Assignment 1

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Suppose a bar of length L of constant cross-sectional area A and Young's Modulus E. The bar has a prescribed displacement at $x = 0$ equal to $u(0) = -g$, and it is subjected to, on the one hand, an axial force F on the right end ($x = L$), and on the other hand, a distributed axial force (per unit area) $q(x) = E(\rho u(x) - s\,x^2)$ ($u = u(x)$ is the displacement field, g is a constant, and

$$\rho = \frac{\pi^2}{L^2}, \quad s = g\rho^2, \quad \frac{F}{AE} = \frac{g\pi^2}{L} \tag{1.0.1}$$

)

PART 1 (7 points)

1. Derive the corresponding Boundary Value Problem (BVP) for the displacement field $u : [0, L] \rightarrow \mathbb{R}$ using the equilibrium equation for 1D problems (strong form).

2. Find the *exact solution* of this BVP. Plot in a Matlab graph the solution of the problem using the following values of the involved constants:

$$L = 1\ m, \quad g = 0.01\ m, \tag{1.0.2}$$

3. Formulate the *Variational (or Weak)* form of the Boundary Value Problem.

4. Derive the corresponding matrix equation in terms of a generic matrix of basis functions $\boldsymbol{N}$ and their corresponding derivatives $\boldsymbol{B} = \dfrac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}x}$.

5. Seek an approximation to the solution of this weak form by using basis polynomial basis functions of increasing order. In particular, try

$$\boldsymbol{N} = [1, x], \quad \boldsymbol{N} = [1, x, x^2], \quad \boldsymbol{N} = [1, x, x^2, x^3] \quad , \boldsymbol{N} = [1, x, x^2, x^3, x^4] \tag{1.0.3}$$

   Plot the approximate solutions together with the exact solution.

   OBSERVATION: You can use the *Symbolic* Math Toolbox for determining both the exact solution and the polynomial approximations.

6. Develop a Matlab program able to solve the weak form using *linear* finite element basis functions. Employ equally sized finite elements. The size of the element —or, equivalently, the number of finite elements n —should be an input of the program.

7. Solve the problem for increasing number of finite elements and plot the corresponding approximate solutions (for at least four discretizations, of 5, 10, 20 and 40 elements) and compare them with the exact solution.

PART 2 (3 points)

- Implement a function able to calculate the approximation error for both u and its derivative u'. Plot the error versus the element size on a log-log plot. The approximation error for a given solution u^h is given by

$$\|e^h\|_{L_2} = \left(\int_0^1 \left(u(x) - u^h(x) \right)^2 \, dx \right)^{\frac{1}{2}}$$

  and for its derivative

$$\|e^{h'}\|_{L_2} = \left(\int_0^1 \left(u'(x) - u'^h(x) \right)^2 \; dx \right)^{\frac{1}{2}}$$

OBSERVATION: The integrals must be computed using Gauss integration rule of appropriate order.

What is the slope of the convergence plot in each case ?