CS 70 Summer 2016 Extra Problems 1

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Abstract

(Well I failed to change the abstract in the title into disclaimer so u probably hve to bear with it...) Disclaimer: This discussion sheet borrowed heavily from CS70 materials in previous semesters and I would like to take this opportunity thank all the hardworking staff for putting these problems together, and as a clarification these problems are **NOT required or officially endorsed/recommended in any sense** , I just thought it might be useful to have a sampler for people who are interested in doing extra/fun problems . (Solutions will be posted at the end of the week.

1.1 Review of Set Theory

We took a peak at some of the fundamental rules of set theory in the first discussion sheet and below are more rules you might find interesting and worth walking through proofs of them.

- (Distributive Laws)
  1. $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
  2. $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
- (Consistency Principle)
  1. $X \subseteq Y$ if and only if $X \cup Y = Y$
  2. $X \subseteq Y$ if and only if $X \cap Y = X$
- (De Morgans Laws) Suppose that S and T are sets. De Morgans Laws state that (where C on the exponent means complement of the set)
  1. $(S \cup T)^C = S^C \cap T^C$
  2. $(S \cap T)^C = S^C \cup T^C$
- Two sets are equal if and only if each is a subset of the other, i.e. $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$

1.2 Propositional logic

For each the following logical equivalence assertions, either prove it is true or give a counterexample showing it is false (i.e., some choices of P and Q such that one side of the equivalence is true and the other is false), together with a one to two sentence justification that it is indeed a counterexample.

1. $\forall x\, P(x) \;\equiv\; \neg\exists x\, \neg P(x)$

2. $\forall x\, \exists y\, P(x,y) \;\equiv\; \forall y\, \exists x\, P(x,y)$
3. $P \Rightarrow \neg Q \;\equiv\; \neg P \Rightarrow Q$
4. $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q) \;\equiv\; P \Leftrightarrow Q$

1.3 Practicing $\sum$ and $\prod$

1.3.1 $\sum$

The expression $\displaystyle\sum_{i=0}^{n} f(i)$ is equal to which of the following expressions (circle all that apply)?

- $f(0) \cdot f(1) \cdot f(2) \cdots f(n-1) \cdot f(n)$
- $\displaystyle\sum_{i=0}^{n-1} (f(i) + f(n))$
- $\displaystyle\left(\sum_{i=0}^{n-1} f(i)\right) + f(n)$
- $f(0) + f(1) + \cdots + f(n)$
- $\displaystyle\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) + \sum_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j)$

1.3.2 $\prod$

The expression $\displaystyle\prod_{i=0}^{n} f(i)$ is equal to which of the following expressions (circle all that apply)?

- $f(0) \cdot f(1) \cdot f(2) \cdots f(n-1) \cdot f(n)$
- $\displaystyle f(n) \prod_{i=0}^{n-1} f(i)$
- $\frac{f(n)}{f(0)}$
- $\displaystyle f(n) \prod_{i=0}^{\frac{n}{2}-1} \prod_{j=2i}^{2i+1} f(j)$ (for n even).
- $\prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) \displaystyle\prod_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j)$

1.4 Cool Proof

Using proof by contradiction, show that the square root of 2 is irrational.

1.5 Mathematical Induction

Use the Well Ordering Principle to prove that $n \leq 3^{\frac{n}{3}}$ for every non-negative integer, n.

1.6 More Induction

Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example, $\frac{10}{9} = \frac{2!\cdot 5!}{3!\cdot 3!\cdot 3!}$ (Hint: you can take it for granted, or proving using induction that every natural number admits a prime factorization, i.e. given any natural number, we can write it as a product of a list of not necessarily distinct primes)

1.7 Even more induction

Later in the class we will introduce binomial coefficients, denoted as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and basically just means (roughly) how many ways are there to choose k items out of n items. While from this definition it is clear that this quantity should be an integer, it is not obvious at all from its arithmetic definition, $\frac{n!}{k!(n-k)!}$, why this should be an integer. In this problem we will explore it using two different tools:

1. Given the recursion $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, prove that $\frac{n!}{k!(n-k)!} \in \mathbb{N}$ for $k \leq n, k, n \in \mathbb{N}$ using induction.

2. (**Bonus**) Can you prove that $\frac{n!}{k!(n-k)!} \in \mathbb{N}$ for $k \leq n, k, n \in \mathbb{N}$ without the recursion relation given above? (Hint: prove that for any natural number k, product of any consecutive k natural numbers is divisible by $k!$.)

Optional: Fun fact about harmonic series

(This is just of general mathematical interest, not related to materials taught in class lol) You have probably seen the divergent harmonic series $H_k = \sum_i^k \frac{1}{i}$ already (either from high school pre-cal or homework 1!), and this question asks you to prove the following pair of generally useful (actually pretty tight) lower and upper bound of the series!

$$ln(k+1) \leq \sum_i^k \frac{1}{i} \leq ln(k) + 1$$

for any positive integer k. (Hint: integral test and bound individual term $\frac{1}{i}$ using $\int_a^b \frac{1}{t} dt$ for appropriate a and b.)