Applied Stochastic Process Problems 2

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1 Question1

In each game played one is equally likely to either win or lose 1 . Let X be your cumulative winnings if you use the strategy that quits playing if you win the first game, and plays two more games and then quits if you lose the first game.
(a) Use Wald's equation to determine $E[X]$.
(b) Compute the probability mass function of X and use it to find $E[X]$.

1.1 Answer of Q1(a)

Firstly, let's define

$$N := \text{number of game playing}$$

then, we argue that N is a stopping time. Recall the definition of stopping time is

$$N \in Z \text{ is a stopping time for } X_1, X_2, ... \text{ if } \{N = n\} \perp X_{n+1}, X_{n+2}, ... \forall n \geq 1$$

Here, when we know the result of X_1, we can tell whether to stop, which is independent of future trials. Therefore, N is a stopping time of this game playing process.
Then, let's define an indicator variable

$$\begin{aligned} I_i &:= I\{\text{whether win for the i th playing}\} \\ &= \begin{cases} 1, & \text{prob.=0.5} \\ 0, & \text{prob.=0.5} \end{cases} \end{aligned}$$

Then, we could write X in this form

$$X = \sum_{i=1}^{N} I_i$$

Applying Wald's equation, we have

$$E[X] = E[N]E[I_i] = (1 \times \frac{1}{2} + 3 \times \frac{1}{2}) \times 0.5 = 1$$

1.2 Answer of Q1(b)

Firstly, note that $X = 0, 1, 2$. Consider the following cases

$$\begin{cases} Pr\{X=0\} = Pr\{lose, lose, lose\} = \frac{1}{8} \\ Pr\{X=1\} = Pr\{\text{win the first game}\} + Pr\{lose, lose, win\} + Pr\{lose, win, lose\} = \frac{3}{4} \\ Pr\{X=2\} = Pr\{lose, win, win\} = \frac{1}{8} \end{cases}$$

Then, we have

$$EX = 0 \times \frac{1}{8} + 1 \times \frac{3}{4} + 2 \times \frac{1}{8} = 1$$

2 Question2

Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two days of travel; door 2 returns him to his room after a four-day journey; and door 3 returns him to his room after a six-day journey. Suppose at all times he is equally likely to choose any of the three doors, and let T denote the time it takes the miner to become free.

(a) Define a sequence of independent and identically distributed random variables $X_1, X_2 \ldots$ and a stopping time N such that

$$T = \sum_{i=1}^{N} X_i$$

Note: You may have to imagine that the miner continues to randomly choose doors even after he reaches safety.

(b) Use Wald's equation to find $E[T]$.

(c) Compute $E\left[\sum_{i=1}^{N} X_i \mid N = n\right]$ and note that it is not equal to $E\left[\sum_{i=1}^{n} X_i\right]$.

(d) Use part (c) for a second derivation of $E[T]$.

2.1 Answer of Q2(a)

Let's define

$$X_i := \text{time of the i th travel}$$

By the assumption of equally likely to choose any doors, we have the pmf of X is

$$\begin{cases} Pr\{X_i = 2\} = \frac{1}{3} \\ Pr\{X_i = 4\} = \frac{1}{3} \\ Pr\{X_i = 6\} = \frac{1}{3} \end{cases}$$

Let's define

$$N := \text{the number of travels until firstly finish a travel choosing door 1}$$

Note that it is a stopping time since after a travel, we could tell whether to stop according to the spending time of this travel. It's independent to the afterward travels.

2.2 Answer of Q2(b)

By Wald's equation, we have

$$ET = ENEX_i$$

Note that N $\sim$ $geom(\frac{1}{3})$, and $EX_i = 4$. Therefore, we have

$$ET = ENEX_i = 12(\text{days})$$

2.3 Answer of Q2(c)

Given $N = n$, we could compute the expectation by property of random sum, that is

$$E[\sum_{i=1} NX_i|N = n] = nEX_i = 4n$$

2.4 Answer of Q2(d)

By conditional expectation and geometric distribution of N, we have

$$\begin{aligned}
ET &= E[E[T|N]] \\
&= \sum_{n=1}^{\infty} 4nPr\{N = n\} \\
&= 4\sum_{n=1}^{\infty} n(\frac{2}{3})^{n-1}\frac{1}{3} \\
&= \frac{4}{3}\sum_{n=1}^{\infty} n(\frac{2}{3})^{n-1} \\
&= \frac{4}{3}9 \\
&= 12
\end{aligned}$$

The result is consistency with previous.

3 Question3

A deck of 52 playing cards is shuffled and the cards are then turned face up one at a time. Let X_i equal 1 if the i th card turned over is an ace, and let it be 0 otherwise, $i = 1, \ldots, 52$. Also, let N denote the number of cards that need be turned over until all four aces appear. That is, the final ace appears on the N th card to be turned over. Is the equation

$$E\left[\sum_{i=1}^{N} X_i\right] = E[N]E\left[X_i\right]$$

valid? If not, why is Wald's equation not applicable?

3.1 Answer of Q3

Here N is a stopping time since once we have the 4 th ace, we could immediately tell stop. However, Wald's equation is not applicable because **the sequence of X_i are not i.i.d.** This violates the assumption of Wald's equation.

4 Question4

An $M/G/\infty$ queueing system is cleaned at the fixed times $T, 2T, 3T, \ldots$ All customers in service when a cleaning begins are forced to leave early and a cost C_1 is incurred for each customer. Suppose that a cleaning takes time $T/4$, and that all customers who arrive while the system is being cleaned are lost, and a cost C_2 is incurred for each one.
(a) Find the long-run average cost per unit time.
(b) Find the long-run proportion of time the system is being cleaned.

4.1 Answer of Q4(a)

The whole process is shown below.

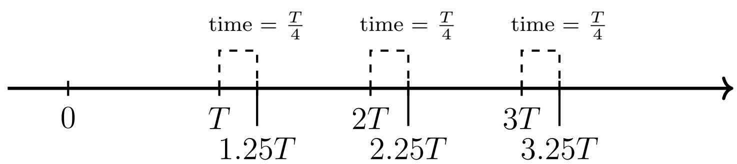

Figure 1: The illustration of the whole process

In the long run, the cycle time is T and it contains $\frac{T}{4}$ cleaning time and then $\frac{3T}{4}$ arrival and service time. To decide the long run cost, we will use renewal reward theorem

$$\lim_{t\to\infty} \frac{ER(t)}{t} = \frac{ER}{EX}$$

Let's define

$$N_1 := \# \text{ of customers in the system at the beginning of a cycle}$$
$$N_2 := \# \text{ of arrivals during the first } \frac{T}{4} \text{ time of a cycle}$$

By $M/G/\infty$, it's obvious that N_2 is a Poisson process with rate λ. Then we have

$$N_2 \sim Poisson(\frac{T}{4})$$

To decide N_1, recall the output process of the $M/G/\infty$ is a non homogeneous Poisson process having intensity function $\lambda(t) = \lambda G(t)$. And in lecture 13, we derived $E[\#$ of customers still in the system by time t$]$

$$E[\# \text{ of customers still in the system by time t}] = \lambda \int_0^t \bar{G}(y)dy$$

Therefore, the expected number of customers still in the system at the end of a cycle is given by

$$E[\# \text{ still here}] = \lambda \int_0^{\frac{3T}{4}} \bar{G}(y)dy$$

Then, the long run cost rate is

$$\text{long run cost rate} = E[\frac{C_1 N_1 + C_2 N_2}{T}] = \frac{C_1 \lambda \int_0^{\frac{3T}{4}} \bar{G}(y)dy + C_2 \frac{T\lambda}{4}}{T}$$

4.2 Answer of Q4(b)

In the long run, the proportion of cleaning time is given by

$$\frac{\frac{T}{4}}{T} = \frac{1}{4}$$

4.3 Supplementary material: Analysis of $M/G/\infty$

5 Question5

37. There are three machines, all of which are needed for a system to work. Machine i functions for an exponential time with rate λ_i before it fails, $i = 1, 2, 3$. When a machine fails, the system is shut down and repair begins on the failed machine. The time to fix machine 1 is exponential with rate 5; the time to fix machine 2 is uniform on $(0, 4)$; and the time to fix machine 3 is a gamma random variable with parameters $n = 3$ and $\lambda = 2$. Once a failed machine is repaired, it is as good as new and all machines are restarted.

(a) What proportion of time is the system working?

(b) What proportion of time is machine 1 being repaired?

(c) What proportion of time is machine 2 in a state of suspended animation (that is, neither working nor being repaired)?

5.1 Answer of Q5(a)

The whole cycle is shown below.

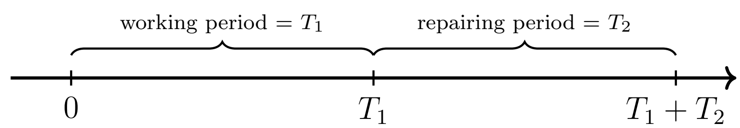

Figure 2: The illustration of a cycle

Let's define

S_i := repairing time of machine i

I_i := the indicator variable that whether machine i is broken when a machine is broken

It's clear that T_1 and T_2 can be written as

$$T_1 \sim exp(\lambda_1 + \lambda_2 + \lambda_3)$$
$$T_2 =_{s.t.} I_1 S_1 + I_2 S_2 + I_3 S_3$$

Let's define

$$T_{cycle} := T_1 + T_2$$

Then, by renewal reward theorem, the proportion of time that the system is working is given by

$$\frac{ET_1}{E[T_1 + T_2]} = \frac{\frac{1}{\lambda_1+\lambda_2+\lambda_3}}{\frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}\frac{1}{5} + \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}2 + \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}\frac{3}{2}}$$

5.2 Answer of Q5(b)

Let's define

$$T_{cycle1} = \text{the time period that machine 1 fails}$$

We could write T_{cycle1} in this form

$$\begin{aligned} T_{cycle1} &= \sum_{i=1}^{N} T_i \\ \text{where } T_i &=_{s.t.} T_{cycle} \\ N &:= \text{the stopping time} \end{aligned}$$

Since the times to fail are all exponential, we have

$$N \sim Geom(\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3})$$

By Wald's equation, we have

$$\begin{aligned} ET_{cycle1} &= ENET_i \\ &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1} \left(\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \frac{1}{5} + \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} 2 + \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \frac{3}{2} \right) \end{aligned}$$

Therefore, the proportion is given by

$$\begin{aligned} \frac{ES_1}{ET_{cycle1}} &= \frac{\frac{1}{5}}{\frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1} \left(\frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \frac{1}{5} + \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} 2 + \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} \frac{3}{2} \right)} \\ &= \frac{\frac{1}{5}}{\frac{1}{\lambda_1} + \frac{1}{5} + \frac{2\lambda_2}{\lambda_1} + \frac{3\lambda_3}{2\lambda_1}} \end{aligned}$$

5.3 Answer of Q5(c)

This proportion is equivalent to the proportion of time that either 1 being repaired or 3 being repaired. Therefore, using the similar method in (b), the answer should be

$$\frac{\frac{1}{5}}{\frac{1}{\lambda_1} + \frac{1}{5} + \frac{2\lambda_2}{\lambda_1} + \frac{3\lambda_3}{2\lambda_1}} + \frac{\frac{3}{2}}{\frac{1}{\lambda_3} + \frac{1}{5\lambda_3} + \frac{2\lambda_2}{\lambda_3} + \frac{3}{2}}$$

6 Question6

For a renewal process with inter-event time X, what's $\lim_{t\to\infty} m(t+ 0.5) - m(t)$ for the following? Justify your answers.
(a) $X = 1$ with probability 1 .
(b) $X \sim \text{unif}(0,1) + 1$

6.1 Answer of Q6(a)

Note that $m(t)$ can be written as

$$m(t) = E[N(t)] = \lfloor t \rfloor$$

Then, we have

$$m(t+0.5) - m(t) = \lfloor t+0.5 \rfloor - \lfloor t \rfloor = \begin{cases} 0, \text{if } t - \lfloor t \rfloor < 0.5 \\ 1, \text{if } t - \lfloor t \rfloor \geq 0.5 \end{cases}$$

Therefore, we have

$$\lim_{t\to\infty} m(t+0.5) - m(t) \; D.N.E$$

Note: Here X is a lattice random variable so Blackwell's Theorem cannot be used.

6.2 Answer of Q6(b)

Note that

$$\mu = EX = 0.5 + 1 = 1.5$$

By Blackwell's Theorem, we have

$$\lim_{t\to\infty} m(t+0.5) - m(t) = \frac{0.5}{\mu} = \frac{1}{3}$$

7 Question (optional)

Consider the gambler's ruin problem where on each bet the gambler either wins 1 with probability p or loses 1 with probability $1-p$. The gambler will continue to play until his winnings are either $N-i$ or $-i$. (That is, starting with i the gambler will quit when his fortune reaches either N or 0.) Let T denote the number of bets made before the gambler stops. Use Wald's equation, along with the known probability that the gambler's final winnings are $N-i$, to find $E[T]$.
Hint: Let X_j be the gambler's winnings on bet $j, j \geq 1$. What are the possible values of $\sum_{j=1}^{T} X_j$? What is $E\left[\sum_{j=1}^{T} X_j\right]$?

7.1 Answer of Question

Recall the result of Gambler's ruin problem

$$p_i := Pr\{\text{starting at i, reach N before reach 0}\} = \begin{cases} \frac{1-(\frac{1-p}{p})^i}{1-(\frac{1-p}{p})^N}, & \text{if } p \neq 0.5 \\ \frac{i}{N}, & \text{if } p = 0.5 \end{cases}$$

Then, if $p = 0.5$ we have

$$\sum_{j=1}^{T} X_j = \begin{cases} N-i, \text{ prob.} = \frac{i}{N} \\ -i, \text{ prob.} = 1 - \frac{i}{N} \end{cases}$$

If $p \neq 0.5$ we have

$$\sum_{j=1}^{T} X_j = \begin{cases} N-i, \text{ prob.} = \frac{1-(\frac{1-p}{p})^i}{1-(\frac{1-p}{p})^N} \\ -i, \text{ prob.} = 1 - \frac{1-(\frac{1-p}{p})^i}{1-(\frac{1-p}{p})^N} \end{cases}$$

Since T is the stopping time of the process, by Wald's equation we have

$$E\left[\sum_{j=1}^{T} X_j\right] = E[T]E[X_j]$$

$$E[T] = \frac{E[\sum_{j=1}^{T} X_j]}{E[X_j]}$$

Therefore, we have

$$E[T] = \begin{cases} \frac{1}{p}\left\{(N-i)\frac{1-(\frac{1-p}{p})^i}{1-(\frac{1-p}{p})^N} + (-i))(1 - \frac{1-(\frac{1-p}{p})^i}{1-(\frac{1-p}{p})^N})\right\}, \text{ if } p \neq 0.5 \\ 2\left\{(N-i)\frac{i}{N} + (-i)(1-\frac{i}{N})\right\}, \text{ if } p = 0.5 \end{cases}$$