Stats 531 Homework 2

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Contents

Question 2.1.

Use the providede two methods to solve the casual AR(1) model with the error term to be a white noise process

$$X_n = \phi X_{n-1} + \epsilon_n,$$

where ϵ_n is white noise and $\text{Var}[\epsilon_n] = \sigma^2$.

A. Based on the distributive property for covariance equation that

$$\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z).$$

Thus we can expand the ACF with lag h in this form:

$$\begin{aligned}\gamma_h = \text{Cov}(X_n, X_{n+h}) &= \text{Cov}(X_n, \phi X_{n+h-1} + \epsilon_{n+h}), \text{ for } h > 0 \\ &= \text{Cov}(X_n, \phi X_{n+h-1}) + \text{Cov}(X_n, \epsilon_{n+h}).\end{aligned}$$

Since the error term based on the stationary process, we can see that $\text{Cov}(X_n, \epsilon_{n+h}) = 0$. Therefore by another property of covariance,

$$\begin{aligned}\gamma_h &= \text{Cov}(X_n, \phi X_{n+h-1}) + \text{Cov}(X_n, \epsilon_{n+h}) \\ &= \text{Cov}(X_n, \phi X_{n+h-1}) \\ &= \phi \text{Cov}(X_n, X_{n+h-1}) \\ &= \phi \gamma_{h-1}.\end{aligned}$$

So we can see the equation can be reduced to a simple the stochastic difference equation $\gamma_h = \phi \gamma_{h-1}$. Then suppose the solution is given by $\gamma_h = A\lambda^h$, we can have

$$\gamma_h = A\lambda^h = \phi \gamma_{h-1} = \phi A \lambda^{h-1}$$

Substituting this general solution and we can get $\gamma_0 = A$. To solve the above equation, we can see that $\lambda = \phi$ holds. So now we need to find the value for A. Again, based on the covariance property,

$$\text{Cov}(aX + Y, aX + Y) = a^2 \text{Cov}(X, X) + \text{Cov}(Y, Y) + 2a\text{Cov}(X, Y),$$

we can do this by relate γ_0 with the covariance form:

$$\begin{aligned}\gamma_0 = A &= \text{Cov}(X_n, X_n) \\ &= \phi^2 \text{Cov}(X_{n-1}, X_{n-1}) + \text{Cov}(\epsilon_n, \epsilon_n) \\ &= \phi^2 \gamma_0 + \sigma^2.\end{aligned}$$

So $\gamma_0 = \frac{\sigma^2}{1-\phi^2} = A$. Therefore, we can solve for $A = \frac{\sigma^2}{1-\phi^2}$ and

$$\gamma_h = A\lambda^h = \frac{\sigma^2}{1-\phi^2}\phi^h = \frac{\sigma^2 \phi^h}{1-\phi^2}.$$

B.

Based on what we derived in class, using the stationary condition, we can write X_n into a summation of error terms:

$$X_n = \sum_{k=0}^{\infty} \phi^k \epsilon_{n-k},$$

since when k approaches to infinity and given that the absolute value of ϕ is smaller than 1, the $\phi^k X_{n-k}$ will vanish.

Try to expand this Taylor series, we can get

$$g(x) = (1-\phi x)^{-1} \approx g(0) + \frac{g'(0)}{1!}x + \frac{g''(0)}{2!}x^2 + ...$$

Since it is a special case of geometric series, where we know, by definition, that $\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + ...$ if $|x| < 1$. So here we can regard ϕx as a whole since ϕ is a constant.

$$\frac{1}{1-\phi x} \approx 1 + \phi x + (\phi x)^2 + (\phi x)^3 + ... = \sum_{n=0}^{\infty} (\phi x)^n.$$

Since we have seen in the slides the following equation holds when h is larger or equal to 0,

$$\gamma_h = \text{Cov}(X_n, X_{n+h}) = \sum_{j=0}^{\infty} \phi_j \phi_{j+h} \sigma^2 = \sum_{j=0}^{\infty} \phi^{2k+h} \sigma^2.$$

So combined the above formulas, we can derive that

$$\begin{aligned}
\gamma_h &= \sum_{j=0}^{\infty} \phi^{2j+h} \sigma^2 \\
&= \sigma^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} \\
&= \sigma^2 \phi^h g(\phi) \\
&= \frac{\sigma^2 \phi^h}{1-\phi^2}.
\end{aligned}$$

C. Since the **ARMAacf** is a function to calculate the autocorrelation function in R, based on the above two problems, we can derive the autocorrelation function to be $\rho_h = \frac{\gamma_h}{\gamma_0} = \phi^h$. So we can compare the values, the one using this formula, and the other purely based on the R result:

```
set.seed(12345678)
# ARMAacf result:
r1 <- ARMAacf(ar = c(0.6), lag.max = 50)
# Formula result:
phi <- 0.6
h <- c(0:50)
r2 <- phi^h
print(r1)
```

```
##            0            1            2            3            4
## 1.000000e+00 6.000000e-01 3.600000e-01 2.160000e-01 1.296000e-01
##            5            6            7            8            9
## 7.776000e-02 4.665600e-02 2.799360e-02 1.679616e-02 1.007770e-02
##           10           11           12           13           14
## 6.046618e-03 3.627971e-03 2.176782e-03 1.306069e-03 7.836416e-04
##           15           16           17           18           19
```

```
## 4.701850e-04 2.821110e-04 1.692666e-04 1.015600e-04 6.093597e-05
##           20           21           22           23           24
## 3.656158e-05 2.193695e-05 1.316217e-05 7.897302e-06 4.738381e-06
##           25           26           27           28           29
## 2.843029e-06 1.705817e-06 1.023490e-06 6.140942e-07 3.684565e-07
##           30           31           32           33           34
## 2.210739e-07 1.326444e-07 7.958661e-08 4.775197e-08 2.865118e-08
##           35           36           37           38           39
## 1.719071e-08 1.031442e-08 6.188655e-09 3.713193e-09 2.227916e-09
##           40           41           42           43           44
## 1.336749e-09 8.020497e-10 4.812298e-10 2.887379e-10 1.732427e-10
##           45           46           47           48           49
## 1.039456e-10 6.236738e-11 3.742043e-11 2.245226e-11 1.347135e-11
##           50
## 8.082813e-12
```

```r
print(r2)
```

```
##  [1] 1.000000e+00 6.000000e-01 3.600000e-01 2.160000e-01 1.296000e-01
##  [6] 7.776000e-02 4.665600e-02 2.799360e-02 1.679616e-02 1.007770e-02
## [11] 6.046618e-03 3.627971e-03 2.176782e-03 1.306069e-03 7.836416e-04
## [16] 4.701850e-04 2.821110e-04 1.692666e-04 1.015600e-04 6.093597e-05
## [21] 3.656158e-05 2.193695e-05 1.316217e-05 7.897302e-06 4.738381e-06
## [26] 2.843029e-06 1.705817e-06 1.023490e-06 6.140942e-07 3.684565e-07
## [31] 2.210739e-07 1.326444e-07 7.958661e-08 4.775197e-08 2.865118e-08
## [36] 1.719071e-08 1.031442e-08 6.188655e-09 3.713193e-09 2.227916e-09
## [41] 1.336749e-09 8.020497e-10 4.812298e-10 2.887379e-10 1.732427e-10
## [46] 1.039456e-10 6.236738e-11 3.742043e-11 2.245226e-11 1.347135e-11
## [51] 8.082813e-12
```

We can say, after comparing the results, that the two results give the same answer to this question.

Question 2.2.

We have learned that the random walk model has an exact solution that $X_n = \sum_{k=1}^{n} \epsilon_k$ for $X_n = X_{n-1} + \epsilon_n$ with $X_0 = 0$.

Therefore, since $\{\epsilon_n\}$ is white noise with variance σ^2, we can expand the acf function $\gamma_{m,n}$ by:

$$\begin{aligned}
\gamma_{m,n} &= \text{Cov}(X_m, X_n) \\
&= \text{Cov}(\sum_{k=1}^{m} \epsilon_k, \sum_{j=1}^{n} \epsilon_j) \\
&= \sum_{k=1}^{m} \sum_{j=1}^{n} \text{Cov}(\epsilon_k, \epsilon_j)
\end{aligned}$$

.

The white noise condition regulates the number of σ^2 to be summatized, since only $m = n$ will give $\text{Cov}(\epsilon_m, \epsilon_n) = \sigma^2 \neq 0$, so

$$\begin{aligned}
\gamma_{m,n} &= \sum_{k=1}^{m} \sum_{j=1}^{n} \text{Cov}(\epsilon_k, \epsilon_j) \\
&= min(m, n)\sigma^2.
\end{aligned}$$

Question 2.3.

Question 2.1 Part A

I review the basic covariance property from the instruction of last homework and the url https://web.calpoly.edu/~dsun09/lessons/properties_of_covariance.pdf.

Question 2.1 Part B

I check the Taylor series expansion from https://en.wikipedia.org/wiki/Taylor_series,

and check the geometric series expansion at

https://ocw.mit.edu/courses/mathematics/18-01sc-single-variable-calculus-fall-2010/unit-5-exploring-the-infinite/part-b-taylor-series/session-99-taylors-series-continued/MIT18_01SCF10_Ses99b.pdf.

Question 2.1 Part C

I learn the usage of **ARMAacf** in R from http://www.craigmile.com/peter/teaching/6550/notes/acf_pacf_in_R.pdf.

Question 2.2.

For this part, I did all the job myself without any onlint reference except the lecture slides.