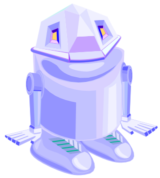


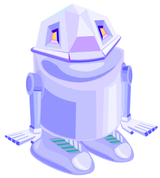
ECE497: Introduction to Mobile Robotics Lecture 2

Dr. Carlotta A. Berry

Spring 06 - 07

Quote of the Week

“In the fifties, it was predicted that in 5 years robots would be everywhere.
In the sixties, it was predicted that in 10 years robots would be everywhere.
In the seventies, it was predicted that in 20 years robots would be everywhere.
In the eighties, it was predicted that in 40 years robots would be everywhere...”
Marvin Minsky



Kinematics

Chapter 3

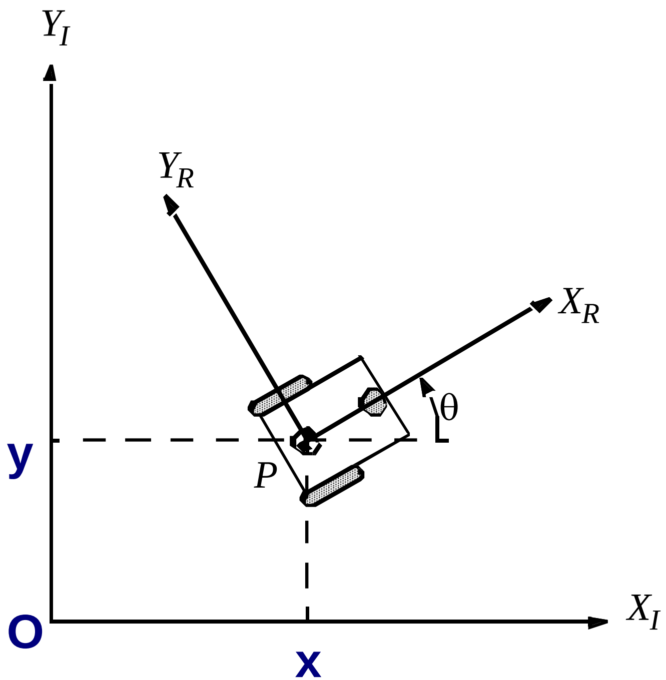
Mobile Robot Kinematics (3.1)

- Mobile Robot Kinematics is the dynamic model of how a mobile robot behaves
- Kinematics is a description of mechanical behavior of the robot for design and control
- Mobile Robot Kinematics is used for:
  - Position estimation
  - Motion estimation

Mobile Robot Kinematics, 2

- Robots move unbounded with respect to their environment
  - There is no direct way to measure robot position
  - Position must be integrated over time
  - The integration leads to inaccuracies in position and motion estimation
- Each wheel contributes to the robot’s motion and imposes constraints on robot’s motion
- All of the constraints must be expressed with respect to the reference frame

Robot Reference Frame (3.2.1)

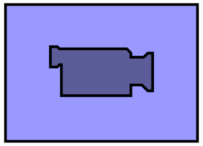


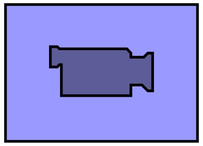
Robot position

- The robot's reference frame is three dimensional including position on the plane and the orientation, **$\{X_R, Y_R, \theta\}$**
- The axes **$\{X_I, Y_I\}$**, define inertial global reference frame with origin, **O**
- The angular difference between the global and reference frames is **θ**
- Point **P** on the robot chassis in the global reference frame is specified by coordinates **(x, y)**

$$\xi_I = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$$

Relative Positioning: Odometry and Kinematics





- Given wheel velocities at any given time, compute position/orientation for any future time
- Advantages
  - Self-contained
  - Can get positions anywhere along curved paths
  - Always provides an “estimate” of position
- Disadvantages
  - Requires accurate measurement of wheel velocities over time, including measuring acceleration and deceleration
  - Position error grows over time

Orthogonal Rotation Matrix

The ***orthogonal rotation matrix*** is used to map motion in the global reference **$\{X_I, Y_I\}$** frame to motion in the robot's local reference frame **$\{X_R, Y_R\}$**

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

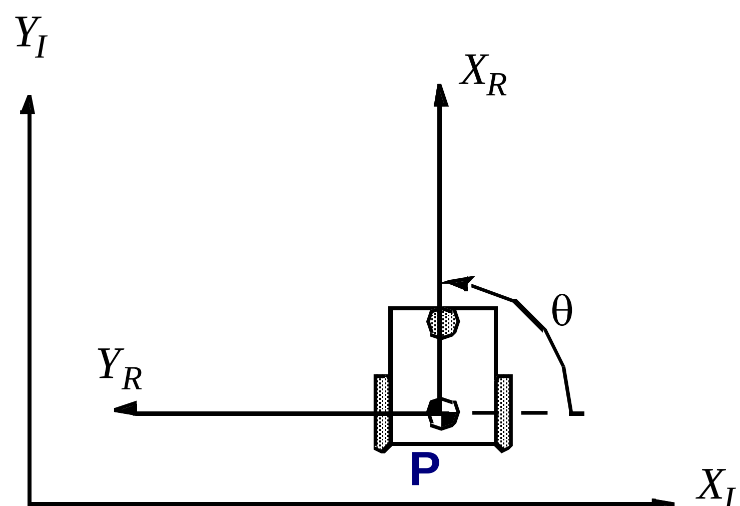
The orthogonal rotation matrix is used to convert robot velocity in the global reference frame to components of motion along the robot's local axes **$\{X_R, Y_R\}$**

$$\dot{\xi}_R = R(\theta)\dot{\xi}_I = R(\theta)\cdot\begin{bmatrix}\dot{x} & \dot{y} & \dot{\theta}\end{bmatrix}^T$$

Rotation Example

- Given some robot velocity $(\dot{x},\dot{y},\dot{\theta})$ in the global reference frame
- Suppose that the robot is at P and $\theta = \pi/2$ with respect to the global reference frame
- The motion along X_R and Y_R due to θ is

$$\dot{\xi}_R = R(\frac{\pi}{2})\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix}$$



Rotation Example 2

If a robot velocity has a velocity of $(\dot{x},\dot{y},\dot{\theta})$ in the global reference frame and is positioned at P and θ = π/3 with respect to the global reference frame. What is the motion along X_R and Y_R due to θ with respect to the robot reference?

$$\dot{\xi}_R = R\left(\frac{\pi}{3}\right)\dot{\xi}_I = \begin{bmatrix} \cos \pi/3 & \sin \pi/3 & 0 \\ -\sin \pi/3 & \cos \pi/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0.5\dot{x}+0.866\dot{y} \\ -0.866\dot{x}+0.5\dot{y} \\ \dot{\theta} \end{bmatrix}$$

This robot will rotate with the same speed with respect to the robot reference frame as the global reference frame. However the linear velocity along the robot's x-axis and y-axis are a combination of the velocities with respect to the global reference frame.

$(\dot{x}, \dot{y}, \dot{\theta})$

Rotation Example 2 (cont.)

What if the robot velocity is (2 cm/s, 3 cm/s, 5 rad/s) with respect to the global reference frame, what is the velocity with respect to the robot's local reference frame?

$$\dot{\xi}_R = R\left(\frac{\pi}{3}\right)\dot{\xi}_I = \begin{bmatrix} \cos \pi/3 & \sin \pi/3 & 0 \\ -\sin \pi/3 & \cos \pi/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3.5981 \\ -0.23205 \\ 5 \end{bmatrix}$$

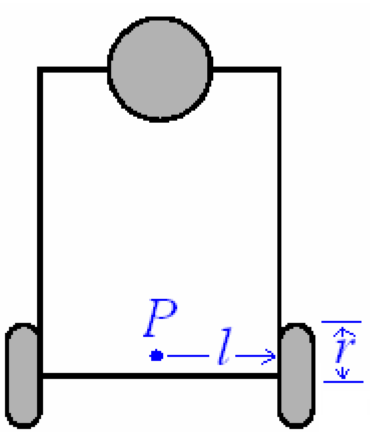
What if the robot velocity is (2 cm/s, 3 cm/s, 5 rad/s) with respect to the robot's local reference frame, what is the robot's velocity with respect to the global reference frame?

$$\dot{\xi}_I = R\left(\frac{\pi}{3}\right)^{-1}\dot{\xi}_R = \begin{bmatrix} \cos \pi/3 & -\sin \pi/3 & 0 \\ \sin \pi/3 & \cos \pi/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -1.5981 \\ 3.2321 \\ 5 \end{bmatrix}$$

Forward Kinematics Model Differential Drive Robot (3.2.2)

- *Forward Kinematics* provides an estimate of the robot's position given its geometry and speed of its wheels
- It requires accurate measurement of the wheel velocities over time
- However, position error (accumulation error) grows with time
- A differential drive robot with wheels that have speeds of $\dot{\varphi}_1$ and $\dot{\varphi}_2$ has the following forward kinematic model

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(l, r, \theta, \dot{\varphi}_1, \dot{\varphi}_2)$$

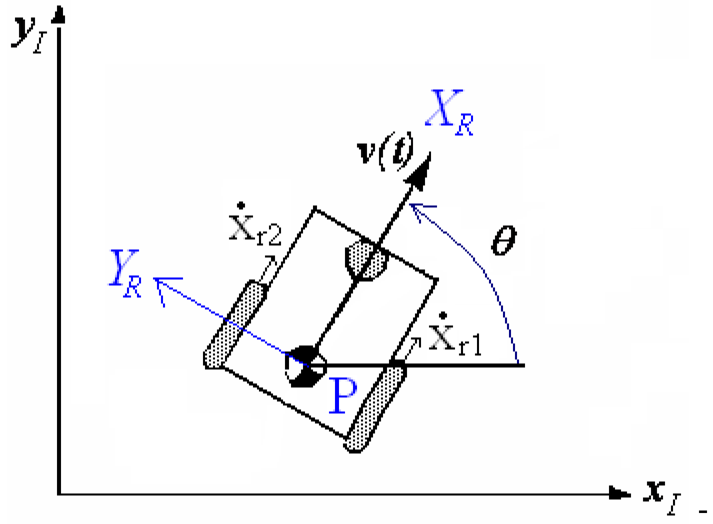


Forward Kinematics Model
Differential Drive Robot (3.2.2)

- To compute the robot's motion in the global reference frame from the local reference frame use $\dot{\xi}_I = R(\theta)^{-1}\dot{\xi}_R$
- To find the linear velocity in the direction of **+x$_R$**, each wheel contributes one half of the total speed.

$$\dot{x}_{r1} = (1/2)r\dot{\varphi}_1 \qquad \dot{x}_{r2} = (1/2)r\dot{\varphi}_2$$

- Since the wheels cannot compute sideways motion, **y$_R$** is zero

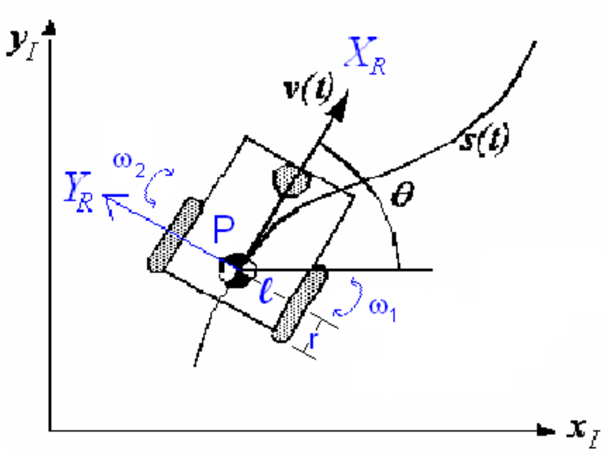


Forward Kinematics, cont.

The angular velocity about θ is calculated from the contribution from the two wheels. The right wheel, ω_1 contributes counterclockwise rotation about point P and the left wheel ω_2 contributes clockwise rotation about point P both with a radius of 2ℓ

$$\omega_1 = \frac{r\dot{\varphi}_1}{2l} \qquad \omega_2 = -\frac{r\dot{\varphi}_2}{2l}$$

$$\dot{\xi}_I = R(\theta)^{-1}\dot{\xi}_R = R(\theta)^{-1}\begin{bmatrix}\dot{x}_r \\ \dot{y}_r \\ \dot{\theta}_r\end{bmatrix} = R(\theta)^{-1}\begin{bmatrix}\dot{x}_{r1}+\dot{x}_{r2} \\ 0 \\ \omega_1+\omega_2\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{r\dot{\varphi}_1}{2}+\frac{r\dot{\varphi}_2}{2} \\ 0 \\ \frac{r\dot{\varphi}_1}{2l}-\frac{r\dot{\varphi}_2}{2l}\end{bmatrix}$$



Forward Kinematics Example

A robot is positioned at a 60° angle with respect to the global reference frame and has wheels with a radius of 1 cm. These wheels are 2 cm from the center of the chassis. If the speeds of wheels 1 and 2, are 4 cm/s and 2 cm/s, respectively. What is the robot velocity with respect to the global reference frame?

$$\theta = \pi / 3$$
$$r = 1$$
$$l = 2$$
$$\dot{\varphi}_1 = 4$$
$$\dot{\varphi}_2 = 2$$

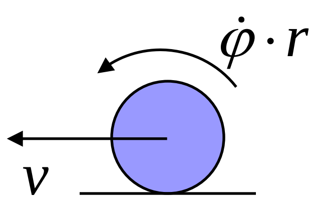
$$\dot{x}_{r1} = \frac{r\dot{\varphi}_1}{2}$$
$$\dot{x}_{r2} = \frac{r\dot{\varphi}_2}{2}$$
$$\omega_1 = \frac{r\dot{\varphi}_1}{2l}$$
$$\omega_2 = -\frac{r\dot{\varphi}_2}{2l}$$

$$\dot{\xi}_I = R(\theta)^{-1}\begin{bmatrix} \dot{x}_{r1} + \dot{x}_{r2} \\ 0 \\ \omega_1 + \omega_2 \end{bmatrix} = \begin{bmatrix} \cos \pi/3 & -\sin \pi/3 & 0 \\ \sin \pi/3 & \cos \pi/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3.0 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.5981 \\ 0.5 \end{bmatrix}$$

This robot will move instantaneously along the global reference frame x-axis with a speed of 1.5 cm/s and along the y-axis at 2.5981 cm/s while rotating with a speed of 0.5 radians/second.

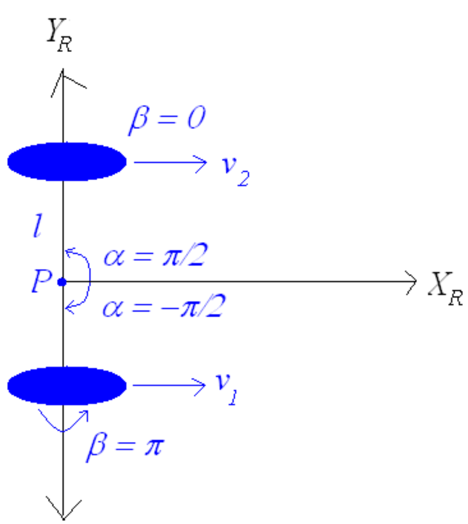
Wheel Kinematic Constraints (3.2.3)

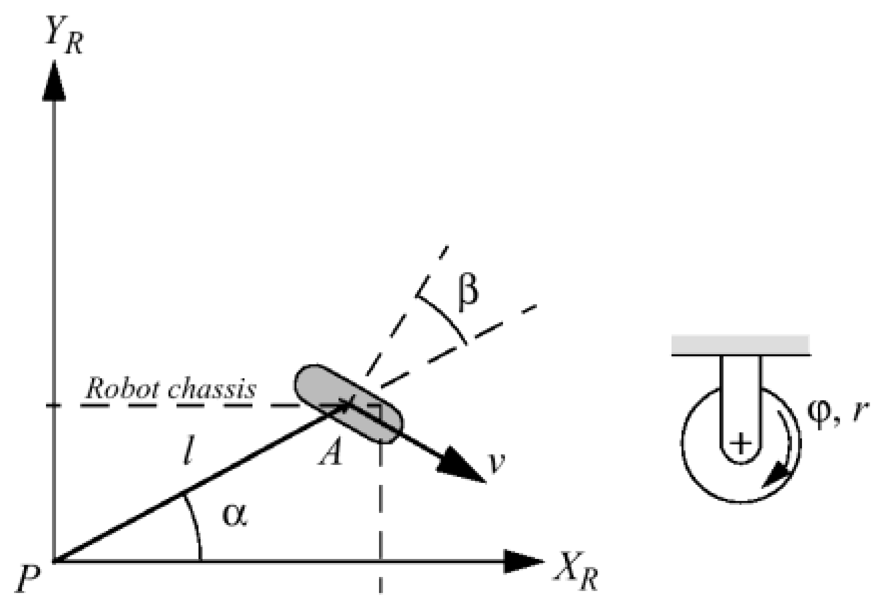
- To create a kinematic model express constraints on the motions of individual wheels
- These motions are combined to compute motion for the whole robot
- Assumptions:
  - The wheel plane must remain vertical at all times
  - There is one single point of contact between the wheel and ground
  - There is no sliding at the single point of contact
  - Movement on a horizontal plane
  - Wheels not deformable
  - Pure rolling (v = 0 at contact point)
  - No slipping, skidding or sliding
  - No friction for rotation around contact point
  - Steering axes orthogonal to the surface
  - Wheels connected by rigid frame (chassis)
- Constraints
  - The wheel must roll when motion takes place in the opposite direction
  - The wheel must not slide orthogonal to the wheel plane



Wheel Kinematic Constraints: Fixed Standard Wheel (3.2.3.1)

The *differential drive fixed standard wheel robot* in the text with right wheel 1 ($\alpha = -\pi/2$, $\beta = \pi$) and left wheel 2 ($\alpha = \pi/2$, $\beta = 0$) has the closest configuration to the tracked mobile robot used in this course and will be the one used for the kinematic analysis.





Fixed Standard Wheel (3.2.3.1)

There are 2 wheel kinematic constraints:

- Rolling – the wheel must roll when motion takes place in the appropriate direction

$$\left[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad (-l)\cos\beta\right] R(\theta)\dot{\xi}_1 - r\dot{\varphi} = 0$$

- Sliding – there should be no lateral slippage (i.e. the wheel must not slide orthogonal to the wheel plane)

$$\left[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta\right] R(\theta)\dot{\xi}_1 = 0$$

Fixed Standard Wheel (3.2.4 – 3.2.5)

- Rolling constraint: $J_1(\beta_s)R(\theta)\dot{\xi}_1 - J_2\dot{\varphi} = 0$
  - J_1 is the matrix for projections for all wheels to their motions along their individual wheel planes
  - J_2 is an N x N matrix whose entries are radii r of all wheels
- Sliding constraint: $C_1(\beta_s)R(\theta)\dot{\xi}_1 = 0$
  - C_1 is the matrix for projections for all wheels to their components of motion orthogonal to their wheel planes
- For the differential-drive robot, the rolling and sliding constraints are

$$\begin{bmatrix} J_1(\beta_s) \\ C_1(\beta_s) \end{bmatrix} R(\theta)\dot{\xi}_I = \begin{bmatrix} J_2\dot{\varphi} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \end{bmatrix} \\ \begin{matrix} 0 & 1 & 0 \end{matrix} \end{bmatrix} R(\theta)\dot{\xi}_I = \begin{bmatrix} J_2\dot{\varphi} \\ 0 \end{bmatrix}$$

$$\dot{\xi}_I = R(\theta)^{-1} \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ 1/2l & -1/2l & 0 \end{bmatrix} \begin{bmatrix} J_2\dot{\varphi} \\ 0 \end{bmatrix}$$

Differential drive robot example (3.2.5.1)

A differential drive robot is positioned at a 60° angle with respect to the global reference frame and has wheels with a radius of 1 cm that are 2 cm from the center of the chassis. If the speeds of wheels 1 and 2, are 4 cm/s and 2 cm/s, respectively. What is the robot velocity with respect to the global reference frame?

(r = 1 cm, l = 2cm, $\theta = \pi/3$, v_1 = 4 cm/s, v_2 = 2 cm/s)

$$\begin{bmatrix} J_1(\beta_s) \\ C_1(\beta_s) \end{bmatrix} R(\theta)\dot{\xi}_I = \begin{bmatrix} J_2\dot{\varphi} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \end{bmatrix} \\ \begin{matrix} 0 & 1 & 0 \end{matrix} \end{bmatrix} R(\theta)\dot{\xi}_I = \begin{bmatrix} r\dot{\varphi}_1 \\ r\dot{\varphi}_2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & -2 \end{bmatrix} \\ \begin{matrix} 0 & 1 & 0 \end{matrix} \end{bmatrix} R\left(\frac{\pi}{3}\right)\dot{\xi}_I = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$\dot{\xi}_I = R\left(\frac{\pi}{3}\right)^{-1} \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ 0.25 & -0.25 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

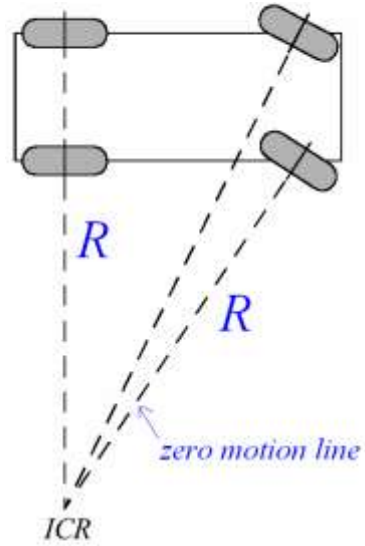
$$\dot{\xi}_I = \begin{bmatrix} 1.5 \\ 2.5981 \\ 0.5 \end{bmatrix}$$

In summary, given a robot's kinematic constraints and wheel motion it is possible to find the motion with respect to the global reference frame. This robot will move instantaneously along the global reference frame x-axis with a speed of 1.5 cm/s and along the y-axis at 2.5981 cm/s while rotating with a speed of 0.5 radians/second **(same as previous example)**

Degree of Mobility (3.3.1)

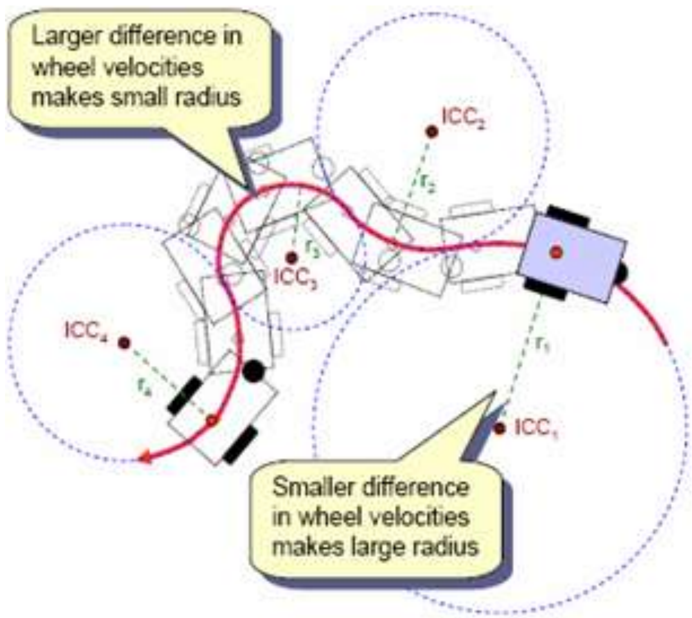
- The ***degree of mobility*** quantifies the degrees of controllable freedom of a mobile robot based on changes to wheel velocity

- The kinematic constraints of a robot with respect to the degree of mobility can be demonstrated geometrically by using the *instantaneous center of rotation (ICR)*



Instantaneous Center of Rotation (ICR)

- The ICR has a *zero motion line* drawn through the horizontal axis perpendicular to the wheel plane
- The wheel moves along a radius R with center on the zero motion line, the center of the circle is the *ICR*
- *ICR* is the point around which each wheel of the robot makes a circular course
- The *ICR* changes over time as a function of the individual wheel velocities



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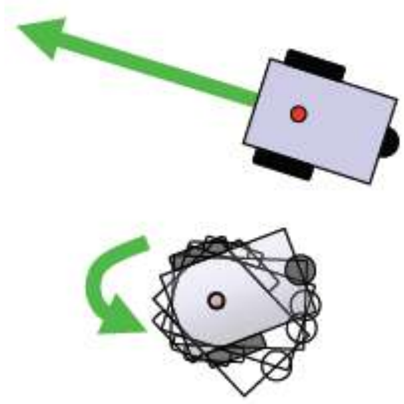
Instantaneous Center of Rotation (ICR)

- When R is infinity, wheel velocities are equivalent and the robot moves in a straight line
- When R is zero, wheel velocities are the negatives of each other and the robot spins in place
- All other cases, R is finite and non-zero and the robot follows a curved trajectory about a point which is a distance R from the robot's center
- Note that differential drive robot's are very sensitive to the velocity differences between the two wheels…making it hard to move in a perfectly straight line



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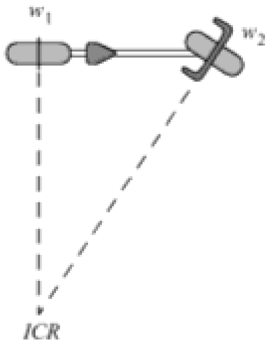
Degree of Mobility (3.3.1)

- *Robot mobility* is the ability of a robot chassis to directly move in the environment
- The *degree of mobility* quantifies the degrees of controllable freedom based on changes to wheel velocity
- *Robot mobility* is a function of the number of constraints on the robot's motion, not the number of wheels
  - A bicycle has 2 independent kinematic constraints
    - each wheel contributes a constraint, or a zero motion line
  - A differential drive robot has one independent kinematic constraint (see Figures 3.12, 3.13)
    - ICR lies along a line, 2nd wheel imposes no additional kinematic constraint



- So the bicycle has 2 independent kinematic constraints while the differential drive robot has only one
- Robot chassis kinematics is a function of the set of independent constraints on the standard wheels
- The rank of the matrix of all of the sliding constraints imposed by wheels of the mobile robot is the number of independent constraints

Mobile Robot Maneuverability: (3.3.1) Degree of Mobility

- Robot chassis kinematics is a function of the set of *independent constraints*, or the rank of the sliding constraints
  - the greater the rank of $C_1(\beta_s)$, the more constrained the mobility
- In the differential drive robot, the matrix has 2 constraints but a rank of one

$$(l_1 = l_2,\ \alpha_1 + \pi = \alpha_2,\ \beta_1 - \pi = \beta_2)$$

$$C_1(\beta_s) = C_{1f} = \begin{bmatrix} \cos(\alpha_1 + \beta_1) & \sin(\alpha_1 + \beta_1) & l\sin(\beta_1) \\ \cos(\alpha_1 + \beta_1) & \sin(\alpha_1 + \beta_1) & l\sin(\beta_1 - \pi) \end{bmatrix}$$

$$rank[C_1(\beta_s)] = 3$$

Mobile Robot Maneuverability: (3.3.1) Degree of Mobility

- no standard wheels means the $\mathrm{rank}[C_1(\beta_s)] = 0$
- all directions constrained means the $\mathrm{rank}[C_1(\beta_s)] = 3$
- therefore, the robot's ***degree of mobility*** is

  $\delta_m = \dim N[C_1(\beta_s)] = 3 - \mathrm{rank}[C_1(\beta_s)]$, where $0 \le \mathrm{rank}[C_1(\beta_s)] \le 3$
- this dimensionality of the nullspace (dimN) of the matrix ($C_1(\beta_s)$) is a measure of the *number of degrees of freedom* of the robot chassis that can be manipulated through changes in wheel velocity

$rank[C_1(\beta_s)] = 3$

Mobile Robot Maneuverability: (3.3.1) Differential Drive Chassis

- Two fixed standard wheels
  - wheels on same axle
  - the 2nd wheel adds no independent kinematic constraints
  - $\text{rank}[C_1(\beta_s)] = 1$ and $\delta_m = 2$
- A differential drive robot can control both the rate of its change in orientation and its forward/reverse speed *by manipulating wheel velocities*
- The ICR lies on the infinite line extending from the wheels horizontal axis

$rank[C_1(\beta_s)] = 3$

Mobile Robot Maneuverability: (3.3.2) Degree of Steerability

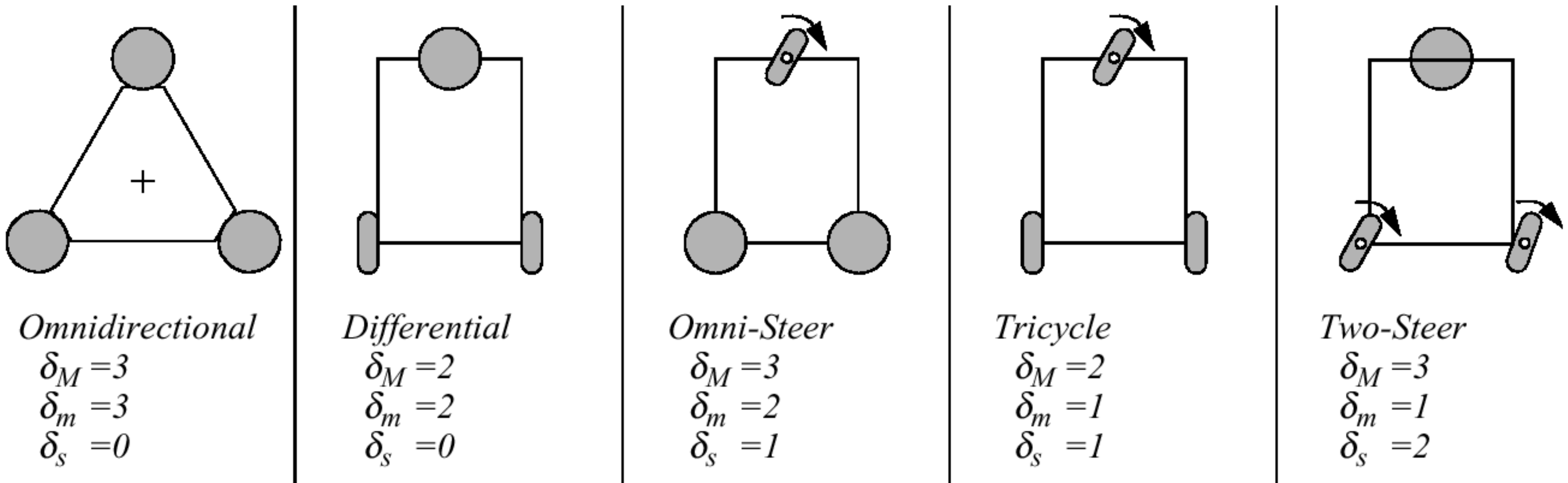
- The ***degree of steerability,*** $\boldsymbol{\delta_s}$, of a mobile robot is defined by the number of independently controllable steering parameters, $\delta_s = rank[C_{1s}(\beta_s)]$
  - An increase in the rank of $C_1(\beta_s)$ implies more kinematic constraints and less mobility
  - Conversely, an increase in the rank of $C_{1s}(\beta_s)$ implies more degrees of steering freedom and greater eventual maneuverability
  - Since $C_1(\beta_s)$ includes $C_{1s}(\beta_s)$, a steered standard wheel can increase steerability and decrease mobility
  - The particular orientation of a steered standard wheel at any instant imposes a kinematic constraint
  - However, the ability to change that orientation can lead to an additional degree of maneuverability
- Range of δ_s: $0 \le \delta_s \le 2$
  - No steerable standard wheels means $\delta_s = 0$
  - No fixed standard wheels, $\delta_s = 2$

Degree of Maneuverability (3.3.3)

- The robot's *overall degrees of freedom (DOF)* or *degree of maneuverability* is defined in terms of mobility and steerability:

$$\delta_M = \delta_m + \delta_s$$

  - Includes degrees of freedom that a robot manipulates through wheel velocity and degrees of freedom that it indirectly manipulates through steering configuration and moving
- Two robots with same δ_M are not necessarily equal (i.e. tricycle and differential drive robot)
- For any robot wit $\delta_M = 2$, the ICR is always constrained to *lie on a line*
- For any robot with $\delta_M = 3$, the ICR is not constrained and can *be set to any point on the plane*



Mobile Robot Workspace (3.4)

- Robot maneuverability is equivalent to its control degrees of freedom
- A robot's space of possible configurations in an environment is the *workspace* and it can exceed the number of control degrees of freedom, δ_M.
- The workspace *degrees of freedom (DOF)* governs the robot's ability to achieve various poses
- The number of dimensions in the velocity space of a robot is called the *differential degrees of freedom (DDOF)*
- The *DDOF* governs a robots ability to achiever various paths
- The *DDOF* is always equal to the degree of mobility, δ_m,

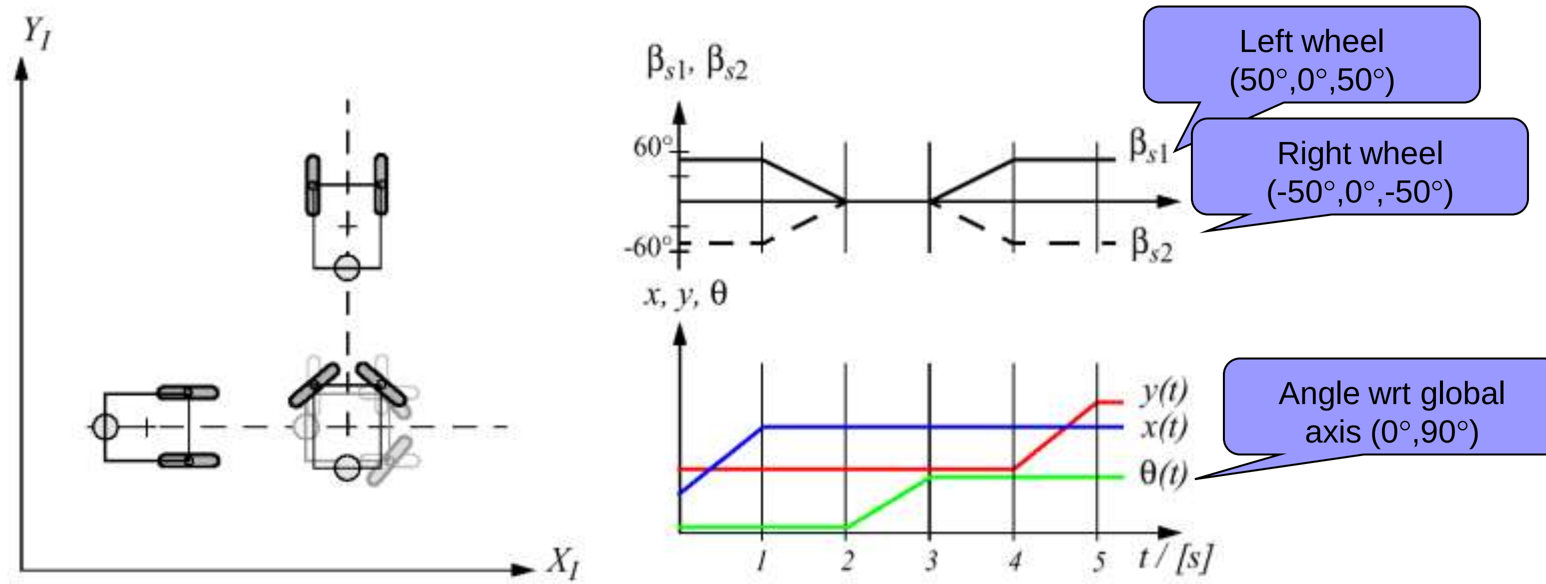
  $$(DDOF \leq \delta_M \leq DOF)$$
- The objective of a *kinematic controller* is to follow a trajectory described by its position and/or velocity profiles as function of time.
- Motion control is not straight forward because mobile robots are non-holonomic systems (may require the derivative of a position variable).
- It is only through non-holonomic constraints (imposed by fixed or steerable wheels) that a robot can achieve a workspace with degrees of freedom exceeding its differential degrees of freedom DOF > DDOF

Path and trajectory considerations (3.4.3)

- There is a difference between DOF granted by steering versus direction control of wheel velocity
- The difference is in the context of *trajectories* rather than paths
- A trajectory is like a path but it has the additional dimension of *time*
- *Motion control (kinematic control)* is not straight forward because mobile robots are non-holonomic systems.

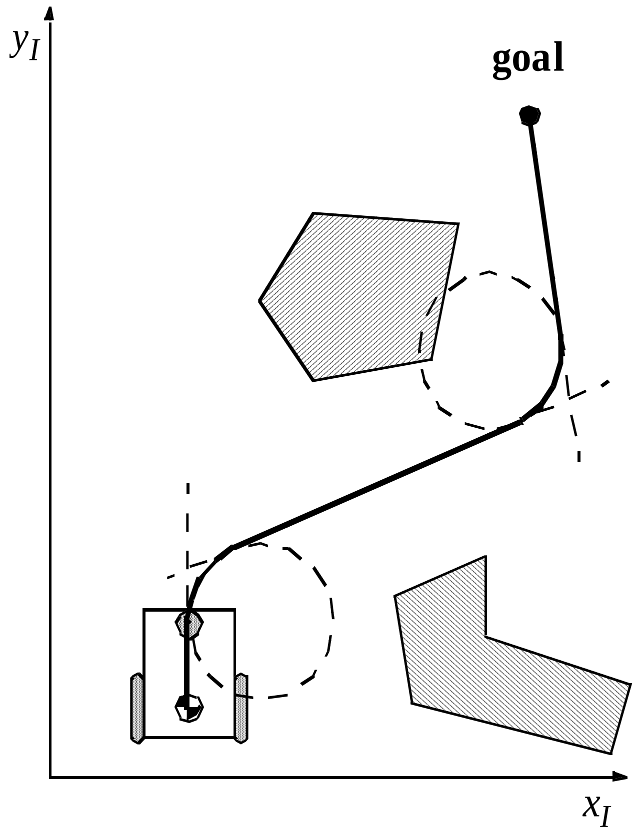
Path and trajectory considerations
Two-steer robot (3.4.3)

A robot has a goal trajectory in which the robot moves along axis X_I at a constant speed of 1 m/s for 1 second. Wheels adjust for 1 second. The robot then turns counterclockwise at 90 degrees in 1 second. Wheels adjust for 1 second. Finally, the robot then moves parallel to axis Y_I for 1 final second.



Motion Control: Open Loop Control (3.6.1)

- The objective of a kinematic controller is to follow a trajectory described by its position and/or velocity profiles as a function of time.
- One method is to divide the trajectory (path) into motion segments of clearly defined shape:
  - straight lines and segments of a circle. (open loop control)
- control problem:
  - pre-compute a smooth trajectory based on line and circle segments

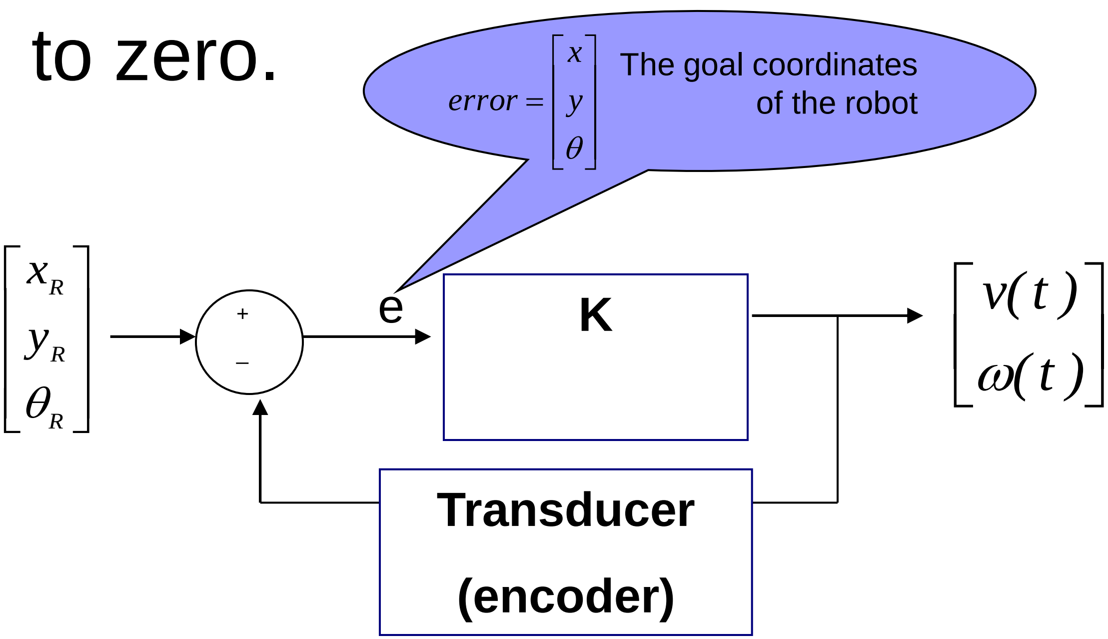


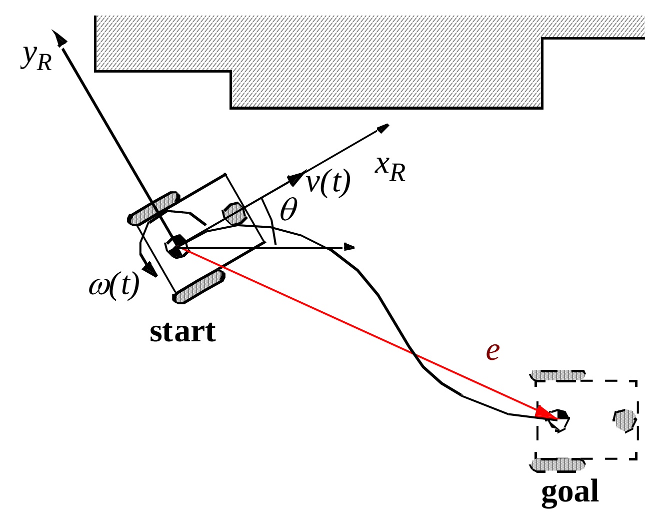
Motion Control:
Open Loop Control (3.6.1)

- Disadvantages:
  - It is not easy to pre-compute a feasible trajectory
  - There are limitations and constraints on the robots velocities and accelerations
  - The robot does not adapt or correct the trajectory if dynamic changes in the environment occur.
  - The resulting trajectories are usually not smooth
  - There are discontinuities in the robot’s acceleration
- A more appropriate approach in motion control is to use a real-state feedback controller

Feedback Control Example (3.6.2.1)

Given a robot with an arbitrary position and orientation and a predefined goal position and orientation. Design a control matrix for a real-state feedback controller to drive the pose error to zero.





Motion Control:
Feedback Control, Problem Statement (3.6.2.1)

- The task of the controller is to find a control matrix *K*, if exists with $k_{ij}=k(t,e)$

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix}$$

- such that the control signals, $v(t)$ and $\omega(t)$

$$\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = K \cdot e = K \cdot {}^{R}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Pose-error = current position – goal position

- drive the position error e to zero: $\lim_{t \to \infty} e(t) = 0$

Kinematic Model (cartesian) (3.6.2)

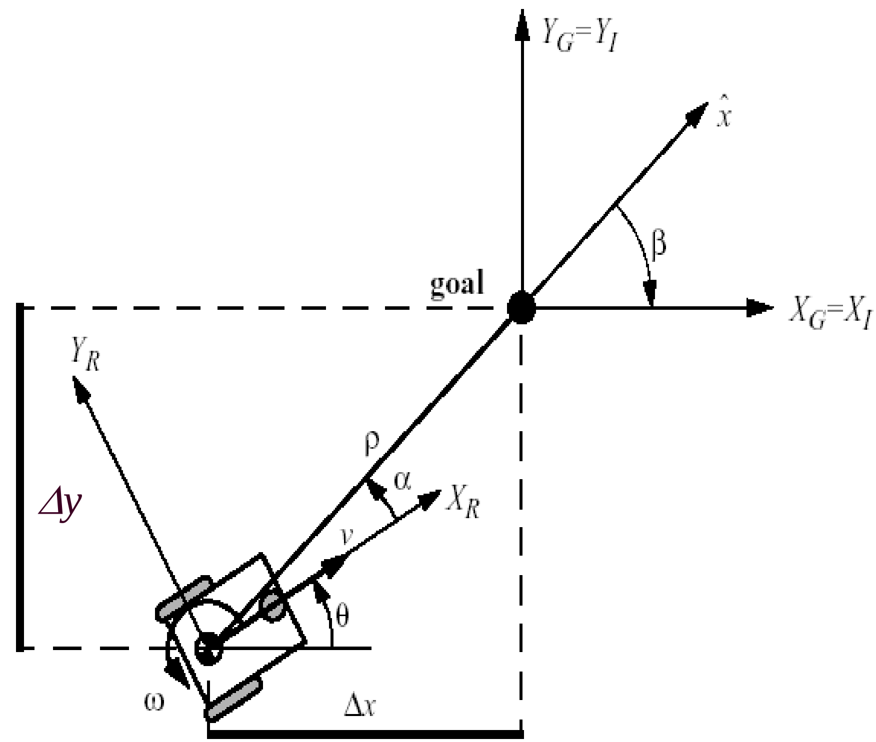
Assume that the goal of the robot is the origin of the global inertial frame. The *kinematics* for the differential drive mobile robot with respect to the global reference frame are:

$$
{}^{I}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}
$$

$$
\rho = \sqrt{\Delta x^2 + \Delta y^2}
$$

$$
\alpha = -\theta + \operatorname{atan2}(\Delta y, \Delta x)
$$

$$
\beta = -\theta - \alpha
$$



Kinematic Model (polar coordinates) (3.6.2)

Robot is facing the goal point

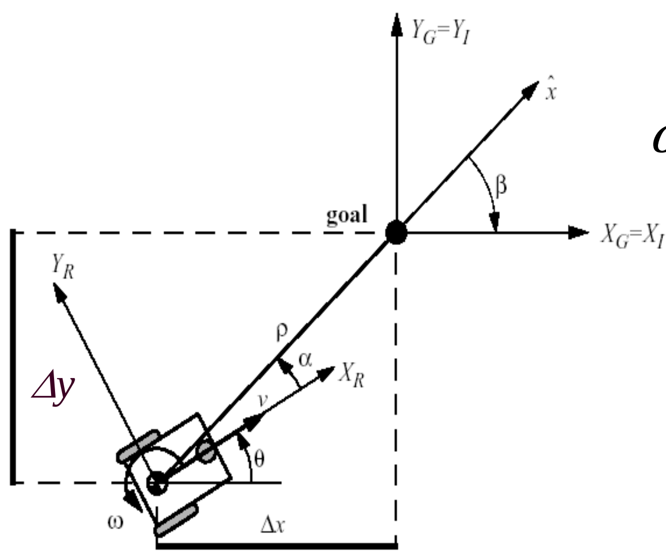
$$\begin{bmatrix}\dot{\rho}\\ \dot{\alpha}\\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}\cos\alpha & 0\\ -\dfrac{\sin\alpha}{\rho} & 1\\ \dfrac{\sin\alpha}{\rho} & 0\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix}$$

$$\alpha \in \left(-\pi/2, \pi/2\right]$$

Robot's back is to the goal point

$$\begin{bmatrix}\dot{\rho}\\ \dot{\alpha}\\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}-\cos\alpha & 0\\ \dfrac{\sin\alpha}{\rho} & -1\\ -\dfrac{\sin\alpha}{\rho} & 0\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix}$$

$$\alpha \in \left(\pi/2, \pi\right] \cup \left(-\pi, -\pi/2\right]$$



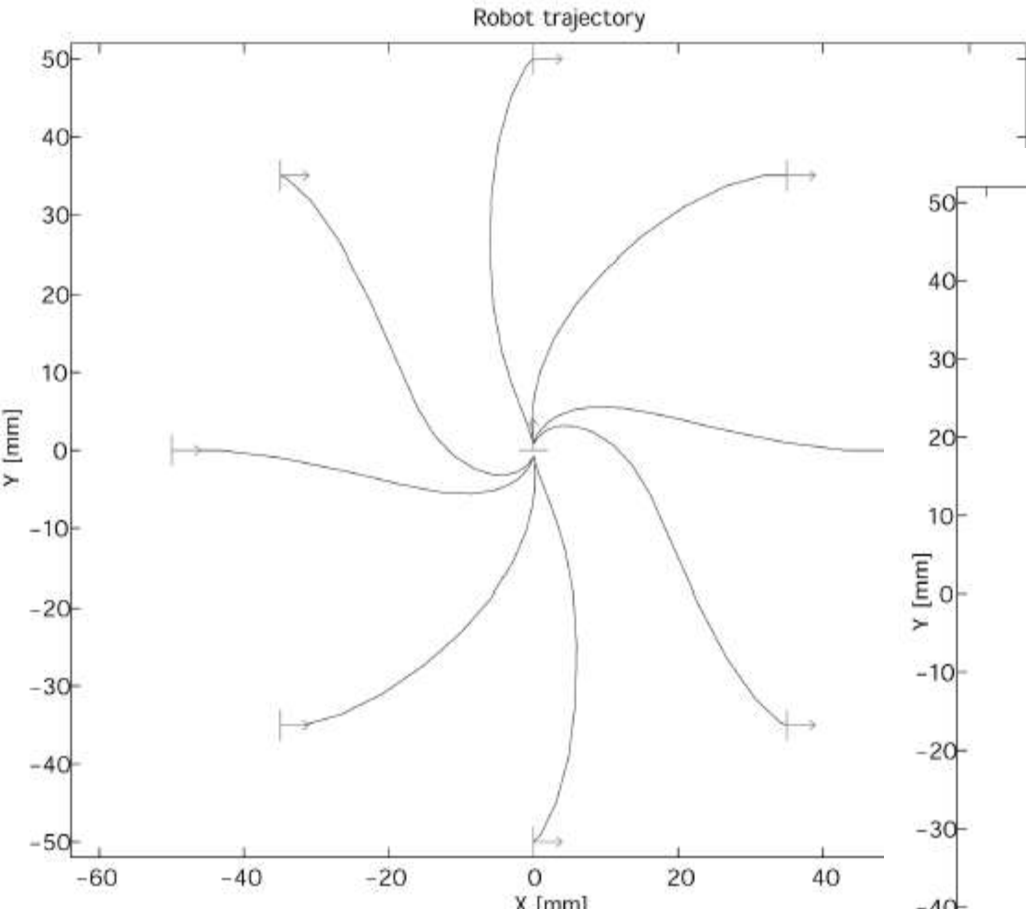
s -

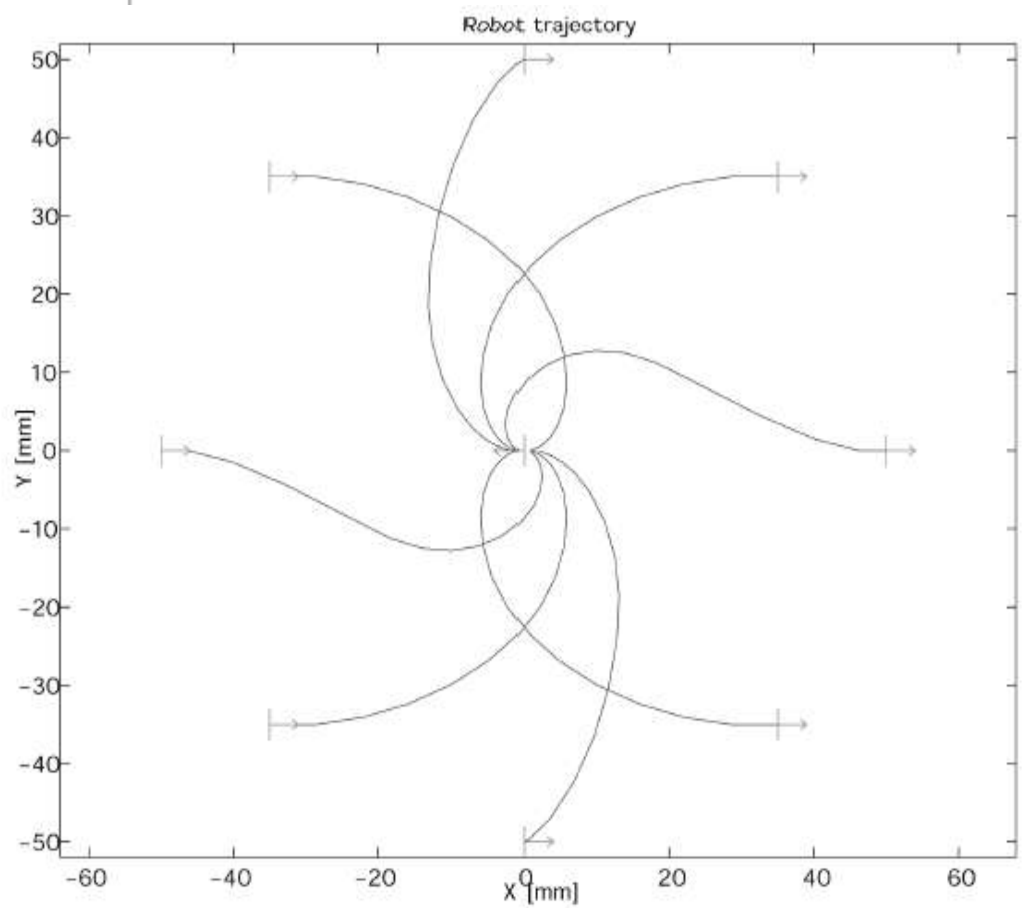
The Control Law (3.6.2.4)

- The controls signals v and ω must be designed to drive the robot from $(\rho_o, \alpha_o, \beta_o)$ to the goal position
- Consider the control law, $v = k_\rho \rho$ and $\omega = k_\alpha \alpha + k_\beta \beta$
- The closed loop system description becomes,

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -k_\rho \rho \cos\alpha \\ k_\rho \sin\alpha - k_\alpha \alpha - k_\beta \beta \\ -k_\rho \sin\alpha \end{bmatrix}$$

Kinematic Position Control: Resulting Path (3.6.2.4)





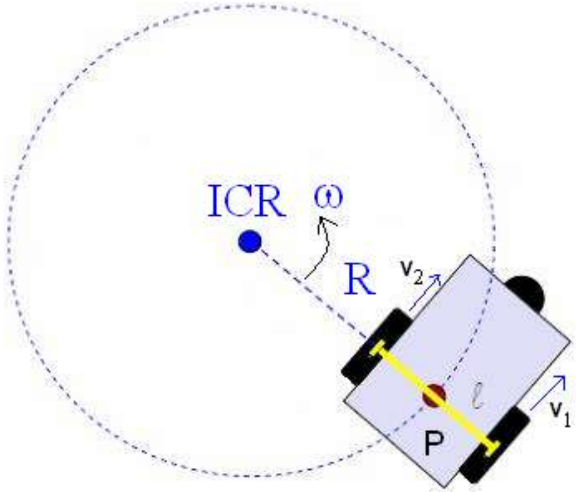
Forward Kinematics

Assume that at each instance of time, the robot is following the **ICR** with radius R at angular rate ω,

$$\omega = \frac{(v_1 - v_2)}{2l} \qquad R = \frac{V}{\omega} = \frac{l(v_1 + v_2)}{(v_1 - v_2)}$$

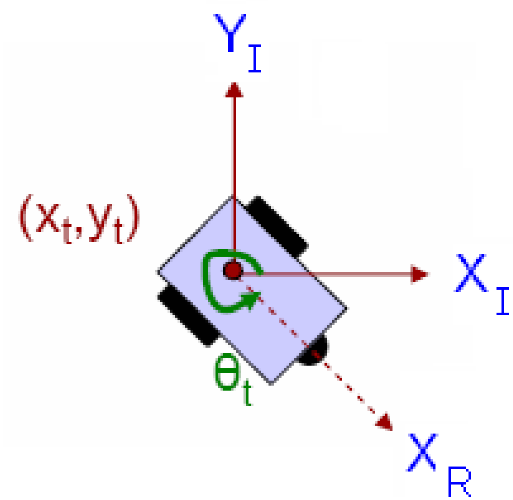
s = rθ
v = rω

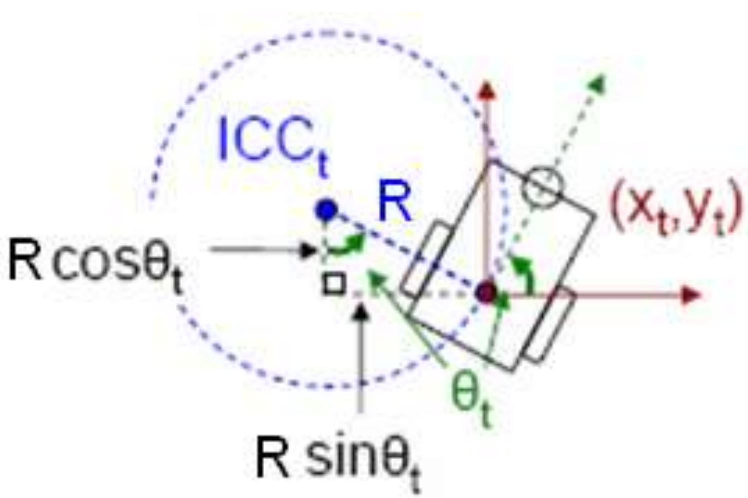
V = robot forward velocity
v_1 – right wheel velocity
v_2 – left wheel velocity
ω - robot angular velocity
ℓ – distance from robot center to wheel



Forward Kinematics

- Given some control parameters (e.g. wheel velocities) determine the poses of the robot
- The position can be determined recursively as a function of the velocity and position, $p(t + \Delta) = F(v_1, v_2)\, p(t)$
- To solve determine the $ICR(t) = (ICR_x, ICR_y) = (x_t - R\sin\theta_t, y_t + R\cos\theta_t)$





Forward Kinematics: instantaneous pose

- At time t + Δ, the robot pose with respect to the ICR is

$$p(t+\Delta) = R(\omega\Delta)^{-1}\, p(t) + ICR(t)$$

$$p(t+\Delta) = \begin{bmatrix} x(t+\Delta) \\ y(t+\Delta) \\ \theta(t+\Delta) \end{bmatrix} = \begin{bmatrix} \cos(\omega\Delta) & -\sin(\omega\Delta) & 0 \\ \sin(\omega\Delta) & \cos(\omega\Delta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R\sin\theta_t \\ -R\cos\theta_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} ICR_x \\ ICR_y \\ \omega\Delta \end{bmatrix}$$

- Since ICR(t) = (x(t) – R sin θ, y(t) + R cos θ)

$$p(t+\Delta) = \begin{bmatrix} R\cos(\omega\Delta)\sin\vartheta_t + R\sin(\omega\Delta)\cos\vartheta_t + (x_t - R\sin\vartheta_t) \\ R\sin(\omega\Delta)\sin\vartheta_t - R\cos(\omega\Delta)\cos\vartheta_t + (y_t + R\cos\vartheta_t) \\ \vartheta_t + \omega\Delta \end{bmatrix}$$

Forward Kinematics: linear displacement

- When $v_1 = v_2 = v_t$, $R = \infty$, the robot moves in a straight line so ignore the *ICR* and use the following equations:
- $x(t + \Delta) = x_t + v_t\Delta \cos \theta_t$
- $y(t + \Delta) = y_t + v_t\Delta \sin \theta_t$
- $\theta(t + \Delta) = \theta_t$

Forward Kinematics Example linear displacement

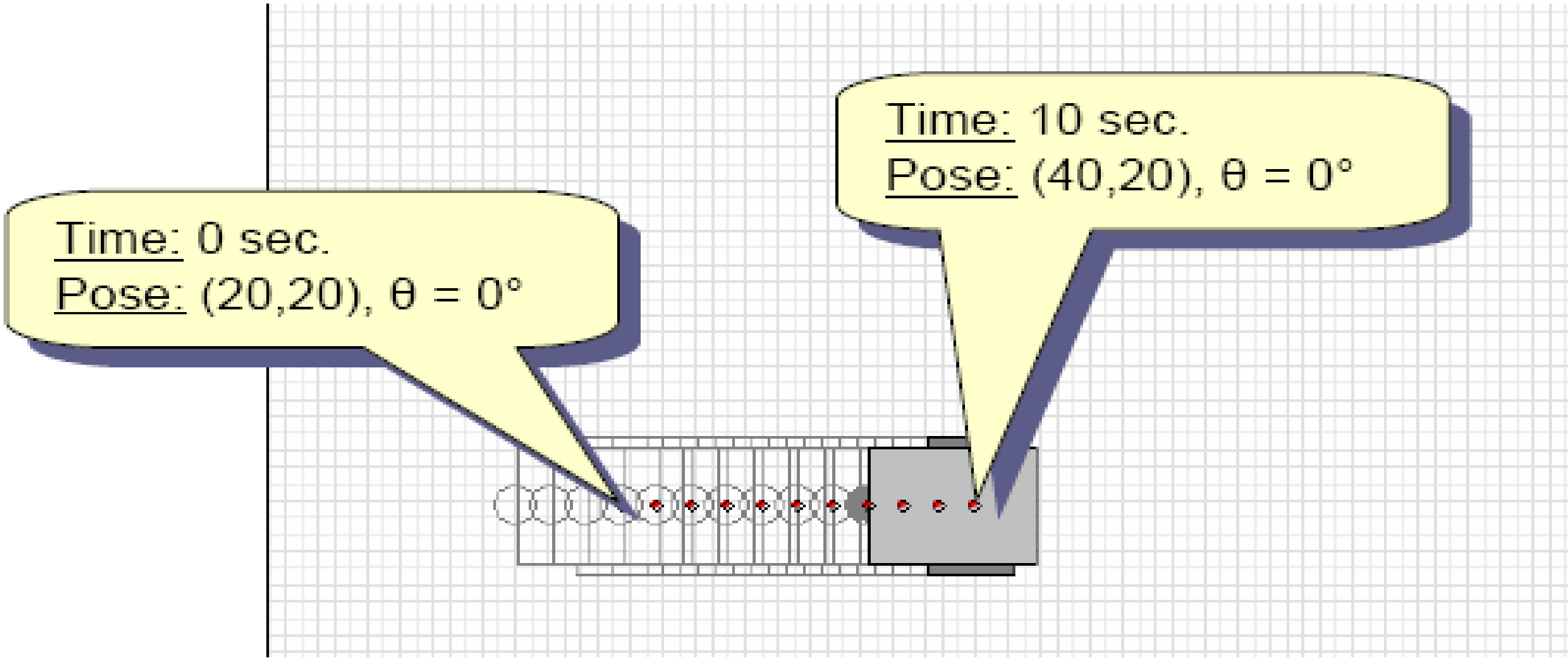
- A differential steering robot with $\ell = 5.3$ cm starts at $(x_o, y_o) = (20 \text{ cm}, 20 \text{ cm})$, $\theta = 0°$, $t = 0$ seconds
- The robot moves both wheels at 2 cm/sec and moves for 10 seconds
- Where is the robot at *t = 10 seconds*?

$$x(t + \Delta) = x_t + v_t\Delta \cos \theta_t \rightarrow x(10) = x(0) + v(0)\cdot(10) \cdot\cos(0°) = 40 \text{ cm}$$

$$y(t + \Delta) = y_t + v_t\Delta \sin \theta_t \rightarrow y(10) = y(0) + v(0)\cdot(10) \cdot\sin(0°) = 20 \text{ cm}$$

$$\theta(t + \Delta) = \theta_t \rightarrow \theta(10) = \theta(0) = 0°$$

Forward Kinematics Example: linear displacement



http://www.scs.carleton.ca/~lanthier/teaching/COMP4900A/Notes/5%20-%20PositionEstimation.pdf

Forward Kinematics Example: counterclockwise turn

- Now the robot sets the right wheel to *3 cm/s* and the left wheel to *2 cm/s* and moves for *10 more seconds*
- Where is the robot at *t = 20 seconds, Δ = 10 seconds*?

$R = \ell(v_1 + v_2)/(v_1 - v_2) = (5.3)(3+2)/(3-2) = 26.5$ cm

$\omega = (v_1 - v_2)/2\ell = 0.094$ rad/s

$x(t + \Delta) = R\cos(\omega\Delta)\sin(\theta_t) + R\sin(\omega\Delta)\cos(\theta t) + x_t - R\sin(\theta_t) \rightarrow$

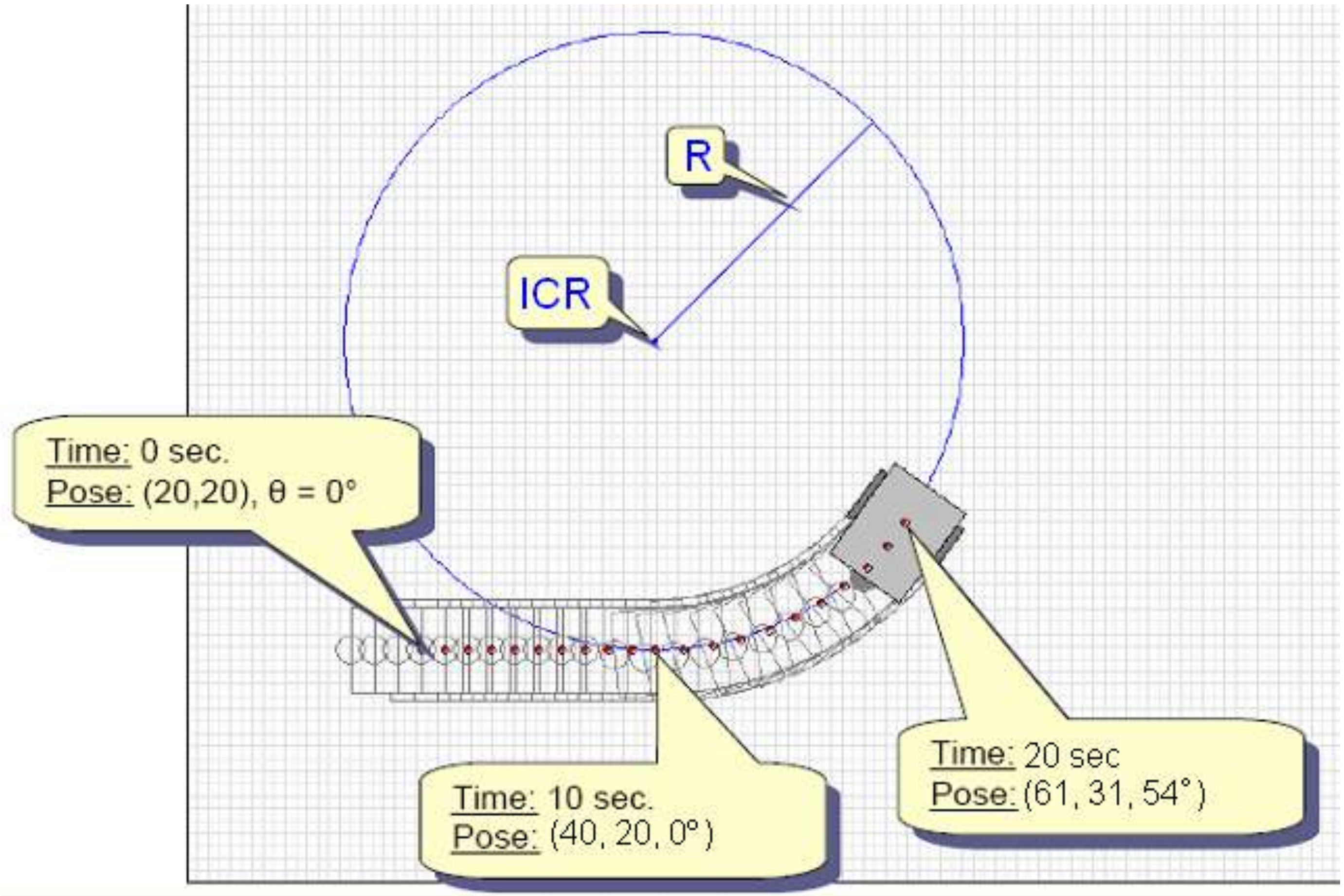
$x(20) = (26.5)(0.587)(0)+26.5(0.810)(1) + 40-26.5(0) =$ ***61.465 cm***

$y(t + \Delta) = R\sin(\omega\Delta)\sin(\theta_t) - R\cos(\omega\Delta)\cos(\theta t) + y_t + R\cos(\theta_t) \rightarrow$

$y(20) = (26.5)(0.810)(0) - 26.5(1)(0.587)+ 20 + 26.5(1) =$ ***30.95 cm***

$\theta(t + \Delta) = \theta_t + \omega\Delta \rightarrow \theta(20) = \theta(10) + \omega(10) = 0^\circ + (0.09433 \text{ rad/s})(10 \text{ sec}) =$ **54°**

Forward Kinematics: counterclockwise turn



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Forward Kinematics: counterclockwise spin

- Now set the robots right wheel to 2 cm/s and the left wheel to -2 cm/s for 5 seconds
- Where is the robot at t = 25 s, Δ = 5 seconds?

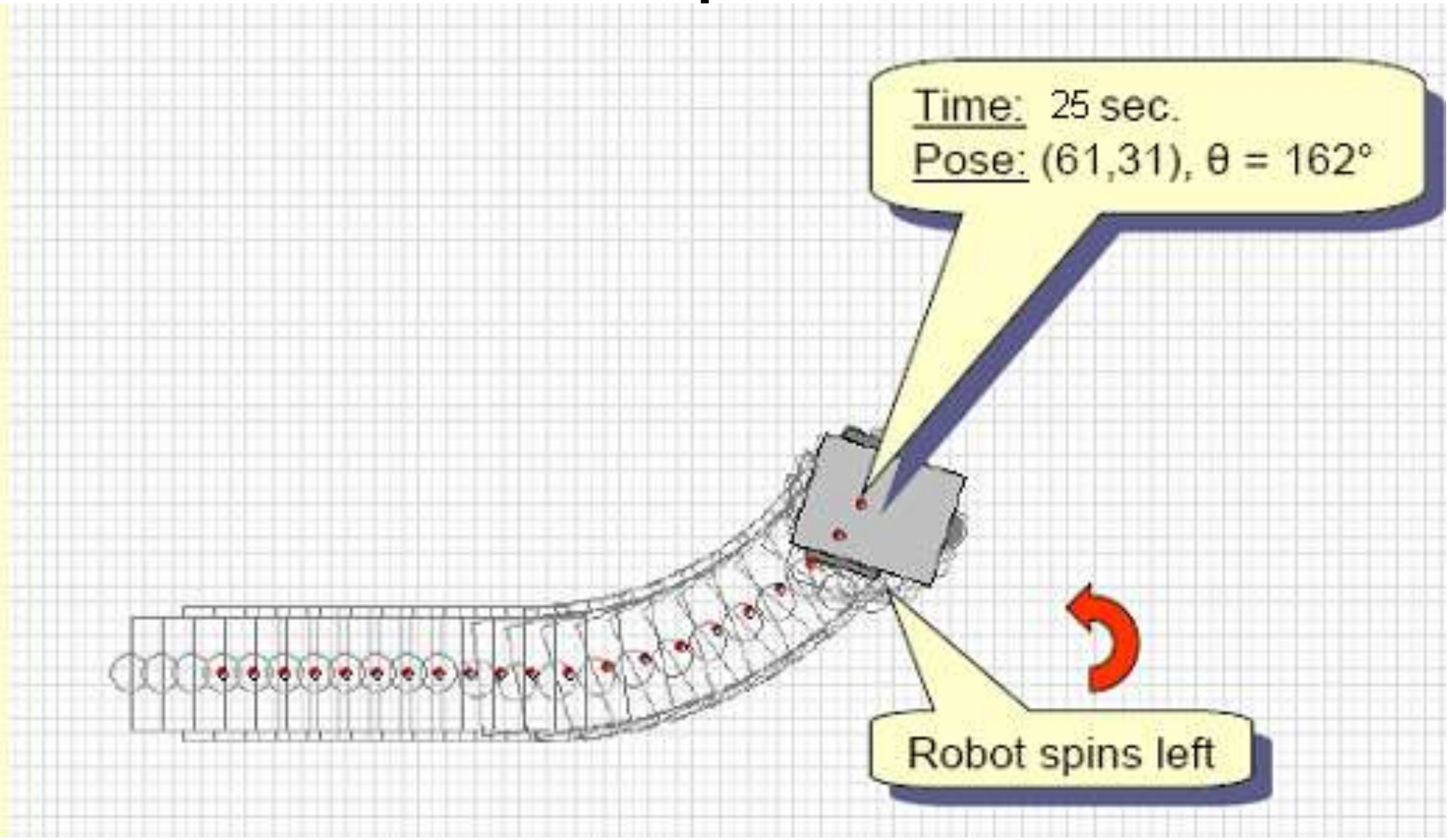
$\omega = (v_1 - v_2)/2\ell = (2 - - 2\text{cm/s})/10.6\text{ cm} = 0.37736\text{ rad/s}$

x(25) = *61.465 cm*

y(25) = *30.95 cm*

$\theta(25) = \theta(20) + \omega\Delta = 54^\circ + (0.377\text{rad/s})(5\text{s}) =$ ***162°***

Forward Kinematics: counterclockwise spin



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Forward Kinematics: clockwise turn

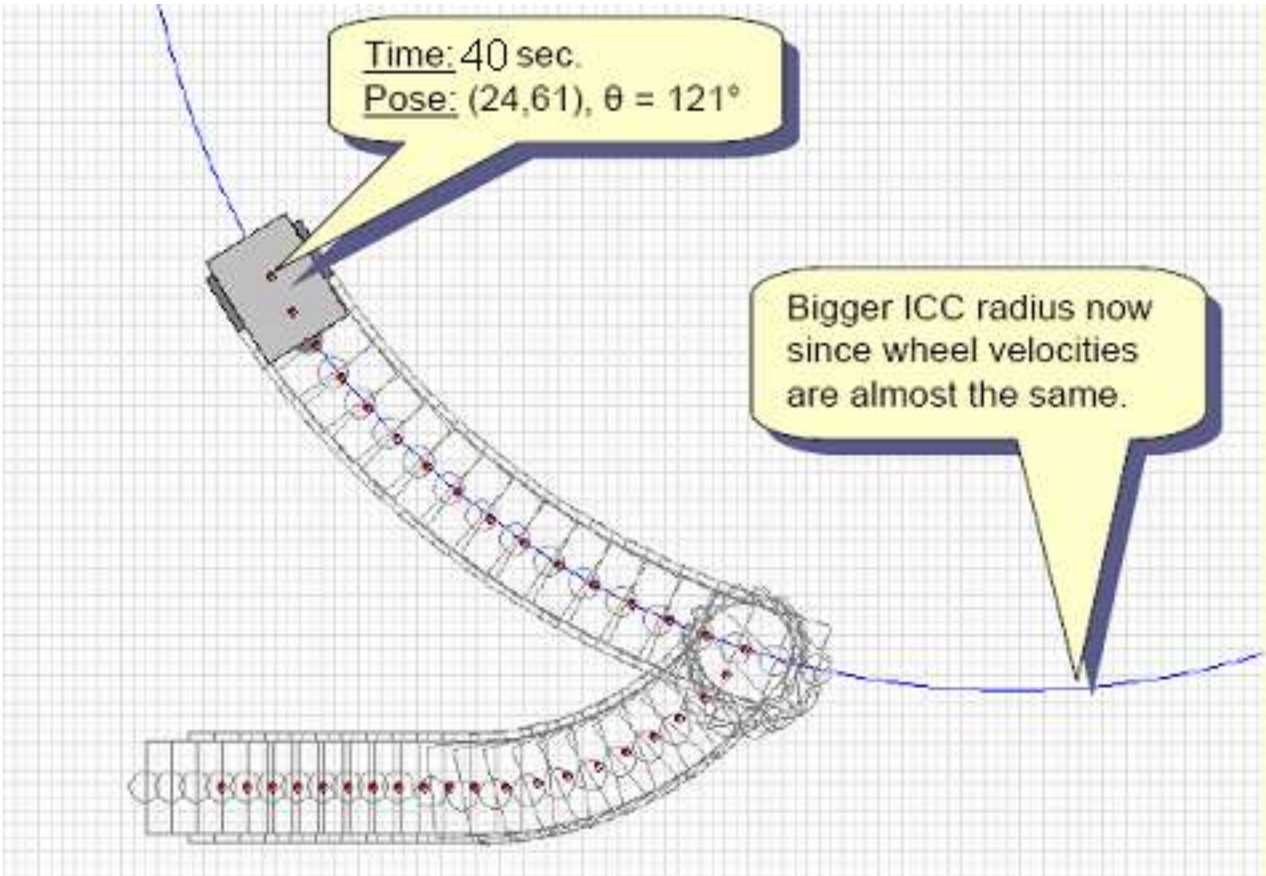
- Now set the robot's right wheel to 3 cm/s and the left wheel to 3.5 cm/sec for 15 seconds
- Where is the robot at $t = 40$ seconds, $\Delta = 15$ seconds?

$x(40) =$ *23.52 cm*

$y(40) =$ *60.55 cm*

$\theta(40) = \theta(25) + \omega\Delta = 162° - (0.047 \text{ rad/s})(15 \text{ sec}) =$ ***121°***

Forward Kinematics: clockwise turn



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Forward Kinematics: clockwise spin

- Finally, the robot sets the right wheel to *0 cm/s* and the left wheel to *3 cm/s* for *10 s*.
- Where is the robot at $t = 50$ s, $\Delta = 10$ s?

$R = (5.3)(3 + 0)/(0 - 3) = -5.3$ cm

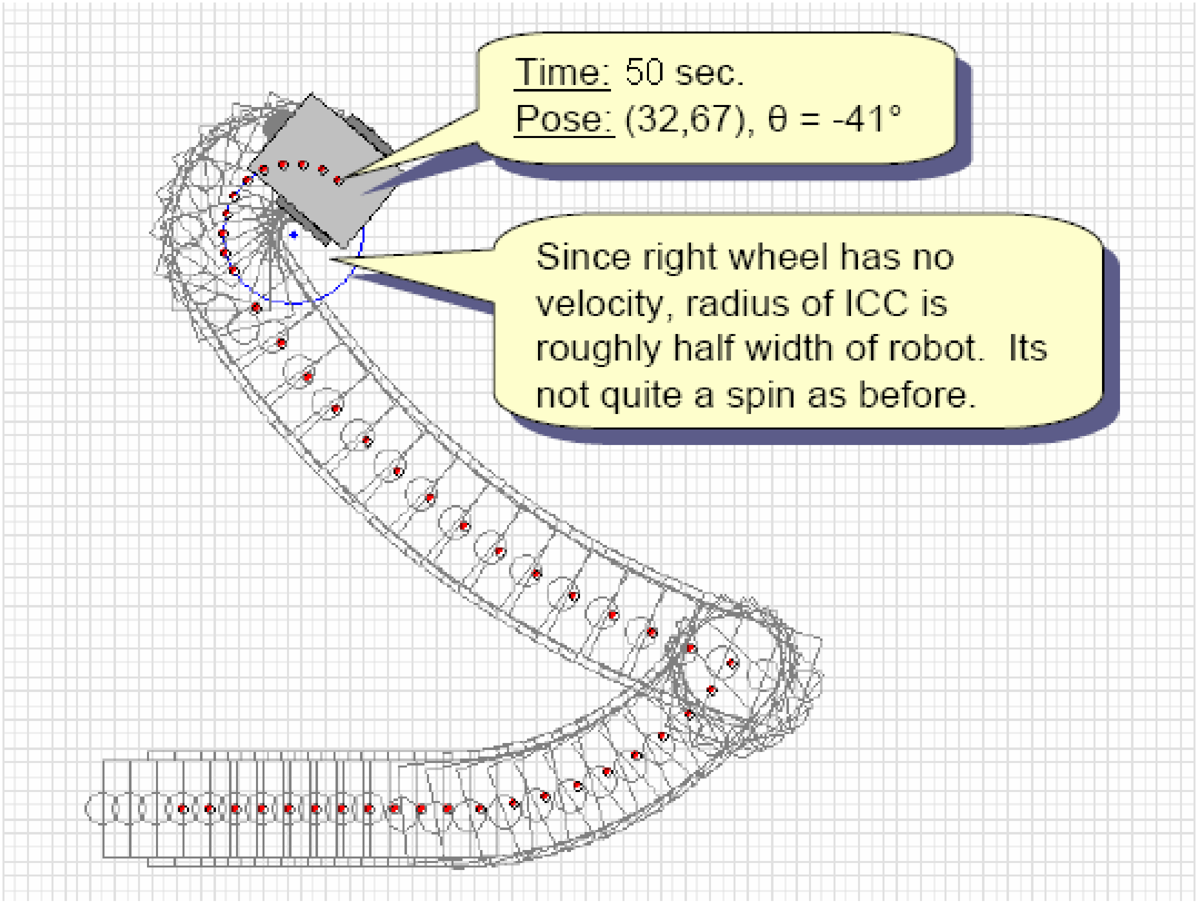
$\omega = (0 - 3)/10.6 = -0.28202$ rad/s

$x(50) =$ ***31.5 cm***

$y(50) =$ ***67.47 cm***

$\theta(50) = \theta(40) + \omega\Delta = 121^\circ - (0.283 \text{ rad/s})(10 \text{ sec}) =$ ***-41°***

Forward Kinematics - Example



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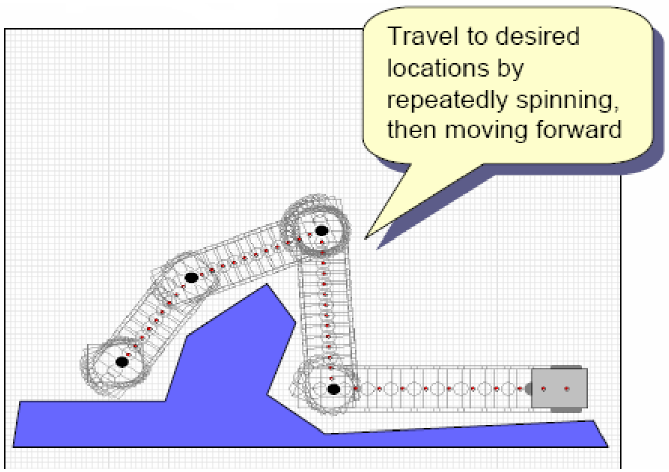
Inverse Kinematics

- *Inverse Kinematics* is determining the control parameters (wheel velocities) that will make the robot move to a new pose from its current pose
- This is a very difficult problem
  - Too many unknowns, not enough equations and multiple solutions
- The easy solution is to
  - Spin the robot to the desired angle
  - Move forward to the desired location



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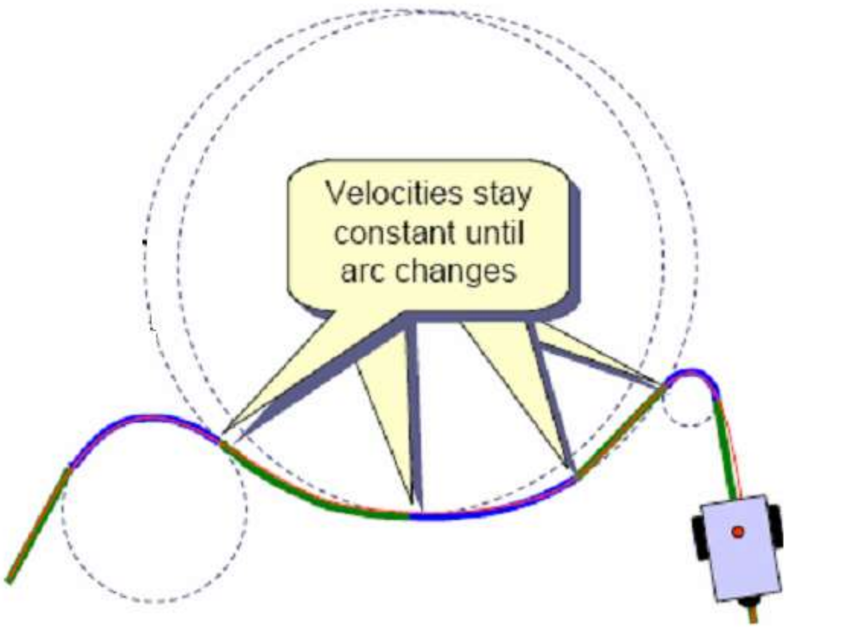
Inverse Kinematics

- Approximate a desired path with arcs based upon computing ICR values
- Result is a set of straight-line paths and ICR arc potions
- Either set the robot drive time and compute velocities for each portion of the path
- Or set velocities and compute drive time for each portion of the path



Inverse Kinematics: spin time and velocities

- The length of time to spin the wheels is determined by the velocity
  - $\theta(t + \Delta) = \theta_t - \omega\Delta \rightarrow \Delta = (\theta(t + \Delta) - \theta(t))/\omega$
  - Since $\omega = (v_1 - v_2)/(2\ell)$ and $v_1 = -v_2$
  - $\Delta = \ell\,(\theta(t + \Delta) - \theta(t))/v_1$
- Alternately, set the spin time and calculate the wheel velocities
  - $v_1 = \ell\,(\theta(t + \Delta) - \theta(t))/\Delta$

Inverse Kinematics: forward time

- The length of time to move forward is determined by the velocity ($v_t = v_1 = v_2$)
- Since $x(t + \Delta) = x_t + v_t\Delta \cos(\theta_t)$ and $y(t + \Delta) = y_t + v_t\Delta \sin(\theta_t)$
  - if $x(t + \Delta) \neq x_t$
    - $\Delta = (x(t+ \Delta) - x_t)/(v_t\cos(\theta_t)$, or
  - if $x(t + \Delta) = x(t)$
    - $\Delta = (y(t+ \Delta) - y_t)/(v_t\sin(\theta_t))$

Inverse Kinematics: forward velocities

- Conversely, the wheel velocities, $v_t = v_1 = v_2$, can be determined by setting the forward move time
- Since $x(t + \Delta) = x_t + v_t\Delta \cos(\theta_t)$ and $y(t + \Delta) = y_t + v_t\Delta \sin(\theta_t)$
  - if $x(t + \Delta) \neq x_t$
    - $v_t = (y(t+ \Delta) – y_t)/(\Delta \cos(\theta (t))$
  - if $x(t + \Delta) = x(t)$
    - $v_t =(y(t+ \Delta) – y_t)/(\Delta \sin(\theta (t))$