Assignment 3

1. Proof. Let $p_1 : A\times B \to A$, $p_2 : A\times B \to B$, $p_3 : A^C\times B^C \to A^C$, $p_4 : A^C \times B^C \to B^C$. Consider morphisms:

$$z_1 = p_1 \circ \overline{1_{(A\times B)^C}} \ : \ (A\times B)^C \times C \longrightarrow A$$

$$z_2 = p_2 \circ \overline{1_{(A\times B)^C}} \ : \ (A\times B)^C \times C \longrightarrow B$$

Then the transpositions are

$$\tilde{z}_1 = \widetilde{(p_1 \circ \overline{1_{(A\times B)^C}})} : \ (A\times B)^C \longrightarrow A^C$$

$$\tilde{z}_2 = \widetilde{(p_2 \circ \overline{1_{(A\times B)^C}})} : \ (A\times B)^C \longrightarrow B^C$$

Consider the following product diagram:

$$\begin{tikzcd}
A^C & A^C\times B^C \arrow[l, "p_3"'] \arrow[r, "p_4"] & B^C \\
& (A\times B)^C \arrow[ul, "\hat{z}_1"] \arrow[u, dashed, "!u"] \arrow[ur, "\tilde{z}_2"'] &
\end{tikzcd}$$

Then exists unique $u = \langle \tilde{z}_1, \tilde{z}_2 \rangle$ such that it commutes.

Now observe that

$$\bar{p}_3 : (A^C\times B^C)\times C \longrightarrow A$$

$$\bar{p}_4 : (A^C\times B^C)\times C \longrightarrow B$$

Similarly, for product $A\times B$, we have

$$\begin{tikzcd}
A & A\times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\
& A^C\times B^C\times C \arrow[ul, "\bar{p}_3"] \arrow[u, dashed, "!v"] \arrow[ur, "\bar{p}_4"'] &
\end{tikzcd}$$

Exists unique $v = \langle \bar{p}_3, \bar{p}_4 \rangle : A^C\times B^C\times C \to A\times B$ making above diagram commute.

Hence the transpose of v is given by

$$\tilde{v} = \widetilde{\langle \bar{p}_3, \bar{p}_4 \rangle} : \ A^C\times B^C \longrightarrow (A\times B)^C$$

Observe that

$$\begin{aligned}
p_3 \circ (u \circ \tilde{v}) &= p_3 \circ \langle \tilde{z}_1, \tilde{z}_2 \rangle \circ \widetilde{\langle \bar{p}_3, \bar{p}_4 \rangle} \\
&= \tilde{z}_1 \circ \widetilde{\langle \bar{p}_3, \bar{p}_4 \rangle} \\
&= \widetilde{(p_1 \circ \overline{1_{(A\times B)^C}})} \circ \widetilde{\langle \bar{p}_3, \bar{p}_4 \rangle} \\
&= \widetilde{(p_1 \circ 1_{A\times B} \circ \langle \bar{p}_3, \bar{p}_4 \rangle)} \\
&= \widetilde{(p_1 \circ \langle \bar{p}_3, \bar{p}_4 \rangle)} \\
&= \tilde{\bar{p}}_3 = p_3
\end{aligned}$$

Similarly,

$$\begin{aligned}
\hat{z}_1 \circ (\tilde{v} \circ u) &= \tilde{z}_1 \circ \left(\widetilde{\langle \bar{p}_3, \bar{p}_4 \rangle} \circ \langle \tilde{z}_1, \tilde{z}_2 \rangle \right) \\
&= \widetilde{(p_1 \circ \overline{1_{(A\times B)^C}})} \circ \widetilde{\langle \bar{p}_3, \bar{p}_4 \rangle} \circ \langle \tilde{z}_1, \tilde{z}_2 \rangle \\
&= \widetilde{(p_1 \circ 1_{A\times B} \circ \langle \bar{p}_3, \bar{p}_4 \rangle)} \circ \langle \tilde{z}_1, \tilde{z}_2 \rangle \\
&= p_3 \circ \langle \tilde{z}_1, \tilde{z}_2 \rangle \\
&= \tilde{z}_1
\end{aligned}$$

Therefore $u \circ \tilde{v} = 1_{A^C\times B^C}$ $\quad \hat{v} \circ u = 1_{(A\times B)^C}$.

Thus u forms an isomorphism between $(A\times B)^C$ and $A^C\times B^C$.

That is $\quad (A\times B)^C \cong A^C \times B^C$

2. Proof: Claim that $1 \times A \cong A$ for all $\mathbb{C}$-object A.

$$\begin{tikzcd}
1 & 1 \times A \arrow[l, "p_1"'] \arrow[r, "p_2"] & A \\
& A \arrow[ul, "a"] \arrow[u, dashed, "\langle a, 1_A \rangle"'] \arrow[ur, "1_A"'] &
\end{tikzcd}$$

Clearly $p_2 \circ \langle a, 1_A \rangle = 1_A$

$$p_1 \circ \left(\langle a, 1_A \rangle \circ p_2 \right) = a \circ p_2 : 1 \times A \to 1$$

As 1 is terminal, $a \circ p_2 = p_1$. Hence $\langle a, 1_A \rangle \circ p_2 = 1_{1 \times A}$

Therefore p_2 forms an isomorphism, $1 \times A \cong A$.

Hence by definition 6.1. clearly that each

$$f : 1 \to B^A \text{ uniquely corresponds to } \bar{f} : 1 \times A \to B$$

by transposition.

As $1 \times A \cong A$, it uniquely corresponds to some $g : A \to B$.

i.e. $\mathrm{Hom}(1, B^A) \cong \mathrm{Hom}(1 \times A, B) \cong \mathrm{Hom}(A, B)$

3. Proof: In Cartision closed category $\mathbb{C}$ with object A and initial object 0, consider the following product diagram:

$$\begin{tikzcd}
& 0 \arrow[dl, "1_0"'] \arrow[d, dashed, "!u"] \arrow[dr, "!g"] & \\
0 \arrow[r, shift right, "!u"'] & 0 \times A \arrow[l, shift right, "p_1"'] \arrow[r, "p_2"] & A \\
& X \arrow[ul, "x_1"] \arrow[u, dashed, "!\langle x_1, x_2 \rangle"'] \arrow[ur, "x_2"'] &
\end{tikzcd}$$

As 0 is initial, $u : 0 \to 0 \times A$ is the only morphism.

By definition of product, we have $p_1 \circ u = 1_0$

For all $\mathbb{C}$-object X with unique $\langle x_1, x_2 \rangle : X \to 0 \times A$,

$$(u \circ p_1) \circ \langle x_1, x_2 \rangle = u \circ (p_1 \circ \langle x_1, x_2 \rangle) = u \circ x_1 = \langle x_1, x_2 \rangle$$

Hence $u \circ p_1 = 1_{0 \times A}$

Therefore $p_1 : 0 \times A \to 0$ forms an isomorphism, $0 \times A \cong 0$. That is $0 \times A$ is also initial.

4. Let $U : \mathrm{Groups} \to \mathrm{Monoids}$ be the forgetful functor.

Proposition U is faithful and full.

Proof For all $G, H \in \mathrm{obj}\,\mathrm{Groups}$, let $f, g \in \mathrm{Hom}(G, H)$ such that

$$U(f) = U(g)$$

Obviously by definition of forgetful functor, we have $f = g$ as U only erases the inverse structure of groups.

Therefore U is faithful.

For all monoid homomorphism $f \in \mathrm{Hom}(U(G), U(H))$, $a, b \in U(G)$,

$$f(a \cdot b) = f(a) f(b)$$

Correspondingly define $g : G \to H$ by

$$g(a \cdot b) = g(a) \cdot g(b) \quad \text{for all } a, b \in G$$

and set $\quad g(a) \cdot g(a^{-1}) = g(a \cdot a^{-1}) = g(e_G) = e_H$

$$g(a^{-1}) \cdot g(a) = g(a^{-1} a) = g(e_G) = e_H$$

Clearly g is a group homomorphism, $g \in \mathrm{Hom}(G, H)$.

Then we have $U(g) = f$, thus $U_{G,H} : \mathrm{Hom}(G, H) \to \mathrm{Hom}(U(G), U(H))$ is surjective, that is U is full.

Proposition U is injective on objects, not surjective on objects.

Proof For all $G, H \in \mathrm{obj}\,\mathrm{Groups}$ such that $U(G) = U(H)$. By theorem that the inverse is unique if exists, every monoid can only generates a unique group if possible. Thus we have $G = H$ as required. That is U is injective on objects. And clearly it's not surjective on objects because not every monoid can form a group. $(\mathbb{Z}, \times, 1)$ is an example.

Proposition U is injective on morphisms, not surjective on morphisms.

Proof For all group homomorphism $g, h \in \mathrm{mor}\,\mathrm{Group}$, if $U(g) = U(h)$, then $g = h$ as U doesn't change the mapping rule. Hence U is injective on morphisms. However, for all $f \in \mathrm{mor}\,\mathrm{Monoid}$, domain and codomain of f not necessarily form a group, so $U(g) \neq f$ for all $g \in \mathrm{mor}\,\mathrm{Groups}$. U is not surjective on morphisms

5. Proof: ($\Rightarrow$) Suppose $\alpha : F \to G$, $F, G : \mathbb{C} \to \mathbb{D}$ and α forms an isomorphism in $\mathrm{Func}(\mathbb{C}, \mathbb{D})$. Then there exists $\beta : G \to F$ such that

$$\alpha \circ \beta = 1_F \quad , \quad \beta \circ \alpha = 1_G$$

By definition of identity in $\mathrm{Func}(\mathbb{C}, \mathbb{D})$, for each $\mathbb{C}$-object X, we have

$$(\alpha \circ \beta)_X = \alpha_X \circ \beta_X = (1_F)_X = 1_{F(X)}$$

$$(\beta \circ \alpha)_X = \beta_X \circ \alpha_X = (1_G)_X = 1_{G(X)}$$

Hence each component α_X is an isomorphism: $\alpha_X : F(X) \cong G(X)$

($\Leftarrow$) Suppose $\alpha : F \to G$, $F, G : \mathbb{C} \to \mathbb{D}$ where each component of α is an isomorphism. For all $X \in \mathbb{C}$, as $\alpha_X : F(X) \to G(X)$ is an isomorphism, exists $\theta_X : G(X) \to F(X)$ such that

$$\alpha_X \circ \theta_X = 1_{F(X)} \quad , \quad \theta_X \circ \alpha_X = 1_{G(X)}$$

Therefore define a natural transformation $\beta : G \to F$ by $\beta_X = \theta_X$ for all $\mathbb{C}$-object X. Then in $\mathrm{Func}(\mathbb{C}, \mathbb{D})$

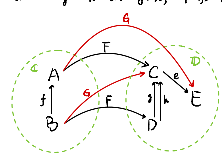
$$(\alpha \circ \beta)_X = \alpha_X \circ \beta_X = 1_{F(X)} = (1_F)_X$$

$$(\beta \circ \alpha)_X = \beta_X \circ \alpha_X = 1_{G(X)} = (1_G)_X$$

Hence α forms a natural isomorphism.

It is not true for monomorphism. Here's the counter example:

Suppose $\mathbb{C}$ is a two-object category, $\mathbb{D}$ is a three-object category as in the following graph with $e \circ g = e \circ h$ but $g \neq h$, $F(f) = g$, $\alpha : F \to G$ is the natural transformation.



Consider another functor $H : \mathbb{C} \to \mathbb{D}$, if $H(B) = D$, $H(A) = E$, since there are no morphisms from E to C, there's no natural transformation from H to F.

If $H(B) = D$, $H(A) = C$, $H(f) = h$, then there's unique natural transformation β from H to F that $\beta_A = 1_C$, $\beta_B = 1_D$. So $\alpha : F \to G$ is monic in $\mathrm{Func}(\mathbb{C}, \mathbb{D})$.

As $e : C \to E$ is the only arrow from C to E, $\alpha_A = e$. However, we have $e \circ g = e \circ h$, $g \neq h$, thus α_A is not monic.

6. Proof: The component at A of $\eta : 1_{Sets} \longrightarrow PP$ is

$$\eta_A : 1_{Sets}(A) \to PP(A) = \eta_A : A \to PP(A)$$

For all $f : A \to B$ in Sets and $a \in A$.

$$\left(\eta_B \circ 1_{Sets}(f)\right)(a) = \left(\eta_B \circ f\right)(a) = \eta_B\left(f(a)\right)$$

Since $\eta_B : B \to PP(B)$, $b \mapsto \{U \subseteq B \mid b \in U\}$, we have

$$\eta_B\left(f(a)\right) = \{U \subseteq B \mid f(a) \in U\}$$

Now consider $\left(PP(f) \circ \eta_A\right)(a) = PP(f)\left(\{U \subseteq A \mid a \in U\}\right)$

Since $P(f)$ maps any subset of B to corresponding preimage, $PP(f)$ maps any set of preimages to the set of images by f. Therefore

$$\begin{aligned}
PP(f)\left(\{U \subseteq A \mid a \in U\}\right) &= \{f(U) \mid U \subseteq A, a \in U\} \\
&= \{V \mid V \subseteq B, f(a) \in V\} \\
&= \{V \subseteq B \mid f(a) \in V\} \\
&= \eta_B(f(a)) \\
&= \left(\eta_B \circ 1_{Sets}(f)\right)(a)
\end{aligned}$$

Therefore such definition of components indeed forms a natural transformation $\eta : 1_{Sets} \to PP$.

7. Proof: Suppose F, G are $Sets^{\mathbb{C}^{op}}$ - objects.

Let $Sets^{\mathbb{C}^{op}}$ - object $H : \mathbb{C}^{op} \to Sets$ be

$$H(C) = F(C) \times G(C) \quad \text{for all } \mathbb{C}\text{-object } C.$$

Then we have canonical natural transformations:

$\beta^{(1)} : H \to F$ with components $\beta^{(1)}_X : H(X) \to F(X)$ be projection

$\beta^{(2)} : H \to G$ with components $\beta^{(2)}_X : H(X) \to G(X)$ be projection

Then for all $Sets^{\mathbb{C}^{op}}$ - object Z, with natural transformation $\alpha^{(1)} : Z \to F$, $\alpha^{(2)} : Z \to G$,

let $u : Z \to H$ be a natural transformation such that each component satisfies:

$$u_X : Z(X) \to H(X), \quad a \mapsto \left(\alpha^{(1)}_X(a), \alpha^{(2)}_X(a)\right)$$

where $\alpha^{(1)}_X$ $\alpha^{(2)}_X$ are the corresponding components at X.

Then fix some arbitrary $\mathbb{C}^{op}$ - object X, for all $a \in Z(X)$, we have

$$\left(\beta^{(1)} \circ u\right)_X(a) = \left(\beta^{(1)}_X \circ u_X\right)(a) = \beta^{(1)}_X\left(u_X(a)\right) = \beta^{(1)}_X\left(\left(\alpha^{(1)}_X(a), \alpha^{(2)}_X(a)\right)\right) = \alpha^{(1)}_X(a)$$

$$\left(\beta^{(2)} \circ u\right)_X(a) = \left(\beta^{(2)}_X \circ u_X\right)(a) = \beta^{(2)}_X\left(u_X(a)\right) = \beta^{(2)}_X\left(\left(\alpha^{(1)}_X(a), \alpha^{(2)}_X(a)\right)\right) = \alpha^{(2)}_X(a)$$

Hence $\beta^{(1)} \circ u = \alpha^{(1)}$, $\beta^{(2)} \circ u = \alpha^{(2)}$, the following diagram commutes.

$$\begin{tikzcd}
& Z \arrow[dl, "\alpha^{(1)}"'] \arrow[d, dashed, "u"] \arrow[dr, "\alpha^{(2)}"] & \\
F & H \arrow[l, "\beta^{(1)}"] \arrow[r, "\beta^{(2)}"'] & G
\end{tikzcd}$$

Now assume there is some $v : Z \to H$ satisfying $\beta^{(1)} \circ v = \alpha^{(1)}$, $\beta^{(2)} \circ v = \alpha^{(2)}$ as well.

Then for all $a \in Z(X)$,

$$\left(\beta^{(1)} \circ v\right)_X(a) = \beta^{(1)}_X\left(v_X(a)\right) = \alpha^{(1)}_X(a)$$

$$\left(\beta^{(2)} \circ v\right)_X(a) = \beta^{(2)}_X\left(v_X(a)\right) = \alpha^{(2)}_X(a)$$

Since $\beta^{(1)}_X$ and $\beta^{(2)}_X$ are projections, $v_X(a) = \left(\alpha^{(1)}_X(a), \alpha^{(2)}_X(a)\right)$, that is $v = u$.

Hence such u is unique, H then forms a product of F and G.

By theorem that products are unique up to isomorphism, we then have

$$(F \times G)(C) \cong F(C) \times G(C)$$

8. Proof: Fix some arbitrary $\mathbb{C}^{op}$-object X, for all $\mathbb{C}$-object A, B, we have

$$y(A\times B)(X) = \left(Hom_{\mathbb{C}}(-, A\times B)\right)(X)$$

$$= Hom_{\mathbb{C}}(X, A\times B)$$

By corollary in Awodey, the representable functor preserves binary products, that is

$$Hom_{\mathbb{C}}(X, A\times B) \cong Hom_{\mathbb{C}}(X, A) \times Hom_{\mathbb{C}}(X, B)$$

Therefore

$$y(A\times B)(X) = Hom_{\mathbb{C}}(X, A\times B)$$

$$\cong Hom_{\mathbb{C}}(X, A) \times Hom_{\mathbb{C}}(X, B)$$

$$\cong y(A)(X) \times y(B)(X)$$

$$\cong \left(y(A)\times y(B)\right)(X)$$

Hence y preserves binary products.

Similarly, for all $\mathbb{C}$-object A, B, we have

$$\left(y(A)^{y(B)}\right)(X) = \left(Hom(-, A)^{Hom(-,B)}\right)(X)$$

$$= Hom(X, A)^{Hom(X, B)}$$

$$\cong Hom(X, A^B)$$

$$\cong y(A^B)(X)$$

Hence y preserves exponential.