# Assignment 4

1. In $\mathbf{Sets}^{\mathbf{2}}$, the subobject classifier is the functor

$$\Omega : \mathbf{2} \to \mathbf{Sets}, \quad 0 \mapsto \{\{0,1\},\{1\}\}, \ 1 \mapsto \{\{1\}\}$$

with natural transformation

$$t : \Delta_{\mathbb{1}} \to \Omega \quad \text{such that}$$

$$t_0(*) = \{0,1\}, \qquad t_1(*) = \{1\}$$

where $*$ is the only element in the singleton set $\mathbb{1}$.

and $\Delta_{\mathbb{1}}$ denotes the constant functor that maps to $\mathbb{1}$.

Now prove that it indeed forms a subobject classifier.

Proof: For any $\mathbf{Sets}^2$-object $F : 2 \to \mathbf{Sets}$ and its subobject $U$ with

$m : U \rightarrowtail F$, let $\alpha : U \to \Delta_{\mathbb{1}}$ be the unique morphism towards the terminal object.

Define $u : F \to \Omega$ such that

$$u_0 : F(0) \to \Omega(0), \quad u_0(a) = \{0,1\} \text{ if } a \in m_0(U(0)), \ \{1\} \text{ otherwise.}$$

$$u_1 : F(1) \to \Omega(1), \quad u_1(a) = \{1\} \text{ for all } a \in F(1)$$

Then for some $a \in U(0)$, $b \in U(1)$

$$(t \circ \alpha)_0(a) = t_0(\alpha_0(a)) = t_0(*) = \{0,1\} = u_0(m_0(a))$$

$$(t \circ \alpha)_1(b) = t_1(\alpha_1(b)) = t_1(*) = \{1\} = u_1(m_1(b))$$

Thus $t \circ \alpha = u \circ m$.

$$\begin{tikzcd}
U \arrow[rr, "\alpha"] \arrow[dd, tail, "m"'] & & \Delta_{\mathbb{1}} \arrow[dd, "t"] \\
& G \arrow[ul, dotted, "\nu"'] \arrow[ur, "\psi"'] \arrow[dl, "\phi"] & \\
F \arrow[rr, "u"'] & & \Omega
\end{tikzcd}$$

Then for all $\mathbf{Sets}^2$-object $G$ with $\phi : G \to F$ and $\psi : G \to \Delta_{\mathbb{1}}$, such that

$$u \circ \phi = t \circ \psi$$

It follows that $\phi_0(G(0)) \subseteq m_0(U(0))$

Therefore for all $x \in G(0)$, $\phi_0(x) \in \phi_0(G(0)) \subseteq m_0(U(0))$. Thus there is $y \in U(0)$ that

$$\phi_0(x) = m_0(y)$$

Thus define $\nu : G \to U$ such that $\nu_0(x) = y$ with $\phi_0(x) = m_0(y)$.

Then $$(m \circ \nu)_0(x) = m_0(\nu_0(x)) = m_0(y) = \phi_0(x)$$

$$\Rightarrow (m \circ \nu)_0 = \phi_0$$

Suppose $! : 0 \to 1$ in poset $\mathbf{2}$. By naturality,

$$\nu_1 \circ G(!) = U(!) \circ \nu_0, \quad m_1 \circ U(!) = F(!) \circ m_0, \quad \phi_1 \circ G(!) = F(!) \circ \phi_0$$

It follows that $$m_1 \circ \nu_1 \circ G(!) = F(!) \circ m_0 \circ \nu_0$$

$$\Rightarrow m_1 \circ \nu_1 \circ G(!) = F(!) \circ \phi_0$$

$$\Rightarrow m_1 \circ \nu_1 \circ G(!) = \phi_1 \circ G(!)$$

As functors on Sets preserve epimorphisms, we have

$$m_1 \circ \nu_1 = \phi_1$$

Therefore $m \circ \nu = \phi$. And clearly $\alpha \circ \nu = \psi$ as both sides map to $*$.

Additionally, the uniqueness of $\nu$ follows from $m$ is monic. Hence such structure actually forms a pullback.

Now suppose $w : F \to \Omega$ also makes it a pullback, and $w \neq u$. Then

$$w \circ m = t \circ \alpha$$

Hence $w_1(x) = \{1\}$ for all $x \in F(1)$,

$$w_0(x) = \{0,1\} \text{ if } x \in R, \ \{1\} \text{ otherwise. where } m_0(U(0)) \subsetneq R \subseteq F(0)$$

Since $m_0(U(0)) \subsetneq R$, pick $r \in R$, $r \notin m_0(U(0))$. $w_0(r) = \{0,1\}$.

For some $\mathbf{Sets}^2$-object $G$ with $\phi : G \to F$ and $\psi : G \to \Delta_{\mathbb{1}}$, such that

$$\phi_0(a_0) = r \quad \text{and} \quad w \circ \phi = t \circ \psi$$

Then there is unique $\nu : G \to U$ that $m \circ \nu = \phi$, $\alpha \circ \nu = \psi$.

However $\phi_0(a_0) = r \notin m_0(U(0))$, yielding a contradiction.

Thus $w = u$, proved uniqueness.

2. Proof: Suppose $\phi: \mathrm{Hom}_{\mathbb{D}}(F(-), *) \to \mathrm{Hom}_{\mathbb{C}}(-, U(*))$ is a natural isomorphism, and define

$$\eta_C = \phi_{C, F(C)}(1_{F(C)})$$

For all $f: X \to Y$ in $\mathbb{C}$, by naturality of $\phi$, we have

$$\eta_Y \circ 1_{\mathbb{C}}(f) = \phi_{Y, F(Y)}(1_{F(Y)}) \circ f = UF(f) \circ \phi_{X, F(X)}(1_{F(X)}) = UF(f) \circ \eta_X$$

Thus $\eta: 1_{\mathbb{C}} \Rightarrow U \circ F$ indeed forms a natural transformation.

Consider arbitrary $\mathbb{C}$-object $C$ and $\mathbb{D}$-object $D$ with $f: C \to U(D)$.

Since $\phi$ is a natural isomorphism, $\phi_{C,D}$ is a bijection, thus there exists unique $g: F(C) \to D$ such that

$$\phi_{C,D}(g) = f$$

By naturality, we have

$$\phi_{C,D}(g) = U(g) \circ \phi_{C, F(C)}(1_{F(C)}) = U(g) \circ \eta_C$$

It follows that $f = U(g) \circ \eta_C$, the required UMP holds.

3. Proof: Consider the natural transformation

$$\eta: 1_{P(A)} \to f^{-1} \circ f$$

Clearly this is well-defined as $X \subseteq (f^{-1} \circ f)(X)$ for all $X \in P(A)$.

Then for all $X \in P(A)$, $Y \in P(B)$ with $h: X \to f^{-1}(Y)$, by definition of poset category, it is equivalent to

$$X \subseteq f^{-1}(Y)$$

$$\iff \forall x \in X,\ x \in f^{-1}(Y)$$

$$\iff \forall x \in X,\ f(x) \in Y$$

$$\iff \forall f(x) \in f(X),\ f(x) \in Y$$

$$\iff f(X) \subseteq Y$$

That is there uniquely exists $g: f(X) \to Y$ by definition of poset.

Thus $f^{-1}(g): f^{-1}f(X) \to f^{-1}(Y)$ exists uniquely, hence the following diagram commutes:

$$\begin{tikzcd} f^{-1}f(X) \arrow[dr, "f^{-1}(g)"'] & X \arrow[l, "\eta_X"'] \arrow[d, "h"] \\ & f^{-1}(Y) \end{tikzcd}$$

It follows that $f \dashv f^{-1}$.

4. Proof: Consider $P : Sets^{op} \to Sets$ and $P' : Sets^{op} \to Sets$.

Then for all sets $A$, $B$. since $P(X) \cong 2^X$ for all set $X$. we have

$$Hom_{Sets}(A, P(B)) \cong Hom_{Sets}(A, 2^B)$$

Since in Sets. $1 \times A \cong A$, by proposition 8.13 in Awodey's, we have

$$\begin{aligned} Hom_{Sets}(A, 2^B) &\cong Hom_{Sets}(1\times A, 2^B) \\ &\cong Hom_{Sets}(1 \times B, 2^A) \\ &\cong Hom_{Sets}(B, 2^A) \\ &\cong Hom_{Sets}(B, P(A)) \end{aligned}$$

By definition of opposite category, we have $Hom_{Sets}(B, P(A)) \cong Hom_{Sets^{op}}(P'(A), B)$ naturally.

Therefore,

$$Hom_{Sets}(A, P(B)) \cong Hom_{Sets^{op}}(P'(A), B)$$

yielding that $P \dashv P'$

5. Proof: Suppose $i : P(I) \to Sets/I$, $U \mapsto (i(U) : U \to I)$

First show that $i$ indeed forms a functor.

For arbitrary functions $f : U \to V$, $g : V \to W$ in $P(I)$.

by definition of slice category, $i(g) : i(V) \to i(W)$ satisfies

$$i(W) \circ i(g) = i(V)$$

Similarly $\qquad i(V) \circ i(f) = i(U)$

And $\qquad i(W) \circ i(g \circ f) = i(U)$

Hence $\qquad i(W) \circ i(g\circ f) = i(U) = i(V) \circ i(f) = i(W) \circ i(g) \circ i(f)$

Since $W$ is a $P(I)$-object, $W \subseteq I$. Thus $i(W) : W \to I$ is an injection,

i.e. monomorphism in the category Sets.

Therefore $\quad i(W) \circ i(g \circ f) = i(W) \circ i(g) \circ i(f)$ implies $i(g\circ f) = i(g) \circ i(f)$.

Additionally, $i(1_U)$ satisfies $\quad i(U) \circ i(1_U) = i(U)$. That is $i(1_U) = 1_{i(U)}$

Hence $i$ is a functor.

Now let $\sigma : Sets/I \to P(I)$, $(f : U \to I) \mapsto imf$.

and for any $Sets/I$-morphism $g : (f_1 : U \to I) \to (f_2 : V \to I)$, define

$$\sigma(g) : imf_1 \to imf_2 \text{ be the inclusion map.}$$

Clearly it is well-defined because $f_2 \circ g = f_1$ implies that $imf_1 \subseteq imf_2$, and

$$\sigma(1_f) = 1_{imf} = 1_{\sigma(f)}.$$

For all $h : (f_0 : U \to I) \to (f_1 : V \to I)$, $g : (f_1 : V \to I) \to (f_2 : W \to I)$,

$$\sigma(g \circ h) = \sigma(g) \circ \sigma(h)$$

as $\sigma(g\circ h)$, $\sigma(g)$, $\sigma(h)$ are all inclusion maps.

Hence $\sigma$ is a functor.

Now let $\eta : 1_{Sets/I} \to i \circ \sigma$ be a natural transformation.

For all $Sets/I$-object $f : U \to I$ and $V \in P(I)$ with $h : f \to i(V)$ in $Sets/I$,

$$i(V) \circ h = f \quad \text{in Sets}$$

Thus $\quad imf = im(i(V) \circ h) \subseteq im(i(V)) = V$

Hence there is a unique $P(I)$-morphism $g : imf \to V$, making following diagram commute:

$$\begin{tikzcd} i(imf) \arrow[r, "i(g)"] & i(V) \\ & f \arrow[ul, "\eta_f"] \arrow[u, "h"'] \end{tikzcd}$$

Therefore $\sigma \dashv i$.

6. Proof. Let $U : \mathbf{Cat} \to \mathbf{Graphs}$ be the forgetful functor.

Fix some graph $G$. Let $\{0\}$ be the solution set, where $0$ is the empty category.

For all small category $\mathbb{C}$ with graph homomorphism $f : G \to U\mathbb{C}$, we have the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, dotted, "\varphi"] \arrow[dr, "f"'] & U(0) \arrow[d, "U(!)"] \\ & U(\mathbb{C}) \end{tikzcd}$$

where $! : 0 \to \mathbb{C}$ is the unique morphism, $\varphi$ is the trivial graph homomorphism from $G$ to the empty graph.

Additionally as $U$ is clearly continuous, by adjoint functor theorem, we have $U$ admits a left adjoint functor $F$.

By UMP definition of adjunction, $F : \mathbf{Graphs} \to \mathbf{Cat}$ exactly produces free categories by UMP of free category.

That is, free category exists.

7. Claim that $\Delta \dashv \mathrm{ver}\varprojlim$, where 'ver' extracts the vertex of some cone.

Proof: Consider the natural transformation

$$\phi : \mathrm{Hom}(\Delta(-), *) \to \mathrm{Hom}(-, \mathrm{ver}\varprojlim(*))$$

For all $\mathbb{C}$-object $X$ and $\mathbb{C}^{\mathbb{J}}$-object $D$, we have

$$\phi_{X,D} : \mathrm{Hom}(\Delta(X), D) \to \mathrm{Hom}(X, \mathrm{ver}\varprojlim D)$$

By definition of diagonal functor $\Delta$, $\Delta(X)$ is the constant functor $\Delta(X) : \mathbb{J} \to \mathbb{C}$ such that $\Delta(X)(j) = X$ for all indices $j$.

Therefore, the only natural transformation $\alpha \in \mathrm{Hom}(\Delta(X), D)$ is just the family of morphisms of the cone of type $\mathbb{J}$ at object $X$.

By definition of limit, $\varprojlim D$ is the terminal object in $\mathrm{Cone}(D)$.

Thus there is only one morphism from the cone at $X$ to $\varprojlim D$, by definition of cone morphisms, correspondingly there's only one morphism from $X$ to $\mathrm{ver}\varprojlim D$.

Hence $\phi_{X,D}$ is a one-to-one (bijective) map for all $X$ and $D$.

Therefore $\phi$ forms a natural isomorphism and $\Delta \dashv \mathrm{ver}\varprojlim D$. $\square$

Thus, we can easily determine the unit and counit by:

$$\eta_C = \phi_{C,\Delta(C)}(1_{\Delta(C)}) = 1_C : C \to \mathrm{ver}\varprojlim \Delta(C) = C \to C$$

$$\varepsilon_D = \phi^{-1}_{\mathrm{ver}\varprojlim D, D}(1_{\mathrm{ver}\varprojlim D}) : \Delta\, \mathrm{ver}\varprojlim D \to D$$ is the natural transformation such that

$$\varepsilon_{D_i} : \mathrm{ver}\varprojlim D \to D_i \quad \text{for all } i \in \mathrm{obj}\,\mathbb{J}.$$

By duality, the left adjoint to $\Delta$ is the colimit:

$$\mathrm{ver}\varinjlim \dashv \Delta$$

And $\psi : \mathrm{Hom}(\mathrm{ver}\varinjlim(*), -) \to \mathrm{Hom}(*, \Delta(-))$ is a natural isomorphism.

Therefore, we can determine the units and counits:

$$\eta_D = \psi_{D, \mathrm{ver}\varinjlim D}(1_{\mathrm{ver}\varinjlim D}) : D \to \Delta\, \mathrm{ver}\varinjlim D$$ is a natural transformation that

$$\eta_{D_i} : D_i \to \mathrm{ver}\varinjlim D \quad \text{for all } i \in \mathrm{obj}\,\mathbb{J}.$$

$$\varepsilon_C = \psi^{-1}_{\Delta(C), C}(1_{\Delta(C)}) = 1_C : C \to C$$