Modeling Stock Price Trajectories Using Geometric Brownian Motion (GBM)

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Introduction

Investing in the stock market involves significant uncertainty, which makes it crucial for investors to adopt strategies that maximize returns while managing risk. One of the fundamental questions in finance is: How should investors allocate their capital across different portfolios to optimize profitability over a given period?

In this paper, we address this question by modeling the price evolution of selected stocks, such as Microsoft (MSFT) and Google (GOOG), as well as BMO (BMO.TO) and CIBC (CM.TO), using Geometric Brownian Motion (GBM), a widely used stochastic process for simulating financial asset prices. Through Monte Carlo simulations, we generate multiple possible trajectories for these stocks over a five-year investment horizon. We then evaluate and compare different investment strategies, including equal investment allocation, proportional investment based on volatility, and a mean-variance optimized strategy inspired by Markowitz Portfolio Theory.

By systematically analyzing these strategies under simulated market conditions, our objective is to identify which approach yields the highest net return, offering insights for investors seeking to enhance their portfolio performance.

Variables and Parameters

The table below summarizes the key variables and parameters used in our stochastic stock price model, based on the Geometric Brownian Motion (GBM) framework.

Symbol	Description	Type
T	Time	Independent variable
X	Stock price at time t	Dependent variable
μ_i	Drift (expected return)	Parameter
σ_i	Volatility (fluctuation level)	Parameter
X_0	Initial stock price	Parameter

Table 1: Key variables and parameters used in the Geometric Brownian Motion model for simulating stock prices.

Assumptions and Constraints

In this study, we make several key assumptions to ensure the validity of our simulation and analysis. First, we assume that investors strictly adhere to the predefined investment strategies without deviating from them. Additionally, stock prices are modeled using the Geometric Brownian Motion (GBM) framework, which provides a stochastic representation of price evolution over time. We further assume that there are no external shocks or unforeseen market events that could disrupt the system. The simulation is conducted in discrete time intervals, allowing for stepwise updates of stock prices and investment values. Lastly, the drift and volatility parameters are considered constant throughout the simulation.

Mathematical Tools and Theory

Stock prices exhibit random fluctuations due to market dynamics, making their future evolution inherently uncertain. To model this behavior, we assume that stock returns follow a **Geometric Brownian Motion (GBM)**, a commonly used stochastic process in financial modeling. Given parameters μ (drift), σ (volatility), and an initial stock price X_0, the price evolution satisfies the stochastic differential equation:

$$dX = \mu X\,dt + \sigma X\,dW, \quad X(0) = X_0.$$

where dW represents a Wiener process (standard Brownian motion).

Net Gain or Loss Computation

To assess the profitability of an investment, we define the Rate of Net Gain or Net Loss:

$$\text{Rate of Net Gain or Net Loss} = \frac{\text{Current Price} - \text{Original Purchase Price}}{\text{Original Purchase Price}} \times 100\%.$$

For a given investment strategy, the total gain or loss for an individual stock is then given by:

$$\text{Amount of Net Gain or Loss} = \text{Rate of Net Gain or Net Loss} \times \text{Initial Investment} \times \text{Weight for One Stock}.$$

Equal Investment Strategy

In this approach, the investor distributes their initial capital equally among all selected stocks. That is, for a portfolio containing n stocks, each stock receives a weight of:

$$w_i = \frac{1}{n}, \quad \text{for each stock } i.$$

Proportional Investment Strategy

This strategy adjusts allocation based on the volatility of each stock. Specifically, stocks with higher volatility receive a lower weight, while more stable stocks receive a higher proportion of the total investment. The weights are computed as:

$$w_{\text{MSFT}} = \frac{\sigma_{\text{GOOG}}}{\sigma_{\text{MSFT}} + \sigma_{\text{GOOG}}}, \quad w_{\text{GOOG}} = \frac{\sigma_{\text{MSFT}}}{\sigma_{\text{MSFT}} + \sigma_{\text{GOOG}}}.$$

Here, MSFT refers to the stock of Microsoft and GOOG refers to the stock of Google. These weights ensure that capital allocation favors stocks with lower relative volatility (the degree of variation in a stock's price over time. It indicates how much a stock's price fluctuates, either going up or down), thereby reducing exposure to excessive risk.

Mean-Variance Optimized Strategy (Markowitz Portfolio Theory)

The Mean-Variance Optimization (MVO) strategy, introduced by Markowitz (1952), aims to construct an optimal portfolio by maximizing expected return while minimizing risk[1]. Given the expected return vector μ and the covariance matrix X, the optimal weight allocation is determined using:

$$w_{\text{optimal}} = \frac{X^{-1}\mu}{\sum(X^{-1}\mu)}.$$

Here:

- X is the covariance matrix of stock returns.
- X^{-1} is its inverse, which accounts for diversification benefits.
- μ is the expected return vector.
- The denominator ensures the weights sum to 1, maintaining full investment allocation.

This strategy allocates more weight to stocks with higher expected returns and lower risk while considering correlations between assets. It is widely used in modern portfolio management for constructing well-balanced, risk-adjusted investment portfolios.

By systematically evaluating these three strategies under simulated stock price trajectories, we aim to determine which allocation method yields the highest profitability over a five-year investment horizon.

[1] Markowitz, H. "Portfolio Selection." *Journal of Finance*, vol. 7, no. 1, 1952.

Discussion

Figure 1 compares stochastic and deterministic stock trajectories for Microsoft and Google. The deterministic paths represent average growth under fixed drift, while the light-colored paths reflect the variability introduced by Geometric Brownian Motion (GBM). The spread of the stochastic simulations illustrates the uncertainty and risk inherent in stock price evolution.

We implemented three investment strategies, equal, proportional (based on volatility) and mean variance optimization for two portfolios: Microsoft-Google and BMO-CIBC.

In the Microsoft-Google portfolio, all strategies performed well, with high average net returns over the 5-year simulated horizon: approximately 361% for the Equal Investment Strategy, 364% for the Proportional Strategy, and 380% for the Mean-Variance Optimized Strategy. Although the Optimized Strategy yielded the highest return on average, the difference among strategies was relatively small. The standard deviations for all three exceeded 330%, indicating significant volatility in returns due to market fluctuations. This suggests that in high-growth portfolios, the choice of allocation strategy plays a secondary role compared to the underlying asset trends.

The BMO-CIBC portfolio presents a contrasting case. The Equal Investment Strategy resulted in a lower average return of 131%, while the Proportional and Optimized strategies achieved higher average returns of 211% and 343%, respectively. In this case, the Mean-Variance Optimized Strategy not only delivered the highest return on average but also demonstrated that strategy choice can have a substantial impact when asset returns are more modest or heterogeneous. Volatility, measured by standard deviation, also increased with average return—again reflecting the trade-off between risk and reward.

In both cases, the Optimized Strategy outperformed the others on average. However, this came with higher variability, which may not suit all investors depending on their risk tolerance.

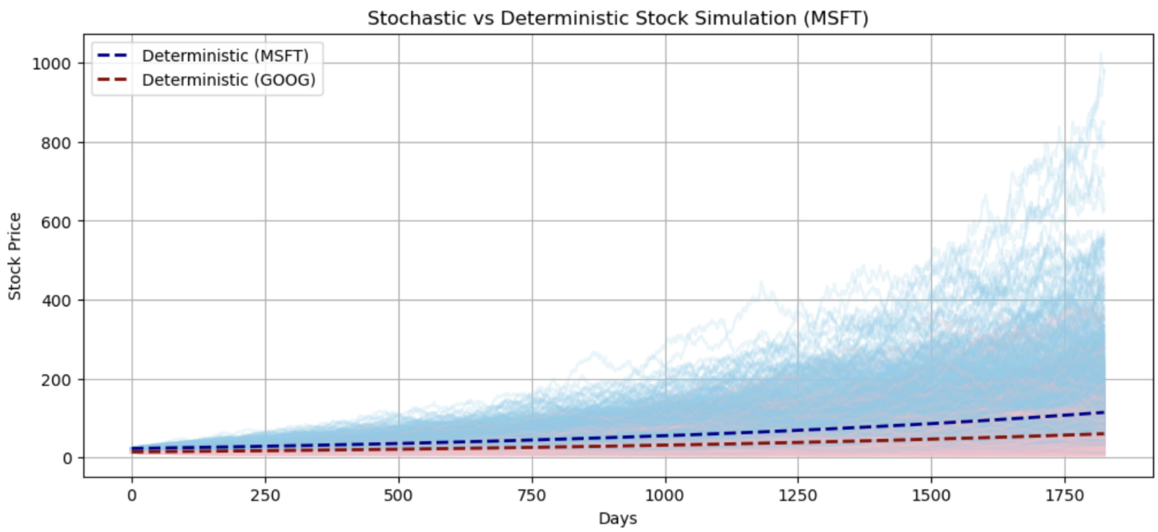

Figure 1: Comparison of stochastic and deterministic stock price simulations for Microsoft (MSFT) and Google (GOOG) over a 5-year period. The light-colored lines represent 2000 stochastic trajectories generated using the Geometric Brownian Motion (GBM) model, while the dashed dark blue and dark red lines show the deterministic growth paths for MSFT and GOOG, respectively, using the expected drift μ.

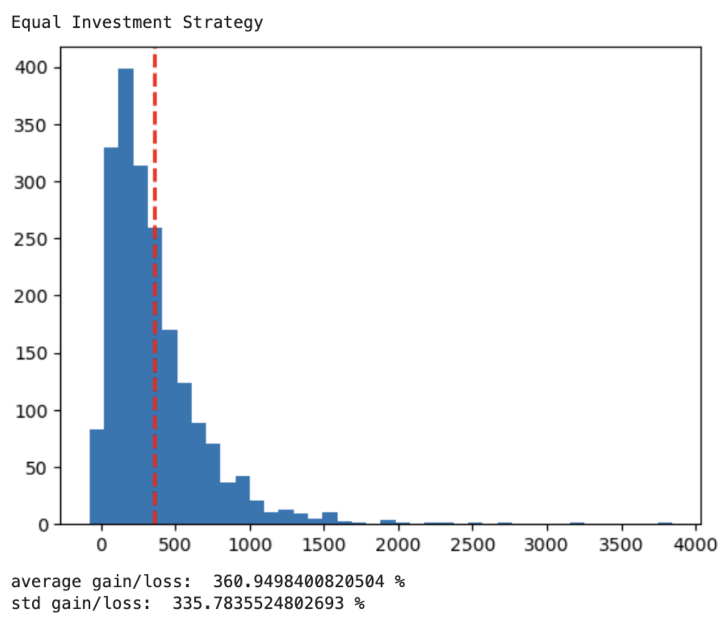

Figure 2: Histogram of simulated net gains/losses (%) for the Equal Investment Strategy over 2000 simulations using the Geometric Brownian Motion (GBM) model on Microsoft-Google portfolio's simulations. The red dashed line indicates the average gain/loss of approximately 361%. The wide spread reflects a standard deviation of approximately 336%, highlighting the variability and risk associated with the strategy.

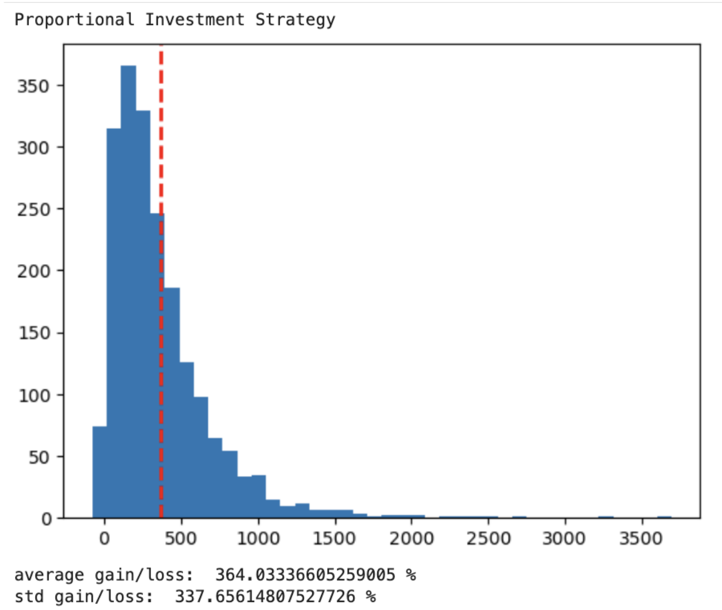

Figure 3: Histogram of simulated net gain/loss (%) for the Equal Investment Strategy over 2000 simulations using Microsoft and Google stock prices. The average gain is approximately 364% with a standard deviation of 338%. A red dashed line indicates the mean.

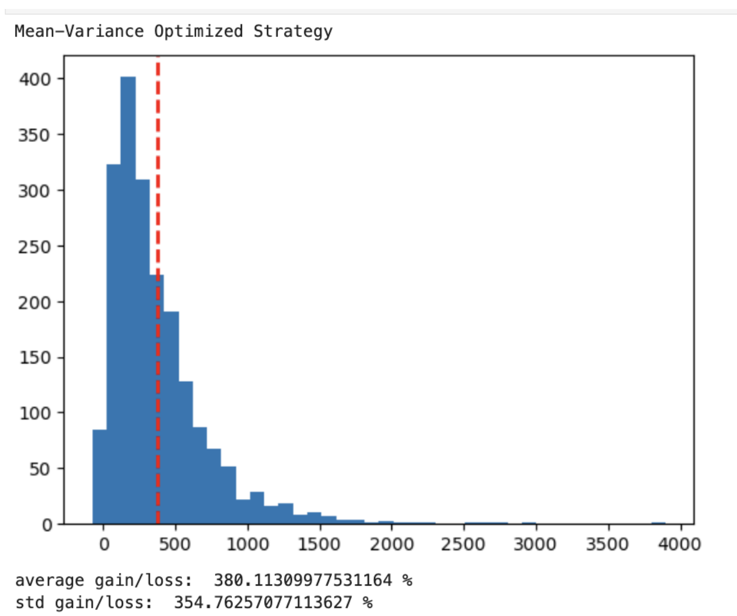

Figure 4: Histogram of simulated net gain/loss (%) for the Mean-Variance Optimized Strategy based on Markowitz portfolio theory on Microsoft-Google portfolio's simulations. This strategy achieved an average gain of approximately 380% with a standard deviation of 355%.

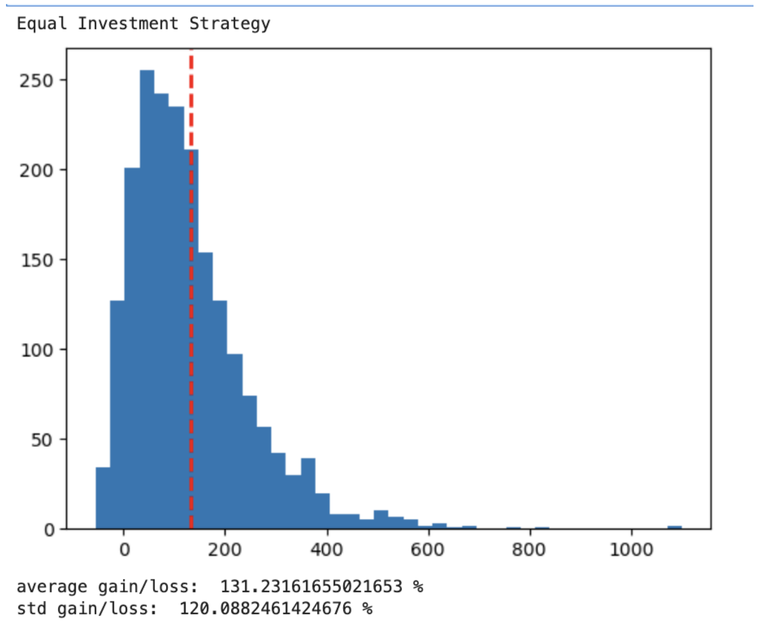

Figure 5: Histogram of net gain/loss for the Equal Investment Strategy over 2000 simulations on BMO-CIBC portfolio's simulations. The average gain is approximately 131%, with a standard deviation of 120%. The red dashed line represents the mean gain.

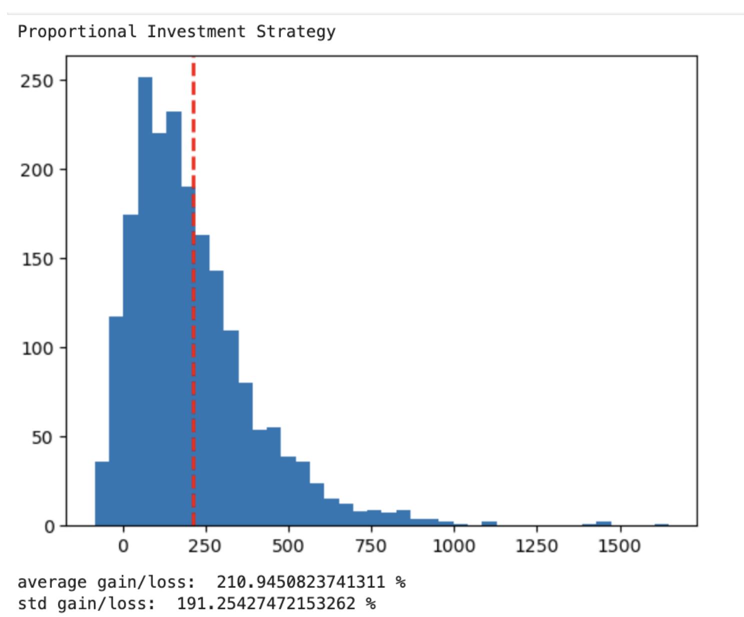

Figure 6: Histogram of net gain/loss for the Proportional Investment Strategy over 2000 simulations on BMO-CIBC portfolio's simulations. The average gain is approximately 211%, with a standard deviation of 191%. The red dashed line indicates the mean gain.

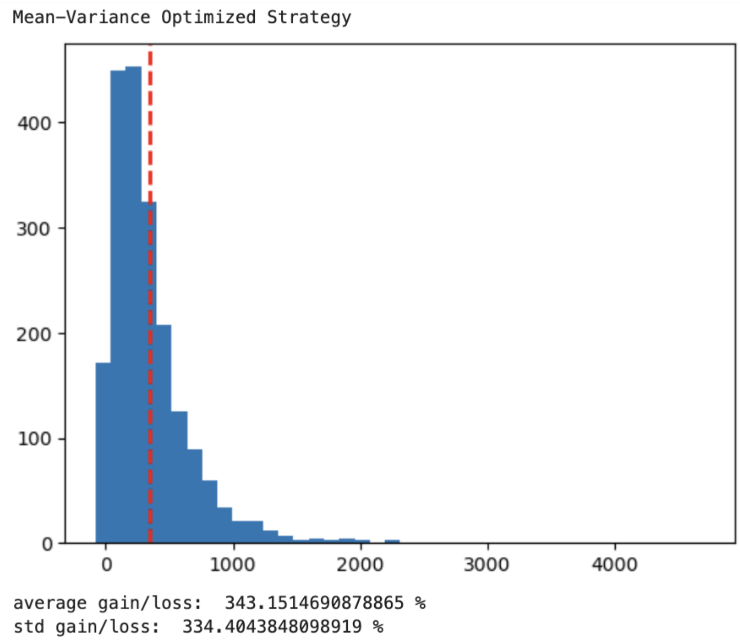

Figure 7: Histogram of net gain/loss for the Mean-Variance Optimized Strategy over 2000 simulations on BMO-CIBC portfolio's simulations. The average gain is around 343%, with a standard deviation of 334%. The red dashed line shows the average gain.

Conclusion

This project used a two-asset Geometric Brownian Motion (GBM) model to simulate five-year price paths for Microsoft-Google and BMO-CIBC portfolios. By running 2000 simulations, we evaluated three investment strategies: Equal Allocation, Proportional to Volatility, and Mean-Variance Optimization.

For the Microsoft-Google portfolio, the Optimized Strategy yielded the highest net return on average (380%), but the performance gap across strategies was modest. In contrast, the BMO-CIBC simulations demonstrated that optimization can significantly improve average returns—from 131% under Equal Investment to 343% with the Optimized Strategy. This highlights the value of strategy selection, particularly for portfolios with lower growth or more diverse volatility characteristics.

Parameter Choices. The drift (μ) and volatility (σ) parameters were estimated from historical daily returns. The covariance matrix was computed from the same return data to capture the joint behavior of the assets. Initial prices were based on actual market values at the start of the simulation window. These values were held constant across all simulations to ensure consistency when comparing strategy performance.

Limitations. The GBM framework assumes constant drift and volatility, and does not account for external market shocks or jumps. While useful for simulation-based strategy testing, it oversimplifies real-world market behavior. Future work could explore stochastic volatility models or jump-diffusion processes for enhanced realism. Future studies could improve upon our work by exploring alternative stochastic models, including jumps and shocks, to better reflect market behaviors.

References

1. Investopedia. "How Do You Calculate a Stock's Gain or Loss?" (2024).
2. Markowitz, H. "Portfolio Selection." *Journal of Finance*, vol. 7, no. 1, 1952.