

13 Godunov's Method

Recall that one-sided methods cannot be used for systems of equations with eigenvalues of mixed sign. For a linear system of equations we previously obtained a natural generalization of the upwind method by diagonalizing the system, yielding the method (10.60). For nonlinear systems the matrix of eigenvectors is not constant, and this same approach does not work directly. In this chapter we will study a generalization in which the local characteristic structure, now obtained by solving a Riemann problem rather than by diagonalizing the Jacobian matrix, is used to define a natural upwind method. This method was first proposed for gas dynamics calculations by Godunov[24].

Since solving a Riemann problem is a nontrivial task, we should first justify the need for upwind methods. An alternative is to use a centered method such as the Lax-Friedrichs method (12.15) which allows eigenvalues of either sign. In the scalar case, this turns out to be stable provided

$$\left| \frac{k}{h} f'(U_j^n) \right| \leq 1 \quad (13.1)$$

for all U_j^n . However, in Chapter 11 we have seen for the linear advection equation that Lax-Friedrichs is generally more dissipative than the upwind method, and gives less accurate solutions. This is not surprising since the theory of characteristics tells us that the solution at x_j depends only on the data to the left of x_j and the upwind method takes advantage of this knowledge. In fact, for the advection equation we know that

$$u(x_j, t_{n+1}) = u(x_j - ak, t_n) \quad (13.2)$$

and both upwind and Lax-Friedrichs can be viewed as interpolation formulas to approximate u at $x_j - ak$ from the values of u at the grid points. Upwind is better because it interpolates using values at the two nearest grid points x_{j-1} and x_j ,

$$U_j^{n+1} = \frac{1}{h} [(h - ak) U_j^n + ak U_{j-1}^n] \quad (13.3)$$

whereas Lax-Friedrichs interpolates using values at x_{j-1} and x_{j+1} ,

$$U_j^{n+1} = \frac{1}{2h} [(h - ak) U_{j+1}^n + (h + ak) U_{j-1}^n]. \quad (13.4)$$

By doing upwind differencing in the appropriate direction for each characteristic component in a system of equations we can hope to obtain a similar improvement and decrease the numerical dissipation.

Looking ahead, there are other reasons for introducing methods based on the solution of Riemann problems. Both Lax-Friedrichs and the upwind method are only first order accurate on smooth data, and even the less dissipative upwind method gives unacceptably smeared shock profiles. We ultimately want to correct these deficiencies by developing "high resolution" methods that are second order accurate in smooth regions and give much sharper discontinuities. However, our experience with the linear advection equation (e.g., in Figure 11.1) indicates that natural second order methods, even one-sided methods like Beam-Warming, give oscillatory solutions. We will be able to cure this only by using more information about the local behavior of the solution. The first order Godunov method introduced here forms a basis for many of the high resolution generalizations that will be studied later.

EXERCISE 13.1. Show that the Lax-Wendroff method on $u_t + au_x = 0$ can be derived by approximating $u(x_j - ak, t_n)$ using quadratic interpolation based on the points $U_{j-1}^n, U_j^n, U_{j+1}^n$. Use this interpretation to explain why oscillations appear near a discontinuity with Lax-Wendroff but not with Upwind or Lax-Friedrichs. Similarly, Beam-Warming corresponds to quadratic interpolation based on $U_{j-2}^n, U_{j-1}^n, U_j^n$.

13.1 The Courant-Isaacson-Rees method

Historically, one of the first attempts at upwinding for the equations of gas dynamics was made by Courant, Isaacson and Rees[14] in 1952. They proposed solving certain equations along the characteristics going back from the point (x_j, t_{n+1}) . To evaluate the characteristic variables at time t_n , this method uses interpolation based on the two nearest grid values, which are (U_{j-1}^n, U_j^n) or (U_j^n, U_{j+1}^n) depending on whether the corresponding characteristic speed is positive or negative. Of course the exact path of the characteristic is not known, but is approximated by a straight line with slope $\lambda_p(U_j^n)$. This is schematically illustrated in Figure 13.1.

For the scalar advection equation this reduces to the upwind method. For a scalar nonlinear problem, in which u is constant along characteristics, it reduces to determining U_j^{n+1} by an approximation to U at the point $x_j - f'(U_j^n)k$ obtained by linear interpolation of the data U^n . For example, if $f'(U_j^n) > 0$ then we would interpolate between U_{j-1}^n and U_j^n :

$$\begin{aligned} U_j^{n+1} &= \frac{1}{h} [(h - f'(U_j^n)k)U_j^n + f'(U_j^n)kU_{j-1}^n] \\ &= U_j^n - \frac{k}{h} f'(U_j^n) [U_j^n - U_{j-1}^n]. \end{aligned} \quad (13.5)$$

genvalues
1 general-
1 (10.60).
approach
the local
1 by diag-
s method

the need
Friedrichs
turns out

(13.1)

that Lax-
accurate
solution
dvantage

(13.2)

approx-
because it

(13.3)

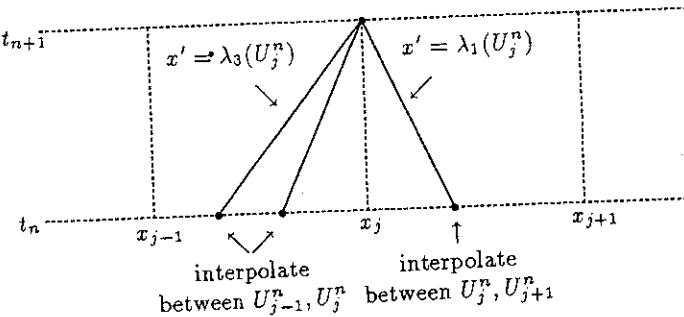


Figure 13.1. In the CIR method, characteristics are traced back from (x_j, t_{n+1}) and linear interpolation used at time t_n .

This is a natural upwind approximation to $u_t + f'(u)u_x = 0$, but this is not a good method for a problem involving shocks, since it is not in conservation form. (Recall our results with the method (12.4), which is precisely the CIR method (13.5) on Burgers' equation.)

13.2 Godunov's method

In 1959, Godunov[24] proposed a way to make use of the characteristic information within the framework of a conservative method. Rather than attempting to follow characteristics backwards in time, Godunov suggested solving Riemann problems forward in time. Solutions to Riemann problems are relatively easy to compute, give substantial information about the characteristic structure, and lead to conservative methods since they are themselves exact solutions of the conservation laws and hence conservative.

In Godunov's method, we use the numerical solution U^n to define a piecewise constant function $\tilde{u}^n(x, t_n)$ with the value U_j^n on the grid cell $x_{j-1/2} < x < x_{j+1/2}$. At time t_n this agrees with the piecewise constant function $U_k(x, t_n)$ that has already been introduced, but the function \tilde{u}^n , unlike U_k , will not be constant over $t_n \leq t < t_{n+1}$. Instead, we use $\tilde{u}^n(x, t_n)$ as initial data for the conservation law, which we now solve exactly to obtain $\tilde{u}^n(x, t)$ for $t_n \leq t \leq t_{n+1}$. The equation can be solved exactly over a short time interval because the initial data $\tilde{u}^n(x, t_n)$ is piecewise constant, and hence defines a sequence of Riemann problems. The exact solution, up to the time when waves from neighboring

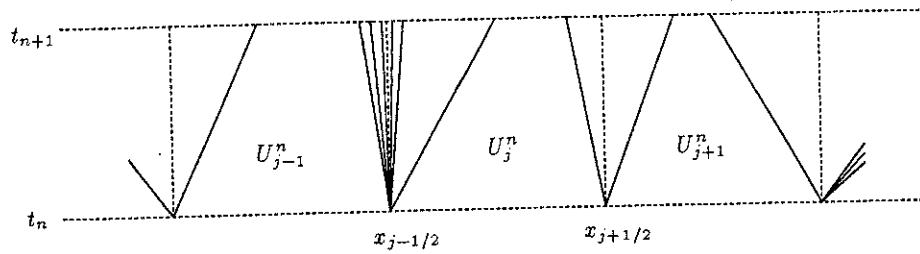


Figure 13.2. Solution of independent Riemann problems for the piecewise constant data $\tilde{u}^n(x, t_n)$ in the case $m = 2$.

Riemann problems begin to interact, is obtained by simply piecing together these Riemann solutions. This is illustrated in Figure 13.2.

After obtaining this solution over the interval $[t_n, t_{n+1}]$, we define the approximate solution U^{n+1} at time t_{n+1} by averaging this exact solution at time t_{n+1} ,

$$U_j^{n+1} = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x, t_{n+1}) dx. \quad (13.6)$$

These values are then used to define new piecewise constant data $\hat{u}^{n+1}(x, t_{n+1})$ and the process repeats.

In practice this algorithm is considerably simplified by observing that the cell average (13.6) can be easily computed using the integral form of the conservation law. Since \tilde{u}^n is assumed to be an exact weak solution, we know that

$$\begin{aligned} \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x, t_{n+1}) dx &= \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x, t_n) dx + \int_{t_n}^{t_{n+1}} f(\tilde{u}^n(x_{j-1/2}, t)) dt \\ &\quad - \int_{t_n}^{t_{n+1}} f(\tilde{u}^n(x_{j+1/2}, t)) dt. \end{aligned}$$

Dividing by h , using (13.6), and noting that $\tilde{u}^n(x, t_n) \equiv U_j^n$ over the cell $(x_{j-1/2}, x_{j+1/2})$, this equation reduces to

$$U_j^{n+1} = U_j^n - \frac{k}{h} [F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)] \quad (13.7)$$

where the numerical flux function F is given by

$$F(U_j^n, U_{j+1}^n) = \frac{1}{k} \int_{t_n}^{t_{n+1}} f(\tilde{u}^n(x_{j+1/2}, t)) dt. \quad (13.8)$$

This shows that Godunov's method can be written in conservation form. Moreover, note that the integral we need to compute in (13.8) is trivial because \tilde{u}^n is constant at the point $x_{j+1/2}$ over the time interval (t_n, t_{n+1}) . This follows from the fact that the solution of the Riemann problem at $x_{j+1/2}$ is a similarity solution, constant along each ray $(x - x_{j+1/2})/t = \text{constant}$.

The constant value of \tilde{u}^n along the line $x = x_{j+1/2}$ depends only on the data U_j^n and U_{j+1}^n for this Riemann problem. If we denote this value by $u^*(U_j^n, U_{j+1}^n)$, then the flux (13.8) reduces to

$$F(U_j^n, U_{j+1}^n) = f(u^*(U_j^n, U_{j+1}^n)) \quad (13.9)$$

and Godunov's method becomes

$$U_j^{n+1} = U_j^n - \frac{k}{h} [f(u^*(U_j^n, U_{j+1}^n)) - f(u^*(U_{j-1}^n, U_j^n))]. \quad (13.10)$$

Note that the flux (13.9) is consistent with f since if $U_j^n = U_{j+1}^n \equiv \bar{u}$ then $u^*(U_j^n, U_{j+1}^n) = \bar{u}$ as well. Lipschitz continuity follows from smoothness of f .

For large t , of course, the solution may not remain constant at $x_{j+1/2}$ because of the effect of waves arising from neighboring Riemann problems. However, since the wave speeds are bounded by the eigenvalues of $f'(u)$ and the neighboring Riemann problems are distance h away, $\tilde{u}^n(x_{j+1/2}, t)$ will be constant over $[t_n, t_{n+1}]$ provided k is sufficiently small. We require that

$$\left| \frac{k}{h} \lambda_p(U_j^n) \right| \leq 1 \quad (13.11)$$

for all eigenvalues λ_p at each U_j^n . The maximum of this quantity over the values of u arising in a particular problem is called the **Courant number**, a natural generalization of (10.56) from the linear problem.

Note that (13.11) allows the interaction of waves from neighboring Riemann problems during the time step, provided the interaction is entirely contained within a mesh cell. See Figure 13.3 for an example. In this case the solution $\tilde{u}^n(x, t)$ would be difficult or impossible to calculate. However, we never explicitly calculate the full solution, since all we require is the cell average (13.6). This is still easy to compute because \tilde{u}^n remains constant on each cell boundary.

13.3 Linear systems

For a constant coefficient linear system $u_t + Au_x = 0$, solving the Riemann problem with left and right states U_j^n and U_{j+1}^n gives an intermediate value $u^*(U_j^n, U_{j+1}^n)$ that can be

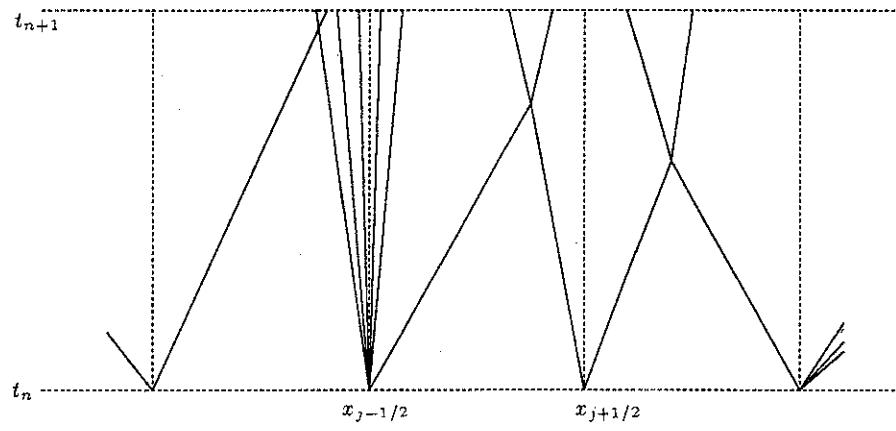


Figure 13.3. Exact solution \tilde{u}^n used in determining U_j^{n+1} by Godunov's method. Note that \tilde{u}^n is constant along the line segments $t_n < t < t_{n+1}$ at $x = x_{j-1/2}$ and $x = x_{j+1/2}$.

written in two different ways, as

$$u^*(U_j^n, U_{j+1}^n) = U_j^n + \sum_{\lambda_p < 0} \alpha_p r_p = U_{j+1}^n - \sum_{\lambda_p > 0} \alpha_p r_p. \quad (13.12)$$

Here r_p is the p th eigenvector of A and α_p is the coefficient of r_p in an eigenvector expansion of $U_{j+1}^n - U_j^n$, so the vector of α_p 's is given by $\alpha = R^{-1}(U_{j+1}^n - U_j^n)$. Using the notation of (10.61), we can write the corresponding flux for Godunov's method as

$$\begin{aligned} F(U_j^n, U_{j+1}^n) &= Au^*(U_j^n, U_{j+1}^n) \\ &= AU_j^n + \sum_{\lambda_p < 0} \alpha_p \lambda_p r_p = AU_{j+1}^n - \sum_{\lambda_p > 0} \alpha_p \lambda_p r_p \\ &= AU_j^n + A^-(U_{j+1}^n - U_j^n) = AU_{j+1}^n - A^+(U_{j+1}^n - U_j^n). \end{aligned} \quad (13.13)$$

If we choose the first of these equivalent expressions for $F(U_j^n, U_{j+1}^n)$ and take the second to define $F(U_{j-1}^n, U_j^n)$, i.e.,

$$F(U_{j-1}^n, U_j^n) = AU_j^n - A^+(U_j^n - U_{j-1}^n), \quad (13.14)$$

then Godunov's method for the linear system takes the form

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{k}{h} [F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)] \\ &= U_j^n - \frac{k}{h} [A^-(U_{j+1}^n - U_j^n) + A^+(U_j^n - U_{j-1}^n)]. \end{aligned}$$

This is simply the upwind method (10.60) for the linear system.

Notice that yet another expression for the flux $F(U_j^n, U_{j+1}^n)$ can be obtained by averaging the two expressions given in (13.13), obtaining

$$\begin{aligned} F(U_j^n, U_{j+1}^n) &= \frac{1}{2}A(U_j^n + U_{j+1}^n) + \frac{1}{2}(A^- - A^+)(U_{j+1}^n - U_j^n) \\ &= \frac{1}{2}A(U_j^n + U_{j+1}^n) - \frac{1}{2}|A|(U_{j+1}^n - U_j^n) \end{aligned} \quad (13.15)$$

where

$$|A| = A^+ - A^- = R|\Lambda|R^{-1} \quad \text{with } |\Lambda| = \text{diag}(|\lambda_1|, \dots, |\lambda_m|). \quad (13.16)$$

If we use this form in Godunov's method we obtain an alternative formulation of the linear upwind method,

$$U_j^{n+1} = U_j^n - \frac{k}{2h}A(U_{j+1}^n - U_{j-1}^n) + \frac{k}{2h}|A|(U_{j+1}^n - 2U_j^n + U_{j-1}^n). \quad (13.17)$$

The first two terms here correspond to the centered method (10.7) which is unconditionally unstable, but the last term is a dissipative term that stabilizes the method. Note that $(U_{j+1}^n - 2U_j^n + U_{j-1}^n) \approx h^2 u_{xx}(x_j, t_n)$ and that the diffusion matrix $|A|$ has nonnegative eigenvalues. This formulation of the upwind method for linear systems will prove useful later.

13.4 The entropy condition

The function $\tilde{u}^n(x, t)$ for $t_n \leq t \leq t_{n+1}$ is assumed to be a weak solution of the conservation law. In situations where this weak solution is not unique, there may be several choices for \tilde{u}^n . Different choices may give different values of $u^*(U_j^n, U_{j+1}^n)$ and hence different numerical solutions. The method is conservative and consistent regardless of what choice we make, but in cases where there is a unique weak solution that satisfies some physically motivated entropy condition, it makes sense to use the entropy-satisfying weak solution for \tilde{u}^n in each time step. We might hope that by doing this the numerical solution will satisfy a discrete version of the entropy condition. This is in fact true, as we verify below. It then follows from the theory of Chapter 12 that any limiting function obtained by refining the grid must then be an entropy-satisfying weak solution. (If, on the other hand, we use Riemann solutions that do not satisfy the entropy condition in defining $u^*(U_j^n, U_{j+1}^n)$, then our numerical solution may converge to a weak solution that does not satisfy the entropy condition.)

Suppose that we have a convex entropy function $\eta(u)$ and entropy flux $\psi(u)$, and that every \tilde{u}^n satisfies the entropy inequality (12.53). Then we wish to derive a discrete entropy

inequality of the form (12.55) for Godunov's method. Since $\tilde{u}^n(x, t)$ for $t_n \leq t \leq t_{n+1}$ represents the *exact* entropy satisfying solution, we can integrate (12.53) over the rectangle $(x_{j-1/2}, x_{j+1/2}) \times (t_n, t_{n+1})$ to obtain

$$(13.15) \quad \begin{aligned} \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \eta(\tilde{u}^n(x, t_{n+1})) dx &\leq \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \eta(\tilde{u}^n(x, t_n)) dx \\ &- \frac{1}{h} \left[\int_{t_n}^{t_{n+1}} \psi(\tilde{u}^n(x_{j+1/2}, t)) dt - \int_{t_n}^{t_{n+1}} \psi(\tilde{u}^n(x_{j-1/2}, t)) dt \right]. \end{aligned}$$

This is almost what we need. Since \tilde{u}^n is constant along three of the four sides of this rectangle, all integrals on the right hand side can be evaluated to give

$$(13.16) \quad \begin{aligned} \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \eta(\tilde{u}^n(x, t_{n+1})) dx &\leq \eta(U_j^n) \\ &- \frac{k}{h} [\psi(u^*(U_j^n, U_{j+1}^n)) - \psi(u^*(U_{j-1}^n, U_j^n))]. \end{aligned}$$

Again u^* represents the value propagating with velocity 0 in the solution of the Riemann problem. If we define the numerical entropy flux by

$$(13.17) \quad \Psi(U_j^n, U_{j+1}^n) = \psi(u^*(U_j^n, U_{j+1}^n)),$$

then Ψ is consistent with ψ , and the right hand side of (13.18) agrees with (12.55).

The left hand side of (13.19) is not equal to $\eta(U_j^{n+1})$, because \tilde{u}^n is not constant in this interval. However, since the entropy function η is convex, we can use *Jensen's inequality* which says that the value of η evaluated at the average value of \tilde{u}^n is less than or equal to the average value of $\eta(\tilde{u}^n)$, i.e.,

$$(13.20) \quad \eta\left(\frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x, t_{n+1}) dx\right) \leq \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \eta(\tilde{u}^n(x, t_{n+1})) dx.$$

The left hand side here is simply $\eta(U_j^{n+1})$ while the right hand side is bounded by (13.18). Combining (13.18), (13.19) and (13.20) thus gives the desired entropy inequality

$$\eta(U_j^{n+1}) \leq \eta(U_j^n) - \frac{k}{h} [\Psi(U_j^n, U_{j+1}^n) - \Psi(U_{j-1}^n, U_j^n)].$$

This shows that weak solutions obtained by Godunov's method satisfy the entropy condition, provided we use entropy-satisfying Riemann solutions.

13.5 Scalar conservation laws

If we apply Godunov's method to a scalar nonlinear equation, we obtain generalizations of the upwind methods defined by (12.13) and (12.14) for the case where the sign of $f'(u)$

varies. Recall that the Riemann problem with data u_l , u_r always has one weak solution consisting simply of the discontinuity propagating with speed $s = (f(u_r) - f(u_l))/(u_r - u_l)$. If we always use this Riemann solution (which may not satisfy the entropy condition), then the intermediate value u^* in the Riemann solution is given by

$$u^*(u_l, u_r) = \begin{cases} u_l & \text{if } s > 0, \\ u_r & \text{if } s < 0. \end{cases} \quad (13.21)$$

If $s = 0$ then this value is not well defined, but note that in this case $f(u_l) = f(u_r)$ and so the resulting flux, which is all that is required in Godunov's method, is the same whether we define $u^* = u_l$ or $u^* = u_r$.

From (13.21) we thus find the flux function

$$\begin{aligned} F(u_l, u_r) &= f(u^*(u_l, u_r)) \\ &= \begin{cases} f(u_l) & \text{if } (f(u_r) - f(u_l))/(u_r - u_l) \geq 0, \\ f(u_r) & \text{if } (f(u_r) - f(u_l))/(u_r - u_l) < 0 \end{cases} \end{aligned} \quad (13.22)$$

Notice that this is precisely the flux (12.51) which, as we have already seen, gives a method that may compute entropy-violating solutions.

To fix this, we should use the entropy-satisfying weak solution in implementing Godunov's method. This may consist of a rarefaction wave rather than a shock (or of some combination of shocks and rarefactions in the nonconvex case).

In the convex case, there are four cases that must be considered. The reader should verify the following results in each case:

1. $f'(u_l), f'(u_r) \geq 0 \implies u^* = u_l$,
2. $f'(u_l), f'(u_r) \leq 0 \implies u^* = u_r$,
3. $f'(u_l) \geq 0 \geq f'(u_r) \implies u^* = u_l$ if $[f]/[u] > 0$ or $u^* = u_r$ if $[f]/[u] < 0$,
4. $f'(u_l) < 0 < f'(u_r) \implies u^* = u_s$ (transonic rarefaction).

In each of the first three cases, the value u^* is either u_l or u_r , and the flux is correctly given by (13.22). Note in particular that in Cases 1 and 2 it is irrelevant whether the solution is a shock or rarefaction, since the value of u^* is the same in either case. This shows that using Godunov's method with entropy-violating Riemann solutions does not necessarily lead to entropy-violating numerical solutions!

It is only in Case 4, the transonic rarefaction, that the value of F differs from (13.22). In this case u^* is neither u_l nor u_r , but is the intermediate value u_s with the property that

$$f'(u_s) = 0. \quad (13.23)$$

This is the value of u for which the characteristic speed is zero, and is called the sonic point.

We can modify (13.22) to include this possibility: if $f'(u_l) < 0 < f'(u_r)$ then we redefine

$$F(u_l, u_r) = f(u_s).$$

The resulting flux function can be written in a simplified form, however, as

$$F(u_l, u_r) = \begin{cases} \min_{u_l \leq u \leq u_r} f(u) & \text{if } u_l \leq u_r \\ \max_{u_r \leq u \leq u_l} f(u) & \text{if } u_l > u_r. \end{cases} \quad (13.24)$$

EXERCISE 13.2. Verify this by considering all possible cases.

More surprisingly, it turns out that (13.24) is valid more generally for any scalar conservation law, even nonconvex ones, and gives the correct Godunov flux corresponding to the weak solution satisfying Olienik's entropy condition (3.46).

EXERCISE 13.3. Verify this claim, using the convex hull construction of the entropy-satisfying Riemann solution presented in Chapter 4.

This also follows from a more general result due to Osher[58], who found a closed form expression for the entropy solution $u(x, t) \equiv w(x/t)$ of a general nonconvex scalar Riemann problem with data u_l and u_r . The solution $w(x/t) = w(\xi)$ satisfies the implicit relation

$$f(w(\xi)) - \xi w(\xi) = g(\xi) \equiv \begin{cases} \min_{u_l \leq u \leq u_r} [f(u) - \xi u] & \text{if } u_l \leq u_r \\ \max_{u_r \leq u \leq u_l} [f(u) - \xi u] & \text{if } u_l > u_r. \end{cases} \quad (13.25)$$

Setting $\xi = 0$ gives $f(w(0)) = f(u^*(u_l, u_r))$ which is precisely the numerical flux (13.24). Osher goes on to show that an explicit expression for $w(\xi)$ is obtained by differentiating the above expression, yielding

$$w(\xi) = -g'(\xi) \quad (13.26)$$

where $g(\xi)$ is the function defined by the right side of (13.25).

14 Approximate Riemann Solvers

Godunov's method, and higher order variations of the method to be discussed later, require the solution of Riemann problems at every cell boundary in each time step. Although in theory these Riemann problems can be solved, in practice doing so is expensive, and typically requires some iteration for nonlinear equations.

Note that most of the structure of the resulting Riemann solver is not used in Godunov's method. The exact solution is averaged over each grid cell, introducing large numerical errors. This suggests that it is not worthwhile calculating the Riemann solutions exactly and that we may be able to obtain equally good numerical results with an approximate Riemann solution obtained by some less expensive means.

There are two distinct ways we could think of generalizing Godunov's method. One is to start with the Godunov flux

$$F(u_l, u_r) = f(u^*(u_l, u_r)) \quad (14.1)$$

where $u^*(u_l, u_r)$ is the intermediate state $w(0)$ arising in the similarity solution $u(x, t) = w(x/t)$ of the Riemann problem, and replace the function $u^*(u_l, u_r)$ by some approximation $\hat{u}^*(u_l, u_r)$. This leads to the approximate Godunov method

$$U_j^{n+1} = U_j^n - \frac{k}{h} [f(\hat{u}^*(U_j^n, U_{j+1}^n)) - f(\hat{u}^*(U_{j-1}^n, U_j^n))]. \quad (14.2)$$

Note that this method will be conservative and consistent for any choice of the function \hat{u}^* , provided it satisfies the natural condition that $\hat{u}^*(\bar{u}, \bar{u}) = \bar{u}$ with appropriate Lipschitz continuity.

We might define the function \hat{u}^* by first defining an approximate Riemann solution $\hat{u}(x, t) = \hat{w}(x/t)$ (see below) and then setting $\hat{u}^*(u_l, u_r) = \hat{w}(0)$.

A second approach to generalizing Godunov's method is to go back to the original description of the method, where U_j^{n+1} is taken to be the cell average of $\tilde{u}^n(x, t_{n+1})$, and now replace $\tilde{u}^n(x, t)$ by an approximate solution $\hat{u}^n(x, t)$. This can be defined by simply piecing together approximate Riemann solutions at each cell interface, just as \tilde{u}^n was defined for the exact Riemann solutions. This requires that we define an approximate

Riemann solution $\hat{w}(x/t)$ with finite propagation speeds, so that

$$\hat{w}(\xi) = \begin{cases} u_l & \text{for } \xi < a_{\min} \\ u_r & \text{for } \xi > a_{\max} \end{cases} \quad (14.3)$$

where a_{\min} and a_{\max} are the minimum and maximum propagation speeds (which typically correspond closely to the range of eigenvalues of $f'(u_l)$ and $f'(u_r)$). We need

$$\left| \frac{ak}{h} \right| < \frac{1}{2} \quad \text{for all } a \text{ between } a_{\min} \text{ and } a_{\max} \quad (14.4)$$

in order for the construction of $\hat{u}^n(x, t)$ described above to work, but once we determine the corresponding flux function this can be relaxed to require only $|ak/h| < 1$ as in Godunov's method with the exact Riemann solution.

Once the approximate solution \hat{u}^n is defined in the strip $t_n \leq t \leq t_{n+1}$, we can take U_j^{n+1} to be the cell average at time t_{n+1} :

$$U_j^{n+1} = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \hat{u}^n(x, t_{n+1}) dx. \quad (14.5)$$

Note that this will not in general be the same method as (14.2) with $\hat{u}^* = \hat{w}(0)$. If \hat{u}^n is not an exact solution, we can no longer integrate the conservation laws over the rectangle $(x_{j-1/2}, x_{j+1/2}) \times (t_n, t_{n+1})$ to obtain (14.2) from (14.5).

The second approach is the one more commonly used, but requires some care in defining the approximate Riemann solution, since for arbitrary choices of \hat{w} there is no guarantee that (14.5) is even conservative. Harten, Lax and van Leer[33] present some of the general theory of such approximate Riemann solvers.

14.1 General theory

To be conservative, the approximate Riemann solution $\hat{u}(x, t) = \hat{w}(x/t)$ must have the following property that for M sufficiently large,

$$\int_{-M}^M \hat{w}(\xi) d\xi = M(u_l + u_r) + f(u_l) - f(u_r). \quad (14.6)$$

Note that the exact Riemann solution $w(x/t)$ has this property, as seen from the integral form of the conservation law over $[-M, M] \times [0, 1]$.

If \hat{w} satisfies (14.6) then we can write the resulting method in the standard conservation form. In order to determine the numerical flux $F(u_l, u_r)$, consider Figure 14.1. Since by consistency the flux should reduce to $f(u)$ wherever u is constant, we can integrate \hat{w} from 0 to M , for M sufficiently large, and expect to get

$$\int_0^M \hat{w}(\xi) d\xi = Mu_r + F(u_l, u_r) - f(u_r). \quad (14.7)$$

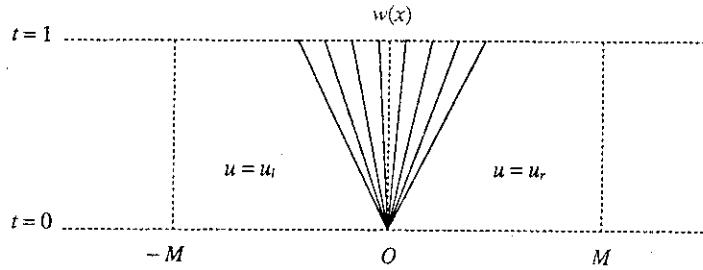


Figure 14.1. Integrating the approximate Riemann solution \hat{w} gives the numerical flux.

Similarly, integrating from $-M$ to 0 gives

$$\int_{-M}^0 \hat{w}(\xi) d\xi = Mu_l + f(u_l) - F(u_l, u_r). \quad (14.8)$$

The condition (14.6) insures that these two requirements on F can be satisfied simultaneously, and solving for F gives two different expressions,

$$F(u_l, u_r) = f(u_r) - Mu_r + \int_0^M \hat{w}(\xi) d\xi, \quad (14.9)$$

or, equivalently,

$$F(u_l, u_r) = f(u_l) + Mu_l - \int_{-M}^0 \hat{w}(\xi) d\xi. \quad (14.10)$$

14.1.1 The entropy condition

Harten and Lax[32] (see also [33]) point out that the Godunov-type method (14.5) is also consistent with the entropy condition provided the approximate Riemann solution satisfies another condition analogous to (14.6):

$$\int_{-M}^M \eta(\hat{w}(\xi)) d\xi \leq M(\eta(u_l) + \eta(u_r)) + (\psi(u_l) - \psi(u_r)). \quad (14.11)$$

From this it can be shown that the discrete entropy condition (12.55) is satisfied with the numerical entropy flux

$$\Psi(u_l, u_r) = \psi(u_l) + M\eta(u_l) - \int_{-M}^0 \eta(\hat{w}(\xi)) d\xi. \quad (14.12)$$

Harten, Lax and van Leer present some approximate Riemann solvers that satisfy this condition.

14.1.2 Modified conservation laws

One natural way to obtain $\hat{w}(x/t)$ is to compute the *exact* Riemann solution to some modified conservation law $\hat{u}_t + \hat{f}(\hat{u})_x = 0$, with a flux function $\hat{f}(u)$ that is presumably easier to work with than the original flux $f(u)$. By using the integral form of this conservation law over $[-M, M] \times [0, 1]$, we see that the condition (14.6) will be satisfied provided that

$$\hat{f}(u_r) - \hat{f}(u_l) = f(u_r) - f(u_l). \quad (14.13)$$

The resulting numerical flux function is then given by

$$F(u_l, u_r) = \hat{f}(\hat{w}(0)) + f(u_r) - \hat{f}(u_r). \quad (14.14)$$

EXERCISE 14.1. Verify (14.14).

14.2 Roe's approximate Riemann solver

One of the most popular Riemann solvers currently in use is due to Roe[64]. The idea is to determine $\hat{u}(x, t)$ by solving a constant coefficient linear system of conservation laws instead of the original nonlinear system, i.e., we solve a modified conservation law as described above with flux $\hat{f}(u) = \hat{A}u$. Of course the coefficient matrix used to define this linear system must depend on the u_l and u_r in order for (14.13) to be satisfied, so we will write the linear system for \hat{u} as

$$\hat{u}_t + \hat{A}(u_l, u_r)\hat{u}_x = 0. \quad (14.15)$$

This linear Riemann problem is relatively easy to solve (see Section 6.5). If \hat{A} has eigenvalues $\hat{\lambda}_i$ and eigenvectors \hat{r}_i , and if we decompose

$$u_r - u_l = \sum_p \alpha_p \hat{r}_p, \quad (14.16)$$

then

$$\hat{w}(\xi) = u_l + \sum_{\hat{\lambda}_p < \xi} \alpha_p \hat{r}_p \quad (14.17)$$

where the sum is over all p for which $\hat{\lambda}_p < \xi$. Equivalently,

$$\hat{w}(\xi) = u_r - \sum_{\hat{\lambda}_p > \xi} \alpha_p \hat{r}_p \quad (14.18)$$

We still have the problem of determining $\hat{A}(u_l, u_r)$ in a reasonable way. Roe suggested that the following conditions should be imposed on \hat{A} :

- i) $\hat{A}(u_l, u_r)(u_r - u_l) = f(u_r) - f(u_l)$
- ii) $\hat{A}(u_l, u_r)$ is diagonalizable with real eigenvalues
- iii) $\hat{A}(u_l, u_r) \rightarrow f'(\bar{u})$ smoothly as $u_l, u_r \rightarrow \bar{u}$.

Condition (14.19*i*) has two effects. First, it is required by (14.13) and guarantees that the condition (14.6) is satisfied. Another effect is that, in the special case where u_l and u_r are connected by a single shock wave or contact discontinuity, the approximate Riemann solution agrees with the exact Riemann solution. This follows from the fact that the Rankine-Hugoniot condition is satisfied for u_l and u_r in this case, so

$$f(u_r) - f(u_l) = s(u_r - u_l)$$

for some s (the speed of the shock or contact). Combined with (14.19*i*), this shows that $u_r - u_l$ must, in this situation, be an eigenvector of \hat{A} with eigenvalue s and so the approximate solution $\hat{u}(x, t)$ also consists of this single jump $u_r - u_l$ propagating with speed s .

Condition (14.19*ii*) is clearly required in order that the problem $\hat{u}_t + \hat{A}\hat{u}_x = 0$ is hyperbolic and solvable.

Condition (14.19*iii*) guarantees that the method behaves reasonably on smooth solutions, since if $\|U_j - U_{j-1}\| = O(h)$ then the linearized equation $u_t + f'(U_j)u_x = 0$ is approximately valid. It is natural to require that the linear system (14.15) agree with the linearization in this case. Since (14.19*i*) guarantees that the method behaves reasonably on an isolated discontinuity, it is only when a Riemann problem has a solution with more than one strong shock or contact discontinuity that the approximate Riemann solution will differ significantly from the true Riemann solution. In practice this happens infrequently — near the point where two shocks collide, for example.

One way to guarantee that both conditions (14.19*ii*) and (14.19*iii*) are satisfied is to take

$$\hat{A}(u_l, u_r) = f'(u_{\text{ave}}) \quad (14.20)$$

for some average value of u , e.g., $u_{\text{ave}} = \frac{1}{2}(u_l + u_r)$. Unfortunately, this simple choice of u_{ave} will not give an \hat{A} that satisfies (14.19*i*) in general. Harten and Lax[32] show (see also Theorem 2.1 in [33]) that for a general system with an entropy function, a more complicated averaging of the Jacobian matrix in state space can be used. This shows that such linearizations exist, but is too complicated to use in practice.

Fortunately, for special systems of equations it is possible to derive suitable \hat{A} matrices that are very efficient to use relative to the exact Riemann solution. Roe[64] showed how to do this for the Euler equations. Later in this chapter we will work through the analogous derivation for the isothermal equations.

14.2.1 The numerical flux function for Roe's solver

For Roe's approximate Riemann solver, the function $\hat{w}(x/t)$ is in fact the exact solution to a Riemann problem for the conservation law (14.15) with flux $\hat{f}(u) = \hat{A}u$. It follows

and guarantees that case where u_l and u_r approximate Riemann in the fact that the

(19i), this shows that value s and so the u_l propagating with

$\hat{u}_t + \hat{A}\hat{u}_x = 0$ is

ably on smooth so $t + f'(U_j)u_x = 0$ is (14.15) agree with method behaves re- blem has a solution proximate Riemann actice this happens

) are satisfied is to

$$(14.20)$$

is simple choice of Lax[32] show (see function, a more d. This shows that

uitable \hat{A} matrices oe[64] showed how hugh the analogous

from (14.14) that

$$\begin{aligned} F(u_l, u_r) &= \hat{A}\hat{w}(0) + f(u_r) - \hat{A}u_r \\ &= f(u_r) - \hat{A} \sum_{\lambda_p > 0} \alpha_p \hat{r}_p \\ &= f(u_r) - \sum_{p=1}^m \hat{\lambda}_p^+ \alpha_p \hat{r}_p \end{aligned} \quad (14.21)$$

where $\hat{\lambda}_p^+ = \max(\hat{\lambda}_p, 0)$. Alternatively, using (14.13) we obtain an equivalent expression,

$$F(u_l, u_r) = f(u_l) + \sum_{p=1}^m \hat{\lambda}_p^- \alpha_p \hat{r}_p \quad (14.22)$$

where $\hat{\lambda}_p^- = \min(\hat{\lambda}_p, 0)$.

Note that if we average the two expressions (14.21) and (14.22), we obtain a third form,

$$\begin{aligned} F(u_l, u_r) &= \frac{1}{2}(f(u_l) + f(u_r)) - \sum_p |\hat{\lambda}_p| \alpha_p \hat{r}_p, \\ &= \frac{1}{2}(f(u_l) + f(u_r)) - |\hat{A}|(u_r - u_l), \end{aligned} \quad (14.23)$$

where the absolute value of the matrix is defined as in (13.16). This form is reminiscent of the Godunov flux (13.15) for a linear system.

EXAMPLE 14.1. For a scalar conservation law the condition (14.19i) determines $\hat{a} = \hat{A}(u_l, u_r)$ uniquely as

$$\hat{a} = \frac{f(u_r) - f(u_l)}{u_r - u_l}. \quad (14.24)$$

The linearized problem is the scalar advection equation $\hat{u}_t + \hat{a}\hat{u}_x = 0$ and the approximate Riemann solution consists of the jump $u_r - u_l$ propagating with speed \hat{a} . But note that \hat{a} in (14.24) is the Rankine-Hugoniot shock speed for the nonlinear problem, and so the "approximate" Riemann solution is in fact an exact weak solution (though one which may violate the entropy condition). Roe's method in the scalar case thus reduces to the method already discussed with flux (13.22), which we can rewrite using (14.22), for example, as

$$F(u_l, u_r) = f(u_l) + \hat{a}^-(u_r - u_l). \quad (14.25)$$

14.2.2 A sonic entropy fix

One disadvantage of Roe's linearization is that the resulting approximate Riemann solution consists of only discontinuities, with no rarefaction waves. This can lead to a violation

er

the exact solution $= \hat{A}u$. It follows

of the entropy condition, as has been observed previously for the scalar method (13.22) with appropriate initial data.

Recall that in the scalar case, the use of an entropy-violating Riemann solution leads to difficulties only in the case of a sonic rarefaction wave, in which $f'(u_l) < 0 < f'(u_r)$. This is also typically true when we use Roe's approximate Riemann solution for a system of conservation laws. It is only for sonic rarefactions, those for which $\lambda_p < 0$ to the left of the wave while $\lambda_p > 0$ to the right of the wave, that entropy violation is a problem.

In the case of a sonic rarefaction wave, it is necessary to modify the approximate Riemann solver in order to obtain entropy satisfying solutions. There are various ways to do this. One approach, discussed by Harten and Hyman[30], is outlined here.

For the wave in the p th family, traveling at speed $\hat{\lambda}_p$ according to the approximate Riemann solver, compute the states to either side of this wave in \hat{u} :

$$u_{pl} = u_l + \sum_{i=1}^{p-1} \alpha_i \hat{r}_i, \quad u_{pr} = u_{pl} + \alpha_p \hat{r}_p. \quad (14.26)$$

Also compute the true characteristic speed in the p th family for each of these states, say $\lambda_{pl} = \lambda_p(u_{pl})$ and $\lambda_{pr} = \lambda_p(u_{pr})$. If $\lambda_{pl} > \lambda_{pr}$ then the characteristics are going into this discontinuity. If $\lambda_{pl} < \lambda_{pr}$ then it should perhaps be a rarefaction wave, but it is only in the transonic case where $\lambda_{pl} < 0 < \lambda_{pr}$ that we need to modify our approximate Riemann solution.

Suppose that there appears to be a sonic rarefaction in the q th family for some q , i.e., $\lambda_{ql} < 0 < \lambda_{qr}$. Then we replace the single jump \hat{r}_q propagating at speed $\hat{\lambda}_q$ by two jumps propagating at speeds λ_{ql} and λ_{qr} , with a new state u_{qm} in between. Conservation requires that

$$(\lambda_{qr} - \lambda_{ql})u_{qm} = (\hat{\lambda}_q - \lambda_{ql})u_{ql} + (\lambda_{qr} - \hat{\lambda}_q)u_{qr}$$

so that the integral of \hat{u} is unaffected by this modification. This determines u_{qm} :

$$u_{qm} = \frac{(\hat{\lambda}_q - \lambda_{ql})u_{ql} + (\lambda_{qr} - \hat{\lambda}_q)u_{qr}}{\lambda_{qr} - \lambda_{ql}}. \quad (14.27)$$

From this we can easily compute that $u_{qm} - u_{ql}$ and $u_{qr} - u_{qm}$ are both scalar multiples of the eigenvector \hat{r}_q ,

$$u_{qm} - u_{ql} = \left(\frac{\lambda_{qr} - \hat{\lambda}_q}{\lambda_{qr} - \lambda_{ql}} \right) \alpha_q \hat{r}_q, \quad (14.28)$$

$$u_{qr} - u_{qm} = \left(\frac{\hat{\lambda}_q - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}} \right) \alpha_q \hat{r}_q, \quad (14.29)$$

and of course the sum of these recovers $u_{qr} - u_{ql} = \alpha_q \hat{r}_q$. We can easily compute the resulting flux function $F(u_l, u_r)$ when the entropy fix is applied to the q th family. If we

define

$$\hat{\lambda}_{ql} = \lambda_{ql} \left(\frac{\lambda_{qr} - \hat{\lambda}_q}{\lambda_{qr} - \lambda_{ql}} \right) \quad (14.30)$$

and

$$\hat{\lambda}_{qr} = \lambda_{qr} \left(\frac{\hat{\lambda}_q - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}} \right) \quad (14.31)$$

then

$$F(u_l, u_r) = f(u_l) + \sum_{p \neq q} \hat{\lambda}_p^- \alpha_p \hat{r}_p + \hat{\lambda}_{ql} \alpha_q \hat{r}_q \quad (14.32)$$

$$= f(u_r) - \sum_{p \neq q} \hat{\lambda}_p^+ \alpha_p \hat{r}_p - \hat{\lambda}_{qr} \alpha_q \hat{r}_q \quad (14.33)$$

This can be written more concisely if we make the following definitions for all $p = 1, 2, \dots, m$:

$$\hat{\lambda}_{pl} = \lambda_{pl}^- \left(\frac{\lambda_{pr}^+ - \hat{\lambda}_p}{\lambda_{pr}^+ - \lambda_{pl}^-} \right) \quad (14.34)$$

and

$$\hat{\lambda}_{pr} = \lambda_{pr}^+ \left(\frac{\hat{\lambda}_p - \lambda_{pl}^-}{\lambda_{pr}^+ - \lambda_{pl}^-} \right). \quad (14.35)$$

In the sonic rarefaction case $\lambda_{pl}^- = \lambda_{pr}$ and $\lambda_{pr}^+ = \lambda_{pr}$ so these reduce to the formulas (14.30) and (14.31). In all other cases it is easy to verify that $\hat{\lambda}_{pl}$ reduces to $\hat{\lambda}_p^-$ while $\hat{\lambda}_{pr}$ reduces to $\hat{\lambda}_p^+$, so that we can write the flux function in the more general case as

$$F(u_l, u_r) = f(u_l) + \sum_p \hat{\lambda}_{pl} \alpha_p \hat{r}_p \quad (14.36)$$

$$= f(u_r) - \sum_p \hat{\lambda}_{pr} \alpha_p \hat{r}_p \quad (14.37)$$

with sonic rarefactions handled properly automatically.

EXERCISE 14.2. Verify these claims.

14.2.3 The scalar case

It is illuminating to consider the effect of this entropy fix in the scalar case. In this case there is only one characteristic family with eigenvalue $f'(u)$, and so we simplify notation by identifying $\lambda_{ql} = f'(u_l)$ and $\lambda_{qr} = f'(u_r)$. Moreover the Roe linearization gives the jump $(u_r - u_l)$ propagating with speed $\hat{\lambda}_q = \hat{a} = (f(u_r) - f(u_l))/(u_r - u_l)$.

In the sonic rarefaction case, $f'(u_l) < 0 < f'(u_r)$ and the entropy fix discussed above replaces the single jump propagating at speed \hat{a} by two jumps propagating at speeds $f'(u_l)$

and $f'(u_r)$, separated by the state

$$\begin{aligned} u_m &= \frac{(\hat{a} - f'(u_l))u_l + (f'(u_r) - \hat{a})u_r}{f'(u_r) - f'(u_l)} \\ &= u_l + \left(\frac{f'(u_r) - \hat{a}}{f'(u_r) - f'(u_l)} \right) (u_r - u_l) \end{aligned} \quad (14.38)$$

using (14.27). The approximate Riemann solution

$$\hat{w}(x/t) = \begin{cases} u_l & x/t < \hat{a} \\ u_r & x/t > \hat{a} \end{cases} \quad (14.39)$$

has been replaced by a modified approximate Riemann solution

$$\hat{w}(x/t) = \begin{cases} u_l & x/t < f'(u_l) \\ u_m & f'(u_l) < x/t < f'(u_r) \\ u_r & x/t > f'(u_r). \end{cases} \quad (14.40)$$

It is interesting to note that we can again interpret $\hat{w}(x/t)$ as the *exact* solution to a modified conservation law, now slightly more complicated than the advection equation $u_t + \hat{a}u_x = 0$ defining (14.39). The function $\hat{w}(x/t)$ in (14.40) is the exact Riemann solution to the problem $u_t + \hat{f}(u)_x = 0$, where

$$\hat{f}(u) = \begin{cases} f(u_l) + (u - u_l)f'(u_l) & u \leq u_m \\ f(u_r) + (u - u_r)f'(u_r) & u \geq u_m. \end{cases} \quad (14.41)$$

This piecewise linear function is shown in Figure 14.2. It is easy to verify that the point u_m where the two tangent lines meet is the same point u_m defined by (14.38).

The flux function $\hat{f}(u)$ is not smooth at u_m , and so our standard theory does not apply directly to determine the solution to this Riemann problem. However, we can approximate $\hat{f}(u)$ arbitrarily well by smooth functions if we round off the corner at u_m , and in this way justify the claim that \hat{w} in (14.40) solves this Riemann problem.

EXERCISE 14.3. Determine the structure of the Riemann solution for a smoothed version of $\hat{f}(u)$ and verify that it consists of two contact discontinuities separated by a rarefaction wave, and approaches (14.40) in the limit of zero smoothing.

The resulting numerical flux $F(u_l, u_r)$ can be evaluated from the general expression (14.32), which in the scalar case reduces to

$$F(u_l, u_r) = f(u_l) + f'(u_l) \left(\frac{f'(u_r) - \hat{a}}{f'(u_r) - f'(u_l)} \right) (u_r - u_l). \quad (14.42)$$

Alternatively, applying formula (14.14) in this case gives

$$F(u_l, u_r) = \hat{f}(u_m), \quad (14.43)$$

14.38)

14.39)

14.40)

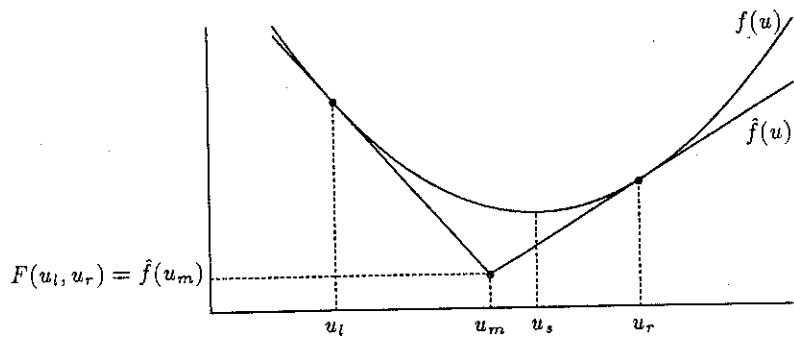


Figure 14.2. The piecewise linear function $\hat{f}(u)$ used to define the approximate Riemann solution in the sonic rarefaction case.

since $\hat{f}(u_r) = f(u_r)$. Using (14.41) we see that this agrees with (14.42).

This gives a nice geometric interpretation to the numerical flux in the scalar case when we apply the entropy fix. Moreover, we see from this that

$$F(u_l, u_r) < f(u_s) \quad (14.44)$$

where u_s is the sonic point at which $f'(u_s) = 0$ and f is a minimum. Recall that the Godunov flux using the correct rarefaction wave solution of the original conservation law is given by

$$F_G(u_l, u_r) = f(u_s) \quad (14.45)$$

in this case. Hence the flux we use in Roe's method with the entropy fix is always less than the Godunov flux. Since the flux for Roe's method always agrees with the Godunov flux in every case except the sonic rarefaction, we see that

$$\begin{aligned} F(u_l, u_r) &\leq F_G(u_l, u_r) \quad \forall u_l \leq u_r, \\ F(u_l, u_r) &\geq F_G(u_l, u_r) \quad \forall u_l \geq u_r, \end{aligned} \quad (14.46)$$

where F_G denotes the Godunov flux. The second inequality of (14.46) is in fact satisfied as an equality for all $u_l \geq u_r$. It is written in the above form since it turns out that any numerical method for a scalar conservation law with a flux satisfying (14.46) automatically satisfies the entropy condition and can only converge to entropy-satisfying solutions. This shows that, in the scalar case at least, the entropy fix presented here does indeed cure the entropy problem.

14.42)

14.43)

Numerical methods satisfying (14.46) are called **E-schemes** (E for entropy), a notion introduced by Osher[58], who also discusses a related entropy fix for Roe's method and some alternative approximate Riemann solvers. See also Tadmor[85] for a discussion of these issues.

14.2.4 A Roe matrix for isothermal flow

As an example, we derive the Roe matrix for the isothermal equations of gas dynamics. Roe[64] presents the analogous formulas for the full Euler equations.

In the isothermal equations,

$$u = \begin{bmatrix} \rho \\ m \end{bmatrix}, \quad f(u) = \begin{bmatrix} m \\ m^2/\rho + a^2\rho \end{bmatrix}, \quad (14.47)$$

and

$$f'(u) = \begin{bmatrix} 0 & 1 \\ a^2 - m^2/\rho^2 & 2m/\rho \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a^2 - v^2 & 2v \end{bmatrix} \quad (14.48)$$

with $m = \rho v$. To derive an averaged Jacobian satisfying (14.19), introduce the new variables

$$z = \rho^{-1/2}u, \quad i.e. \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \rho^{1/2} \\ m/\rho^{1/2} \end{bmatrix} = \begin{bmatrix} \rho^{1/2} \\ \rho^{1/2}v \end{bmatrix}. \quad (14.49)$$

Then

$$u = z_1 z = \begin{bmatrix} z_1^2 \\ z_1 z_2 \end{bmatrix}, \quad f(u) = \begin{bmatrix} z_1 z_2 \\ a^2 z_1^2 + z_2^2 \end{bmatrix}. \quad (14.50)$$

The motivation for this change of variables is that both $(u_l - u_r)$ and $(f(u_l) - f(u_r))$ can be expressed in the form of some matrix times $(z_l - z_r)$. The averaging is done in terms of the variables z :

$$\bar{z} = \frac{1}{2}(z_l + z_r) = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \rho_l^{1/2} + \rho_r^{1/2} \\ m_l/\rho_l^{1/2} + m_r/\rho_r^{1/2} \end{bmatrix}. \quad (14.51)$$

Then it is easy to check that

$$\begin{aligned} u_l - u_r &= \begin{bmatrix} 2\bar{z}_1 & 0 \\ \bar{z}_2 & \bar{z}_1 \end{bmatrix} (z_l - z_r) \equiv \hat{B}(z_l - z_r), \\ f(u_l) - f(u_r) &= \begin{bmatrix} \bar{z}_2 & \bar{z}_1 \\ 2a^2\bar{z}_1 & 2\bar{z}_2 \end{bmatrix} (z_l - z_r) \equiv \hat{C}(z_l - z_r). \end{aligned}$$

Using $[\cdot]$ to denote the jump in a quantity, we have $[u] = \hat{B}[z]$ and $[f] = \hat{C}[z]$, which can be combined to give

$$[f] = \hat{C}\hat{B}^{-1}[u]. \quad (14.52)$$

entropy), a notion
Roe's method and
for a discussion of

s of gas dynamics.

(14.47)

introduce the new

(14.49)

$f(u_l) - f(u_r)$) can
is done in terms

(14.50)

Condition (14.19*i*) is thus satisfied if we take

$$\hat{A}(u_l, u_r) = \hat{C} \hat{B}^{-1} = \begin{bmatrix} 0 & 1 \\ a^2 - \bar{z}_2^2/\bar{z}_1^2 & 2\bar{z}_2^2/\bar{z}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a^2 - \bar{v}^2 & 2\bar{v} \end{bmatrix} \quad (14.53)$$

where we have defined the average velocity \bar{v} by

$$\bar{v} = \frac{\bar{z}_2}{\bar{z}_1} = \frac{\rho_l^{1/2} v_l + \rho_r^{1/2} v_r}{\rho_l^{1/2} + \rho_r^{1/2}}. \quad (14.54)$$

This is often called the rho-averaged (Roe-averaged?) velocity. We can view \hat{A} as being the Jacobian matrix $f'(u)$ from (14.48) evaluated at the averaged velocity \bar{v} . Clearly, when $u_l = u_r$, and $\bar{z} = z_l = z_r$, the matrix $\hat{A}(u_l, u_l)$ reduces to $f'(u_l)$. Since the coefficients of \hat{A} are smooth functions of u_l and u_r , this shows that (14.19*iii*) is satisfied. Finally, condition (14.19*ii*) is also clearly satisfied since the eigenvalues and eigenvectors of \hat{A} are given by:

$$\hat{\lambda}_1 = \bar{v} - a, \quad \hat{\lambda}_2 = \bar{v} + a \quad (14.55)$$

$$\hat{r}_1 = \begin{bmatrix} 1 \\ \bar{v} - a \end{bmatrix}, \quad \hat{r}_2 = \begin{bmatrix} 1 \\ \bar{v} + a \end{bmatrix}. \quad (14.56)$$

Again, note the similarity to the expressions (7.6) and (7.7) for the eigenvalues and eigenvectors of $f'(u)$.

The approximate solution to the Riemann problem with data u_l and u_r is thus

$$\hat{u}(x, t) = \begin{cases} u_l & x/t < \bar{v} - a \\ \hat{u}_m & \bar{v} - a < x/t < \bar{v} + a \\ u_r & x/t > \bar{v} + a \end{cases} \quad (14.57)$$

where \hat{u}_m is the intermediate state. We can obtain this state by decomposing

$$u_r - u_l = \alpha_1 \hat{r}_1 + \alpha_2 \hat{r}_2 \quad (14.58)$$

and then

$$\hat{u}_m = u_l + \alpha_1 \hat{r}_1. \quad (14.59)$$

EXERCISE 14.4. Find a Roe matrix for the one-dimensional shallow water equations (5.38).

= $\hat{C}[z]$, which can

(14.52)