Assignment 2

1. Proof: ($\Rightarrow$) Suppose $A \xrightarrow[c_1]{} C \xleftarrow[c_2]{} B$ forms a coproduct.

Consider $g: \mathrm{Hom}(C,Z) \to \mathrm{Hom}(A,Z) \times \mathrm{Hom}(B,Z)$, $f \mapsto \langle f\circ c_1, f\circ c_2\rangle$.

Let $h: \mathrm{Hom}(A,Z)\times\mathrm{Hom}(B,Z) \to \mathrm{Hom}(C,Z)$, $\langle z_1, z_2\rangle \mapsto [z_1, z_2]$

where $[z_1,z_2]: C \to Z$ is the unique morphism determined by z_1 and z_2 according to the coproduct.

Then for all $\langle z_1, z_2\rangle \in \mathrm{Hom}(A,Z)\times\mathrm{Hom}(B,Z)$, $f \in \mathrm{Hom}(A,Z)$

$(g\circ h)(\langle z_1,z_2\rangle) = g([z_1,z_2]) = \langle [z_1,z_2]\circ c_1, [z_1,z_2]\circ c_2\rangle = \langle z_1, z_2\rangle$

$(h\circ g)(f) = h(\langle f\circ c_1, f\circ c_2\rangle) = [f\circ c_1, f\circ c_2]$

By definition of coproduct, $[f\circ c_1, f\circ c_2]$ is unique with

$$[f\circ c_1, f\circ c_2]\circ c_1 = f\circ c_1, \quad [f\circ c_1, f\circ c_2]\circ c_2 = f\circ c_2$$

Thus $f = [f\circ c_1, f\circ c_2]$, yielding that $(h\circ g)(f) = f$

Therefore $g\circ h = 1_{\mathrm{Hom}(A,Z)\times\mathrm{Hom}(B,Z)}$, $h\circ g = 1_{\mathrm{Hom}(C,Z)}$.

It follows that g is an isomorphism.

($\Leftarrow$) Conversely, suppose g is an isomorphism. Then

$$g^{-1}: \mathrm{Hom}(A,Z)\times\mathrm{Hom}(B,Z) \to \mathrm{Hom}(C,Z) \text{ exists.}$$

For all Z with $z_1: A\to Z$, $z_2: B \to Z$, we have

$$\langle z_1, z_2\rangle = (g\circ g^{-1})(\langle z_1,z_2\rangle) = (g^{-1}(z_1,z_2)\circ c_1,\ g^{-1}(z_1,z_2)\circ c_2)$$

$$\Rightarrow \quad z_1 = g^{-1}(z_1,z_2)\circ c_1 \quad \text{and} \quad z_2 = g^{-1}(z_1,z_2)\circ c_2$$

Now assume there is some $u: C \to Z$ such that

$$z_1 = u\circ c_1 \quad \text{and} \quad z_2 = u\circ c_2$$

Then $u = (g^{-1}\circ g)(u) = g^{-1}(g(u)) = g^{-1}(\langle u\circ c_1, u\circ c_2\rangle) = g^{-1}(\langle z_1, z_2\rangle)$

Therefore there exists unique $g^{-1}(\langle z_1,z_2\rangle)$ such that

$$z_1 = g^{-1}(z_1,z_2)\circ c_1 \quad \text{and} \quad z_2 = g^{-1}(z_1,z_2)\circ c_2$$

which yields that $A \xrightarrow{c_1} C \xleftarrow{c_2} B$ forms a coproduct.

2. Proof: In the category Ab of abelian groups, suppose $f, g: A \to B$ are group homomorphisms, denote group composition law as "$+$". Let

$$h: \ker(f-g) \to A, \quad a \mapsto a.$$

For all $a \in \ker(f-g)$, $(f-g)(a) = 0_B$. It follows that

$f(a) = g(a)$, and $f(h(a)) = g(h(a))$, hence $f\circ h = g\circ h$.

For all $Z \in \mathrm{obj}\,\mathrm{Ab}$ with $z: Z \to A$, $f\circ z = g\circ z$, let

$$u: Z \to \ker(f-g), \quad x \mapsto z(x)$$

Observe that this u is well-defined as $f\circ z = g\circ z$ implies that

$z(x) \in \ker(f-g)$ for all $x \in Z$.

and it is a group homomorphism as z is.

We have $(h\circ u)(x) = h(u(x)) = u(x) = z(x)$ for all $x \in Z$.

Hence $h\circ u = z$.

Now suppose there is some $v: Z \to \ker(f-h)$ with $h \circ v = z$.

Then $u(x) = z(x) = (h\circ v)(x) = h(v(x)) = v(x)$

That is such $u: Z \to \ker(f-g)$ is unique.

Hence by definition $h: \ker(f-g) \to A$ forms an equalizer to $A \overset{f}{\underset{g}{\rightrightarrows}} B$.

3. Proof: Suppose $R \subseteq A \times A$ is a binary relation, and we have two projections:

$$R \overset{f}{\underset{g}{\rightrightarrows}} A$$

Let $\pi: A \longrightarrow A/\langle R \rangle$, $a \mapsto [a]$ be the canonical map.

Then $(\pi \circ f)((a_1, a_2)) = \pi(a_1) = [a_1]$

$(\pi \circ g)((a_1, a_2)) = \pi(a_2) = [a_2]$

Since $\langle R \rangle$ is generated by R, for all $(a_1, a_2) \in R$, we have

$(a_1, a_2) \in \langle R \rangle$, thus $a_1 \sim_{\langle R \rangle} a_2$, $[a_1] = [a_2]$

Hence $(\pi \circ g)((a_1, a_2)) = (\pi \circ f)((a_1, a_2))$ for all $(a_1, a_2) \in R$,

yielding that $\pi \circ g = \pi \circ f$.

For all set Z with $z: A \to Z$, $z \circ f = z \circ g$. let

$u: A/\langle R \rangle \to Z$, $[a] \longmapsto z(a)$

Clearly since $z \circ f = z \circ g$, for all $a_1 \sim_{\langle R \rangle} a_2$, $(a_1, a_2) \in R$,

$z(a_1) = z(f(a_1, a_2)) = z(g(a_1, a_2)) = z(a_2)$

So u is a well-defined function.

For all $a \in A$, $(u \circ \pi)(a) = u(\pi(a)) = u([a]) = z(a)$

Hence $u \circ \pi = z$.

Now suppose $v: A/\langle R \rangle \longrightarrow Z$ also satisfies $v \circ \pi = z$.

Then $u([a]) = z(a) = (v \circ \pi)(a) = v([a])$ for all $[a] \in A/\langle R \rangle$

That is $u = v$, u is unique.

Therefore the canonical map π with $A/\langle R \rangle$ form a coequalizer for $R \overset{f}{\underset{g}{\rightrightarrows}} A$.

4. Proof: Clearly $f \sim g$ implies $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$.

Suppose $f, g: X \to Y$ with $f \sim g$. For all $a: A \to X$, $b: Y \to B$,

clearly, $\mathrm{dom}(b \circ f \circ a) = A = \mathrm{dom}(b \circ g \circ a)$

$\mathrm{cod}(b \circ f \circ a) = B = \mathrm{cod}(b \circ g \circ a)$

Then for all category $\mathbb{E}$ and functor $H: \mathbb{D} \to \mathbb{E}$ such that $H \circ F = H \circ G$,

we have $H(f) = H(g)$ as $f \sim g$. Then,

$H(b \circ f \circ a) = H(b) \circ H(f) \circ H(a) = H(b) \circ H(g) \circ H(a) = H(b \circ g \circ a)$

Therefore $b \circ f \circ a \sim b \circ g \circ a$, $\sim$ defines a congruence.

Now in $\mathbf{Cat}$, consider $\mathbb{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathbb{D} \xrightarrow{\Pi} \mathbb{D}/\sim$

where $\Pi: \mathbb{D} \to \mathbb{D}/\sim$ is the canonical functor.

For all $C \in \mathrm{obj}\,\mathbb{C}$, $f: C \to C' \in \mathrm{mor}\,\mathbb{C}$

$(\Pi \circ F)(C) = \Pi(F(C)) = \Pi(G(C)) = (\Pi \circ G)(C)$

Observe that $F(f) \sim G(f)$ as $H \circ F = H \circ G$ implies $H(F(f)) = H(G(f))$

for all functor H. Hence $[F(f)] = [G(f)]$. Then,

$(\Pi \circ F)(f) = \Pi(F(f)) = [F(f)] = [G(f)] = \Pi(G(f)) = (\Pi \circ G)(f)$

Thus $\Pi \circ F = \Pi \circ G$.

Now for all category $\mathbb{Z}$ with $H: \mathbb{D} \to \mathbb{Z}$. such that

$H \circ F = H \circ G$.

Define $U: \mathbb{D}/\sim \longrightarrow \mathbb{Z}$, such that

$U(D) = H(D)$ for all $D \in \mathrm{obj}\,\mathbb{D}$

$U([f]) = H(f)$ for all $[f] \in \mathrm{mor}\,\mathbb{D}$

Since $H \circ F = H \circ G$, for all $f \sim g$ in $\mathbb{D}$, $H(f) = H(g)$, U is well-defined.

For all $D \in \mathrm{obj}\,\mathbb{D}$, $f \in \mathrm{mor}\,\mathbb{D}$,

$(U \circ \Pi)(D) = U(\Pi(D)) = U(D) = H(D)$

$(U \circ \Pi)(f) = U(\Pi(f)) = U([f]) = H(f)$

Hence $U \circ \Pi = H$. Now assume $W: \mathbb{D}/\sim \longrightarrow \mathbb{Z}$ also satisfies

$W \circ \Pi = H$

Then for all $D \in \mathrm{obj}\,\mathbb{D}$, $f \in \mathrm{mor}\,\mathbb{D}$,

$U(D) = H(D) = (W \circ \Pi)(D) = W(D)$

$U([f]) = H(f) = (W \circ \Pi)(f) = W([f])$

That is $U = W$, thus U is unique. Therefore $\mathbb{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathbb{D} \xrightarrow{\Pi} \mathbb{D}/\sim$ forms a coequalizer.

5. Proof: In any category, suppose in a pullback square m' is monic.

$$\begin{tikzcd} M' \arrow[r, "g"] \arrow[d, "m'"'] & M \arrow[d, "m"] \\ A' \arrow[r, "f"'] & A \end{tikzcd}$$

Consider any $x, y : X \to M$, such that $m \circ x = m \circ y$

Let $z : X \to A'$ such that $f \circ z = m \circ x = m \circ y$

$$\begin{tikzcd} X \arrow[dr, dashed, "u_1"', shift right] \arrow[dr, dashed, "u_2", shift left] \arrow[drr, "x"] \arrow[drr, bend left, "y"] \arrow[ddr, bend right, "z"'] & & \\ & M' \arrow[r, "g"] \arrow[d, "m'"'] & M \arrow[d, "m"] \\ & A' \arrow[r, "f"'] & A \end{tikzcd}$$

By UMP of pullback square, there exists unique $u_1, u_2 : X \to M'$

with $x = g \circ u_1$, $z = m' \circ u_1$ and $y = g \circ u_2$, $z = m' \circ u_2$

Hence $z = m' \circ u_1 = m' \circ u_2$

Since m' is monic, $u_1 = u_2$

Thus $x = g \circ u_1 = g \circ u_2 = y$

Therefore m is also monic.

6. Proof: (Identity) For each $A \in \mathrm{obj}\, Par(\mathbb{C})$, with $|g| : U_g \to B \in \mathrm{mor}\, \mathbb{C}$.

Consider the morphism $(1_A, A)$:

$$\begin{tikzcd} A \times_A U_g \arrow[r, "p_2"] \arrow[dd, "p_1"'] & & U_g \arrow[r, "|g|"] \arrow[dd, "m"] & B \\ & U_g \arrow[ul, dashed, "u"'] \arrow[ur, "1_{U_g}"] \arrow[dl, "m"] & & \\ A \arrow[rr, "|1_A|"'] \arrow[d, "1_A"'] & & A \arrow[uur, "g"'] & \\ A \arrow[urr, "1_A"'] & & & \end{tikzcd}$$

Since $A \times_A U_g$ is a pullback, for $1_{U_g} : U_g \to U_g$ and $m : U_g \to A$,

there exists unique $u : U_g \to A \times_A U_g$ such that

$$\begin{cases} 1_{U_g} = p_2 \circ u \\ m = p_1 \circ u \\ m \circ p_2 = |1_A| \circ p_1 = p_1 \end{cases}$$

Thus $p_1 \circ u \circ p_2 = p_1$, it follows that $u \circ p_2 = 1_{A \times_A U_g}$

Therefore p_2 forms an isomorphism, $A \times_A U_g \cong U_g$.

Hence $(|g|, U_g) \circ (|1_A|, A) = (|g \circ 1_A|, A \times_A U_g) = (|g|, U_g)$, showing that

$(|1_A|, A)$ is a well-defined identity in $Par(\mathbb{C})$.

(Associativity) Consider the following diagram in $\mathbb{C}$:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U_g \times_C U_h \arrow[r] \arrow[d] & U_h \arrow[r, "|h|"] \arrow[d] & D \\ U_f \times_B U_g \arrow[r] \arrow[d] & U_g \arrow[r, "|g|"'] \arrow[d] & C \arrow[ur, "h"'] & \\ U_f \arrow[r, "|f|"'] \arrow[d] & B \arrow[ur, "g"'] & & \\ A \arrow[ur, "f"'] & & & \end{tikzcd}$$

By two-squares lemma,

$U \cong (U_f \times_B U_g) \times_C U_h \cong (U_f \times_B U_g) \times_{U_g} (U_g \times_C U_h) \cong U_f \times_B (U_g \times_C U_h)$

Therefore, $\big((|f|, U_f) \circ (|g|, U_g)\big) \circ (|h|, U_h)$

$= (|f \circ g|, U_f \times_B U_g) \circ (|h|, U_h)$

$= (|(f \circ g) \circ h|, (U_f \times_B U_g) \times_C U_h)$

$= (|f \circ (g \circ h)|, U_f \times_B (U_g \times_C U_h))$

$= (|f|, U_f) \circ \big((|g|, U_g) \circ (|h|, U_h)\big)$

That is, the composition is associative. Hence $Par(\mathbb{C})$ is a category.

7. Proof: Suppose $f, g : B \to C$ such that $c_j \circ f = c_j \circ g$ for all $c_j : C \to D_j$.

Let $b_j = c_j \circ f = c_j \circ g$. Then (B, b_j) forms a cone object in $\mathrm{Cone}(D)$.

$$\begin{tikzcd}
B \arrow[r] \arrow[dr, "b_j"'] & C \arrow[d, "c_j"] \\
 & D_j
\end{tikzcd}$$

And by definition, f, g form morphisms from (B, b_j) to (C, c_j)

Since (C, c_j) is a limiting cone, (C, c_j) is a terminal object in $\mathrm{Cone}(D)$, thus there's a unique morphism from (B, b_j) to (C, c_j).

Therefore, $f = g$.