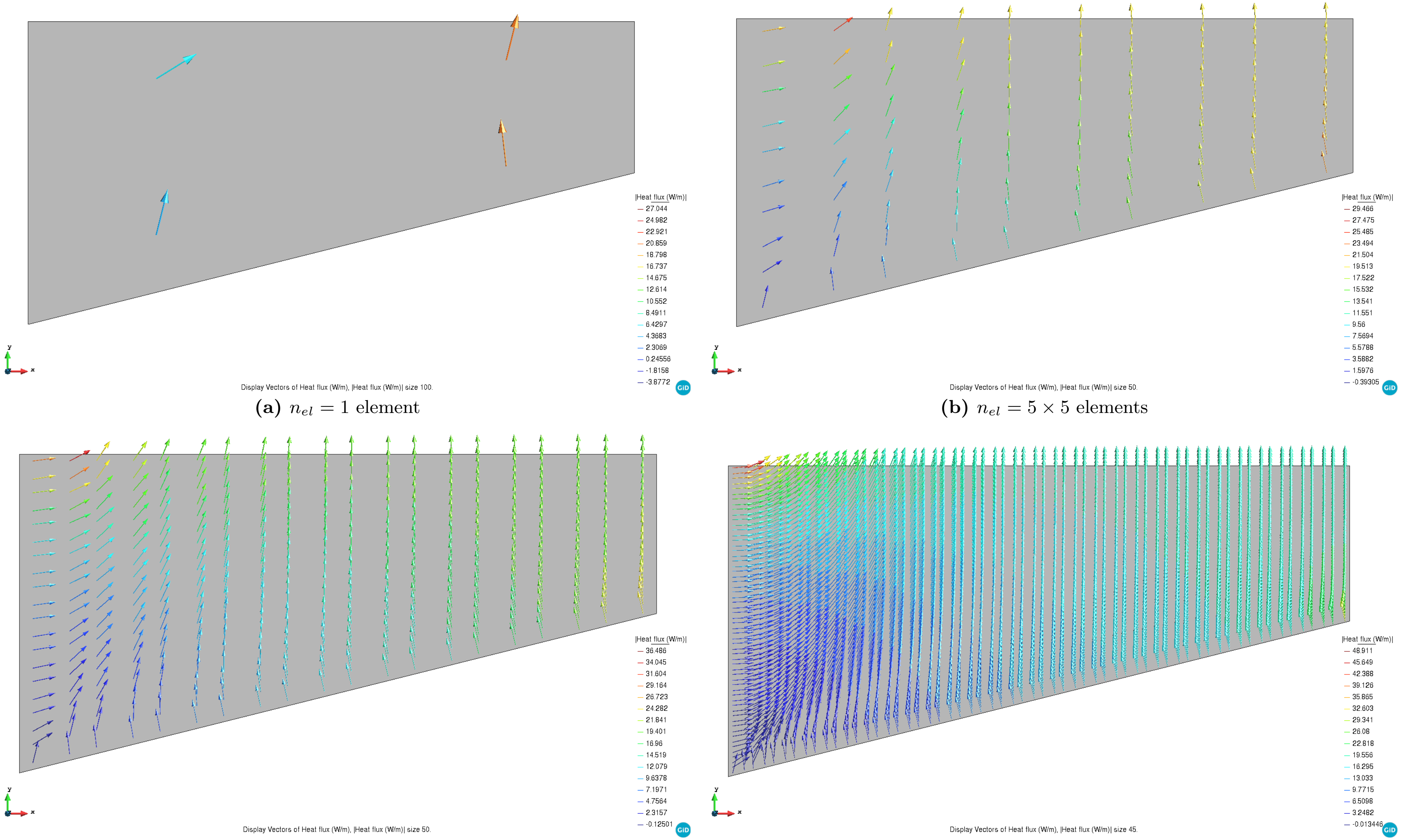


Figure 12 Heat Flux distribution for $n_{el} = 30 \times 30$ elements



(a) $n_{el} = 1$ element

(b) $n_{el} = 5 \times 5$ elements

(c) $n_{el} = 10 \times 10$ elements

(d) $n_{el} = 30 \times 30$ elements

Figure 13 Heat Flux distribution for all elements

2 Part 2 (advanced)

2.1 Convective heat transfer

2.1.1 Modified conductance matrix and nodal heat flux vector (see Video10_heat.avi)

The weak form of the prototypical linear heat conduction problem is given by (see page 18 of the slides):

$$\text{W.F.} \begin{cases} \text{Given } f : \Omega \to \mathbb{R}, \bar{q} : \Gamma_b \to \mathbb{R}, \quad \text{find } u \in \mathcal{S} \text{ such that} \\ \displaystyle\int_\Omega \nabla v^T \kappa \nabla u \; d\Omega = \int_\Omega v f \; d\Omega + \int_{\Gamma_b} v \bar{q} \; d\Gamma, \quad \forall v \in \mathcal{V} \end{cases}$$

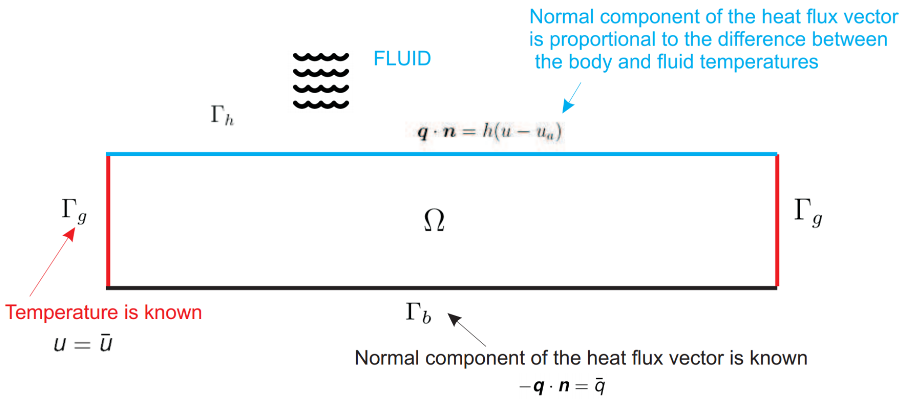


Figure 14 Structure 2

This boundary value problem assumes that the boundary is split into a portion on which the temperature is known (Γ_g) and another portion on which the normal component of the heat flux vector is known (Γ_b). However, there are physical processes in which the conditions at the boundary cannot be described by either of these boundary conditions. This happens, for instance, when we have convective heat transfer between an external fluid and the body under study (see Figure 3). Mathematically, this condition is described in strong form as

$$\boldsymbol{q} \cdot \boldsymbol{n} = h\,(u - u_a) \qquad \text{on } \Gamma_h$$

where $u - u_a$ is the temperature difference between the wall and the fluid, and h is the convection heat transfer coefficient. Upon introduction of this boundary condition into the formulation of the problem, the corresponding expressions for the coefficient matrix $\boldsymbol{K}$ and the vector of external actions $\boldsymbol{F}$ change with respect to the prototypical problem. In particular, they adopt the forms

$$\boldsymbol{K} = \int_\Omega \boldsymbol{B}^T \boldsymbol{\kappa} \boldsymbol{B} \; \mathrm{d}\Omega + \boldsymbol{K}_h$$

$$\boldsymbol{F} = \int_\Omega \boldsymbol{N}^T f \; d\Omega + \int_{\Gamma_b} \boldsymbol{N}^T \bar{q} \; d\Gamma + \boldsymbol{F}_h$$

Which are the expressions for K_h **and** F_h?. Explain the procedure to arrive at such expressions.

2.2 Convective heat transfer resolution

Let's set some definitions at first:

- Cartesian components of the **heat flux vector**: $q_i : \bar{\Omega} \to \mathbb{R}, i = 1, 2 \ldots n_{sd}$
- **Temperature**: $u : \bar{\Omega} \to \mathbb{R}$
- **Heat supply** per unit volume: $f : \Omega \to \mathbb{R}$
- Heat flux vector is defined in terms of the temperature gradient by the generalized ***Fourier law***:

$$q_i = -\kappa_{ij} \frac{\partial u}{\partial x_j} \tag{1}$$

  where $\kappa_{ij} : \bar{\Omega} \to \mathbb{R}$ are the **conductivities** of the material. The **conductivity matrix** $\kappa \in \mathbb{R}^{n_{sd}} \times n_{sd}$ is assumed positive definite. The most common situation in practice is the **isotropic case** ($\kappa_{ij} = \kappa\delta_{ij}$)

Indicial notation
The problem stated can be described as, given a heat source f defined within all the domain Ω, and known the value of the temperature in some parts of the domain $\overline{u}$ as well as the heat flux at the boundary $\overline{q}$, the problem states to find $u : \Omega \to \mathbb{R}$ that complies the requirements listed here-under:

$$\boxed{\begin{array}{l} \text{Given} \quad f : \Omega \to \mathbb{R}, \bar{u} : \Gamma_g \to \mathbb{R},\ \bar{q} : \Gamma_b \to \mathbb{R}, \quad \text{find } u : \Omega \to \mathbb{R} \quad \text{such that} \\ \begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{q} - f = \rho c \frac{\partial u}{\partial t}, & \qquad \text{in } \Omega \\ u = \bar{u} & \qquad \text{on } \Gamma_g \\ -\boldsymbol{q} \cdot \boldsymbol{n} = \bar{q} & \qquad \text{on } \Gamma_b \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \qquad \text{on } \bar{\Omega} \text{ (initial condition)} \end{aligned} \\ \text{where} \quad \boldsymbol{q} = -\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} u. \end{array}}$$

($\nabla \cdot q$ is the divergence of the heat flux vector, while ∇u denotes the gradient of the temperature field. The functions $\bar{u}$ and $\bar{q}$ are the prescribed boundary temperature and heat flux, respectively. It is noticeable to mention that this problem possesses a unique solution for appropriate restrictions on the given data

Analogously to the 1D problem, in order to formulate the weak boundary problem, test and trial functions must be defined:

- **Test functions**: $\nu = v : \overline{\Omega} \to \mathbb{R}|$ continuous with square-integrable derivative, $v = 0$ on Ω_g.
- **Trial functions**: $\mathcal{S} = u : \overline{\Omega} \to \mathbb{R}|$ continuous with square-integrable derivative, $u = \overline{u}$ on Γ_g.

The weak form is obtained by integrating over the domain the product of the balance equation with the test functions and using the *divergence theorem* to shift the derivatives on q to the test functions v:

$$\int_\Omega v(\nabla \cdot q - f)\ \mathrm{d}\Omega = 0 \quad \Longrightarrow \quad \int_\Omega v(\nabla \cdot q)\ \mathrm{d}\Omega - \int_\Omega vf\ \mathrm{d}\Omega = 0 \tag{3}$$

Another time, applying the divergence theorem to elaborate the $v(\nabla \cdot q)$ term,

$$\int_\Omega v(\nabla \cdot q)\ \mathrm{d}\Omega = \int_\Gamma v(\mathrm{q} \cdot \mathbf{n})\ \mathrm{d}\Gamma - \int_\Omega \nabla v \cdot \mathbf{q}\ \mathrm{d}\Omega \tag{4}$$

The first term of the right-hand side can be expressed as the sum over the integrals over Γ_g, Γ_b and Γ_h as follows

$$\int_\Gamma v(\mathbf{q} \cdot \mathbf{n})\ \mathrm{d}\Gamma = \int_{\Gamma_g} v(\mathbf{q} \cdot \mathbf{n})\ \mathrm{d}\Gamma + \int_{\Gamma_b} v(\mathbf{q} \cdot \mathbf{n})\ \mathrm{d}\Gamma + \int_{\Gamma_h} v(\mathbf{q} \cdot \mathbf{n})\ \mathrm{d}\Gamma \tag{5}$$

The integral over Γ_g vanishes in virtue of the property $v = 0$ on Γ_g, so the above expression is reduced to

$$\int_{\Gamma} v(\mathbf{q}\cdot\mathbf{n})\ \mathrm{d}\Gamma = \int_{\Gamma_b} v(\mathbf{q}\cdot\mathbf{n})\ \mathrm{d}\Gamma + \int_{\Gamma_h} v(\mathbf{q}\cdot\mathbf{n})\ \mathrm{d}\Gamma \tag{6}$$

Next, since the values of the heat flux vector times the normal on Γ_b is an input data $[q\cdot n = -\overline{q}]$ then:

$$\int_{\Gamma_b} v(\mathbf{q}\cdot\mathbf{n})\ \mathrm{d}\Gamma = \int_{\Gamma_b} v(-\overline{q})\ \mathrm{d}\Gamma \tag{7}$$

$$\int_{\Gamma_h} v(\mathbf{q}\cdot\mathbf{n})\ \mathrm{d}\Gamma = \int_{\Gamma_h} h(u-u_a)\ \mathrm{d}\Gamma \tag{8}$$

$$\int_{\Omega} v(\nabla\cdot q)\ \mathrm{d}\Omega = \int_{\Gamma_b} v(-\overline{q})\ \mathrm{d}\Gamma + \int_{\Gamma_h} h(u-u_a)\ \mathrm{d}\Gamma - \int_{\Omega} \nabla v\cdot\mathbf{q}\ \mathrm{d}\Omega \tag{9}$$

Replacing the above expressions into equation (4),

$$\int_{\Gamma_b} v(-\overline{q})\ \mathrm{d}\Gamma + \int_{\Gamma_h} h(u-u_a)\ \mathrm{d}\Gamma - \int_{\Omega} \nabla v\cdot\mathbf{q}\ \mathrm{d}\Omega - \int_{\Omega} vf\ \mathrm{d}\Omega = 0 \tag{10}$$

$$\int_{\Omega} \nabla v\cdot\mathbf{q}\ \mathrm{d}\Omega = \int_{\Gamma_b} v(-\overline{q})\ \mathrm{d}\Gamma + \int_{\Gamma_h} h(u-u_a)\ \mathrm{d}\Gamma - \int_{\Omega} vf\ \mathrm{d}\Omega \tag{11}$$

Since the convection heat flux vector times the normal is a known expression as well $[q\cdot n = h(u-u_a)]$,

$$\int_{\Omega} \nabla v\cdot\kappa\nabla u\ \mathrm{d}\Omega = \int_{\Gamma_b} v(-\overline{q})\ \mathrm{d}\Gamma + \int_{\Gamma_h} h(u-u_a)\ \mathrm{d}\Gamma - \int_{\Omega} vf\ \mathrm{d}\Omega \tag{12}$$

The weak form of the convective heat transfer is obtained

$$\text{W.F.}\ \left\{ \begin{array}{l} \text{Given } f:\Omega\to\mathbb{R}, \overline{q}:\Gamma_b\to\mathbb{R}, u_a:\Gamma_h\to\mathbb{R} \\ \displaystyle\int_{\Omega} \nabla v\cdot\kappa\nabla u\ \mathrm{d}\Omega = \int_{\Gamma_b} v(-\overline{q})\ \mathrm{d}\Gamma + \int_{\Gamma_h} h(u-u_a)\ \mathrm{d}\Gamma - \int_{\Omega} vf\ \mathrm{d}\Omega \end{array}\right.$$

Discretizating the domain using the finite-dimensional approximation of $\mathcal{V}$ and $\mathcal{S}$:

$$\text{W.F.}\ \left\{ \begin{array}{l} \text{Given } f:\Omega\to\mathbb{R}, \overline{q}:\Gamma_b\to\mathbb{R}, u_a:\Gamma_h\to\mathbb{R} \\ \displaystyle\int_{\Omega} \nabla v^h\cdot\kappa\nabla u^h\ \mathrm{d}\Omega = \int_{\Gamma_b} v^h(-\overline{q})\ \mathrm{d}\Gamma + \int_{\Gamma_h} h(u^h-u_a)\ \mathrm{d}\Gamma - \int_{\Omega} v^h f\ \mathrm{d}\Omega \end{array}\right.$$

Defining

$$u = \mathbf{N}\cdot d \tag{13}$$

$$v = \mathbf{N}\cdot c = c^T N^T \tag{14}$$

$$\nabla u = \mathbf{B}\cdot d \tag{15}$$

$$\nabla v = \mathbf{B}\cdot c = c^T B^T \tag{16}$$

$$\mathbf{B} = \nabla\mathbf{N} \tag{17}$$

Applying the definition above,

$$\boldsymbol{c}^T\left(\int_{\Omega}\boldsymbol{B}^T\kappa\boldsymbol{B}\ \mathrm{d}\Omega\right)\boldsymbol{d} = \boldsymbol{c}^T\left[\int_{\Omega}\boldsymbol{N}^T f\mathrm{d}\Omega + \int_{\Gamma_b}\boldsymbol{N}^T\overline{q}\ \mathrm{d}\Gamma + \int_{\Gamma_h}\mathbf{N}^T h\cdot u_a\ \mathrm{d}\Gamma - \left(\int_{\Gamma_h}\mathbf{N}^T h\mathbf{N}\ \mathrm{d}\Gamma\right)\cdot d\right] \tag{18}$$

Finally,

$$\boldsymbol{c}^T\left(\int_{\Omega}\boldsymbol{B}^T\kappa\boldsymbol{B}\ \mathrm{d}\Omega + \int_{\Gamma_h}\mathbf{N}^T h\mathbf{N}\ \mathrm{d}\Gamma\right)\boldsymbol{d} = \boldsymbol{c}^T\left[\int_{\Omega}\boldsymbol{N}^T f\mathrm{d}\Omega + \int_{\Gamma_b}\boldsymbol{N}^T\overline{q}\ \mathrm{d}\Gamma + \int_{\Gamma_h}\mathbf{N}^T h\cdot u_a\ \mathrm{d}\Gamma\right] \tag{19}$$

Therefore, the **global conductance matrix** shall be expressed as:

$$\boxed{\mathbf{K} := \int_{\Omega} \boldsymbol{B}^T \kappa \boldsymbol{B} \, \mathrm{d}\Omega + \underbrace{\int_{\Gamma_h} \mathbf{N}^T h \mathbf{N} \, \mathrm{d}\Gamma}_{K_h}} \tag{20}$$

Also, the **global flux vector** is expressed as:

$$\boxed{\mathbf{F} := \int_{\Omega} \boldsymbol{N}^T f \mathrm{d}\Omega + \int_{\Gamma_b} \boldsymbol{N}^T \bar{q} \, \mathrm{d}\Gamma + \underbrace{\int_{\Gamma_h} \mathbf{N}^T h \cdot u_a}_{F_h} \, \mathrm{d}\Gamma} \tag{21}$$

2.3 Time-dependent heat conduction

2.3.1 Formulation

So far we have only addressed finite element analysis of heat conduction for steady-state conditions. No time dependence has been included in such analyses, as we have assumed conditions in which a steady state has been reached. Certainly, transient, time-dependent effects are often of paramount important, since such effects determine whether a steady state is achieved and, furthermore, what that steady state will be.

To account for transient effects, the strong form of the boundary value problem presented in Slide 14 of the theory document has to be modified as follows (the new terms are highlighted in blue):

Given $f : \Omega \times [0,T] \to \mathbb{R}, \bar{u} : \Gamma_g \times [0,T] \to \mathbb{R}, \bar{q} : \Gamma_b \times [0,T] \to \mathbb{R},$ find $u : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{q} - f = \rho c \frac{\partial u}{\partial t}, &\qquad \text{in } \Omega \times [0,T] \\
u = \bar{u} &\qquad \text{on } \Gamma_g \times [0,T] \\
-\boldsymbol{q} \cdot \boldsymbol{n} = \bar{q} &\qquad \text{on } \Gamma_b \times [0,T] \\
u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) &\qquad \text{on } \bar{\Omega} \text{ (initial condition)}
\end{aligned}$$

where $\boldsymbol{q} = -\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} u$.

In the preceding equations, t is the time, $[0,T]$ represents the time domain, whereas $\rho(x)$ and $c(x)$ denote the density and the specific heat of the material, respectively. Problem 2.1 represents the prototypical second-order parabolic partial differential [1] equation (PDE). Notice that the only difference with respect to the steady state equation is the derivative with respect time appearing in the right-hand side of the differential equation. since the equation is first-order in time, it requires the specification of the temperature of the body at time $t = 0$ (the initial condition $u(\boldsymbol{x}, 0) = u_0$). It can be shown that the variational expression corresponding to problem 2.1 is given

$$\boldsymbol{c}^T (\boldsymbol{M}\dot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} - \boldsymbol{F}) = \boldsymbol{0} \tag{23}$$

where M is the capacitance matrix, whereas $\dot{d}$ denotes the derivative with respect time of the nodal temperature vector $\boldsymbol{d}$. **Find the expression of the capacitance matrix in terms of the shape functions employed for the interpolation.**

2.4 Time-dependent heat conduction resolution

In this case, the parabolic partial differential equation changes the weak form where both the test and trial functions are to be taken into consideration.

$$\int_\Omega v(\nabla \cdot \mathbf{q} - f)\, d\Omega = \int_\Omega v\rho c \frac{\partial u}{\partial t}\, d\Omega \tag{24}$$

$$\int_\Omega v(\nabla \cdot \mathbf{q})\, d\Omega - \int_\Omega v f d\Omega = \int_\Omega v\rho c \frac{\partial u}{\partial t}\, d\Omega \tag{25}$$

We elaborate now on the integral of $v(\nabla \cdot \boldsymbol{q})$

$$\int_\Omega v(\nabla \cdot \boldsymbol{q})\, d\Omega = \int_\Gamma v(\boldsymbol{q} \cdot \boldsymbol{n})\, d\Gamma - \int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{q}\, d\Omega \tag{26}$$

The first term of the right-hand side of the preceding equation can be expressed as the sum of the integrals over Γ_g and Γ_b as follows

$$\int_\Gamma v(\boldsymbol{q} \cdot \boldsymbol{n})\, d\Gamma = \int_{\Gamma_g} v(\boldsymbol{q} \cdot \boldsymbol{n})\, d\Gamma + \int_{\Gamma_b} v(\boldsymbol{q} \cdot \boldsymbol{n})\, d\Gamma \tag{27}$$

The integral over Γ_g vanishes in virtue of the property $v = 0$ on Γ_g . Thus:

$$\int_\Gamma v(\boldsymbol{q} \cdot \boldsymbol{n})\, d\Gamma = \int_{\Gamma_b} v(\boldsymbol{q} \cdot \boldsymbol{n})\, d\Gamma \tag{28}$$

since the value of heat flux vector times the normal on Γ_b is an input data, we have

$$\int_\Gamma v(\boldsymbol{q} \cdot \boldsymbol{n})\, d\Gamma = \int_{\Gamma_b} v(-\bar{q})\, d\Gamma \tag{29}$$

Inserting this equation into (26), and the resulting expression into (25), we finally arrive at

$$\int_\Omega \nabla v \cdot \mathbf{q}\, d\Omega = -\int_\Omega v f\, d\Omega - \int_{\Gamma_b} v\bar{q}\, d\Gamma - \int_\Omega v\rho c \frac{\partial u}{\partial t}\, d\Omega \tag{30}$$

which results in a matrix notation after replacing $\mathbf{q}$ by $-\kappa \cdot \nabla u$

$$\int_\Omega \nabla v^T \kappa \nabla u\, d\Omega = \int_\Omega v f d\Omega + \int_{\Gamma_b} v\bar{q}\, d\Gamma + \int_\Omega v\rho c \frac{\partial u}{\partial t}\, d\Omega, \quad \forall v \in \mathcal{V} \tag{31}$$

Then, replacing u and v by $\mathrm{u^h}$ and $\mathrm{v^h}$ respectively in a finite element formulation with a global point of view

$$\int_\Omega \nabla {v^h}^T \kappa \nabla u^h\, d\Omega = \int_\Omega v^h f d\Omega + \int_{\Gamma_b} v^h \bar{q}\, d\Gamma + \int_\Omega v^h \rho c \cdot \frac{\partial u}{\partial t}\, d\Omega, \quad \forall\ v^h \in \mathcal{V}^h \tag{32}$$

where

$$\frac{\partial u}{\partial t} = \frac{\partial(\mathbf{Nd})}{\partial t} = \mathbf{N} \cdot \frac{\partial \mathbf{d}}{\partial t} + \mathbf{d} \cdot \frac{\partial \mathbf{N}}{\partial t} \tag{33}$$

However, as the $\mathbf{N}$ matrix is not time-dependant, the second right-hand term is null and the previous expressions results in

$$\frac{\partial u}{\partial t} = \frac{\partial(\mathbf{N}\mathbf{d})}{\partial t} = \mathbf{N} \cdot \frac{\partial \mathbf{d}}{\partial t} = \mathbf{N} \cdot \dot{d} \tag{34}$$

which is then introduced in (32) to obtain

$$\int_\Omega \nabla {v^h}^T \kappa \nabla u^h \, d\Omega = \int_\Omega v^h f d\Omega + \int_{\Gamma_b} v^h \bar{q} \, d\Gamma + \int_\Omega v^h \rho c \cdot \mathbf{N} \cdot \dot{d} \, d\Omega, \quad \forall \, v^h \in \mathcal{V}^h \tag{35}$$

Defining $\mathbf{B} = \nabla \mathbf{N}$, the above expression can be recast as

$$\boldsymbol{c}^T \left(\int_\Omega \boldsymbol{B}^T \kappa \boldsymbol{B} \, d\Omega \right) \boldsymbol{d} = \boldsymbol{c}^T \left(\int_\Omega \boldsymbol{N}^T f \, d\Omega + \int_{\Gamma_b} \boldsymbol{N}^T \bar{q} \, d\Gamma + \int_\Omega \mathbf{N}^T \rho c \mathbf{N} \dot{d} \, d\Omega \right) \tag{36}$$

which if it is set as the problem expression (23) results in

$$\boldsymbol{c}^T \left(\int_\Omega \boldsymbol{B}^T \kappa \boldsymbol{B} \, d\Omega \boldsymbol{d} - \int_\Omega \boldsymbol{N}^T f \, d\Omega - \int_{\Gamma_b} \boldsymbol{N}^T \bar{q} \, d\Gamma - \int_\Omega \mathbf{N}^T \rho c \mathbf{N} \dot{d} \, d\Omega \right) = 0 \tag{37}$$

so that the following matrices and vectors are obtained

Capacitance matrix

$$\boxed{\boldsymbol{M} := - \int_\Omega \mathbf{N}^T \rho c \mathbf{N} \, d\Omega} \tag{38}$$

Global conductance matrix

$$\boxed{\boldsymbol{K} := \int_\Omega \boldsymbol{B}^T \kappa \boldsymbol{B} \, d\Omega} \tag{39}$$

Global flux vector (source and boundary contributions)

$$\boxed{\boldsymbol{F} := \int_\Omega \boldsymbol{N}^T f \, d\Omega + \int_{\Gamma_b} \boldsymbol{N}^T \bar{q} \, d\Gamma} \tag{40}$$

References

[1] Hernández, Joaquín. “Classical Linear Heat Conduction”. In: 1st ed. UPC, 2020, pp. 1–91.