STATS 551

Basics of Bayesian Inference

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Overview

1. Bayesian Data Analysis
2. Three Steps of Bayesian Analysis
3. Examples

Bayesian Data Analysis

Quantities we observe

Data.

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Data.

Quantities we wish to learn

Parameters.

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- Bayesian methods provide
  - models for rational, quantitative learning
  - estimators that work for small and large sample sizes
  - methods for generating statistical procedures in complicated problems

Bayesian Data Analysis: Example

A clinical trial of a new cancer drug.

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- Bayesian methods:

  quantify uncertainty in the estimated survival probabilities.

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Greek letters: parameters, e.g. θ.
Roman letters: observations, e.g. y_i, $1 \leq i \leq n$.

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For $1 \le i \le n$, $y_i = 1$ if alive and 0 otherwise. θ is probability of survival.

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Mathematically

$$y_i \stackrel{i.i.d.}{\sim} p(\cdot|\theta), 1 \leq i \leq n; \quad \theta \sim p(\cdot).$$

$p(\cdot|\theta)$: conditional probability density (distribution); $p(\cdot)$: marginal distribution. Same notation for continuous & discrete densities.

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Bayes Rule

$$p(\theta|y) = \frac{p(\theta, y)}{p(y)} = \frac{p(\theta)p(y|\theta)}{p(y)},$$

where $p(y) = \sum_\theta p(\theta)p(y|\theta)$, $y = \{y_1, y_2, \cdots\}$.

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likelihood: $p(y|\theta)$ — clinical trial example, $p(y|\theta) = \prod_{i=1}^n p(y_i|\theta)$.

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Numerical formulation of joint beliefs about y and θ:

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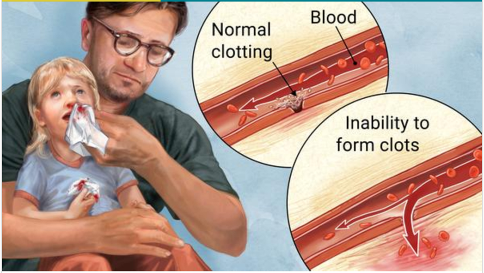
Once we obtain data y, the last step is to update our beliefs about θ:

- For each value of θ, our posterior distribution $p(\theta|y)$ describes our belief that θ is the true value, after having observed data y.

Example: inference about a genetic status

Hemophilia

ABOUT SYMPTOMS TREATMENTS

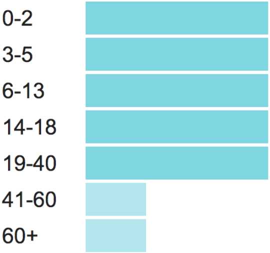


A disorder in which blood doesn't clot normally.

Rare

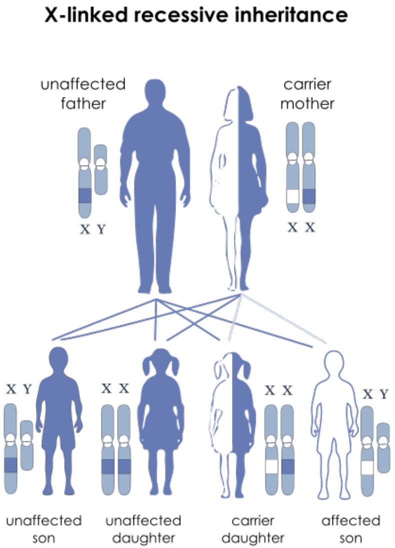
Fewer than 200,000 US cases per year

Ages affected



Genders affected

Example: inference about a genetic status



Humans

Male: XY chromosome.

Female: XX chromosome.

Hemophilia

- Male with the disease-causing gene on X: affected.
- Female with the disease-causing gene on one of two X: not affected.
- Female with the disease-causing gene on both two X: affected.

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  She has fifth-fifty chance of having the gene.
- Prior distribution for θ:

$$P(\theta = 1) = P(\theta = 0) = \frac{1}{2}.$$

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Neither of her two sons is affected ($y_1 = y_2 = 0$).

- Two sons: independent and not identical twins.

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Likelihood

$$P(y_1 = y_2 = 0|\theta = 1) = 0.5 \times 0.5 = 0.25,$$
$$P(y_1 = y_2 = 0|\theta = 0) = 1 \times 1 = 1.$$

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\begin{aligned}
P(\theta = 1|y) &= \frac{P(y|\theta = 1)P(\theta = 1)}{P(y|\theta = 1)P(\theta = 1) + P(y|\theta = 0)P(\theta = 0)} \\
&= \frac{0.25 * 0.5}{0.25 * 0.5 + 1 * 0.5} = 0.2,
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which is smaller than 0.5 (given by the prior).

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Another unaffected son.

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$$P(\theta = 1|y_1, y_2, y_3) = \frac{0.5 * 0.2}{0.5 * 0.2 + 1 * 0.8} = 0.111.$$

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- misspelling or mistyping of “random” or “radon”?
- intentional typing of “radom”

What is the probability that “radom” actually means “random”?

Label y as the data and θ as the word that the person was intending to type, then

$$Pr(\theta|y = “radom'') \propto p(\theta)Pr(y =' radom'|\theta).$$

This product is the unnormalized posterior density.

Example: spelling correction

Let (θ_1 = random, θ_2 = radon, and θ_3 = radom), then

$$p(random|``radom") = \frac{p(\theta_1)p(``radom"|\theta_1)}{\sum_{j=1}^{3} p(\theta_j)p(``radom"|\theta_j)}.$$

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Specify "Priors" $p(\theta)$ and "Likelihood" $p(``radom"|\theta)$.

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Prior: relative word frequency

θ	random	radon	radom
$p(\theta)$	7.60×10^{-5}	6.05×10^{-6}	3.12×10^{-7}

Example: spelling correction

Likelihood: Google spelling & tying error model

θ	random	radon	radom
p("radom" $\|\theta$)	0.00193	0.000143	0.975

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θ	random	radon	radom
p(“radom” $\vert\theta$)	0.00193	0.000143	0.975

Posterior distribution.

θ	random	radon	radom
$p(\theta)$ p(“radom” $\vert\theta$)	1.47×10^{-7}	8.65×10^{-10}	3.04×10^{-7}
p($\theta\vert$ “radom”)	0.325	0.002	0.673

Example: estimating percentage of Dunkin' lovers

Prior Knowledge + Data = Current Knowledge