A Mathematical Model of Germany's Population Growth with Periodic Environmental Forcing

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1 Introduction

In this report, we employ periodic forced logic equations to simulate the population changes in Germany, attempting to identify the patterns of its population changes. According to the classic logistic model, the number of living organisms will keep rising until they reach the limit that the materials in the environment can accommodate and then stop. The growth rate contains an added periodic term that captures population behavior changes that stem from outside influences. The new method enhances our ability to match empirical data while identifying repeating patterns and boosting model accuracy beyond what static models can achieve. We aim to apply this advanced model to studying Germany's population trends across extended periods while examining the effects of recurring external factors on demographic changes.

2 Problem Statement

Long-term planning efforts are based on understanding and predicting population growth to guide resource allocation and infrastructure development and shape social policy. The traditional population model explains how the population reaches saturation and stops growing when resources are limited. However, these models assume that the growth conditions are constant and will continue over time, but the actual population growth rate is not constant, it is constantly changing over time.

Our study examines Germany's population dynamics through a logistic model with growth rate adjustments that capture cyclical factors, including economic trends, policy changes, and migration patterns. By incorporating a sinusoidal term into the growth rate, the model allows us to describe observed variations in population change over time more accurately.

Our objective is to use real demographic data to estimate key parameters in the model, including the intrinsic growth rate, the carrying capacity, and the characteristics of the periodic fluctuation (amplitude and frequency). This method aims to improve the mathematical model of the population in more fit with the actual situation by using long-term data.

3 Assumptions and Constraints

In this project, we will consider that the population dynamics of Germany can be treated as a periodically forced logistic growth model. Carrying capacity is defined as the maximum sustainable population for the duration of the study period, suggesting that the availability of resources, infrastructure, and environmental constraints does not fluctuate significantly over the study period. In addition, we assume there are no major shocking events (e.g., wars, pandemics, steep economic collapse, etc.) over this time period that would greatly affect population path. The periodic term in the model helps capture more general cyclical patterns in growth rates, rather than localized sharp or nonlinear deviations. Further, we consider the population as a continuous quantity, appropriate for a differential equation model, and assume that the data available are accurate and reflect the real population.

4 Variables and Parameters

The variables and parameters used in the periodic forced logistic population model are summarized below.

Variable	**Description**	**Unit**
P	Population size at time t	people
t	Time (years since 1955)	years

Table 1: Model variables used in the population growth equation.

Parameter	**Description**	**Unit**
r_0	Baseline intrinsic growth rate (no oscillation)	year^{-1}
A	Amplitude of seasonal variation via sine function	year^{-1}
B	Amplitude of seasonal variation via cosine function	year^{-1}
ω	Angular frequency of oscillation	radians/year
ϕ	Phase shift (horizontal displacement of oscillation)	radians
K	Carrying capacity: maximum sustainable population	people

Table 2: Model parameters governing growth rate dynamics and oscillatory forcing.

5 Mathematical Tools

To model the population dynamics of Germany, we start with the classic logistic growth model:

$$\frac{dP}{dt} = r \cdot P\left(1 - \frac{P}{K}\right), \quad [1]$$

where:

- $P(t)$ is the population at time t,
- r is the intrinsic growth rate,
- K is the carrying capacity of the population.

However, historical population data for Germany show clear signs of periodic fluctuations (see Figure 2), influenced by factors such as economic cycles, policy changes, migration patterns, and demographic changes. To account for these oscillations, we extend the logistic model by allowing the growth rate r to vary periodically over time.

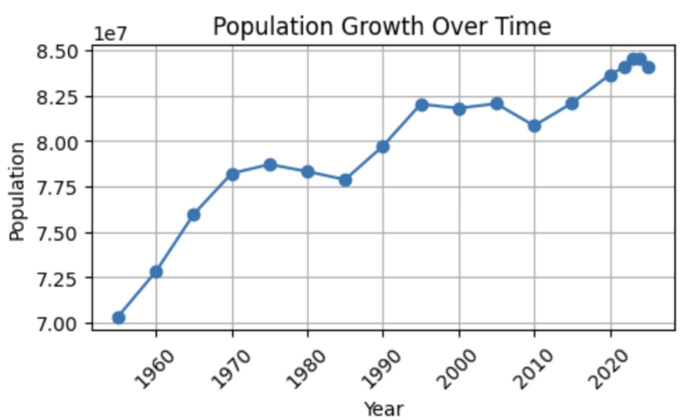

Figure 1: Population growth of Germany from 1955 to 2025 showing oscillatory behavior.

The modified growth rate $r(t)$ is defined as:

$$r(t) = r_0 + A\sin(\omega t + \phi) + B\cos(\omega t + \phi),$$

where:

- r_0 is the baseline growth rate,
- A and B are the amplitudes of the sine and cosine components,

- ω is the angular frequency,
- ϕ is the phase shift, representing a horizontal shift in the oscillation.

This equation introduces periodic changes in the growth rate, enabling the model to better capture the real changes in population trends. We added sine, cosine, and phase shift ϕ to the equation to make this logical equation better fit the real population data.

The final differential equation becomes

$$\frac{dP}{dt} = [r_0 + A\sin(\omega t + \phi) + B\cos(\omega t + \phi)] \cdot P\left(1 - \frac{P}{K}\right), \text{(Equation 2)}$$

and is solved numerically to fit the observed population data using least-squares optimization.

6 Analysis and Assessment

In the chapter on the mathematical approach, we introduced r(t) as a time variable in the original logistic growth model to represent the changing trends of the total population of Germany over time, as highlighted in the dataset. Also, due to the addition of r(t), the optimal parameters we need to find an increase from the initial K to r_0, A, B, ω, ϕ and K. We created a cost function by using equation (2) and used the least-squares optimization to find the optimal parameters, enabling our equation to represent the characteristics and trends shown by the data.

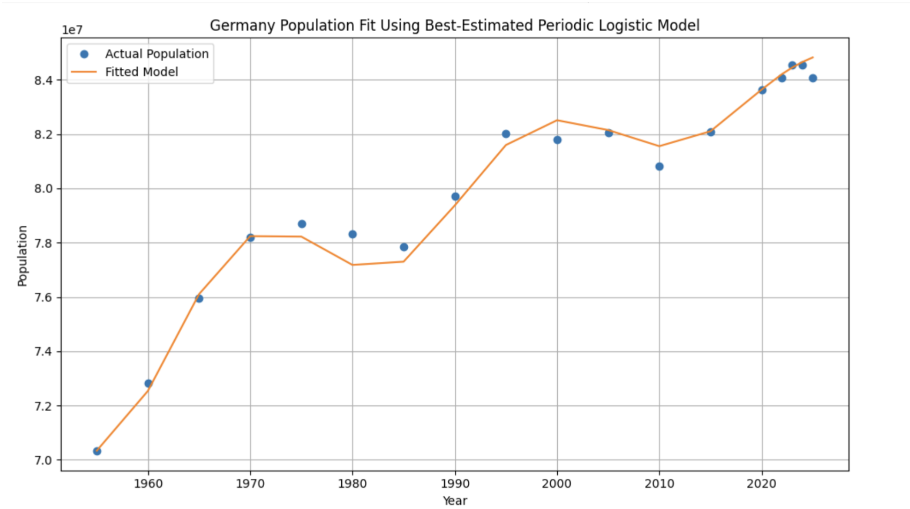

Figure 2: Fitted population curve using the optimized periodically forced logistic model. The model is fitted to Germany's population data using a least-squares procedure.

Figure 2 shows the curve of Equation 2 after optimizing the parameters using least-squares. We find that after using the optimal parameters, our equation captures the rising and falling patterns exhibited by the data points very well. The increase and decrease of these populations correspond to the sine and cosine parameters in our equation, indicating that the underlying patterns of population changes may be due to the decline in the birth rate caused by economic growth or political reasons. Meanwhile, the values of both the starting point and the ending point of the expression are highly approximate to the data points. This also proves that our curves and data are fit with each other.

Through the least-square method, we obtain the optimal parameters belonging to our expression as: r_0 : 0.0206 $year^{-1}$, A : 0.0185 $year^{-1}$, B : -0.0418 $year^{-1}$, ω : 0.2245 $radians/year$, ϕ : 0.8578 $radians$ and K : 8.8×10^7 $people$. When using the least-squares method, the starting values and the range of values used are all obtained through a large number of manual tests. We observe the changes in the curve caused by different parameters by entering different initial values and value ranges, and summarize the appropriate initial values and value ranges.

Despite this improvement, there are limitations. In Figure 2, it can be found that the curve in our model is not smooth, which indicates the possibility of overfitting in our model. Because we overly pursue the degree of fit between the model and the data, these parameters may perform poorly if applied to the model with more data. Meanwhile, the data points we selected are also relatively few, and the data we used has a time span of five years. This will cause many details of data features to be overlooked, which is also a factor we need to consider in future models. Additionally, the chosen periodic form assumes smooth oscillations, which may not capture abrupt demographic changes caused by one-time events.

7 Conclusion

In this report, we used an extended classical logistic model by introducing a sinusoidally varying growth rate to capture the cyclical changes—economic fluctuations, policy shifts and migration trends—of Germany' s demographic trajectory. We obtained (by fitting the logistic equation to historical population data using least-squares optimization) a baseline growth rate of approximately 0.0206 $year^{-1}$, oscillation amplitudes of 0.0185 and –0.0418 $year^{-1}$, an angular frequency near 0.2245 $rad/year$, a phase shift of 0.8578 rad, and a carrying capacity on the order of 8.8×10^7. The resulting curve closely traces both the long-term trend and the regular ups and downs showed by the data.

Future work might incorporate stochastic components to account for random perturbations, use higher-frequency or auxiliary data sources to refine parameter estimates, and explicitly model the impact of singular events in a hybrid deterministic–stochastic setting. Those will create much more nuanced and predictive demographic models of Germany's population growth.

References

[1] Mark Kot, *Elements of Mathematical Ecology*, Cambridge University Press, Cambridge, 2001.