## III. PPE IMPLEMENTATION

$$\Phi(f) = \begin{cases} \Phi_{\rm ins}(f) = \Phi_{\rm TF2}(f) + \Phi_{\rm Phenom}(f), & f \le f_1, \\ \Phi_{\rm int} = \alpha_0 + \alpha_1 f + \ldots, & f_1 \le f \le f_2, \\ \Phi_{\rm MR} = \beta_0 + \beta_1 f + \ldots, & f \ge f_2 \end{cases} \tag{18}$$

If $\Phi_{\rm ins}(f) \to \bar{\Phi}_{\rm ins}(f) = \Phi_{\rm ins}(f) + \Delta\Phi(f)$, where $\Delta\Phi(f) = \beta u^b$, then $\Phi_{\rm int/MR}(f) \to \bar{\Phi}_{\rm int/MR}(f)$ via $C^1$ continuity at $f = f_1$ and $f = f_2$. Suppose that

$$\bar{\Phi}_{\rm int}(f) = \Phi_{\rm int}(f) + \Delta\alpha_0 + \Delta\alpha_1 f, \tag{19}$$
$$\bar{\Phi}_{\rm MR}(f) = \Phi_{\rm MR}(f) + \Delta\beta_0 + \Delta\beta_1 f. \tag{20}$$

Enforcing $C^1$ at $f = f_1$,

$$\begin{aligned} \bar{\Phi}_{\rm ins}(f_1) &= \bar{\Phi}_{\rm int}(f_1), \\ \bar{\Phi}'_{\rm ins}(f_1) &= \bar{\Phi}'_{\rm int}(f_1) \\ \implies \cancel{\Phi_{\rm ins}(f_1)} + \beta(\pi\mathcal{M}f_1)^{b/3} &= \cancel{\Phi_{\rm int}(f_1)} + \Delta\alpha_0 + \Delta\alpha_1 f_1 \\ \cancel{\Phi'_{\rm ins}(f_1)} + \beta\frac{b}{3}(\pi\mathcal{M})^{b/3} f_1^{b/3-1} &= \cancel{\Phi'_{\rm int}(f_1)} + \Delta\alpha_1 \\ \implies \Delta\alpha_0 + \Delta\alpha_1 f_1 &= \beta(\pi\mathcal{M}f_1)^{b/3} \\ \Delta\alpha_1 f_1 &= \beta\frac{b}{3}(\pi\mathcal{M})^{b/3} f_1^{b/3-1} f_1 \\ \implies \Delta\alpha_0 &= \beta\left(1 - \frac{b}{3}\right)(\pi\mathcal{M}f_1)^{b/3} \\ \Delta\alpha_1 &= \beta\frac{b}{3}(\pi\mathcal{M}f_1)^{b/3}/f_1. \end{aligned}$$

Enforcing $C^1$ at $f = f_2$,

$$\begin{aligned} \bar{\Phi}_{\rm int}(f_2) &= \bar{\Phi}_{\rm MR}(f_2), \\ \bar{\Phi}'_{\rm int}(f_2) &= \bar{\Phi}'_{\rm MR}(f_2) \\ \implies \cancel{\Phi_{\rm int}(f_2)} + \Delta\alpha_0 + \Delta\alpha_1 f_2 &= \cancel{\Phi_{\rm MR}(f_2)} + \Delta\beta_0 + \Delta\beta_1 f_2 \\ \Phi'_{\rm int}(f_2) + \Delta\alpha_1 &= \Phi'_{\rm MR}(f_2) + \Delta\beta_1 \\ \implies \Delta\alpha_1 &= \Delta\beta_1 \\ \implies \Delta\alpha_0 + \cancel{\Delta\alpha_1 f_2} &= \Delta\beta_0 + \cancel{\Delta\beta_1 f_2}. \end{aligned}$$