2020-

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BDA. 4.9

Point Estimation:

Suppose a measurement y is recorded with a $N(\theta, \sigma^2)$

here σ^2 known and $\theta \in [0,1]$

Consider two point estimation for θ:

(i) MLE

(ii) the posterior mean based on the assumption of a uniform prior distribution on θ.

(i) First let us find MLE of θ.

Given $y \sim N(\theta, \sigma^2)$ $\theta \in [0,1]$, σ^2 known

Then likelihood function is.

$$f(y|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\theta)^2\right\}$$

Since we want to max $f(y|\theta)$.

Hence it is sufficient to $\arg\min (y-\theta)^2$

discuss: if $y < 0$. then choose $\hat{\theta}_{MLE} = 0$

if $y \in [0,1]$. the choose $\hat{\theta}_{MLE} = y$

if $y > 1$ then choose $\hat{\theta}_{MLE} = 1$

because θ has restriction $[0,1]$

Then, putting everything together.

$$\hat{\theta}_{MLE} = \begin{cases} 0 & \text{if } y < 0 \\ y & \text{if } y \in [0,1] \\ 1 & \text{if } y > 1 \end{cases}$$

(ii) Next. let us find the posterior.

Assume prior: $\theta \sim \text{uniform}([0,1])$

$y|\theta \sim \text{normal}(\theta, \sigma^2)$

Then the posterior is

$$
\begin{aligned}
P(\theta|y) &\propto P(\theta)\cdot P(y|\theta) \\
&= \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) \\
&= \exp\left(-\frac{1}{2\sigma^2}(y^2-2y\theta+\theta^2)\right) \\
&= \exp\left(-\frac{1}{2\sigma^2}y^2+\frac{1}{\sigma^2}y\theta-\frac{1}{2\sigma^2}\theta^2\right) \\
&\propto \exp\left(\frac{1}{\sigma^2}y\theta\right)\exp\left(-\frac{1}{2\sigma^2}\theta^2\right)
\end{aligned}
$$

take $\sigma\to\infty$ $\quad \to \exp(0)\exp(-0) = 1$

Thus we conclude as $\sigma\to\infty$, the posterior tends to uniform dist.

So posterior mean estimator $\hat{\theta}_{pos} = \frac{1}{2}$.

(iii) let us compute MSE for both estimators.

$MSE(\hat{\theta}_{MLE}) = \mathbb{E}(\hat{\theta}_{MLE}-\theta)^2$ $\quad$ write $\hat{\theta}_1$ as $\hat{\theta}_{MLE}$

$$
\begin{aligned}
&= \int_0^1 (\hat{\theta}_1-\theta)^2\,d\theta \\
&= \int_0^1 \theta_1^2-2\hat{\theta}_1\theta+\theta^2\,d\theta. \\
&= \hat{\theta}_1^2\theta-\hat{\theta}_1\theta^2+\frac{1}{3}\theta^3\Big|_0^1 \\
&= \hat{\theta}_1^2-\hat{\theta}_1+\frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
MSE(\hat{\theta}_{pos}) &= \mathbb{E}(\hat{\theta}_{pos}-\theta)^2 \\
&= \mathbb{E}\left(\frac{1}{2}-\theta\right)^2 \\
&= \int_0^1\left(\frac{1}{2}-\theta\right)^2 d\theta \\
&= \int_0^1 \frac{1}{4}-\theta+\theta^2\,d\theta \\
&= \frac{1}{4}-\frac{1}{2}\theta^2+\frac{1}{3}\theta^3\Big|_0^1 \\
&= \frac{1}{4}-\frac{1}{2}+\frac{1}{3} = \frac{3-6+4}{12} = \frac{1}{12}
\end{aligned}
$$

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