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Beam element: Computation of the global stiffness matrix

1 Geometry

The following terms are defined:

n_d : Problem dimension (e.g. 2 for 2D, 3 for 3D).
n_{el} : Total number of bars.
n_{nod} : Total number of nodes (joints).
n_{ne} : Number of nodes in a bar, i.e. $n_{ne} = 2$.
n_i : Degrees of freedom per node, i.e. $n_i = 2$ (1D), $n_i = 3$ (2D).
n_{dof} : Total number of degrees of freedom, i.e. $n_{dof} = n_{nod} \times n_i$.

The following matrices are defined (as in the bar element case):

$\mathbf{x}$: Nodal coordinates array ($n_{nod} \times n_d$).
$\mathbf{T}_n$: <u>Nodal</u> connectivity table ($n_{el} \times n_{ne}$).
$\mathbf{T}_d$: <u>Degrees of freedom</u> connectivity table ($n_{el} \times (n_{ne} \times n_i)$).

2 Computation of the element stiffness matrices

For each beam $e = 1 \ldots n_{el}$

$$x_1^e = \mathbf{x}(\mathbf{T}_n(e,1),1), \quad x_2^e = \mathbf{x}(\mathbf{T}_n(e,2),1)$$

$$y_1^e = \mathbf{x}(\mathbf{T}_n(e,1),2), \quad y_2^e = \mathbf{x}(\mathbf{T}_n(e,2),2)$$

$$l^e = \sqrt{(x_2^e - x_1^e)^2 + (y_2^e - y_1^e)^2}$$

$$\mathbf{R}^e = \frac{1}{l^e}\begin{bmatrix} x_2^e - x_1^e & y_2^e - y_1^e & 0 & 0 & 0 & 0 \\ -(y_2^e - y_1^e) & x_2^e - x_1^e & 0 & 0 & 0 & 0 \\ 0 & 0 & l^e & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2^e - x_1^e & y_2^e - y_1^e & 0 \\ 0 & 0 & 0 & -(y_2^e - y_1^e) & x_2^e - x_1^e & 0 \\ 0 & 0 & 0 & 0 & 0 & l^e \end{bmatrix}$$

$$\mathbf{K}^{e\prime} = \frac{I^e E^e}{l^{e3}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6l^e & 0 & -12 & 6l^e \\ 0 & 6l^e & 4l^{e2} & 0 & -6l^e & 2l^{e2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6l^e & 0 & 12 & -6l^e \\ 0 & 6l^e & 2l^{e2} & 0 & -6l^e & 4l^{e2} \end{bmatrix} + \frac{A^e E^e}{l^e}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}^e(:,:,e) = \mathbf{R}^{e\mathrm{T}} \mathbf{K}^{e\prime} \mathbf{R}^e$$

Next beam e

3 Computation of the element force vector

For each beam $e = 1 \dots n_{el}$

$$x_1^e = \mathbf{x}(\mathbf{T}_n(e,1),1), \quad x_2^e = \mathbf{x}(\mathbf{T}_n(e,2),1)$$

$$y_1^e = \mathbf{x}(\mathbf{T}_n(e,1),2), \quad y_2^e = \mathbf{x}(\mathbf{T}_n(e,2),2)$$

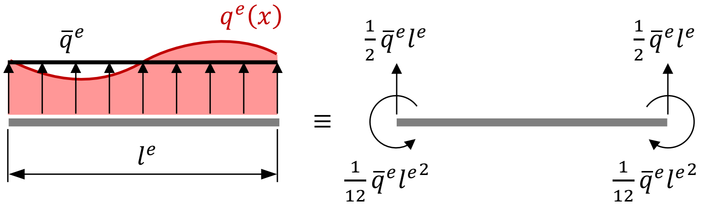
$$l^e = \sqrt{(x_2^e - x_1^e)^2 + (y_2^e - y_1^e)^2}$$

$$\mathbf{R}^e = \frac{1}{l^e}\begin{bmatrix} x_2^e - x_1^e & y_2^e - y_1^e & 0 & 0 & 0 & 0 \\ -(y_2^e - y_1^e) & x_2^e - x_1^e & 0 & 0 & 0 & 0 \\ 0 & 0 & l^e & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2^e - x_1^e & y_2^e - y_1^e & 0 \\ 0 & 0 & 0 & -(y_2^e - y_1^e) & x_2^e - x_1^e & 0 \\ 0 & 0 & 0 & 0 & 0 & l^e \end{bmatrix}$$

$$\mathbf{F}^{e\prime} = \frac{\bar{q}_{y\prime}^e l^e}{2}\begin{bmatrix} 0 \\ 1 \\ l^e/6 \\ 0 \\ 1 \\ -l^e/6 \end{bmatrix} + \frac{\bar{q}_{x\prime}^e l^e}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{F}^e(:,e) = \mathbf{R}^{e\mathrm{T}}\mathbf{F}^{e\prime}$$

Next beam e



4 Global force vector and stiffness matrix assembly

For each beam $e = 1 \dots n_{el}$

For each local degree of freedom $i = 1 \dots n_{ne} \times n_i$ (rows)

$I = \mathbf{T}_d(e,i)$ (corresponding global degree of freedom)

$\widehat{\mathbf{F}}^{ext}(I) = \widehat{\mathbf{F}}^{ext}(I) + \mathbf{F}^e(i,e)$

For each local degree of freedom $j = 1 \dots n_{ne} \times n_i$ (columns)

$J = \mathbf{T}_d(e,j)$ (corresponding global degree of freedom)

$\mathbf{K}_G(I,J) = \mathbf{K}_G(I,J) + \mathbf{K}_{el}(i,j,e)$

Next local degree of freedom j

Next local degree of freedom i

Next beam e

5 Global system of equations

a) Global system

$$\mathbf{K}_G \hat{\mathbf{u}} = \widehat{\mathbf{F}}^{ext} + \hat{\mathbf{R}}$$

Remember to add point loads, if any, in the corresponding degrees of freedom of the global force vector!

b) Apply conditions

$\nu_R = [\ldots]$: Array with the imposed degrees of freedom.

$\nu_L = [\ldots]$: Array with the free degrees of freedom.

c) Partitioned system of equations

$$\begin{bmatrix} \mathbf{K}_{RR} & \mathbf{K}_{RL} \\ \mathbf{K}_{LR} & \mathbf{K}_{LL} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_R \\ \hat{\mathbf{u}}_L \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{F}}_R^{ext} \\ \widehat{\mathbf{F}}_L^{ext} \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{R}}_R \\ \mathbf{0} \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{K}_{LL} &= \mathbf{K}_G(\nu_L, \nu_L) \\ \mathbf{K}_{LR} &= \mathbf{K}_G(\nu_L, \nu_R) \\ \mathbf{K}_{RL} &= \mathbf{K}_G(\nu_R, \nu_L) \\ \mathbf{K}_{RR} &= \mathbf{K}_G(\nu_R, \nu_R) \\ \widehat{\mathbf{F}}_L^{ext} &= \widehat{\mathbf{F}}^{ext}(\nu_L, 1) \\ \widehat{\mathbf{F}}_R^{ext} &= \widehat{\mathbf{F}}^{ext}(\nu_R, 1) \end{aligned}$$

Data:

$\hat{\mathbf{u}}_R$: Imposed displacement/rotations vector

$\widehat{\mathbf{F}}^{ext}$: External force vector

Unknowns:

$\hat{\mathbf{u}}_L$: Free displacement/rotations vector

$\hat{\mathbf{R}}_R$: Reactions vector

d) System resolution

$$\mathbf{K}_{LL}\hat{\mathbf{u}}_L = \widehat{\mathbf{F}}_L^{ext} - \mathbf{K}_{LR}\hat{\mathbf{u}}_R \quad \rightarrow \quad \hat{\mathbf{u}}_L$$

$$\hat{\mathbf{R}}_R = \mathbf{K}_{RR}\hat{\mathbf{u}}_R + \mathbf{K}_{RL}\hat{\mathbf{u}}_L - \widehat{\mathbf{F}}_R^{ext}$$

e) Obtain generalized displacement/rotations vector

$$\hat{\mathbf{u}}(\nu_L, 1) = \hat{\mathbf{u}}_L$$

$$\hat{\mathbf{u}}(\nu_R, 1) = \hat{\mathbf{u}}_R$$

6 Computation of the internal forces and bending moment

For each beam $e = 1 \dots n_{el}$

a) Compute the rotation matrix

$$x_1^e = \mathbf{x}(\mathbf{T}_n(e,1),1), \quad x_2^e = \mathbf{x}(\mathbf{T}_n(e,2),1)$$

$$y_1^e = \mathbf{x}(\mathbf{T}_n(e,1),2), \quad y_2^e = \mathbf{x}(\mathbf{T}_n(e,2),2)$$

$$l^e = \sqrt{(x_2^e - x_1^e)^2 + (y_2^e - y_1^e)^2}$$

$$\mathbf{R}^e = \frac{1}{l^e}\begin{bmatrix} x_2^e - x_1^e & y_2^e - y_1^e & 0 & 0 & 0 & 0 \\ -(y_2^e - y_1^e) & x_2^e - x_1^e & 0 & 0 & 0 & 0 \\ 0 & 0 & l^e & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2^e - x_1^e & y_2^e - y_1^e & 0 \\ 0 & 0 & 0 & -(y_2^e - y_1^e) & x_2^e - x_1^e & 0 \\ 0 & 0 & 0 & 0 & 0 & l^e \end{bmatrix}$$

b) Obtain element's displacement in <u>global</u> coordinates

For each local degree of freedom $i = 1 \dots n_{ne} \times n_i$

$$I = \mathbf{T}_d(e,i)$$

$$\hat{\mathbf{u}}^e(i,1) = \hat{\mathbf{u}}(I,1)$$

Next degree of freedom i

c) Compute internal forces in <u>local</u> coordinates

$$\widehat{\mathbf{F}}_{int}^e = \mathbf{K}_{el}\hat{\mathbf{u}}^e$$

$$\widehat{\mathbf{F}}_{int}^{e'} = \mathbf{R}^e\widehat{\mathbf{F}}_{int}^e$$

d) Compute axial and shear forces and bending moment at element's nodes

$$\left.\begin{aligned} \hat{F}_{x'}^e(e,1) &= -\widehat{\mathbf{F}}_{int}^{e'}(n_i - 2) \\ \hat{F}_{x'}^e(e,2) &= \widehat{\mathbf{F}}_{int}^{e'}(2n_i - 2) \end{aligned}\right\} \text{Axial force}$$

$$\left.\begin{aligned} \hat{F}_{y'}^e(e,1) &= -\widehat{\mathbf{F}}_{int}^{e'}(n_i - 1) \\ \hat{F}_{y'}^e(e,2) &= \widehat{\mathbf{F}}_{int}^{e'}(2n_i - 1) \end{aligned}\right\} \text{Shear force}$$

$$\left.\begin{aligned} \widehat{M}_{z'}^e(e,1) &= -\widehat{\mathbf{F}}_{int}^{e'}(n_i) \\ \widehat{M}_{z'}^e(e,2) &= \widehat{\mathbf{F}}_{int}^{e'}(2n_i) \end{aligned}\right\} \text{Bending moment}$$

Next beam e

7 Computation of the axial strain, deflection and section rotation

For each beam $e = 1 \dots n_{el}$

a) Compute the rotation matrix

$$x_1^e = \mathbf{x}(\mathbf{T}_n(e,1),1), \quad x_2^e = \mathbf{x}(\mathbf{T}_n(e,2),1)$$

$$y_1^e = \mathbf{x}(\mathbf{T}_n(e,1),2), \quad y_2^e = \mathbf{x}(\mathbf{T}_n(e,2),2)$$

$$l^e = \sqrt{(x_2^e - x_1^e)^2 + (y_2^e - y_1^e)^2}$$

$$\mathbf{R}^e = \frac{1}{l^e}\begin{bmatrix} x_2^e - x_1^e & y_2^e - y_1^e & 0 & 0 & 0 & 0 \\ -(y_2^e - y_1^e) & x_2^e - x_1^e & 0 & 0 & 0 & 0 \\ 0 & 0 & l^e & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2^e - x_1^e & y_2^e - y_1^e & 0 \\ 0 & 0 & 0 & -(y_2^e - y_1^e) & x_2^e - x_1^e & 0 \\ 0 & 0 & 0 & 0 & 0 & l^e \end{bmatrix}$$

b) Obtain element's displacement in <u>global</u> coordinates

For each local degree of freedom $i = 1 \dots n_{ne} \times n_i$

$$I = \mathbf{T}_d(e,i)$$

$$\hat{\mathbf{u}}^e(i,1) = \hat{\mathbf{u}}(I,1)$$

Next degree of freedom i

c) Obtain element's displacement in <u>local</u> coordinates

$$\hat{\mathbf{u}}^{e\prime} = \mathbf{R}^e \hat{\mathbf{u}}^e$$

d) Compute element's deflection and rotation polynomial coefficients:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{1}{l^{e3}}\begin{bmatrix} 0 & 2 & l^e & 0 & -2 & l^e \\ 0 & -3l^e & -2l^{e2} & 0 & 3l^e & -l^{e2} \\ 0 & 0 & l^{e3} & 0 & 0 & 0 \\ 0 & l^{e3} & 0 & 0 & 0 & 0 \end{bmatrix}\hat{\mathbf{u}}^{e\prime}$$

$$\hat{p}_{u_y^e}(e,[1,2,3,4]) = [a,b,c,d] \qquad \left(u_{y'}^e(x') = a(x')^3 + b(x')^2 + cx' + d, \ \forall x \in [0,l^e]\right)$$

$$\hat{p}_{\theta_z^e}(e,[1,2,3]) = [3a,2b,c] \qquad \left(\theta_{z'}^e(x') = 3a(x')^2 + 2bx' + c, \ \forall x \in [0,l^e]\right)$$

e) Compute element's axial strain

$$\hat{\varepsilon}(e,1) = \frac{1}{l^e}[-1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0]\hat{\mathbf{u}}^{e\prime}$$

Next beam e