Applied Stochastic Process Problems 3

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1 Question1

Determine the long-run proportion of time that $X_{N(t)+1} < c$.

1.1 Answer of Q1

This is equivalent to the long-run proportion of time that the age is smaller than c. It can be written as

$$\lim_{t\to\infty} \frac{\int_0^t I\{X_{N(S)+1} < c\}ds}{t} = \frac{E[min(X,c)]}{EX}$$

For non-negative random variable X, we have

$$EX = \int_0^\infty Pr\{X > s\}ds$$

Therefore, we have

$$\begin{aligned} E[min(X,c)] &= \int_0^\infty Pr\{min(X,c) > x\}dx \\ &= \int_0^c Pr\{X > x\}dx + \int_c^\infty Pr\{c > x\}dx \\ &= \int_0^c 1 - F(x)dx \end{aligned}$$

Thus, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{\int_0^t I\{X_{N(S)+1} < c\}ds}{t} &= \frac{E[min(X,c)]}{EX} \\ &= \frac{\int_0^c 1 - F(x)dx}{EX} \end{aligned}$$

2 Question2

For an interarrival distribution F having mean μ, we defined the equilibrium distribution of F, denoted F_e, by

$$F_e(x) = \frac{1}{\mu}\int_0^x [1 - F(y)]dy$$

(a) Show that if F is an exponential distribution, then $F = F_e$.
(b) If for some constant c,

$$F(x) = \begin{cases} 0, & x < c \\ 1, & xc \end{cases}$$

show that F_e is the uniform distribution on $(0, c)$. That is, if interarrival times are identically equal to c, then the equilibrium distribution is the uniform distribution on the interval $(0, c)$.

(c) The city of Berkeley, California, allows for two hours parking at all nonmetered locations within one mile of the University of California. Parking officials regularly tour around, passing the same point every two hours. When an official encounters a car he or she marks it with chalk. If the same car is there on the official's return two hours later, then a parking ticket is written. If you park your car in Berkeley and return after three hours, what is the probability you will have received a ticket?

2.1 Answer of Q2(a)

Since F follows exponential, we have

$$F(x) = 1 - e^{-\lambda x} E[X] = \mu = \frac{1}{\lambda}$$

Then, to compute F_e, we have

$$\begin{aligned} \frac{1}{\mu}\int_0^x (1 - F(y))dy &= \lambda \int_0^x (1 - 1 + e^{-\lambda})dy \\ &= 1 - e^{\lambda x} \end{aligned}$$

Therefore, $F_e = F$

2.2 Answer of Q2(b)

According to $F_(x)$, we have

$$F_e(x) = \begin{cases} \frac{1}{\mu}\int_0^x (1 - F(y))dy = \frac{x}{\mu}, \text{for } x < c \\ \frac{1}{\mu}(\int_0^c 1dx + \int_c^x 0dx) = \frac{c}{\mu}, \text{for } x \geq c \end{cases}$$

Since $\mu = EX = c$, we have

$$F_e(x) = \begin{cases} \frac{x}{c}, \text{for } x < c \\ 1, \text{for } x \geq c \end{cases}$$

By definition of uniform distribution, we have F_e is a uniform distribution on (0,c).

2.3 Answer of Q2(c)

From (b), we have $F_e \sim uniform(0, 2)$, therefore with probability $\frac{1}{2}$ it will happen in 1 hour and therefore received a ticket. Otherwise, it will not receive a ticket.

3 Question3

Consider a renewal process having interarrival distribution F such that

$$\bar{F}(x) = \frac{1}{2}e^{-x} + \frac{1}{2}e^{-x/2}, \quad x > 0$$

That is, interarrivals are equally likely to be exponential with mean 1 or exponential with mean 2.

(a) Without any calculations, guess the equilibrium distribution F_e.

(b) Verify your guess in part (a).

3.1 Answer of Q3(a)

Since on average, the exponential with mean 2 will have more proportion than exponential with mean 1, we could guess the equilibrium distribution as the following

$$\bar{F}_e(x) = \frac{2}{3}e^{-\frac{x}{2}} + \frac{1}{3}e^{-x}$$

3.2 Answer of Q3(b)

By definition, we have

$$\begin{aligned} F_e(x) &= \frac{1}{\mu}\int_0^x (1 - F(y))dy \\ &= \frac{2}{3}\int_0^x (\frac{1}{2}e^{-y} + \frac{1}{2}e^{-\frac{y}{2}})dy \\ &= 1 - \frac{1}{3}e^{-x} - \frac{2}{3}e^{\frac{x}{2}} \end{aligned}$$

This verifies our guess in (a).

4 Question4

In Example 7.20, let π denote the proportion of passengers that wait less than x for a bus to arrive. That is, with W_i equal to the waiting time of passenger i, if we define

$$X_i = \begin{cases} 1, & \text{if } W_i < x \\ 0, & \text{if } W_i x \end{cases}$$

then $\pi = \lim_{n\to\infty} \sum_{i=1}^{n} X_i/n$.

(a) With N equal to the number of passengers that get on the bus, use renewal reward process theory to argue that

$$\pi = \frac{E\left[X_1 + \cdots + X_N\right]}{E[N]}$$

(b) With T equal to the time between successive buses, determine $E\left[X_1 + \cdots + X_N \mid T = t\right]$

(c) Show that $E\left[X_1 + \cdots + X_N\right] = \lambda E[\min(T, x)]$.

(d) Show that

$$\pi = \frac{\int_0^x P(T > t)dt}{E[T]} = F_e(x)$$

(e) Using that $F_e(x)$ is the proportion of time that the excess of a renewal process with interarrival times distributed according to T is less than x, relate the result of (d) to the PASTA principle that "Poisson arrivals see the system as it averages over time". 1. (optional) If the time between renewals is either 5 with probability 1/3 or 15 with probability 2/3, what is the equilibrium distribution? What is its mean? Describe the equilibrium distribution as a mixture of Unif$(0, 5)$ and Unif$(0, 15)$. Why do the mixing probabilities make sense?

4.1 Answer of Q4(a)

We could define a cycle time as the inter-arrival time between two buses. Then, it is a renewal process. The reward of the process is the number of passengers that wait less than x. Applying the RRT, we have

$$\pi = \frac{E[\text{reward in a cycle}]}{E[\text{cycle time}]} = \frac{E[X_1 + X_2 + ... + X_N]}{E[N]}$$

4.2 Answer of Q4(b)

Given $T = t$, there are two cases. If $t < x$, we have

$$E[X_1 + X_2 + ... + X_N | T = t] = EN = \lambda t$$

If $t \geq x$, the number of customers arrived before x follows $P.P.(\lambda x)$

$$E[X_1 + X_2 + ... + X_N | T = t] = \lambda x$$

Combine them together, we have

$$E[X_1 + X_2 + ... + X_N | T = t] = \lambda min(x, t)$$

4.3 Answer of Q4(c)

By conditioning, we have

$$\begin{aligned} E[X_1 + X_2 + ... + X_N] &= E[E[X_1 + X_2 + ... + X_N | T]] \\ &= \lambda(TPr\{T < x\} + xPr\{T \geq x\}) \\ &= \lambda E[min(T, x)] \end{aligned}$$

4.4 Answer of Q4(d)

From (a), (c), we have

$$\pi = \frac{E[X_1 + X_2 + ... + X_N]}{EN} = \frac{\lambda E[min(T, x)]}{EN} = \frac{\lambda \int_0^x 1 - F(t)dt}{EN}$$

Since $N \sim P.P.(\lambda T)$, we have $EN = \lambda ET$. Therefore,

$$\pi = \frac{\lambda \int_0^x 1 - F(t)dt}{EN} = \frac{\int_0^x 1 - F(t)dt}{ET} = F_e(x)$$

4.5 Answer of Q4(e)

The PASTA principle says that a system as seen by Poisson arrivals is the same as the system as averaged over all time. In our scenario, the system seen by Poisson arrivals is the waiting time that less than x. We've shown that it equals to proportion of time that the excess is less than x, which is the average result. This is consistent with PASTA principle.

In the class we have a proposition that

$$Z_n = \sum_{i=1}^{n} Y_i \to t \ \textit{w.p.1 as } n \to \infty$$

I tried to prove it by SLLN but it seems little strange because

$$Y_i \sim exp(\frac{n}{t}), \ \text{ the mean is related to } n$$

In this case, directly apply SLLN would have

$$Pr\{\lim_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n} = \frac{t}{n}\} = 1$$

It seems little weird. However, if we just think about the mean and variance of Z_n, which follows Erlang distribution,

$$E[Z_n] = \frac{n}{\frac{n}{t}} = t$$

$$Var(Z_n) = \frac{t^2}{n}$$

As $n \to \infty$, Z_n is almost equal to t because the variance goes to zero. Could we argue the propostition in this way? And could we apply SLLN in this case?