COMPUTATIONAL AEROSPACE ENGINEERING

ASSIGNMENT 1

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January 20, 2021

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1 Problem description

Suppose a bar of length L of constant cross-sectional area A and Young's Modulus E. The bar has a prescribed displacement at $x = 0$ equal to $u(0) = -g$, and is subject to, on the one hand, an axial force F on the right end $(x = L)$, and on the other hand, a distributed axial force (per unit area) $q(x) = E(\rho u(x) - sx^2)$.

$u = u(x)$ is the displacement field, g is a constant, and

$$\rho = \frac{\pi^2}{L^2}\,, \quad s = g\rho^2\,, \quad \frac{F}{AE} = \frac{g\pi^2}{L}$$

Part 1

1. Derive the corresponding Boundary Value Problem (BVP) for the displacement field u: $[0, L] \to \mathbb{R}$ using the equilibrium equation for 1D problems (strong form).

Let $\overline{\Omega}$ be a closed biunit interval between 0 and L, that is, $\overline{\Omega} = [0, L]$. And let $f : \overline{\Omega} \to \mathbb{R}$ be a given smooth, scalar-valued function.

A representation of the illustration of the problem described is:

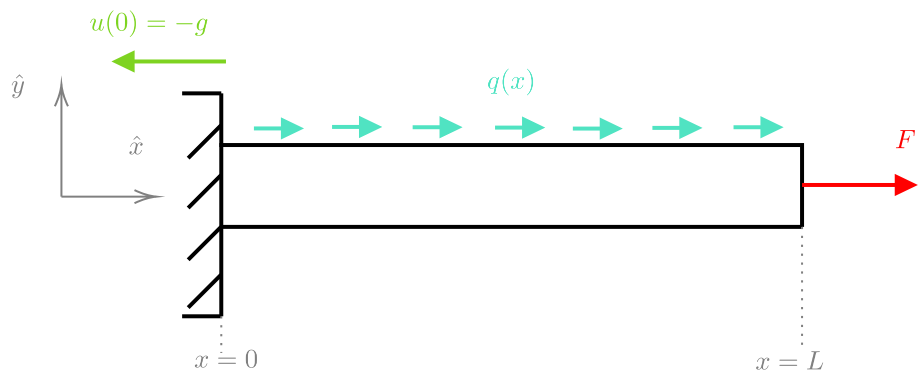


Figure 1 Problem scheme

It is considered a bar of unit length L fixed at $x = 0$ (which means that g=0), of constant cross-sectional area A and Young Modulus E. The bar is subjected to an axial force F on the right end (x = L) as well as a distributed axial force (per unit area) q: $\overline{\Omega} \to \mathbb{R}$, $q(x) = E(\rho u(x) - sx^2)$ where $u = u(x)$ is the displacement field, g is a constant and

$$\rho = \frac{\pi^2}{L^2}\,, \quad s = g\rho^2\,, \quad \frac{F}{AE} = \frac{g\pi^2}{L}$$

Taking a differential axial slice of the bar and representing the internal stress for each slice we get the following diagram [Figure 2].

Applying *Newton's Second Law* to the problem above and considering the bar be an in an equilibrium state, then:

$$\sum F_x = 0 \tag{1}$$

$$\underbrace{\sigma + \frac{d\sigma}{dx}}_{\text{right stress}} - \underbrace{\sigma}_{\text{left stress}} + \underbrace{(x)}_{\substack{\text{distributed}\\ \text{axial force}}} = 0 \tag{2}$$

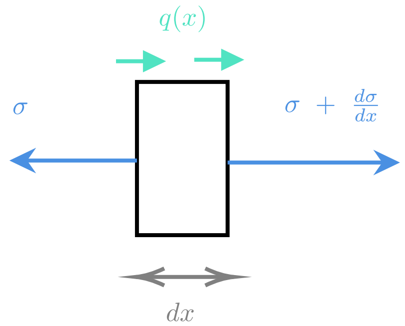


Figure 2 Differential slice

The axial equilibrium equations reads

$$\frac{d\sigma}{dx} + q(x) = 0 \tag{3}$$

$$\frac{d\sigma}{dx} + E(\rho u(x) - sx^2) = 0 \tag{4}$$

where $\sigma : \overline{\Omega} \to \mathbb{R}$ stands for the stress field. By virtue of Hooke's law, we know that

$$\sigma = E\frac{du}{dx} = Eu' \tag{5}$$

Thus, inserting (5) into (4), we get

$$\frac{d}{dx}\left(E\frac{du}{dx}\right) + E\rho u(x) - Esx^2 = 0 \tag{6}$$

Subsequently, as the Young's modulus E is constant and $\neq 0$, the function is divided by E obtaining

$$\boxed{\begin{aligned} \frac{d^2u}{dx^2} + \rho u(x) - sx^2 = 0 \quad (7) \\ u''(x) + \rho u(x) - sx^2 = 0 \quad (8) \end{aligned}}$$

where we define

$$f(x) = \frac{q}{E} = \rho u(x) - sx^2 \tag{9}$$

As for the condition on the right end $x = L$, we have that

$$F = A\sigma = AEu'(L) \quad \Rightarrow \quad u'(L) = \frac{F}{AE} = \frac{g\pi^2}{L} \tag{10}$$

and we obtain that

$$\boxed{b = u'(L) = \frac{g\pi^2}{L}} \tag{11}$$

And as the bar is fixed but has a prescribed displacement in $x = 0$, then

$$\boxed{u(0) = -g} \tag{12}$$

2. Find the *exact solution* of this BVP. Plot in a Matlab graph the solution of the problem using the following values of the involved constants:

$$L = 1 \text{ m}, \quad g = 0.01 \text{ m}$$

The differential equation to solve is:

$$\boxed{\frac{d^2u}{dx^2} + \rho u(x) - sx^2 = 0} \quad \text{on } \Omega \tag{13}$$

where $u'' = \dfrac{d^2u}{dx}$ and is subjected to boundary conditions

$$u(0) = -g \tag{14}$$
$$u'(L) = b \tag{15}$$

and b and g are given constants.

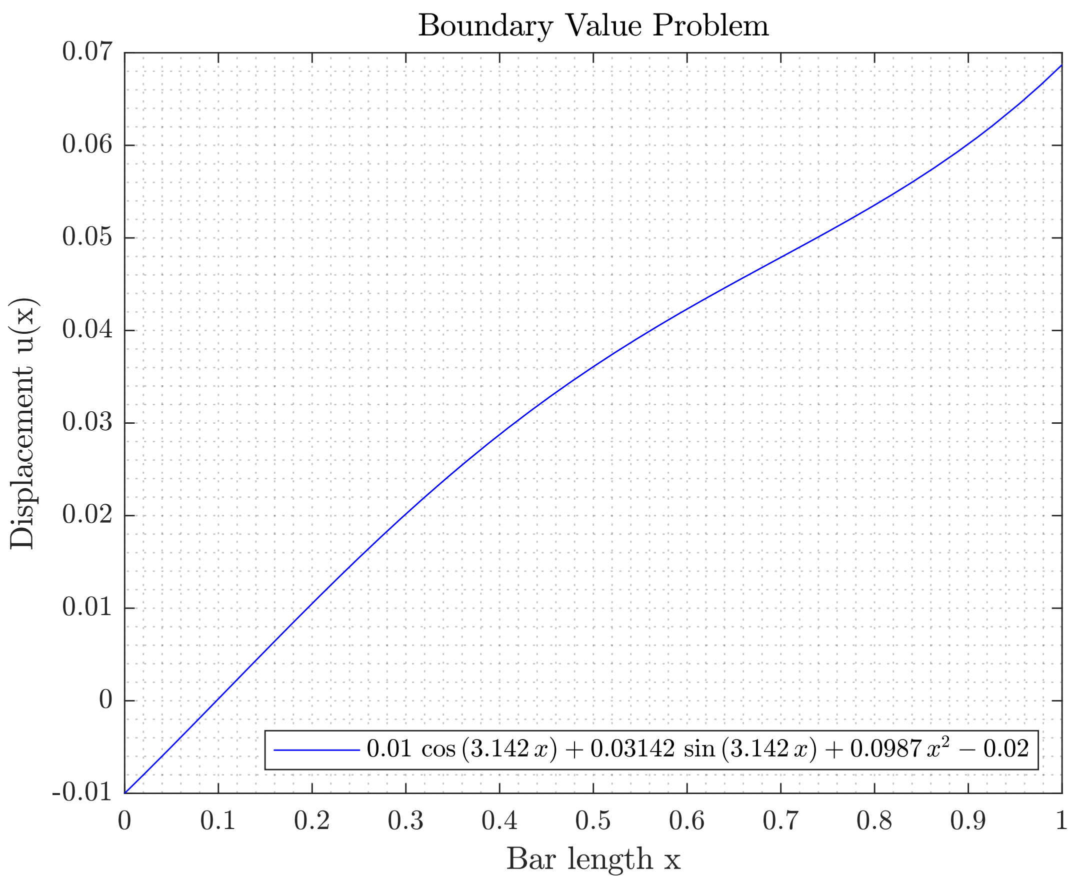


Figure 3 Boundary Value Problem

3. Formulate the *Variational* (or Weak) form of the Boundary Value Problem (*Boundary Value Problem*).

The strong form form consists of governing equations and the boundary conditions for a physical system. The governing equations are usually partial differential equations but in one-dimensional problems they become ordinary differential equations. Whereas the weak form consists of the integral form of these equations, which are necessary for the finite element method.

The residual of the differential equation associated to functions N(x) is defined by

$$r(x) := \frac{d^2\underline{\underline{N}}(x)}{dx^2}\underline{d} + f(x) \tag{16}$$

The residual vanishes for all $x \in \Omega$ when the actual solution of the problem is a linear combination of the basis functions $\underline{\underline{N_i}}(x)$. If it is not possible to impose the residual to vanish at all point of the domain, the most intuitive route is to simply make the residual as small as possible, more specifically, to make its norm as small as possible. Were the residual a vector in an Euclidean space, its norm would be computed as $\|r(x:\underline{d})\| = \sqrt{\sum_i r_i^2}$. Thus, since the residual is a function, its norm has to be computed using the concept of integral.

$$\|r(x:\underline{d})\| := \left(\int_0^L r(x:\underline{d})^2 dx\right)^{\frac{1}{2}} \tag{17}$$

However, this method involves the second derivatives of the trial functions, which results in infinite values due to a jump discontinuity. What is more, Lagrangian multipliers would be required to find a solution adding a considerable computational cost overrun. As a result, a new method that allows the usage of finite element bases is to be taken advantage of.

In our case of analysis, the residual $r = r(x)$ can be regarded can be regarded as a residual force arising from the incapability of capturing the exact solution.

In this scenario, we could interpret that it is more physically appropriate to make the work performed by such residual forces as small as possible (in absolute value). The work done by the residual is given by

In order to formulate the *Variational* form of the Boundary Value Problem, we shall begin with considering the work done by a residual force as

$$\int_0^L u(x)r(x;u)\ dx \tag{18}$$

this means, that the objective is the minimization of the work done by the residual forces. This is a more physical interpretation of the initial differential equation. Then, by making $u(x) = g + (u(x) - g)$, we can write

$$\int_0^L u(x)r(x;u)\ dx = \int_0^L gr(x;u)\ dx + \int 0^L(u(x) - g)r(x;u)\ dx \tag{19}$$

Taking a deeper look on the above expression, notice that the first integral of the right hand side only vanishes if $g = 0$ or the residual term is null, that is to say, the solution is exact. However, generally, it will not be null. Thereby, we must work on the rightmost integral and we want to minimize this term to obtain the most exact solution as possible. This means, the problem translates to find a function $u(x)$ with $u(0) = g$ and $u'(L) = b$ that renders this term zero from the following equation:

$$\int_0^L (u(x) - g)r(x;u)\ dx = 0 \tag{20}$$

The equation above is the integral of the product of two functions and it can be interpreted as a scalar product between, and $(u(x) - g)$ and $r(x;u)$ are to be orthogonal. To enforce this condition, a plane in $\mathbb{R}^3$ has dimension 2. In order to set the orthogonality condition, one has to take 2 linearly independent vectors N_1 and N_2 and enforce the conditions $N_1 \cdot r = 0$ and $N_2 \cdot r = 0$. Therefore, assuming $u(x) \approx N(x) \cdot d$.

Hence, $(u(x)-g)$ pertains to a space of functions of the form $v = Nc$, with $v(0) = u(0) - g = 0$. With this consideration, the problem now translates again to finding $u(x) = N(x)d$ with $u(0) = N(0)d = g$ such that

$$\int_0^L v(x)r(x;u)\ dx = 0 \quad \forall \quad v(x) = N(x)c \quad (\text{with} \ \ N(0)c = 0) \tag{21}$$

Finally, to find the algebraic solution it is necessary to get rid of the second derivative, this can bu done by using the chains rule:

$$(vu')' = v'u' + vu'' \quad \Longrightarrow \quad vu'' = (vu')' - v'u' \tag{22}$$

Therefore,

$$\int_0^L v(u''+f)\ dx = \int_0^L (u')'\ dx - \int_0^L (v'u')\ dx + \int_0^L vf\ dx \tag{23}$$

Using the general scenario where $u'' + f = 0$, we get rid of the second derivative by virtue of the chain rule and we arrive at the general variational or weak form of the Boundary Value Problem:

$$\int_0^L v'(x)u'(x)dx = \int_0^L v(x)f(x)dx + bv(1) \tag{24}$$

Now, taking into consideration that the distributed axial form corresponds to the value of $f(x)$ (Eq. 9), the following statement is obtained:

$$\boxed{\int_0^L v'(x)u'(x)dx - \int_0^L v(x)\rho u(x)dx = bv(1) - \int_0^L v(x)sx^2dx} \tag{25}$$

4. Derive the corresponding matrix equation in terms of a generic matrix of basis functions N and their corresponding derivatives $B = \dfrac{dN}{dx}$.

In order to formulate the variational for of the BVP, for the first approach, the solutions sought are constructed by assuming that the solution is a linear combination of known functions:

$$u(x) \approx \sum_{i=1}^{n} N_i(x)d_i = \overbrace{[N_1(x) \ \ N_2(x) \ \ ... \ \ N_n(x)]}^{=\underline{\underline{\mathrm{N}}}(\mathrm{x})} \overbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}}^{=\underline{\mathrm{d}}} = \underline{\underline{N}}(x)\underline{d} \tag{26}$$

where $N_i(x)$ is th i-th function and d_i is its corresponding coefficient (unknown, meant to be determined). Next, replacing $u(x) \approx \underline{\underline{N}}(x)\underline{d}$ in $u(x) \approx \underline{\underline{N}}\,\underline{d}$ in (13) :

$$u'' + \rho u - sx^2 = \frac{d^2u}{dx^2} + \rho u - sx^2 \approx \frac{d^2\underline{\underline{N}}(x)}{dx^2}\underline{d} + \rho u - sx^2 \tag{27}$$

where

$$\frac{d^2\underline{\underline{N}}(x)}{dx^2} = \left[\frac{d^2N_1(x)}{dx^2} \quad \frac{d^2N_2(x)}{dx^2} \quad \cdots \quad \frac{d^2N_n(x)}{dx^2}\right] \tag{28}$$

Once we known how to proceed, it is essential to notice that the equation above (25) does not take into account the axial stiffness of the bar at each point $(E(x)A(x))$. Taking them into account, thus, the above equation becomes

$$\int_0^L v'(x)(EA)u'(x)dx - \int_0^L v(x)\rho u(x)dx = bv(1) - \int_0^L v(x)sx^2dx \tag{29}$$

Since $F_{ax}(x) = E(x)A(x)u'(x) = E(x)A(x)\epsilon(x)$ is the axial force at point x, we can write

$$\underbrace{\int_0^L v'(x)F_{ax}(x)\,dx}_{\text{Virtual internal work}} = \underbrace{\int_0^L v(x)\rho u(x)\,dx - \int_0^L v(x)sx^2\,dx + Fv(1)}_{\text{Virtual external work}} \tag{30}$$

Regarding equation (30), it can be interpreted as a **balance of virtual work**. The left-hand side of the equation is the *internal* work done by the axial force F_{ax} for a virtual strain $v'(x)$, whereas the right-hand side can be viewed as the *virtual work* developed by the external forces $q(x)$ (distributed) and F (point load acting on the end of the bar). Additionally, another interpretation of the original formula can be alternatively formulated as:

Finding a displacement field u(x) (with $u(0) = g$) such that the balance of virtual work holds for any virtual displacement field $v(x)$ (which it translates to $v(0) = 0$).

Continuing with the formulation, replacing u and v by the linear expressions:

$$u = \underline{\underline{N}}\,\underline{d} \tag{31}$$

$$v = \underline{\underline{N}}\,\underline{c} = c^T \underline{\underline{N}}^T \text{[1]} \tag{32}$$

$$u' = \underline{\underline{N}}'\underline{d} \tag{33}$$

$$v' = \underline{c}^T \underline{\underline{B}}^T \tag{34}$$

in Eq. (25), we arrive at

$$\int_0^L \underline{c}^T \underline{\underline{B}}^T \underline{\underline{B}}\,\underline{d}\,dx - \int_0^L \underline{c}^T \underline{\underline{N}}^T \rho \underline{\underline{N}}\,\underline{d}\,dx = \underline{c}^T \underline{\underline{N}}(1)^T b - \int_0^L \underline{c}^T \underline{\underline{N}}^T sx^2\,dx \tag{35}$$

that rearranging the terms is equal to

$$c^T \left(\int_0^L \underline{\underline{B}}^T \underline{\underline{B}}\,dx\,\underline{d} - \int_0^L \underline{\underline{N}}^T \rho \underline{\underline{N}}\,\underline{d}\,dx - \underline{\underline{N}}(1)^T b + \int_0^L \underline{\underline{N}}^T sx^2\,dx \right) = 0 \tag{36}$$

where the displacement column matrix multiplies the stiffness constant such that

$$c^T \left(\overbrace{\int_0^L \left(\underline{\underline{B}}^T \underline{\underline{B}} - \underline{\underline{N}}^T \rho \underline{\underline{N}} \right) dx}^{:=K}\,\underline{d} - \overbrace{\left(\underline{\underline{N}}(1)^T b - \int_0^L \underline{\underline{N}}^T sx^2\,dx \right)}^{:=F} \right) = 0 \tag{37}$$

Rewriting the above expression we get

$$c^T (Kd - F) = 0 \tag{38}$$

where $B(x) = N'(x)$ This equations must hold for all coefficients $\underline{c}$ obeying the condition $v(0) = \underline{\underline{N}}(0)\underline{c} = 0$. For polynomial and finite element basis functions, $v(0) = c_1 = 0$, and thus $\underline{c}$ has the form

$$\underline{c} = \begin{bmatrix} c_r \\ c_l \end{bmatrix} = \begin{bmatrix} 0 \\ c_l \end{bmatrix} \tag{39}$$

where $l = 2, 3, .., n$

Therefore, it can be rewritten as

$$c_l^T \,(K(l,:)d - F) = 0, \;\; \forall \;\; c_l \quad \Longrightarrow \quad K(l,:)d = F(l) \tag{40}$$

where $K(l,:)$ is the block matrix of K formed by rows $l = 2, 3, ...n$. Additionally, since

$$\underline{d} = \begin{bmatrix} d_r \\ d_l \end{bmatrix} = \begin{bmatrix} g \\ d_l \end{bmatrix} \tag{41}$$

we end up with the equation

$$\boxed{K(l,l)d_l = F(l) - K(l,r)g} \tag{42}$$

5. Seek an approximation to the solution of this weak form by using basis polynomial basis functions of increasing order. In particular, try

$$N = [1, x]\,, \quad N = [1, x, x^2]\,, \quad N = [1, x, x^2, x^3]\,, \quad N = [1, x, x^2, x^3, x^4]$$

Plot the approximate solutions together with the exact solution. OBSERVATION: You can use the *Symbolic* Math Toolbox for determining both the exact solution and the polynomial approximations.

Using the polynomial approximations and the exact solution the resulting plot is shown in the Figure 4.

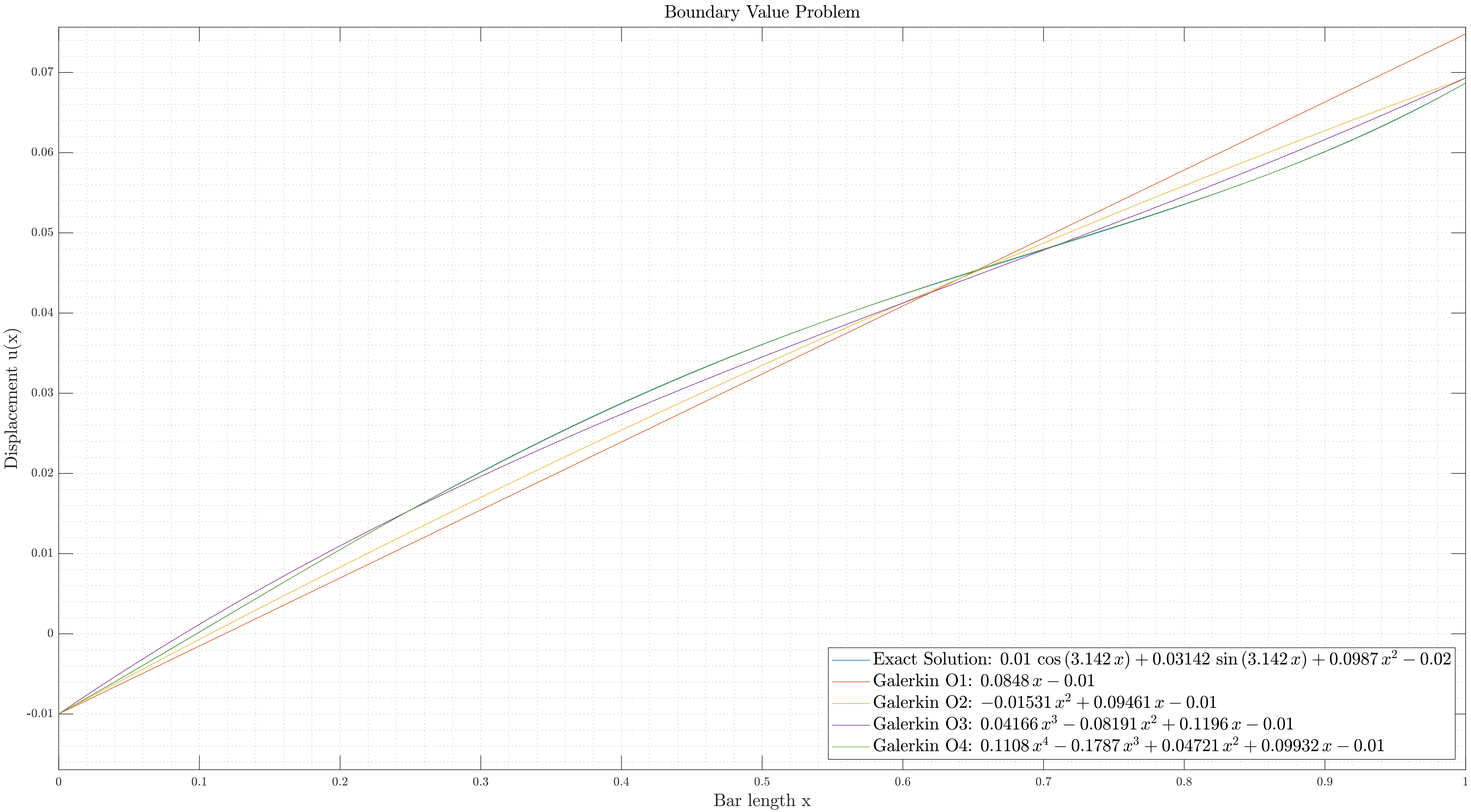


Figure 4 Boundary Value Problem with the Exact Solution and Galerkin's method

6. Develop a Matlab program able to solve the weak form using *linear* finite element basis functions. Employ equally sized finite elements. The size of the element —or, equivalently, the number of finite elements n —should be an input of the program

So far we have developed the finite element method as simply as a particular **Galerkin's approximation** procedure. Now, in this section we aim to develop the **local or element point of view**. The *global method* is useful for calculating properties and compare them to, for instance, Galerkin's method using polynomials. The finite element is more computational efficient than the Galerkin's method.

To develop the element point of view it is assumed that the model consists of n_{el} elements where the letter e will be the variable index for the elements so that $1 \leq e \leq n_{el}$.

It is beneficial to introduce a local set of quantities, corresponding to the global ones, so that calculations for a typical element may be standardised. These are given as follows:

1. Domain $[\epsilon_1, \epsilon_2]$
2. Degrees of freedom ($\sim$ unknowns) $\Rightarrow \{\mathrm{d}_1^2, \mathrm{d}_2^\mathrm{e}\}$
3. Shape functions $\Rightarrow \{\mathrm{N}_1^\mathrm{e}(\xi), \mathrm{N}_2^\mathrm{e}(\xi)\}$
4. Interpolated functions

$$u^e(\xi) = \left[N_1^e(\xi) N_2^e(\xi)\right] \cdot \begin{bmatrix} d_1^e \\ d_2^e \end{bmatrix} = \underline{\underline{N}}^e(\xi) \cdot \underline{d}^e \tag{43}$$

Mapping from the parent domain $[-1, 1]$ to the physical domain $\mathrm{x}_1^\mathrm{e}, \mathrm{x}_2^\mathrm{e}$ (isoparametric elements).

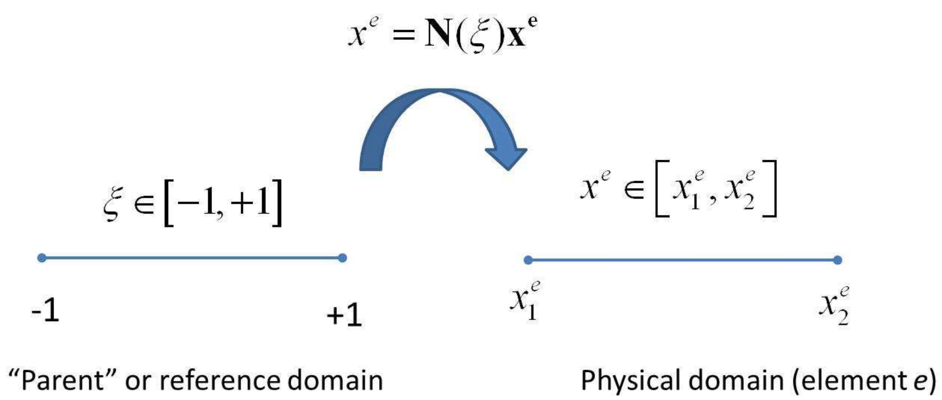


Figure 5 Parent and Physical domain.

$$x^e = \left[N_1^e(\xi) N_2^e(\xi)\right] \cdot \begin{bmatrix} x_1^e \\ d_2^e \end{bmatrix} = \underline{\underline{N}}^e(\xi) \cdot \underline{x}^e \tag{44}$$

Where the matrix of elemental shape functions (for linear a element) is

$$\underline{\underline{N}}^e(\xi) := \frac{1}{2} \begin{bmatrix} (1-\xi) & (1+\xi) \end{bmatrix} \tag{45}$$

and the matrix of derivatives of shape functions is obtained with the following method

$$\underline{\underline{B}}^e := \frac{d\underline{\underline{N}}^e}{dx} = \frac{d\underline{\underline{N}}^e}{d\xi}\frac{d\xi}{dx} = \frac{1}{2}\begin{bmatrix} -1 & 1 \end{bmatrix} \frac{d\xi}{dx} \tag{46}$$

Notice that,

$$\frac{d\xi}{dx} = \left(\frac{dx}{d\xi}\right)^{-1} = \left(\frac{d\underline{\underline{N}}^e}{d\xi}\underline{x}^e\right)^{-1} = 2\left(-x_1^2 + x_2^2\right)^{-1} = \frac{2}{h^e} \tag{47}$$

Thus:

$$\underline{\underline{B}}^e = \frac{1}{2}\begin{bmatrix}-1 & 1\end{bmatrix}\frac{2}{h^e} = \frac{1}{h^e}\begin{bmatrix}-1 & 1\end{bmatrix} \tag{48}$$

Assembly of the K matrix

For linear elements we introduce (45) and (48) in (49):

$$\underline{\underline{K}}^e := \int_{\Omega^e} \underline{\underline{B}}^{eT}\underline{\underline{B}}^e \, dx - \int_{\Omega^e} \underline{\underline{N}}^{eT}\rho \underline{\underline{N}}^e \, dx \tag{49}$$

$$\underline{\underline{K}}^e = \int_{\Omega^e} \underbrace{\left(\frac{1}{h^e}\begin{bmatrix}-1 & 1\end{bmatrix}\right)^T}_{\underline{\underline{B}}^{eT}} \underbrace{\frac{1}{h^e}\begin{bmatrix}-1 & 1\end{bmatrix}}_{\underline{\underline{B}}^e} dx - \int_{\Omega^e} \underbrace{\frac{1}{2}\begin{bmatrix}(1-\xi) & (1+\xi)\end{bmatrix}^T}_{\underline{\underline{N}}^{eT}} \rho \underbrace{\frac{1}{2}\begin{bmatrix}(1-\xi) & (1+\xi)\end{bmatrix}}_{\underline{\underline{N}}^e} dx \tag{50}$$

$$\underline{\underline{K}}^e = \frac{1}{(h^e)^2}\begin{bmatrix}-1\\+1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}\int_{\Omega^e} dx - \int_{\Omega^e}\frac{\rho}{4}\begin{bmatrix}(1-\xi)^2 & (1-\xi)(1+\xi)\\(1+\xi)(1-\xi) & (1+\xi)^2\end{bmatrix} dx \tag{51}$$

and then using (47) relation between ξ and x we perform a change of variable

$$\underline{\underline{K}}^e = \frac{1}{(h^e)^2}\begin{bmatrix}-1\\+1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}\int_{\Omega^e} dx - \int_{-1}^{1}\frac{h^e}{2}\frac{\rho}{4}\begin{bmatrix}(1-\xi)^2 & (1-\xi)(1+\xi)\\(1+\xi)(1-\xi) & (1+\xi)^2\end{bmatrix} d\xi \tag{52}$$

$$\underline{\underline{K}}^e = \frac{1}{h^e}\begin{bmatrix}+1 & -1\\-1 & +1\end{bmatrix} - \frac{h^e\rho}{8}\begin{bmatrix}(\xi-\xi^2+\frac{\xi^3}{3}) & (\xi-\frac{\xi^3}{3})\\(\xi-\frac{\xi^3}{3}) & (\xi+\xi^2+\frac{\xi^3}{3})\end{bmatrix}_{-1}^{1} \tag{53}$$

$$\underline{\underline{K}}^e = \frac{1}{h^e}\begin{bmatrix}+1 & -1\\-1 & +1\end{bmatrix} - \frac{h^e\rho}{8}\begin{bmatrix}\frac{8}{3} & \frac{4}{3}\\\frac{4}{3} & \frac{8}{3}\end{bmatrix} \tag{54}$$

which simplifying results in

$$\boxed{\underline{\underline{K}}^e = \frac{1}{h^e}\begin{bmatrix}+1 & -1\\-1 & +1\end{bmatrix} - \frac{h^e\rho}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}} \tag{55}$$

The above equation is known as the **coefficient matrix for an element**.

The element definitions are stored in the element connectivity matrix. This is a matrix of integer numbers where each row of the matrix contains the connectivity of an element. In 1D, this matrix boils down to

$$CN = \begin{bmatrix}1 & 2\\2 & 3\\\vdots & \vdots\\n & n+1\end{bmatrix} \tag{56}$$

Definition of the Boolean connectivity matrix of the $e-th$ element:

$$\underline{d}^e = \underline{\underline{L}}^e\underline{d} \tag{57}$$

where

$$\underline{d} := \begin{bmatrix}d_1\\d_2\\\vdots\\d_{n+1}\end{bmatrix} \tag{58}$$

(recall that $d_1 = g$). Thus, L^e is a $2 \times (n+1)$ matrix consisting of the integers 0 and 1 that relates element nodal variables $\boldsymbol{d}^e \in \mathbb{R}^2$ with the global nodal vector d. More specifically, L^e is given by

$$[\boldsymbol{L}^e]_{aA} = \begin{cases} 1 & \text{if } \boldsymbol{CN}(e, \boldsymbol{a}) = A \\ 0 & \text{otherwise} \end{cases} \tag{59}$$

Likewise, for the global vector of nodal variations

$$\boldsymbol{c} := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N+1} \end{bmatrix} \tag{60}$$

(in this case $c_1 = 0$), we may write

$$c^e = L^e c \tag{61}$$

After substituting (61) and (57), we obtain the global K matrix (complete, $(n+1) \times (n+1)$):

$$K = \sum_{e=1}^{n_{el}} L^{eT} \boldsymbol{K}^e \boldsymbol{L}^e \tag{62}$$

Definition of assembly operator

$$\boxed{K = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \boldsymbol{K}^e := \sum_{e=1}^{n_{el}} \boldsymbol{L}^{e^T} \boldsymbol{K}^e \boldsymbol{L}^e} \tag{63}$$

Assembly of the force vector

Recall that F was defined in (37):

$$F = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b \end{bmatrix} + \underline{F} = \underline{\underline{N}}(1)^T b - \int_0^L \underline{\underline{N}}^T s x^2 \, dx \tag{64}$$

Now let the elemental force vector be defined as:

$$\underline{F}^e := \int_{\Omega^e} \underline{\underline{N}}^{eT} f \; dx = \int_{\Omega^e} \underline{\underline{N}}^T s x^2 \; dx \tag{65}$$

$$\underline{F}^e = \begin{bmatrix} \int_{\Omega^e} N_1^{eT} s x^2 \; dx \\ \int_{\Omega^e} N_2^{eT} s x^2 \; dx \end{bmatrix} \tag{66}$$

replacing (45) in (66) the expressions becomes

$$\underline{F}^e = \begin{bmatrix} \int_{\Omega^e} \frac{1}{2} \left[1 - \xi\right] s x^2 \; dx \\ \int_{\Omega^e} \frac{1}{2} \left[1 + \xi\right] s x^2 \; dx \end{bmatrix} \tag{67}$$

Now the above expression can be solved. The assembly of the $\underline{\underline{F}}$ must be performed. Nevertheless, we need a function to calculate the integral (65) due to the fact that there are 2 variables involved x and ξ. The procedure for performing this calculations is based on **Gauss' Quadrature**.

Gauss' Quadrature is an approximation of the definite integral of a function (any function), usually stated as a weighted sum of function values at specified points within the domain of integration. Hence,

$$I = \int_{-1}^{1} q(\xi)\ d\xi \approx \sum_{g=1}^{m} w_g q(\xi_g) \tag{68}$$

In other words, Gauss' quadrature consists in finding a set of m points ξ_g and corresponding weights that approximates the solution of the before-mentioned integral.

The main idea is to choose the weights and integration points so that the highest possible polynomial is integrated exactly. Accuracy improves as more Gauss points are used. Supposing the is a polynomial of order p:

$$q(\xi) = \sum_{i=0}^{p} \alpha_i \xi^i \tag{69}$$

Substitution of the above equation into (68) leads to:

$$I = \sum_{i=0}^{p} \alpha_i \int_{-1}^{1} \xi^i\ d\xi = \sum_{i=0}^{p} \alpha_i \left(\sum_{g=1}^{m} w_g \xi_g^i \right) \tag{70}$$

Developing and evaluating I, one has to integrate exactly each monomial ξ^i $(i = 1, 2 \dots p)$ i.e.:

$$\sum_{g=1}^{m} w_g \xi_g^i \begin{cases} 0 & \text{if} \quad i = 1, 3, 5, \dots \\ \dfrac{2}{i+1} & \text{if} \quad i = 2, 4, 6, \dots \end{cases} \tag{71}$$

The enforcement of the preceding conditions leads to a nonlinear system of $p+1$ equations with 2m unknowns. Therefore, m **Gauss' points integrate exactly a polynomial of ordre**

$$p \leq 2m - 1 \tag{72}$$

Back to our problem, we want to evaluate $\underline{\underline{F}}^e$

$$\underline{F}^e = \int_{\Omega^e} \overbrace{\underline{\underline{N}}^{eT} f}^{q(\xi, x(\xi))}\ dx = \int_{\Omega^e} q(\xi, x(\xi))\ dx \tag{73}$$

where q is an arbitrary function and $q \in \mathbb{R}^2$. The number of *quadrature points* (m) is to be chose according to the following criterion: m points integrate *exactly* polynomials of order $p \leq 2m - 1$.

Returning to the bar analysis, it is known that Gauss quadrature is carried out in the parent domain so

$$\underline{F}^e = \int_{\Omega^e} \underline{q}(\xi, x(\xi)) dx = \int_{-1}^{1} \frac{dx}{d\xi} \underline{q}(\xi, x(\xi)) d\xi \tag{74}$$

Since (47), we have that

$$\underline{F}^e = \frac{h^e}{2} \int_{-1}^{1} \underline{q}(\xi) d\xi \tag{75}$$

Gauss quadrature consists in evaluating the integral as the sum of the product of the integrand at the **Gauss points** times the corresponding weights.

$$\underline{F}^e = \frac{h^e}{2} \int_{-1}^{1} \underline{q}(\xi) d\xi = \frac{h^e}{2} \sum_{g=1}^{m} w_g \underline{q}(\xi_g) \tag{76}$$

Keeping in mind that $\underline{q}(\xi_g) = \underline{\underline{N}}^{eT}(\xi_g)f(\xi_g)$, (45) is

$$\underline{\underline{N}}^{e}(\xi_g) := \frac{1}{2}\left[(1-\xi_g) \quad (1+\xi_g)\right] \tag{77}$$

and the value of f comes from

$$f(\xi_g) = f(x^e(\xi_g)) = sx(\xi)^2 \tag{78}$$

with

$$x^e = \left[N_1^e(\xi) \quad N_2^e(\xi)\right]\overbrace{\begin{bmatrix} x_1^e \\ x_2^e \end{bmatrix}}^{\underline{x}^e} = \underline{\underline{N}}^{e}(\xi)\underline{x}^e \tag{79}$$

So the final expression of $\underline{F}^e$, applying the change of variable (47):

$$\underline{F}^e = \begin{bmatrix} \int_{-1}^{1} \frac{1}{2}\left[1-\xi\right] sx^2 \; dx \\ \int_{-1}^{1} \frac{1}{2}\left[1+\xi\right] sx^2 \; dx \end{bmatrix} = \frac{h^e}{4}\begin{bmatrix} \int_{\Omega^e} \left[1-\xi\right] s(N_1(\xi)x_1^e + N_2(\xi)x_2^e)^2 \; d\xi \\ \int_{\Omega^e} \left[1+\xi\right] s(N_1(\xi)x_1^e + N_2(\xi)x_2^e)^2 \; d\xi \end{bmatrix} \tag{80}$$

Developing $N_1(\xi)$ and $N_2(\xi)$. We arrive to the expression below:

$$\underline{F}^e = \frac{h^e}{4}\begin{bmatrix} \int_{-1}^{1} \left[1-\xi\right] s \left[\frac{1}{2}\left(x_1^e + x_2^e + \overbrace{(x_2^e - x_1^e)}^{h_e}\,\xi\right)\right]^2 \; d\xi \\ \int_{-1}^{1} \left[1+\xi\right] s \left[\frac{1}{2}\left(x_1^e + x_2^e + \overbrace{(x_2^e - x_1^e)}^{h_e}\,\xi\right)\right]^2 \; d\xi \end{bmatrix} \tag{81}$$

Thus, the procedure to calculate $\underline{F}^e$ is as follows:

1. From Table 5 [1] the weights $\{w_g\}_{g=1}^{m}$ and the position of Gauss point $\{\xi_g\}_{g=1}^{m}$ in the parent domain:
2. Evaluation of the element shape functions at each Gauss point: $\underline{\underline{N}}^{e}(\xi_g)$ $(g = 1, 2 \ldots m)$:
3. Computation the position of each Gauss point in the physical domain $x^e(\xi_g) = \underline{\underline{N}}^{e}(\xi_g)\underline{x}^e$ $(g = 1, 2, \ldots m)$:
4. Calculation of the value of the function f at each Gauss point $f_g = f(x^e(\xi_g))$ $(g = 1, 2, \ldots m)$:
5. Computation the desired vector F^e though formula (76)

Solution of the system of algebraic equations

The nodes corresponding to Dirichlet Boundary conditions are defined as **r=1** while the remaining nodes, which are unknowns, are characterised by **I = [2,3 ... n+1]**

The block decomposition is finally

$$\begin{bmatrix} K_{rr} & K_{rl} \\ Klr & Kll \end{bmatrix}\begin{bmatrix} d_r \\ d_l \end{bmatrix} = \begin{bmatrix} F_r \\ F_l \end{bmatrix} \tag{82}$$

where the unknown nodal displacements, $\underline{d_l}$, can be obtained as follows

$$d_l = (K_{ll})^{-1}(F_l - K_{lr}d_r) \tag{83}$$

The Matlab code used to solve the weak form either with Matlab symbolic and the Gauss quadrature can be found in the Github link in the Annex **??**.

7. Solve the problem for increasing number of finite elements and plot the corresponding approximate solutions (for at least four discretizations, of 5, 10, 20 and 40 elements) and compare them with the exact solution.

Using the Gauss quadrature values, which have been compared to the results obtained through Matlab symbolic, and the exact solution the resulting plot is shown in the Figures 6, 7, 8, 9, 10 and the overall comparative in 11.

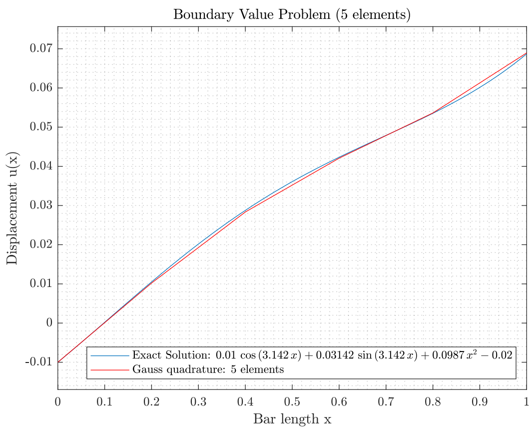


Figure 6 Gauss quadrature with 5 elements

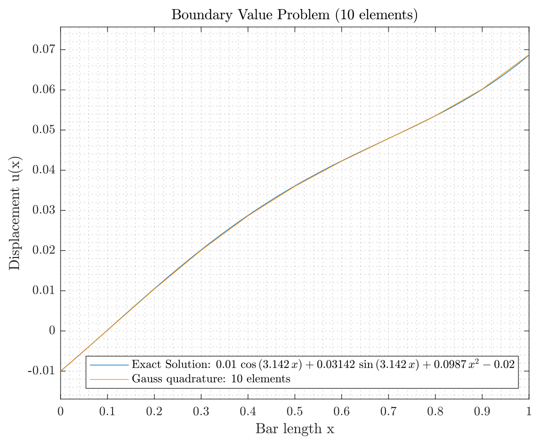


Figure 7 Gauss quadrature with 10 elements

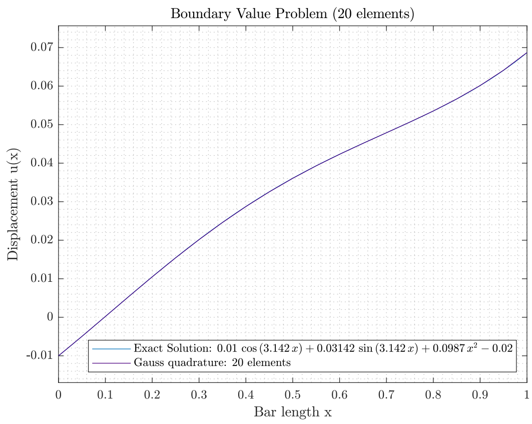


Figure 8 Gauss quadrature with 20 elements

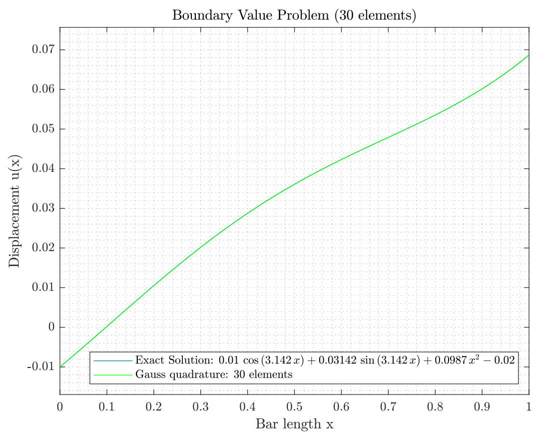


Figure 9 Gauss quadrature with 30 elements

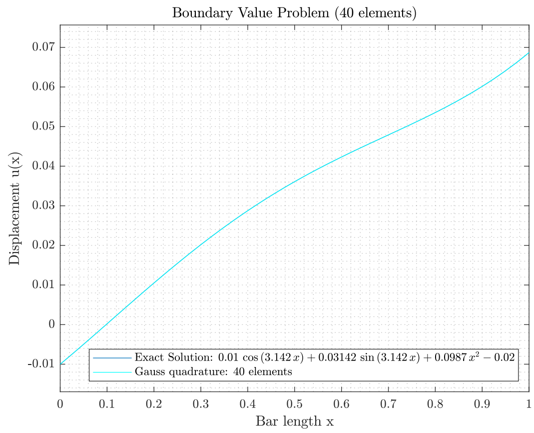


Figure 10 Gauss quadrature with 40 elements

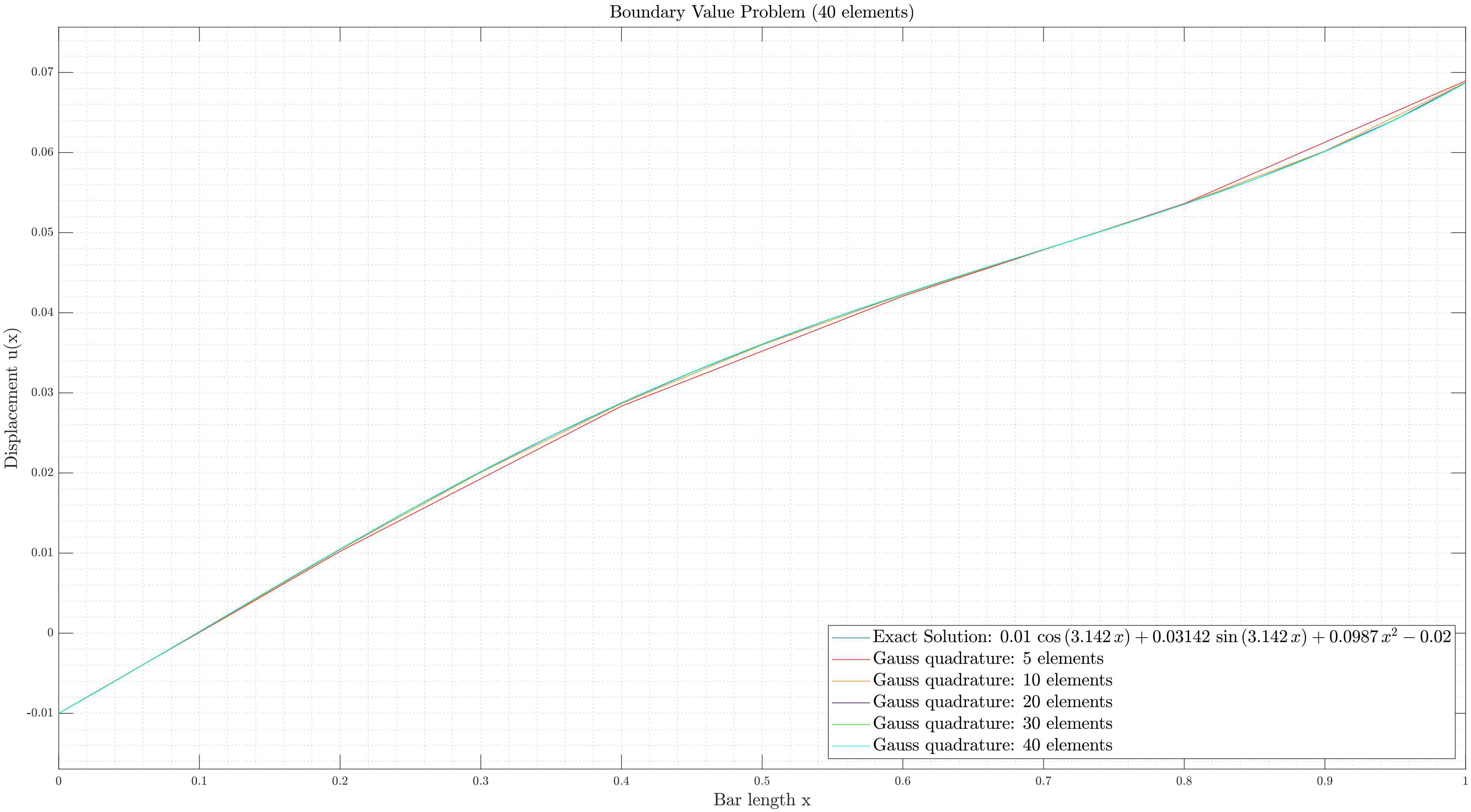


Figure 11 Boundary Value Problem with the Exact Solution and Gauss quadrature with several elements

8. Implement a function able to calculate the approximation error for both u and its derivative u'. Plot the error versus the element size on a *log-log* plot. The approximation error for a given solution u^h is given by:

$$\|e\|\Big|L_2 = \|u - u^h\|\Big||L_2 = \left(\int_\Omega \left(u - u^h\right)^2 dx\right)^{\frac{1}{2}} \tag{84}$$

and for its derivative

$$\|e'\|\Big|L_2 = \|u' - u^{h'}\|\Big|L_2 = \left(\int_\Omega \left(u' - u^{h'}\right)^2 dx\right)^{\frac{1}{2}} \tag{85}$$

What is the sole of the convergence plot in each case?

Equation (84) is a measure of the distance between the exact and the FE displacement solution. The error at any point in the interval contributes to this measure of error because the integrand is the square of the error at any point.

We are looking to approximate those expressions using Gauss' quadrature. Hence, to compute the first integral, the additive property of integrals is applied,

$$\int_\Omega \left(u(x) - u^h(x)\right)^2 dx = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \left[u((x) - \underline{\underline{N}}(\xi)\underline{d}^e\right]^2 dx \quad \text{where } \Omega \in [0,l] \text{ and } \Omega^e \in [-1,1] \tag{86}$$

Then, as shown in (47), we change variable spaces from x to ξ,

$$\int_\Omega \left(u(x) - \underline{N}(\xi)\underline{d}^e\right)^2 dx = \frac{h^e}{2}\int_{\Omega^e} \left[u(x(\xi)) - \underline{N}(\xi)\underline{d}^e\right]^2 d\xi \quad \text{where } \Omega \in [0,l] \text{ and } \Omega^e \in [-1,1] \tag{87}$$

Thus,

$$\int_\Omega \left(u(x) - u^h(x)\right)^2 dx = \frac{h^e}{2}\sum_{e=1}^{n_{el}} \int_{\Omega^e} \left[u(x(\xi)) - \underline{\underline{N}}(\xi)\underline{d}^e\right]^2 d\xi \quad \text{where } \Omega \in [0,l] \text{ and } \Omega^e \in [-1,1] \tag{88}$$

Recall that $u(x)$ is the exact solution and was calculated in exercise 1. And expressing x as a function of xi,

$$x = \underline{\underline{N}}^e(\xi)\underline{x}^e = \frac{x_1^e + x_2^e}{2} + \frac{x_2^e - x_1^e}{2}\xi \tag{89}$$

Hence, the resulting integral will be a polynomial and trigonometric functions. The maximum order of the integrand is 4, the solution can be calculated using Gauss' quadrature with 3 points the weights w and parent coordinates ξ of which are:

$$\begin{cases} w_1 = \frac{5}{9} \\ w_2 = \frac{8}{9} \\ w_3 = \frac{5}{9} \end{cases} ; \quad \begin{cases} \xi_1 = \sqrt{\frac{3}{5}} \\ \xi_2 = 0 \\ \xi_3 = -\sqrt{\frac{3}{5}} \end{cases} \tag{90}$$

The same process applies for the derivative, the additive property of integrals shall be applied:

$$\int_\Omega \left(u'(x) - u^{h'}(x)\right)^2 dx = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \left[u'(x) - \underline{\underline{B}}\underline{d}^e\right]^2 dx \quad \text{where } \Omega \in [0,l] \text{ and } \Omega^e \in [-1,1] \tag{91}$$

Analogously, developing and applying the change of variables from x to ξ,

$$\int_\Omega \left(u'(x) - \underline{\underline{N}}d^e\right)^2 dx = \frac{h^e}{2}\int_{\Omega^e} \left[u'(x(\xi)) - \underline{\underline{B}}\underline{d}^e\right]^2 d\xi \quad \text{where } \Omega \in [0,l] \text{ and } \Omega^e \in [-1,1] \tag{92}$$

Hence,

$$\int_{\Omega}\left(u'(x)-u^{h'}(x)\right)^2 dx=\frac{h^e}{2}\sum_{e=1}^{n_{el}}\int_{\Omega^e}\left[u'(x(\xi))-\underline{\underline{B}}\underline{d}^e\right]^2 d\xi \quad \text{where} \;\; \Omega\in[0,l] \;\; \text{and} \;\; \Omega^e\in[-1,1] \tag{93}$$

In general, the **logarithm of the error varies linearly with element size** and the **slope** depends on the **order of the element** and whether the error is in the **function or its derivative**.

$$\log(\|e\|_{L2})=C^*+\alpha\log h=C^*+\log h^\alpha \tag{94}$$

Where α is the slope of the $\|e_{L2}\|$ curve (the **rate of convergence of the element**), which is $\alpha=2$ for linear two-node elements and, C^* is an arbitrary constant. Taking the power of both sides gives

$$\|e\|_{L_2}=Ch^\alpha \tag{95}$$

What is interesting about this expression is that the error increases and decreases exponentially with the element size.

In general, if the finite element contains the complete polynomial of order p, then the error in the L2 norm of the displacement varies according to $\|e\|_{L_2}=Ch^{p+1}$, while the slope of the convergence plot for derivatives is one order lower"

Implementing the Gauss quadrature method to analyse the convergence, the results can be shown in Figure 12.

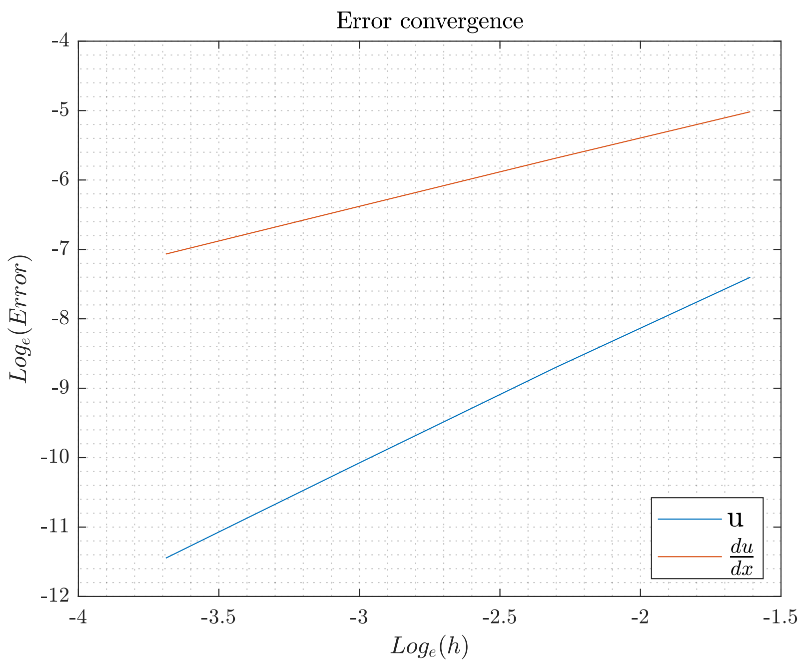


Figure 12 Error convergence (Logarithmic in e base).

In this case, by fitting the graph with a linear approximation, the first order polynomials are as follows:

$$u:\log\|e\|_{L2}=1.9489\log h^e-4.2421 \tag{96}$$

$$u':\log\|e\|'_{L2}=0.9868\log h^e-3.4224 \tag{97}$$

2 Optional Assignment

A bar of length L and varying cross-sectional area

$$A(x) = A_0 \left(1 + 2\frac{x}{L}\left(\frac{x}{L} - 1\right)\right) \tag{98}$$

where $A_0 > 0$, is fixed at one end while the other end is subjected to a linearly increasing displacement $u_L(t) = u_m \frac{t}{T}$, where $u_m > 0$ and $t \in [0, T]$, $T > 0$ being the interval to analyze – this interval is considered sufficiently large so as to ignore inertial effects. On the other hand, the material of the bar obeys the following (exponential) constitutive equation (relation between stress σ and strain ε).

$$\sigma = \sigma_0 \left(1 - e^{-\frac{E}{\sigma_o}\varepsilon}\right) \tag{99}$$

Using the Finite Element method, find the displacement solution u = u(x, t) as a function of t ∈ [0,T] and x ∈ [0,X]. To this end, follow the steps outlined below.

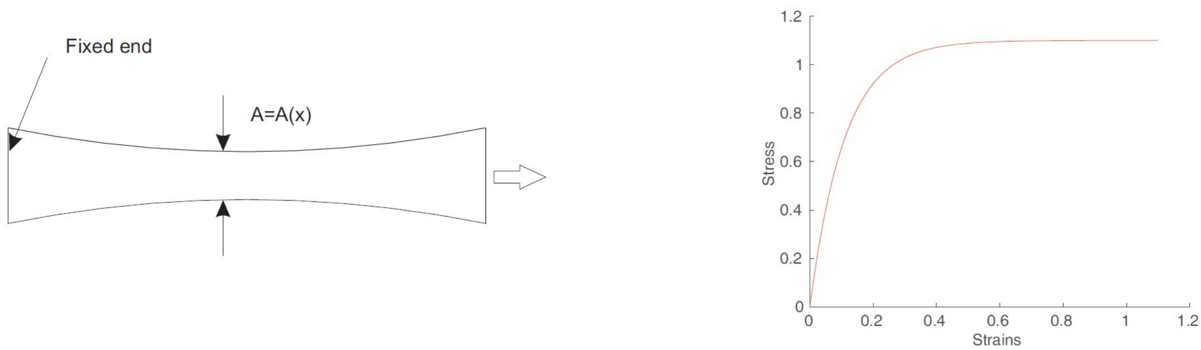


Figure 13 Geometry and constitutive equation

DATA: $E = 10$ MPa; $\sigma_0 = 1$ MPa; $L = 1$ m; $A_0 = 10^{-2}$ m^2; $u_m = 0.2L$.

1. Formulate the strong form of the boundary value problem for the 1D equilibrium of a bar with varying cross-sectional area.

The problem above is different from the previous exercise. In this case, the problem is no longer linear since the section is no longer constant throughout the geometry.

In general nonlinear problems, the function f represents an external action over the domain $\bar{\Omega} = [0, L]$ which also depends on a "loading parameter" $t \in [0, T]$ (that can be interpreted, for instance, as the time itself). Formally, the mathematical definition shall be written as $f : \bar{\Omega} \times [0, T] \to \mathbb{R}$. Likewise, the boundary conditions are also considered function of such loading parameter, i.e.: $u_0 : [0, T] \to \mathbb{R}$, and $\sigma_L : [0, T] \to \mathbb{R}$.

A prototypical nonlinear boundary value problem can be formulated as follows:

Find $u : \bar{\Omega} \times [0, T] \to \mathbb{R}$ such that

$$\frac{\partial \left(A\sigma\left(u'\right)\right)}{\partial x} + f = 0 \quad \text{on } \Omega \times [0, T] \tag{100}$$

subject to the boundary conditions

$$\begin{aligned} u(0, t) &= 0 && \text{on} \quad [0, T] \\ u(L, t) &= u_m \cdot \tfrac{t}{T} && \text{on} \quad [0, T] \end{aligned} \tag{101}$$

Here, $A : \bar{\Omega} \rightarrow \mathbb{R}$, while σ is considered a nonlinear function of the derivative of the solution, that is

$$\sigma = \sigma\left(u'\right) \tag{102}$$

where $u' = \frac{\partial u}{\partial x}$, that is to say $u' = u(x,t)$.

The expression (100) represents the **strong form** of the differential equation. The 3 main differences between the linear and non-linear formulation is that, in this latter one, it introduces the following parameters:

1. Area parameter A, which is no longer constant because it depends on the position.
2. The stress is no longer proportional to the deformation ($\sigma \neq Eu'$) and it is replaced by the general expression $\sigma = \sigma(u')$.
3. The domain is no longer Ω, it is now added $\Omega \times [0,T]$.

Hence, we shall derive the weak form of the non-linear boundary problem. Thereby, by means of the chain rule, multiplying the above-mentioned equation by the test function, $v = v(x)$ and integrate over the domain:

$$\int_0^L \left(v\frac{\partial(A\sigma)}{\partial x} + vf \right) dx = 0 \tag{103}$$

By using the chain rule,

$$\frac{\partial(v(A\sigma))}{\partial x} = \frac{\partial v}{\partial x}(A\sigma) + v\frac{\partial(A\sigma)}{\partial x} \quad \Rightarrow \quad v\frac{\partial(A\sigma)}{\partial x} = \frac{\partial(v(A\sigma))}{\partial x} - \frac{\partial v}{\partial x}(A\sigma) \tag{104}$$

Thus,

$$\int_0^L v'A(x)\sigma(u')dx = \int_0^L \frac{\partial(v(A\sigma))}{\partial x}dx + \int_0^L vf dx \quad \Rightarrow \quad \int_0^L v'A\sigma dx = v(L)A(L)\sigma_L + \int_0^L vf dx \tag{105}$$

2. Determine the weak form of the problem formulated above.

The **weak form** of the Boundary Value Problem can be thus stated as follows: given f: Ω x $[0, \mathrm{T}] \rightarrow \mathbb{R}$, $u_0 : [0, \mathrm{T}] \rightarrow \mathbb{R}$ and $\sigma_L : [0, \mathrm{T}] \rightarrow \mathbb{R}$, find u $\in \mathcal{S}$ such that for all v $\in \mathcal{V}$

$$\boxed{\int_0^L v'A(x)\sigma(u')dx = \int_0^L vf dx + v(L)A(L)\sigma_L} \tag{106}$$

where σ is in general a nonlinear function of u$'$ as in equation 102.

To derive the matrix equations, we shall replace u, v, u' and v' in the preceding equations by

$$\underline{c}^T \overbrace{\int_0^L \underline{\underline{B}}^T A\sigma \; \mathrm{d}x}^{\hat{F}:\text{ Internal Forces}} = c^T \overbrace{\left(\int_0^L \underline{\underline{N}}^T f + \underline{\underline{N}}(L)A(L)\sigma_L \right)}^{F:\text{ External Forces}} \tag{107}$$

However, this problem does not take into account external loads, neither distributed forces nor punctual ones. As a result, the sum of the internal forces of the slice will be null.

$$\int_0^L \underline{\underline{B}}^T A(x)\sigma(u')dx = 0 \tag{108}$$

3. Formulate the corresponding matrix equations.

Following the same procedure as the linear problem, that led to the matrix form of the principle of virtual work, it is obtained the system of nonlinear equations in $\underline{d_l}$:

$$\underline{\hat{F}_l}(\underline{d_l}) = \underline{F_l} \tag{109}$$

where

$$\underline{\hat{F}}(t) := \int_0^L \underline{\underline{B}}^T A\sigma \,\mathrm{dx} \quad \text{where} \quad \sigma = \sigma(\underline{\underline{B}}\,\underline{d}(t)) \tag{110}$$

$$\underline{F}(t) := \int_0^L \underline{\underline{N}}^T f(x,t)\,\mathrm{dx} + \underline{\underline{N}}^T(L)A(L)\sigma_L(t) \tag{111}$$

and l represents the indexes of the unknown values of d.

The integral can be calculated via finite element assembly:

$$\hat{F} = \sum_{e=1}^{n_{el}} L^{eT}\hat{F}^e = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \hat{F}^e \tag{112}$$

where

$$\hat{F}^e := \int_{\Omega^e} B^{e^T} A^e \sigma^e dx, \text{ with } \sigma^e = \sigma\left(B^e d^e\right) \tag{113}$$

$$\hat{F}^e = \int_{\Omega^e} \underbrace{\frac{1}{h^e}\begin{bmatrix}-1\\1\end{bmatrix}}_{B^{eT}} \cdot \underbrace{\left[A_0\left(1+2\frac{x}{L}\left(\frac{x}{L}-1\right)\right)\right]}_{A^e} \cdot \underbrace{\left[\sigma_0\left(1-e^{\frac{-E}{\sigma_0}\frac{1}{h^e}\begin{bmatrix}-1 & 1\end{bmatrix}\cdot\begin{bmatrix}d_1^e\\d_2^e\end{bmatrix}}\right)\right]}_{\sigma^e} \mathrm{d}x \tag{114}$$

Then, transforming all the values into the ξ domain, the following is obtained:

$$\hat{F}^e = \int_{\Omega^e} \underbrace{\frac{1}{h^e}\begin{bmatrix}-1\\1\end{bmatrix}}_{B^{eT}} \cdot \underbrace{\left[A_0\left(1+2\frac{N_1^e(\xi)x_1^e + N_2^e(\xi)x_2^e}{L}\left(\frac{N_1^e(\xi)x_1^e + N_2^e(\xi)x_2^e}{L}-1\right)\right)\right]}_{A^e} \cdot \underbrace{\left[\sigma_0\left(1-e^{\frac{-E}{\sigma_0}\frac{1}{h^e}\begin{bmatrix}-1 & 1\end{bmatrix}\cdot\begin{bmatrix}d_1^e\\d_2^e\end{bmatrix}}\right)\right]}_{\sigma^e} \mathrm{d}x \tag{115}$$

The problem also requires the extraction of the $\underline{\underline{K}}$ which is explained on the next page.

4. Solve the resulting system of nonlinear equations by means of a *Newton-Raphson algorithm*

Newton-Raphson algorithm

The Newton-Raphson method is a powerful iterative method for solving equation numerically. Behind his technique, the fundamental principle behind it is based on the idea of linear approximation.

The problem at hand consists in finding $d_l : [0, T] \to \mathbb{R}^n$ by solving 117.

In practice, $\underline{d}_l$ is only determined at a finite set of values of t. Thus, to do so the interval $[0, T]$ is divided in a finite set of discrete intervals:

$$[0, T] = [0, t_1] \cup [t_1, t_2] \cup \cdots [t_n, t_{n+1}] \cdots [t_{M-1}, T] \tag{116}$$

The problem is as follows:

Given $\underline{d}\,(t_n)$ (initial condition) together with $\underline{d_r}\,(t_{n+1}) = u_0\,(t_{n+1})$ and $\underline{F}\,(t_{n+1})$ (external actions at time t_{n+1}), compute $\underline{d_l}\,(t_{n+1})$ by solving the nonlinear system of equations

$$\underline{\hat{F}_l}\left(\underline{d_l}\,(t_{n+1})\right) = \underline{F_l}\,(t_{n+1}) \tag{117}$$

The above equation is clearly nonlinear since $\sigma = \sigma(u')$ is implicitly inside the expression. This expression represents the equilibrium between internal and external forces.

For notational simplicity, let $\underline{R}$ denote the residual of the above equation by

$$\underline{R} := \underline{\hat{F}_l}\left(\underline{d_l}\,(t_{n+1})\right) - \underline{F_l}\,(t_{n+1}) \tag{118}$$

the sought-after solution by $\underline{y} = \underline{d}_l(t_{n+1})$ and the initial condition by $\underline{y}^{(0)} = \underline{d}_l(t_n)$.

The objective is to **solve the system of nonlinear equations** $\underline{R}(y) = 0$.

To do so, **Newton-Raphson algorithm** is presented to solve the nonlinear equations. This methodology consists in an iterative method which updates the tentative solution at each iteration by solving the system of linear equations:

$$\underline{R}\left(\underline{y}^{(k)}\right) + \left[\frac{\partial \underline{R}}{\partial \underline{y}}\right]_{y^{(k)}} \left(\underline{y}^{(k+1)} - \underline{y}^{(k)}\right) = 0 \tag{119}$$

Accordingly, the solution at the k+1 -th iteration is given by

$$\underline{y}^{k+1} = \underline{y}^{k} - (\underline{\underline{J}}^{(k)})^{-1} \underline{R}(\underline{y}^{k}) \tag{120}$$

where

$$\underline{\underline{J}}^{(k)} = \left[\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{y}}\right]_{y^{(k)}} = \begin{bmatrix} \frac{\partial R_1}{\partial y_1} & \frac{\partial R_1}{\partial y_2} & \cdots & \frac{\partial R_1}{\partial y_n} \\ \frac{\partial R_2}{\partial y_1} & \frac{\partial R_2}{\partial y_2} & \cdots & \frac{\partial R_2}{\partial y_n} \\ \cdots & & \ddots & \cdots \\ \frac{\partial R_n}{\partial y_1} & \frac{\partial R_n}{\partial y_2} & \cdots & \frac{\partial R_n}{\partial y_n} \end{bmatrix} \tag{121}$$

is the Jacobian matrix of the system of equations.

The expression of this Jacobian matrix can be deduced as follows

$$\underline{J} = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{y}} = \frac{\partial \hat{\boldsymbol{F}}_1}{\partial \boldsymbol{d}_{\mathrm{I}}} = \boldsymbol{K}_{\mathrm{II}} \tag{122}$$

where

$$\underline{K} := \frac{\partial \underline{\hat{F}}}{\partial \underline{d}} \tag{123}$$

To obtain $\underline{K}$, the above expression can be reduced to

$$\underline{K} = \sum_{e=1}^{n_{el}} \underline{L}^{e^T} \frac{\partial \underline{\hat{F}}^e}{\partial d^e} \underline{L}^e \tag{124}$$

which means

$$\underline{\underline{K}} = \underset{e=1}{\overset{n_{el}}{\mathbf{A}}} \hat{K}^e \tag{125}$$

where

$$K^e := \frac{\partial \underline{\hat{F}}^e}{\partial d^e} \tag{126}$$

Elaborating now on K^e. This term can be determined as

$$\underline{\underline{K}}^e = \frac{\partial}{\partial \underline{d}^e} \int_{\Omega^e} B^{e^T} A^e \sigma^e dx = \int_{\Omega^e} B^{e^T} A^e \frac{\partial \sigma^\Theta}{\partial \underline{d}^e} \,\mathrm{dx} \tag{127}$$

Recall that $\sigma^e = \sigma^e \left(u'^e\right)$ therefore, by the chain rule, we have that

$$\frac{\partial \sigma^e}{\partial \underline{d}^e} = \overbrace{\frac{\partial \sigma^e}{\partial u'}}^{E^e} \frac{\partial u'^e}{\partial \underline{d}^e} = E^e \underline{\underline{B}}^e \tag{128}$$

where we have used the expression $u'^e = \underline{\underline{B}}^e \underline{d}^e$. Thus, we can finally define the elemental coefficient matrix $\underline{\underline{K}}^e$ as

$$\boxed{\boldsymbol{K}^e = \int_{\Omega^e} \underline{\underline{B}}^{e^T} \left(A^e E^e\right) \underline{\underline{B}}^e dx} \tag{129}$$

In order to implement this method in the MATLAB environment the following steps should be taken:

5. Check that the solution converges upon increasing the number of elements and time steps.

5.1. Results using $n_{el} = 5$

The following results analyzes the stress σ, displacement d of the bar with varying cross-sectional area with respect to is coordinate x.

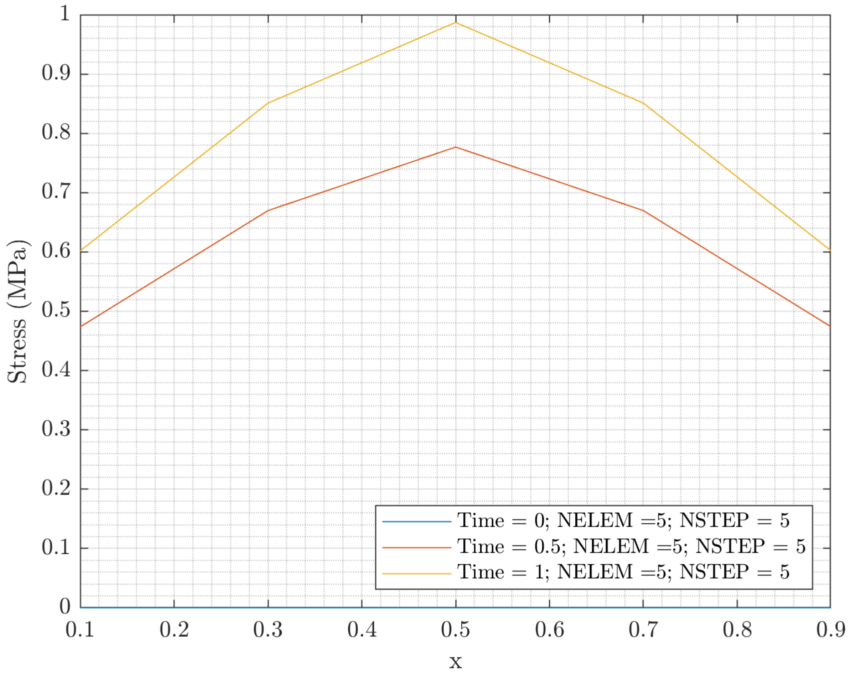


Figure 14 Stress versus x coordinate for $n_{el} = 5$ elements and $n_{step} = 5$

The above Figure 14 shows the stress that for every coordinate x of the bar. It is noticeable that using only $n_{el} = 5$ the results give a hint on the behaviour that the structure is going to face. However, the lack of finite elements results in a non smooth stress transition between nodes. Thus, it is recommended to increase the number of elements n_{el} for a more detailed study.

The above figure presents the evolution of the stress on the bar along its x coordinate. The particularity of the plot resides in the maximum value of stress. This happens to be in the middle of the bar. Nevertheless, the given cross-sectional area shows that the middle of the bar has the least area. Thereby, the most critical point of failure is also the middle of the bar at $x = 0.5$m. The resultant plot, eventually, points out the stated arguments. For $t = 0$ there is no stress, and at as the time increases, the stress gains more importance and reaches almost 1 MPa of stress in the mid section at $t = 1$.

Next, the displacement of the each node is also interesting to analyse. The prior Figure 15 enable us to see the displacement that every node has. This problem shows the displacement in the x axis. As Dirichlet condition was imposed at $x = 0$ m, the node does not presents any displacement although time increases. Nevertheless, the other nodes undergoes horizontal displacements. As the distance increases with respect to x, the displacement effect is major. The effect deepens in the most exterior node suffers the greatest displacement effect, with a maximum displacement almost reaching $d = 0.2$ m.

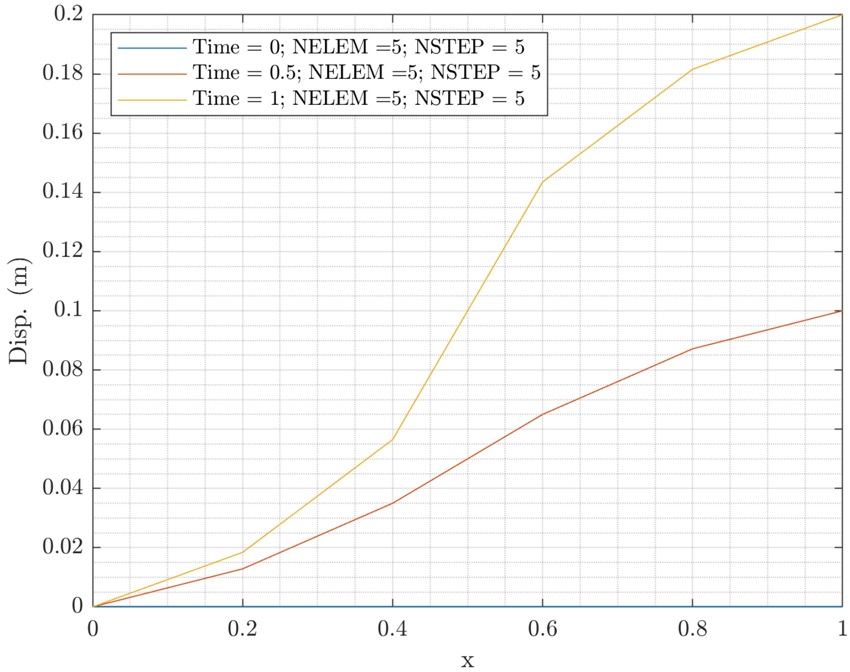


Figure 15 Displacement versus x coordinate for $n_{el} = 5$ elements and $n_{step} = 5$

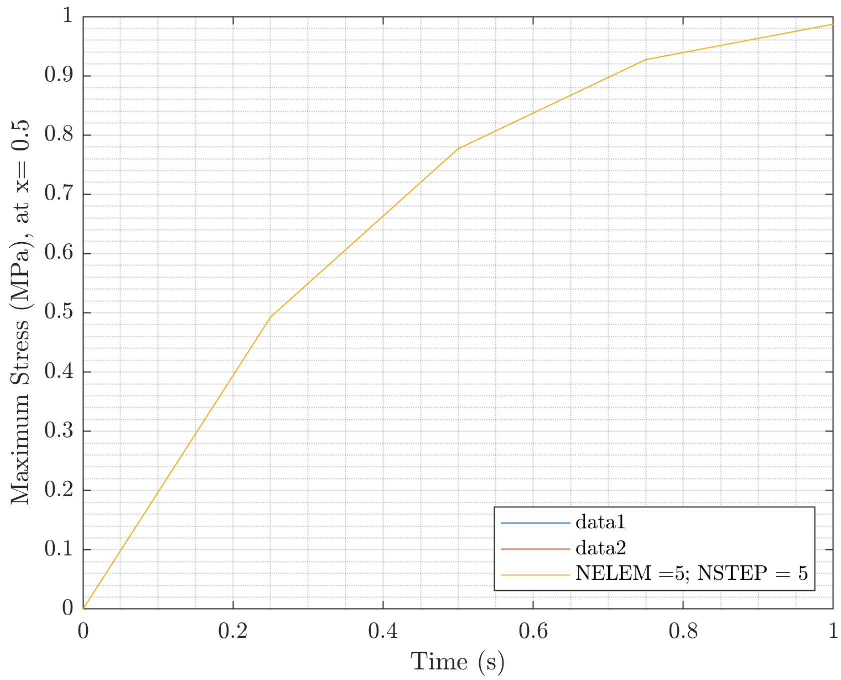


Figure 16 Maximum Stress versus x coordinate for $n_{el} = 5$ elements and $n_{step} = 5$

Finally, the last graph indicates the maximum stress at the middle node $n = 0.5$ m. Immediately after $t = 0$ s the stress increases gradually as time increases. The fact that only 5 time steps were used to analyse the problem results in steep and rough changes.

5.2. Results using $n_{el} = 5$

The following plots shows the same analysis with $n_{el} = 50$, which results in a more uniform and smooth variation.

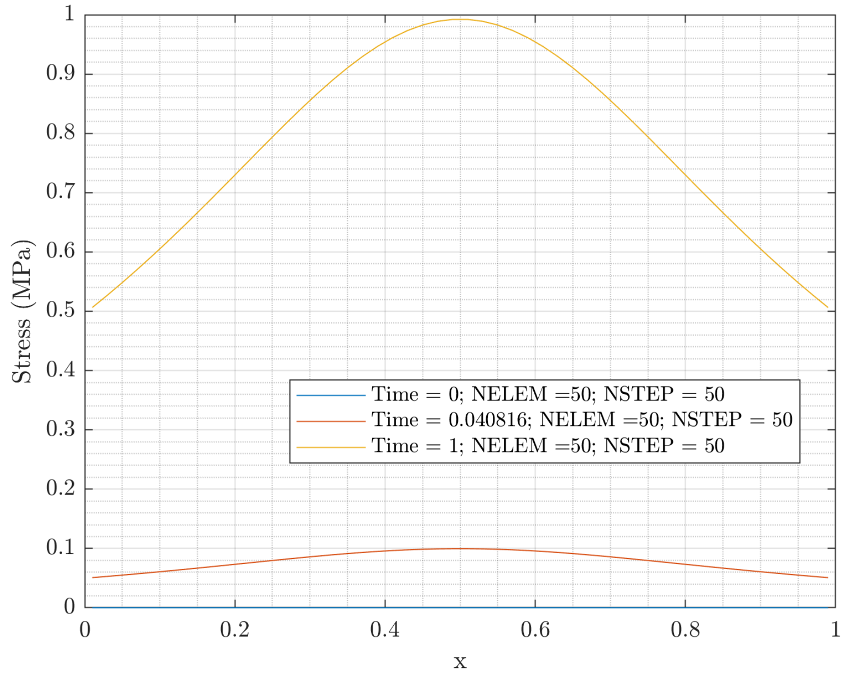


Figure 17 Stress versus x coordinate for $n_{el} = 50 \times 50$ elements and $n_{step} = 50$

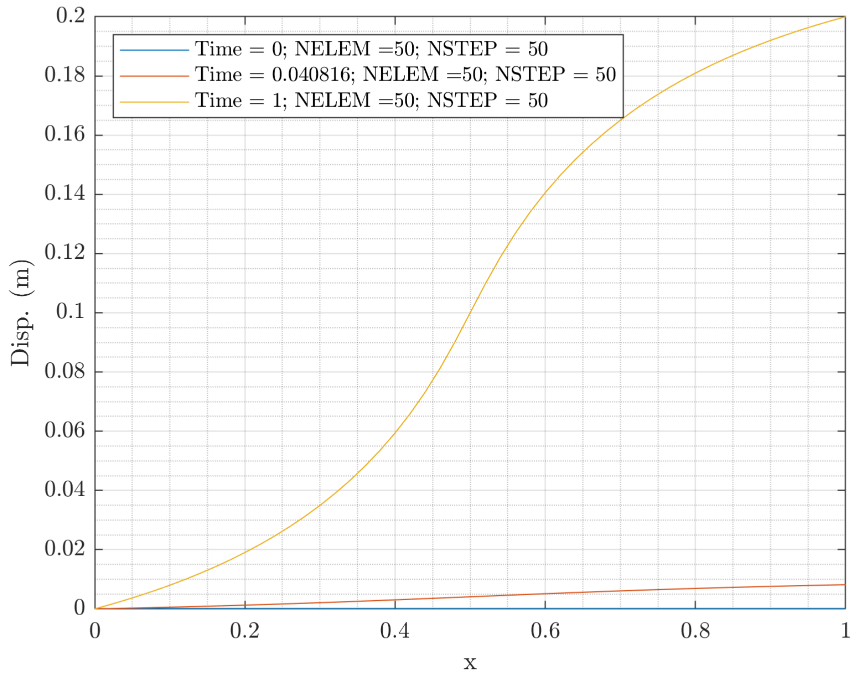


Figure 18 Displacement versus x coordinate for $n_{el} = 50 \times 50$ elements and $n_{step} = 50$

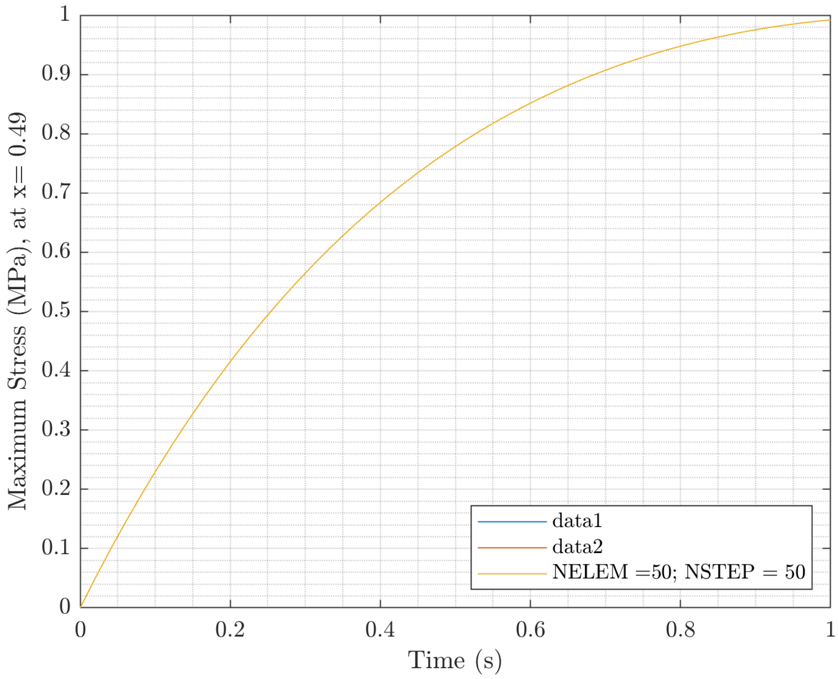
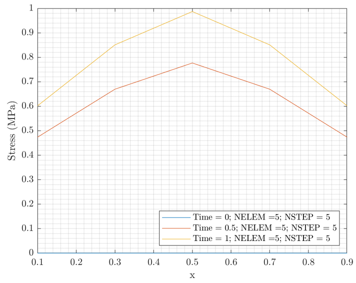


Figure 19 Maximum Stress versus x coordinate for $n_{el} = 50 \times 50$ elements and $n_{step} = 50$

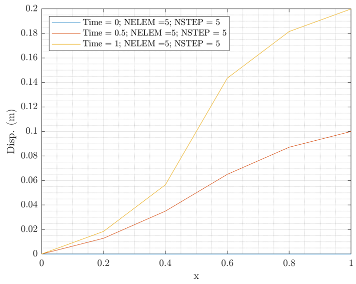
5.3. Convergence comparison

The plots below intends to analyse the convergence for different number of elements n_{el} and time steps t (see Figures 20).

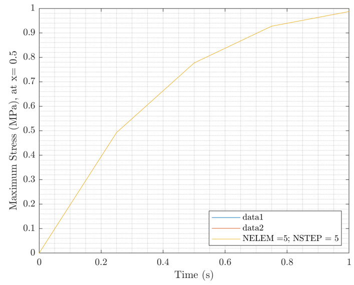
Figures 21, 22, 23 presents $n_{el} = 5$ and $n_{el} = 50$ in a single plot to have a more detailed view of the convergence of the values.



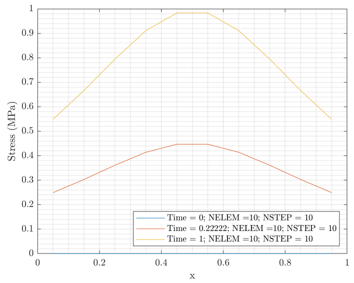
(a) Stress vs x for $n_{el} = 5 \times 5$ and $n_{step} = 5$



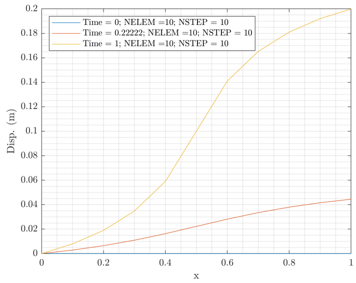
(b) Displacement vs x for $n_{el} = 5 \times 5$ and $n_{step} = 5$



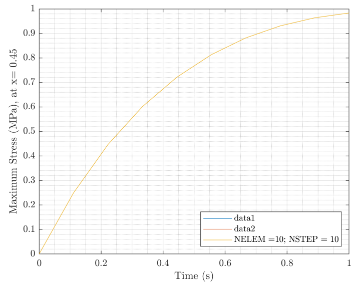
(c) Max. Stress vs x for $n_{el} = 5 \times 5$ and $n_{step} = 5$



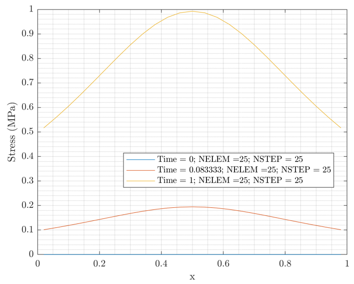
(d) Stress vs x for $n_{el} = 10 \times 10$ and $n_{step} = 10$



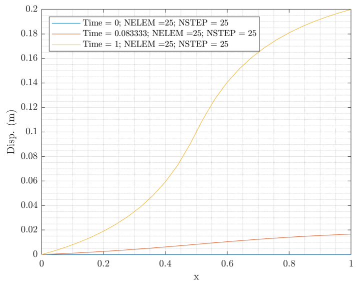
(e) Displacement vs x for $n_{el} = 10 \times 10$ and $n_{step} = 10$



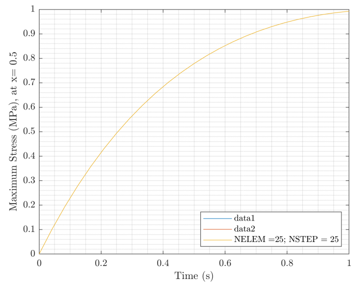
(f) Max. Stress vs x for $n_{el} = 10 \times 10$ and $n_{step} = 10$



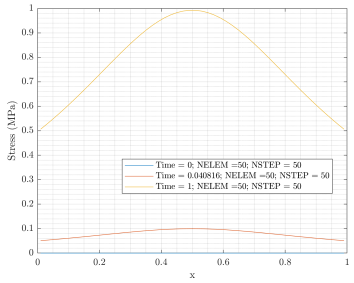
(g) Stress vs x for $n_{el} = 25 \times 25$ and $n_{step} = 25$



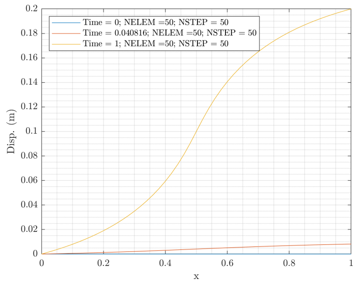
(h) Displacement vs x for $n_{el} = 25 \times 25$ and $n_{step} = 25$



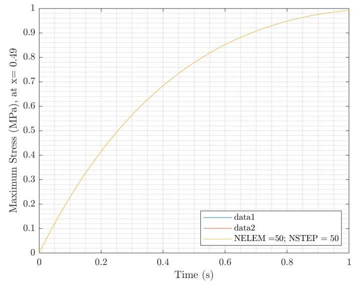
(i) Max. Stress vs x for $n_{el} = 25 \times 25$ and $n_{step} = 25$



(j) Stress vs x for $n_{el} = 50 \times 50$ and $n_{step} = 50$



(k) Displacement vs x for $n_{el} = 50 \times 50$ and $n_{step} = 50$



(l) Max. Stress vs x for $n_{el} = 50 \times 50$ and $n_{step} = 50$

Figure 20 Convergence comparison for different sizes of mesh n_{el}

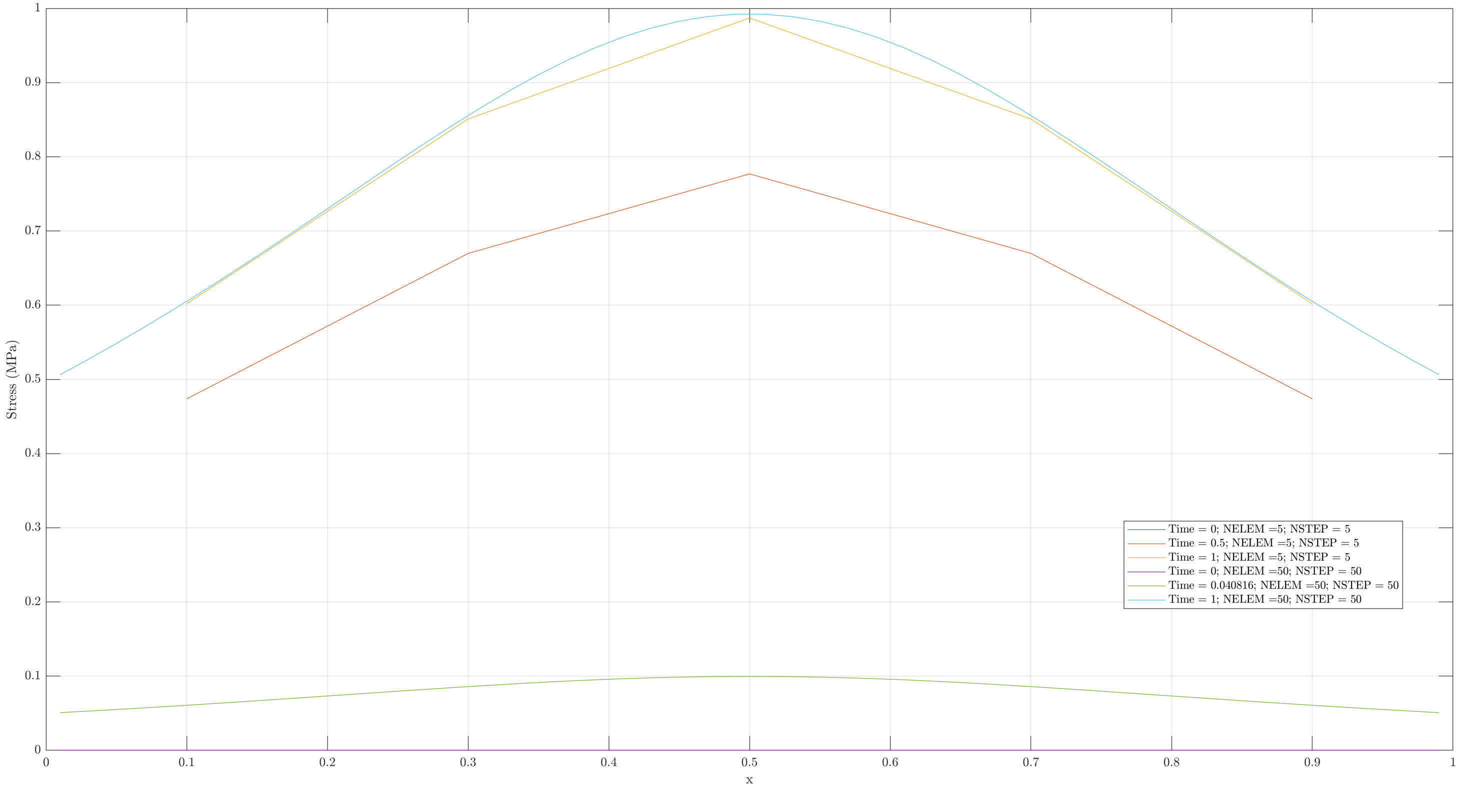


Figure 21 Maximum Stress versus x coordinate for $n_{el} = 5 \times 5$ elements and $n_{el} = 5 \times 5$ elements

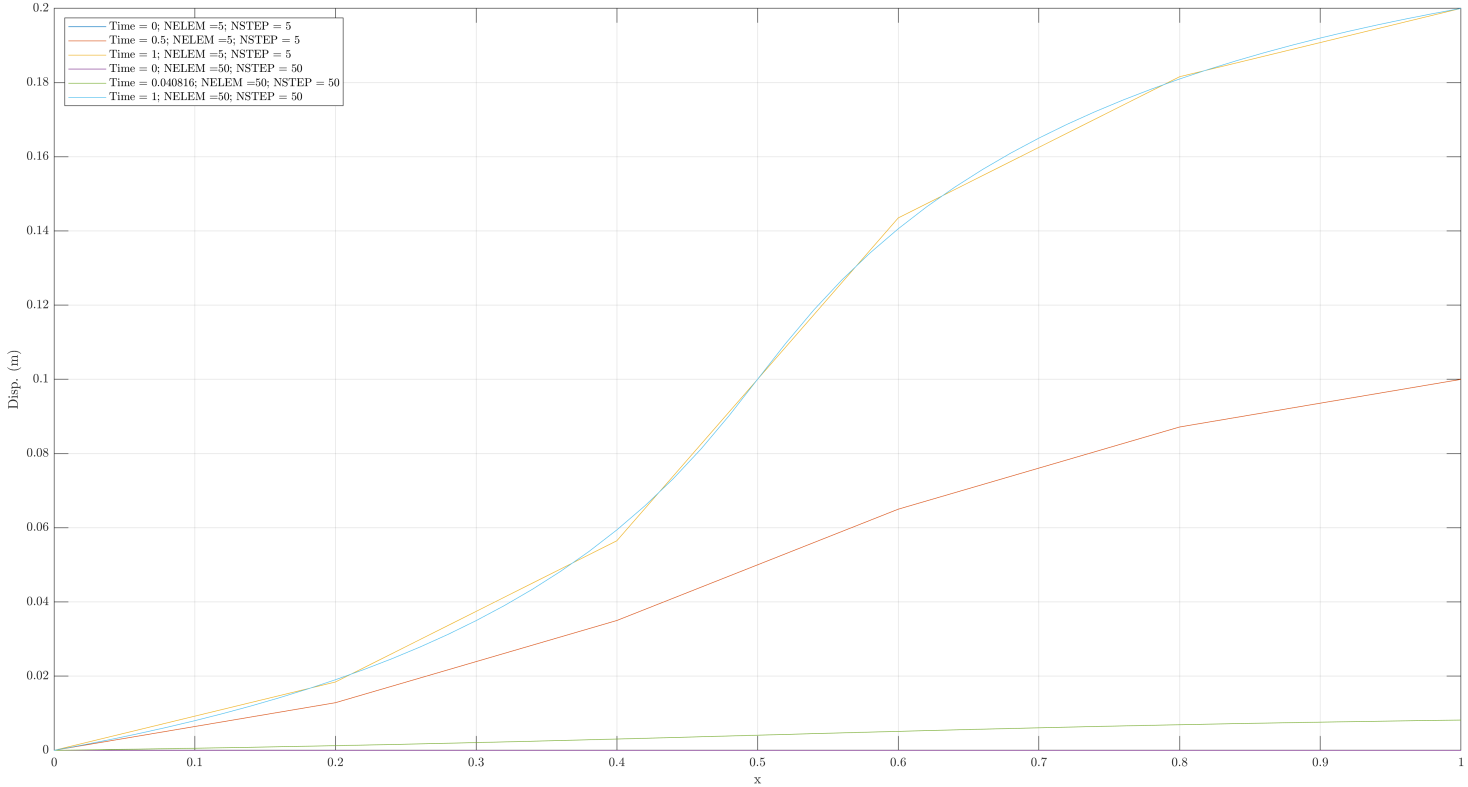


Figure 22 Maximum Stress versus x coordinate for $n_{el} = 5 \times 5$ elements and $n_{el} = 5 \times 5$ elements

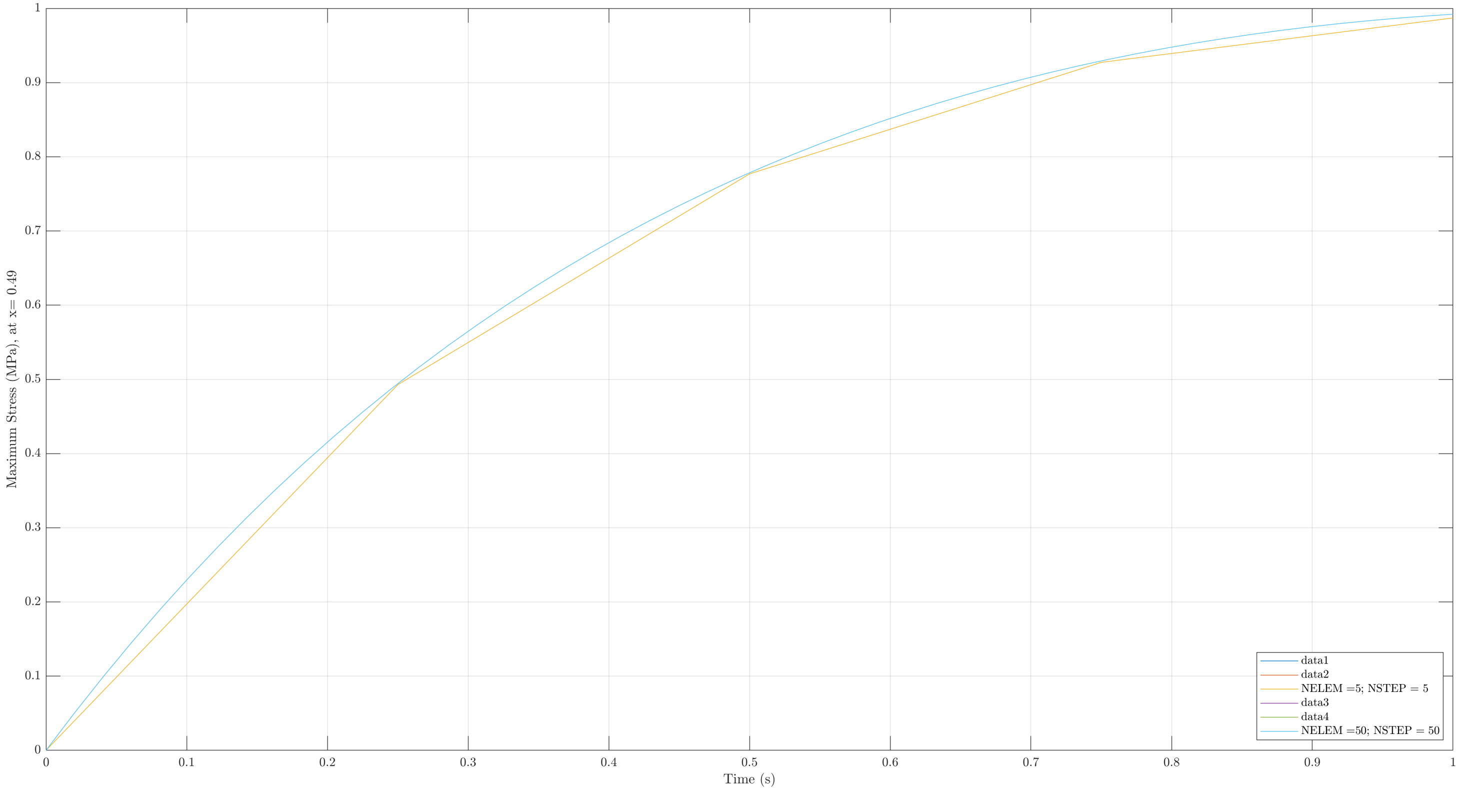


Figure 23 Maximum Stress versus x coordinate for $n_{el} = 5 \times 5$ elements and $n_{el} = 5 \times 5$ elements

3 Annex

The plots below analyses the results for $n_{el} = 10$ and $n_{el} = 25$ which were previously discussed in Figures 11.

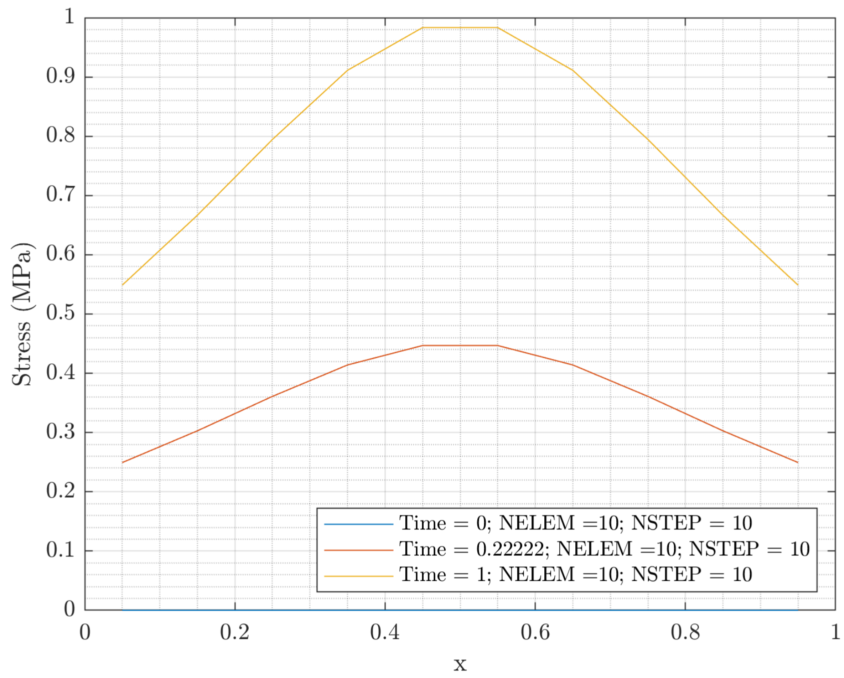


Figure 24 Stress versus x coordinate for $n_{el} = 10 \times 10$ elements and $n_{step} = 10$

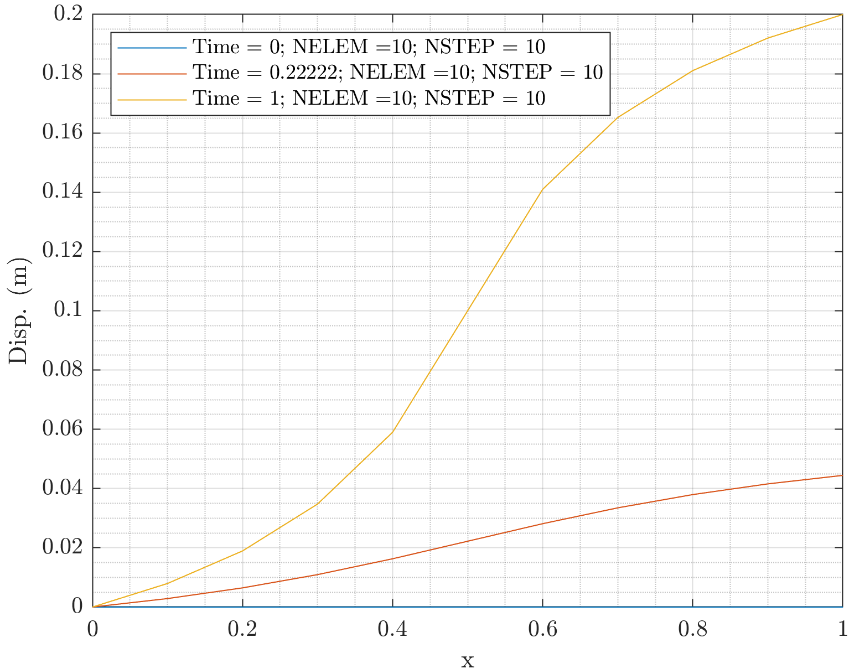


Figure 25 Displacement versus x coordinate for $n_{el} = 10 \times 10$ elements and $n_{step} = 10$

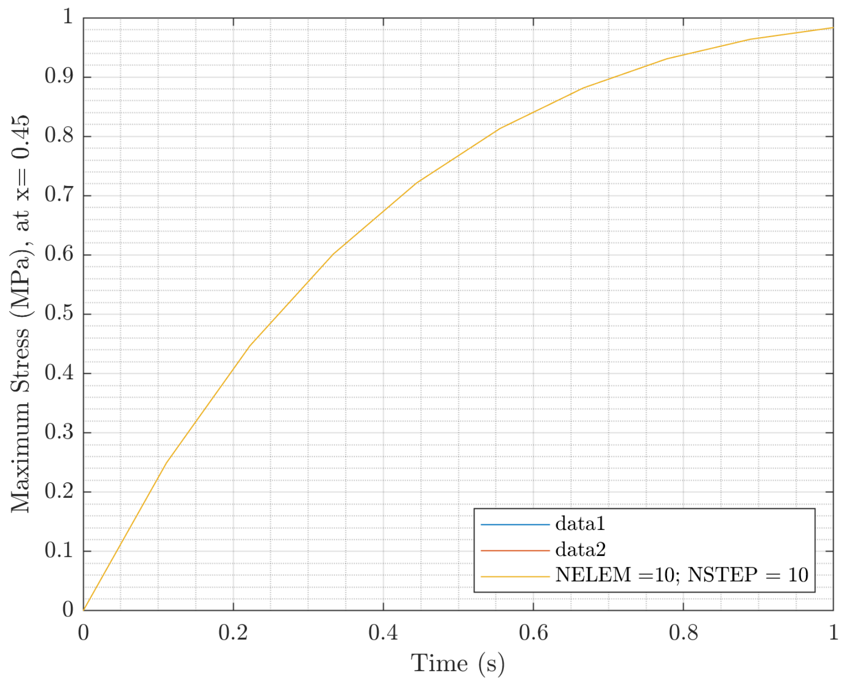


Figure 26 Maximum Stress versus x coordinate for $n_{el} = 10 \times 10$ elements and $n_{step} = 10$

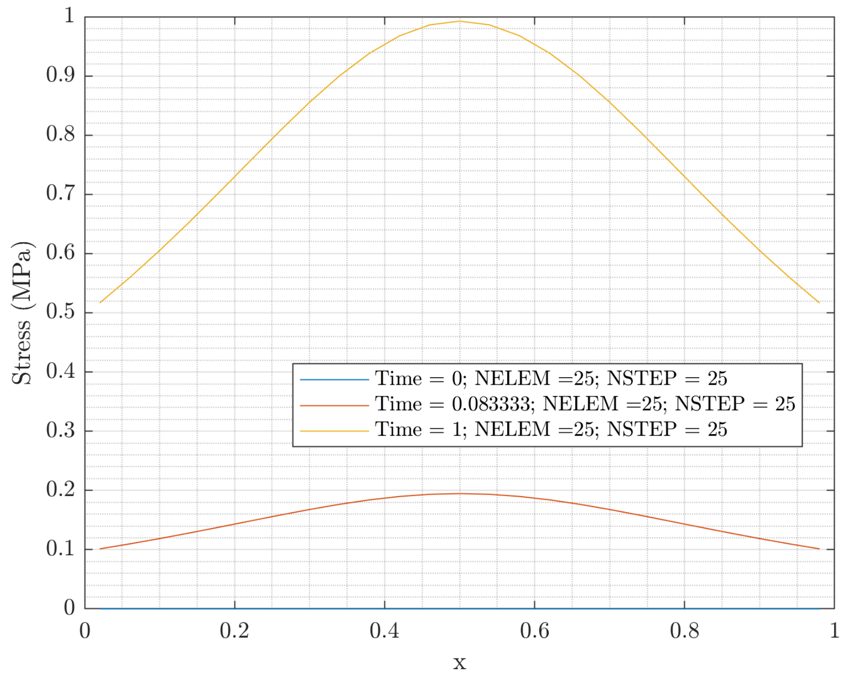


Figure 27 Stress versus x coordinate for $n_{el} = 25 \times 25$ elements and $n_{step} = 25$

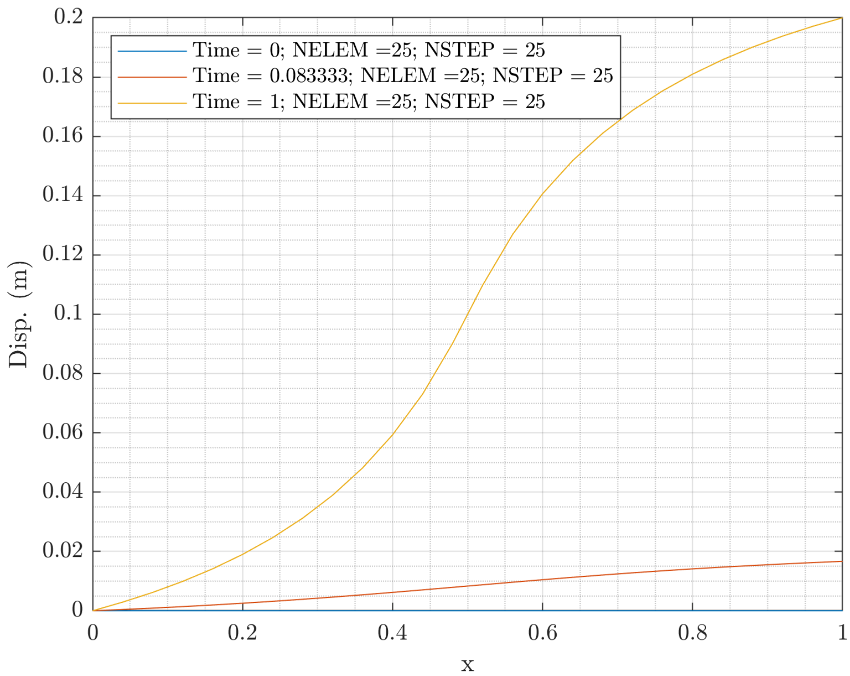


Figure 28 Displacement versus x coordinate for $n_{el} = 25 \times 25$ elements and $n_{step} = 25$

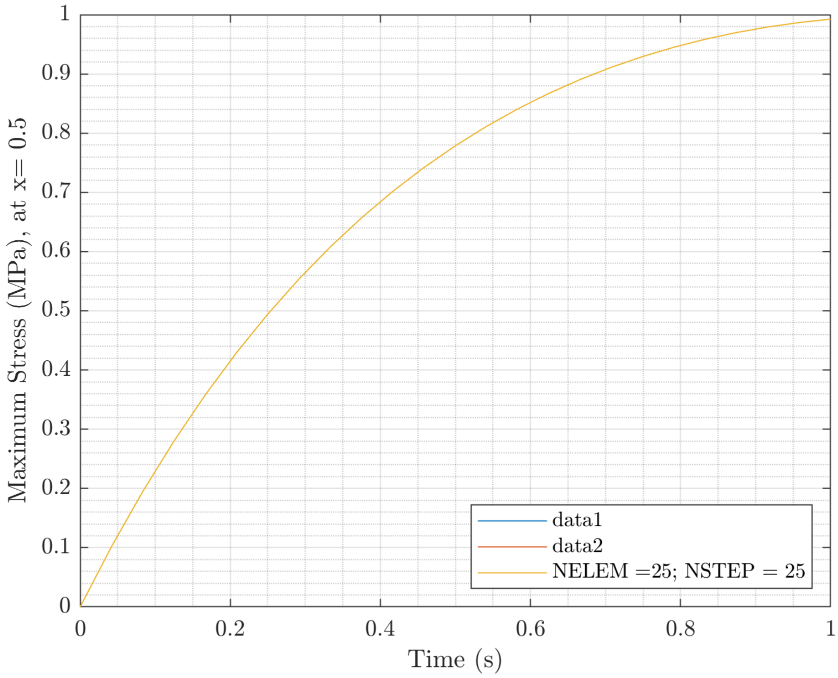


Figure 29 Maximum Stress versus x coordinate for $n_{el} = 25 \times 25$ elements and $n_{step} = 25$

References

[1] Hernández, Joaquín. "Tema 1D FE". In: 1st ed. UPC, 2020, pp. 1–75.