CS 70 Summer 2016 Solutions to Extra Problems 1

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1.1 Review of Set Theory

We took a peak at some of the fundamental rules of set theory in the first discussion sheet and below are more rules you might find interesting and worth walking through proofs of them.

- (Distributive Laws)
  1. $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
  2. $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
- (Consistency Principle)
  1. $X \subseteq Y$ if and only if $X \cup Y = Y$
  2. $X \subseteq Y$ if and only if $X \cap Y = X$
- (De Morgans Laws) Suppose that S and T are sets. De Morgans Laws state that (where C on the exponent means complement of the set)
  1. $(S \cup T)^C = S^C \cap T^C$
  2. $(S \cap T)^C = S^C \cup T^C$
- Two sets are equal if and only if each is a subset of the other, i.e. $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$

1.2 Propositional logic

For each the following logical equivalence assertions, either prove it is true or give a counterexample showing it is false (i.e., some choices of P and Q such that one side of the equivalence is true and the other is false), together with a one to two sentence justification that it is indeed a counterexample.

1. $\forall x\, P(x) \;\equiv\; \neg\exists x\, \neg P(x)$
2. $\forall x\, \exists y\, P(x,y) \;\equiv\; \forall y\, \exists x\, P(x,y)$
3. $P \Rightarrow \neg Q \;\equiv\; \neg P \Rightarrow Q$
4. $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q) \;\equiv\; P \Leftrightarrow Q$

1. True. We use the fact that $\neg(\neg A) \equiv A$ for any proposition A. The full proof is:

$$\forall x\, P(x) \;\equiv\; \neg\left(\neg\left(\forall x\, P(x)\right)\right) \;\equiv\; \neg\left(\exists x\, \neg P(x)\right).$$

2. False. Take the universe of both x, y to be $\mathbb{N}$, and take $P(x, y)$ to be the statement "$x > y$". Then the left hand side $\forall x\, \exists y\, P(x, y)$ claims that for every $x \in \mathbb{N}$ we can find another natural number $y \in \mathbb{N}$ that is strictly less than x; this is false, since when $x = 0$ we cannot find such a y. The right hand side $\forall y\, \exists x\, P(x, y)$ claims that for all $y \in \mathbb{N}$ we can find $x \in \mathbb{N}$ that us strictly larger than y; this is true, e.g., we can take $x = y + 1$.

3. False. Take P and Q to be any true propositions (e.g., P is "$1+1 = 2$" and Q is "$1+2 = 3$"). Then $P \Rightarrow \neg Q$ is false while $\neg P \Rightarrow Q$ is true.

4. True. Recall that $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Since $Q \Rightarrow P$ is equivalent to its contraposition $\neg P \Rightarrow \neg Q$, we conclude that $P \Leftrightarrow Q$ is also equivalent to $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q)$.

1.3 Practicing $\sum$ and $\prod$

1.3.1 $\sum$

The expression $\displaystyle\sum_{i=0}^{n} f(i)$ is equal to which of the following expressions (circle all that apply)?

- $f(0) \cdot f(1) \cdot f(2) \cdots f(n-1) \cdot f(n)$
- $\displaystyle\sum_{i=0}^{n-1} (f(i) + f(n))$
- $\displaystyle\left(\sum_{i=0}^{n-1} f(i)\right) + f(n)$
- $f(0) + f(1) + \cdots + f(n)$
- $\displaystyle\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) + \sum_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j)$

- $f(0) \cdot f(1) \cdot f(2) \cdots f(n-1) \cdot f(n)$

  Not equal. This expression is the product of $f(0), f(1), \ldots, f(n)$, whereas we were looking for the sum. As a counter example take $n = 1$, and $f(0) = 0$ and $f(1) = 1$. Then $\sum_{i=0}^{n} f(i) = 1$, whereas $f(0) \cdot f(1) = 0$.

- $\displaystyle\sum_{i=0}^{n-1} (f(i) + f(n))$

  Not equal. The term $f(n)$ gets repeated $n-1$ times (once for each value of i) in this expression whereas it should appear just once. As a counter example take $n = 2$, and f to be the constant function 1. Then $\sum_{i=0}^{n} f(i) = 3$. But $\sum_{i=0}^{n-1} (f(i) + f(n)) = \sum_{i=0}^{1} 2 = 4$.

- $\left(\sum_{i=0}^{n-1} f(i)\right) + f(n)$

  Equal. The sum of n terms is always the sum of the first $n-1$ terms added with the last term. This expression is just that.

- $f(0) + f(1) + \cdots + f(n)$

  Equal. This is the sum of the n terms expanded in an explicit way.

- $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) + \sum_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j)$

  Equal. For any integer $0 \leq c < n$, we can break the sum $\sum_{i=0}^{n} f(i)$ into two parts: the sum of the terms where $i \leq c$ and the sum of the terms where $i \geq c+1$. Thus $\sum_{i=0}^{n} f(i) = \sum_{i=0}^{c} f(i) + \sum_{j=c+1}^{n} f(j)$. The expression in this statement is the special case where $c = \lfloor \frac{n}{2} \rfloor$

1.3.2 $\prod$

The expression $\prod_{i=0}^{n} f(i)$ is equal to which of the following expressions (circle all that apply)?

- $f(0) \cdot f(1) \cdot f(2) \cdots f(n-1) \cdot f(n)$
- $f(n) \prod_{i=0}^{n-1} f(i)$
- $\frac{f(n)}{f(0)}$
- $f(n) \prod_{i=0}^{\frac{n}{2}-1} \prod_{j=2i}^{2i+1} f(j)$ (for n even).
- $\prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) \prod_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j)$

- $f(0) \cdot f(1) \cdot f(2) \cdots f(n-1) \cdot f(n)$

  Equal. This is simply the product of $f(i)$ for $0 \leq i \leq n$ written in an explicit way.

- $f(n) \prod_{i=0}^{n-1} f(i)$

  Equal. The product of n terms is simply the product of the first $n-1$ terms multiplied by the last term. This expression is just that.

- $\frac{f(n)}{f(0)}$

  Not equal. As a counter example take $n = 1$ and let f be the constant 2 function. Then $\prod_{i=0}^{n} f(i) = 2 \times 2 = 4$, but $\frac{f(n)}{f(0)} = 1$.

- $f(n) \prod_{i=0}^{\frac{n}{2}-1} \prod_{j=2i}^{2i+1} f(j) \qquad$ (for n even).

  Equal. Expanding the innermost product we get

  $$f(n) \prod_{i=0}^{\frac{n}{2}-1} f(2i)f(2i+1).$$

  If we now expand the product further we get

  $$f(n)\left((f(0)f(1))(f(2)f(3))\ldots(f(n-2)f(n-1))\right)$$

  So we can see the expanded product contains every term from $f(0)$ through $f(n-1)$, and the extra $f(n)$ that is multiplied makes the whole product $\prod_{i=0}^{n} f(i)$.

- $\prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) \prod_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} f(j)$

  Equal. The first product is $f(0)f(1)\ldots f(\lfloor \frac{n}{2} \rfloor)$ and the second product is $f(\lfloor \frac{n}{2} \rfloor+1)f(\lfloor \frac{n}{2} \rfloor+2)\ldots f(n)$. So multiplied together they give us $f(0)\ldots f(\lfloor \frac{n}{2} \rfloor)f(\lfloor \frac{n}{2} \rfloor+1)\ldots f(n)$ which is just $\prod_{i=0}^{n} f(i)$.

1.4 Cool Proof

Using proof by contradiction, show that the square root of 2 is irrational.

There are probably tens of solutions but refer to the notes for the simplest one

1.5 Mathematical Induction

Use the Well Ordering Principle to prove that $n \leq 3^{\frac{n}{3}}$ for every non-negative integer, n.

You can use induction instead.

1.6 More Induction

Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example, $\frac{10}{9} = \frac{2!\cdot 5!}{3!\cdot 3!\cdot 3!}$ (Hint: you can take it for granted, or proving using induction that every natural number admits a prime factorization, i.e. given any natural number, we can write it as a product of a list of not necessarily distinct primes)

1.7 Even more induction

Later in the class we will introduce binomial coefficients, denoted as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and basically just means (roughly) how many ways are there to choose k items out of n items. While from this definition it is clear that this quantity should be an integer, it is not obvious at all from its arithmetic definition, $\frac{n!}{k!(n-k)!}$, why this should be an integer. In this problem we will explore it using two different tools:

1. Given the recursion $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, prove that $\frac{n!}{k!(n-k)!} \in \mathbb{N}$ for $k \leq n, k, n \in \mathbb{N}$ using induction.

   The claim we want to prove is that for any fixed n and $0 \leq k \leq n$, we have $\binom{n}{k} \in \mathbb{Z}$.

   - Base Case: Clearly when $n = 1$, $\binom{1}{0} = \binom{1}{1} = 1 \in \mathbb{Z}$.
   - Induction Hypothesis: Assume that for any fixed n and $0 \leq k \leq n$, we have $\binom{n}{k} \in \mathbb{Z}$
   - Induction Step: Now we proceed to $n+1$ case. Using the induction formula given, we know $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ which means $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ and by our induction hypothesis both of these numbers are integers and sum of integers are integers. Q.E.D.

2. (**Bonus**) Can you prove that $\frac{n!}{k!(n-k)!} \in \mathbb{N}$ for $k \leq n, k, n \in \mathbb{N}$ without the recursion relation given above? (Hint: prove that for any natural number k, product of any consecutive k natural numbers is divisible by $k!$.)

We first cancel the $(n-k)!$ part at the denominator and numerator(hidden in $n!$. $\frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ and following the hint, we want to show that our numerator is divisible by the denominator and not surprisingly we are gonna use inductions!

- Base case:
- Induction Hypothesis:
- Induction Step:

And I will leave this as an exercise, feel free to comment or ask questions.

Optional: Fun fact about harmonic series

(This is just of general mathematical interest, not related to materials taught in class lol) You have probably seen the divergent harmonic series $H_k = \sum_i^k \frac{1}{i}$ already (either from high school pre-cal or homework 1!), and this question asks you to prove the following pair of generally useful (actually pretty tight) lower and upper bound of the series!

$$ln(k+1) \leq \sum_i^k \frac{1}{i} \leq ln(k) + 1$$

for any positive integer k. (Hint: integral test and bound individual term $\frac{1}{i}$ using $\int_a^b \frac{1}{t} dt$ for appropriate a and b.)

We first observe that the following inequality holds for $n \in \mathbb{N}, n \geq 2$: $\int_n^{n+1} \frac{dt}{t} \leq \frac{1}{n} \leq \int_{n-1}^{n} \frac{dt}{t}$ and this is by the integral test like argument in high school/Math 1A,B. If you are not clear why this is the case, draw a picture!

Now we just add the inequalities up to k of them (varying the n's) and we get $\int_1^{k+1} \frac{dt}{t} \leq \sum_{i=1}^k \frac{1}{i} \leq \int_1^k \frac{dt}{t} + 1$ and we just evaluate the integrals.