Applied Stochastic Process Problems 1

Yisen Du (Eason)

1 Question1

A population of organisms consists of both male and female members. In a small colony any particular male is likely to mate with any particular female in any time interval of length h, with probability $\lambda h + o(h)$. Each mating immediately produces one offspring, equally likely to be male or female. Let $N_1(t)$ and $N_2(t)$ denote the number of males and females in the population at t. Derive the parameters of the continuous-time Markov chain $\{N_1(t), N_2(t)\}$, i.e., the v_i, P_{ij} of Section 6.2.

1.1 Answer of Q1

Note that the state is defined as $\{N_1(t), N_2(t)\}$, and the state changes whenever a new offspring is produced. The time that stays in state $(N_1, N_2) \sim exp(N_1 N_2 \lambda)$ since any particular male could mate with any particular female. Thus, $v_{N_1,N_2} = N_1 N_2 \lambda$

Since it's equally like to produce male or female, then

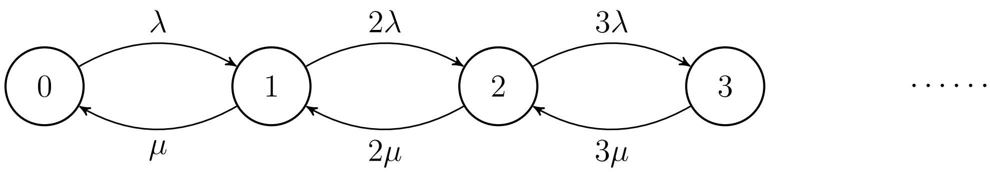
$$P_{i,j} = \begin{cases} 0.5, N_1 \text{ increases by } 1 \\ 0.5, N_2 \text{ increases by } 1 \end{cases}$$

2 Question2

Consider a birth and death process with birth rates $\lambda_i = (i+1)\lambda, i \geq 0$, and death rates $\mu_i = i\mu, i \geq 0$.
(a) Determine the expected time to go from state 0 to state 4 .
(b) Determine the expected time to go from state 2 to state 5 .
(c) Determine the variances in parts (a) and (b).
(d) Determine $E[\text{time to go from state 4 to 0}]$ assuming $\lambda_i \equiv \lambda, \mu_i \equiv \mu$, and $\lambda < \mu$.

2.1 Answer of Q2(a)

The CTMC of this question is given by



Firstly, let's define $T_{i,j}$:= time go from i to j. For state 0, it's obvious that

$$E[T_{0,1}] = \frac{1}{\lambda} \tag{1}$$

For $T_{1,2}$, let's define T_1 := time stays in state 1, and $I := I\{$Starting from state 1, whether go to state 2
Then, by first step analysis, we could write this equation

$$T_{1,2} =_{s.t.} T_1 + (1-I)(T+0,1+T^{'}_{1,2})$$

Note that $T^{'}_{1,2} \sim T_{1,2}$, and $T^{'}_{1,2} \perp T_{1,2}$. Also, $I \perp T_{0,1}$, $T^{'}_{1,2}$. Thus, we have

$$ET_{1,2} = \frac{1}{\mu+2\lambda} + \frac{\mu}{\mu+2\lambda}[ET_{0,1} + ET_{1,2}] \tag{2}$$

After the simplification, we have

$$ET_{1,2} = \frac{1}{2\lambda} + \frac{\mu}{2\lambda}ET_{0,1}$$

Similarly, we have

$$ET_{2,3} = \frac{1}{3\lambda} + \frac{2\mu}{3\lambda}ET_{1,2} \tag{3}$$

$$ET_{3,4} = \frac{1}{4\lambda} + \frac{3\mu}{4\lambda}ET_{2,3} \tag{4}$$

$$ET_{4,5} = \frac{1}{5\lambda} + \frac{4\mu}{5\lambda}ET_{3,4} \tag{5}$$

Finally, by "skip-free" property, we have

$$T_{0,4} =_{s.t.} T_{0,1} + T_{1,2} + T_{2,3} + T_{3,4}$$

Since the terms in the RHS are pair-wisely independent, we have

$$ET_{0,4} = ET_{0,1} + ET_{1,2} + ET_{2,3} + ET_{3,4}$$

By the above equations, we could solve $ET_{0,4}$

2.2 Answer of Q2(b)

According to the recursive relation and initial condition we derived in (a), we could solve $ET_{2,5}$ similarly.

2.3 Answer of Q2(c)

By independence, we have

$$Var(T_{0,4}) = Var(T_{0,1}) + Var(T_{1,2}) + Var(T_{2,3}) + Var(T_{3,4})$$

Similarly to what we do in (a), we could also find the recursive relation between $Var(T_{i,i+1})$ and $Var(T_{i+1,i+2})$. The detailed derivation can be found in p.365 of textbook. The recursive relation is given by

$$Var(T_{i,i+1}) = \frac{1}{\lambda_i(\lambda_i + \mu_i)} + \frac{\mu_i}{\Lambda_i} Var(T_{i-1,i}) + \frac{\mu_i}{\mu_i + \lambda_i}(ET_{i-1,i} + ET_{i,i+1})^2$$

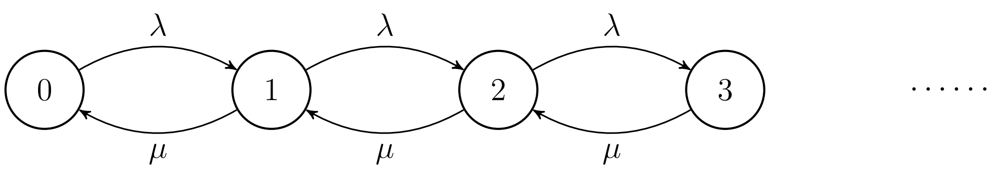
And the initial condition is given by

$$Var(T_{0,1}) = \frac{1}{\lambda^2}$$

Using the above formulae, we could recursively find $Var(T_{0,4})$ and $Var(T_{2,5})$.

2.4 Answer of Q2(d)

For (d), the CTMC is given by



By the skip-free and symmetric properties, we have

$$\begin{aligned} T_{4,0} &=_{s.t.} T_{4,3} + T_{3,2} + T_{2,1} + T_{1,0} \\ T_{4,3} &=_{s.t.} T_{3,2} =_{s.t.} T_{2,1} =_{s.t.} T_{1,0} \end{aligned}$$

Therefore, we have

$$T_{4,0} =_{s.t.} 4T_{1,0} \Longrightarrow ET_{4,0} = 4ET_{1,0}$$

Then, consider $T_{1,0}$ by first step analysis. Let's define a indicator variable

$$I := I\{\text{Starting at state 1, whether go to state 0 directly }\} = \begin{cases} 1, \text{ prob.} = \frac{\lambda}{\lambda+\mu} \\ 0, \text{ prob.} = \frac{\mu}{\lambda+\mu} \end{cases}$$

We have

$$T_{1,0} =_{s.t.} IT_1 + (1-I)T_{2,0}$$

Note that here $ET_1 = \frac{1}{\mu}$ since given it will go to state 0 directly, the time it stays in state 1 follows $exp(\mu)$.
Again by the skip-free and symmetric properties, we have

$$\begin{cases} T_{2,0} =_{s.t.} T_{2,1} + T_{1,0} \\ T_{2,1} =_{s.t.} T_{1,0} \end{cases} \implies T_{2,0} =_{s.t.} 2T_{1,0}$$

Using the above equations, we have

$$\begin{aligned} ET_{1,0} &= \frac{\mu}{\lambda+\mu} ET_1 + \frac{\lambda}{\lambda+\mu} ET_{2,0} \\ &= \frac{\mu}{\lambda+\mu}\frac{1}{\mu} + \frac{\lambda}{\lambda+\mu} 2ET_{1,0} \end{aligned}$$

After simplification, we have

$$ET_{1,0} = \frac{1}{\mu - \lambda}$$

Therefore,

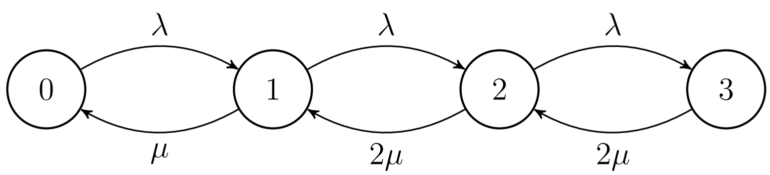
$$ET_{4,0} = \frac{4}{\mu - \lambda}$$

3 Question3

15. A service center consists of two servers, each working at an exponential rate of two services per hour. If customers arrive at a Poisson rate of three per hour, then, assuming a system capacity of at most three customers,
(a) what fraction of potential customers enter the system?
(b) what would the value of part (a) be if there was only a single server, and his rate was twice as fast (that is, $\mu = 4$)?

3.1 Answer of Q3(a)

The CTMC is given by



From birth and death view,

$$\begin{cases} P_1\mu = P_0\lambda \\ P_2 2\mu = P_1\lambda \\ P_3 2\mu = P_2\lambda \\ P_1 + P_2 + P_3 + P_0 = 1 \end{cases}$$

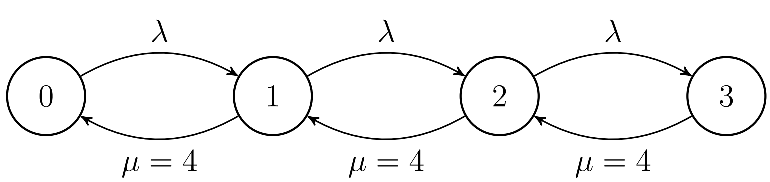
After solving the above system, we have

$$P_3 = \frac{27}{143}$$

Note that when the system stays in state 3, it will loss customers since the capacity is three. Therefore, the actual enter rate is equal to $1 - P_3 = \frac{116}{143}$.

3.2 Answer of Q3(b)

The CTMC is given by



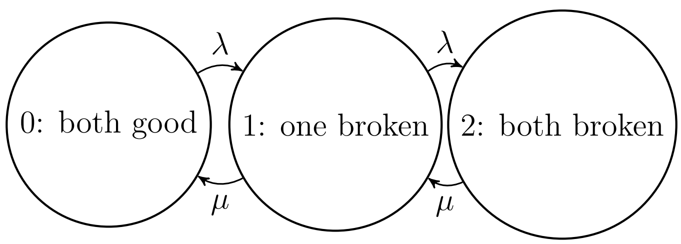
Using the same method, we have $1 - P_3 = \frac{148}{175}$

4 Question4

There are two machines, one of which is used as a spare. A working machine will function for an exponential time with rate λ and will then fail. Upon failure, it is immediately replaced by the other machine if that one is in working order, and it goes to the repair facility. The repair facility consists of a single person who takes an exponential time with rate μ to repair a failed machine. At the repair facility, the newly failed machine enters service if the repairperson is free. If the repairperson is busy, it waits until the other machine is fixed; at that time, the newly repaired machine is put in service and repair begins on the other one. Starting with both machines in working condition, find
(a) the expected value and
(b) the variance of the time until both are in the repair facility.
(c) In the long run, what proportion of time is there a working machine?
(d) Give the generator matrix
(e) Give v and P^* for the uniformized process

4.1 Answer of Q4(a)

The CTMC is given by



Note that it essentially is a birth and death process. Using the same method in Q2, we have

$$ET_{0,2} = \frac{1}{\lambda} + \frac{1}{\lambda} + \frac{\mu}{\lambda^2}$$

4.2 Answer of Q4(b)

To compute variance, firstly note that

$$\begin{aligned} Var(T_{0,2}) &= Var(T_{0,1} + T_{1,2}) \\ &= Var(T_{0,1}) + Var(T_{1,2}) \\ &= \frac{1}{\lambda^2} + Var(T_{1,2}) \end{aligned}$$

To compute $Var(T_{1,2})$, we will use the conditioning technique. Let's define

$$I := I\{\text{Starting at 1, go to 2 directly}\}.$$

By the conditioning variance formula, we have

$$Var(T_{1,2}) = E[Var(T_{1,2}|I)] + Var(E[T_{1,2}|I])$$

- (i) Compute $E[Var(T_{1,2}|I)]$
  Firstly, we have
  $$E[Var(T_{1,2}|I)] = Pr\{I=1\}Var(T_{1,2}|I=1) + Pr\{I=0\}Var(T_{1,2}|I=0)$$
  Let's define
  $$T_1 := \text{time to leave state 1}$$
  We have $T_1 \sim exp(\mu+\lambda)$.
  The conditional variances are given by
  $$\begin{aligned} Var(T_{1,2}|I=1) &= Var(T_1) = (\frac{1}{\mu+\lambda})^2 \\ Var(T_{1,2}|I=0) &= Var(T_1 + T_{0,2}) = Var(T_1) + Var(T_{0,1}) + Var(T_{1,2}) \end{aligned}$$
  Then, we have
  $$E[Var(T_{1,2}|I)] = \frac{\lambda}{\lambda+\mu}\frac{1}{(\lambda+\mu)^2} + \frac{\mu}{\lambda+\mu}[\frac{1}{(\lambda+\mu)^2} + \frac{1}{\lambda^2} + Var(T_{1,2})]$$
- (ii) Compute $Var(E[T_{1,2}|I])$
  Firstly, note that
  $$E[T_{1,2}|I] = \frac{1}{\mu+\lambda} + (1-I)[ET_{0,1} + ET_{1,2}]$$
  Then,
  $$\begin{aligned} Var(T_{1,2}|I) &= (ET_{0,1} + ET_{1,2})^2 Var(1-I) \\ &= (\frac{1}{\lambda} + \frac{1}{\lambda} + \frac{\mu}{\lambda^2})^2 \frac{\lambda\mu}{(\mu+\lambda)^2} \end{aligned}$$

Using the above result, we have
$$Var(T_{1,2}) = \frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda^3} + (\frac{1}{\lambda} + \frac{1}{\lambda} + \frac{\mu}{\lambda^2})^2 \frac{\mu}{(\mu+\lambda)}$$
Therefore, we have
$$Var(T_{0,2}) = \frac{1}{\lambda^2} + \frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda^3} + (\frac{1}{\lambda} + \frac{1}{\lambda} + \frac{\mu}{\lambda^2})^2 \frac{\mu}{(\mu+\lambda)}$$

4.3 Answer of Q4(c)

Solving the following system
$$\begin{cases} P_1\mu = P_0\lambda \\ P_2\mu = P_1\lambda \\ P_0 + P_1 + P_2 = 1 \end{cases}$$
We have
$$P_0 + P_1 = \frac{1 + \frac{\lambda}{\mu}}{1 + \frac{\lambda}{\mu} + (\frac{\lambda}{\mu})^2}$$
This is what we want.

4.4 Answer of Q4(d)

The generator matrix is given by

$$\begin{Vmatrix} -\lambda & \lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & \mu & -\mu \end{Vmatrix}$$

4.5 Answer of Q4(e)

Let's define

$$\gamma := \lambda + \mu$$

Using the formula

$$P^* = I + \frac{Q}{\gamma}$$

We have

$$P^* = \begin{bmatrix} \frac{\mu}{\mu+\lambda} & \frac{\lambda}{\mu+\lambda} & 0 \\ \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} \\ 0 & \frac{\mu}{\mu+\lambda} & \frac{\lambda}{\mu+\lambda} \end{bmatrix}$$

5 Question5

Suppose that the interarrival distribution for a renewal process is Poisson distributed with mean μ. That is, suppose

$$P\{X_n = k\} = e^{-\mu}\frac{\mu^k}{k!}, \quad k = 0, 1, \ldots$$

(a) Find the distribution of S_n.
(b) Calculate $P\{N(t) = n\}$.

5.1 Answer of Q5(a)

By definition,

$$S_n := \sum_{i=1}^{n} X_i$$

By the property of Poisson distribution, we have

$$S_n = \sum X_i \sim Poisson(n\mu)$$

5.2 Answer of Q5(b)

Let's define

$$F_n(t) := \text{cdf of } Poisson(n\mu)$$

Then, we have

$$PrN(t) = n = F_n(t) - F_{n+1}(t) = \sum_{k=0}^{t} \frac{(n\mu)^k e^{-n\mu}}{k!} - \sum_{k=0}^{t} \frac{((n+1)\mu)^k e^{-(n+1)\mu}}{k!}$$

Note: If t is not integer, we should use $\lfloor t \rfloor$.

6 Question6

4. Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be independent renewal processes. Let $N(t) = N_1(t) + N_2(t)$.
(a) Are the interarrival times of $\{N(t), t \geq 0\}$ independent?
(b) Are they identically distributed?
(c) Is $\{N(t), t \geq 0\}$ a renewal process?

6.1 Answer of Q6(a)

No, they aren't. Let's consider a counter example. First, let's define

$$\begin{aligned} M_i &:= \text{inter-arrival time of } N_1(t) \equiv 1 \\ N_i &:= \text{inter-arrival time of } N_2(t) \sim uniform\{0.6, 1.5\} \end{aligned}$$

If we are not given any information, then

$$PrX_2 = 0.4 < 1$$

However, if we are given that $X_1 = 0.6$, then

$$PrX_2 = 0.4|X_1 = 0.6 = 1$$

Therefore, they are not independent.

6.2 Answer of Q6(b)

Using the above example, note that

$$X_1 = \begin{cases} 0.6 \\ 1 \end{cases}$$
$$X_2 = \begin{cases} 0.4 \\ 0.6 \end{cases}$$

Therefore, they are not identically distributed.

6.3 Answer of Q6(c)

Due to the above analysis, by definition of renewal process, it's not a renewal process.

7 Question7

A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to Poisson process with rate λ. Whenever a shock occurs, the worker discontinues working on the present job and starts a new one. In the long run, at what rate are jobs completed?

7.1 Answer of Q9

Firstly, let's define

$$\begin{aligned} X_i &:= \text{the time of completing the i th work} \\ R_i &:= \text{the required time to finish the i th job} \sim F \\ T_i &:= \text{the time that a shock happens} \sim exp(\lambda) \end{aligned}$$

Our goal is to compute $E[X]$

Let's first conditioning on R then conditioning on T to solve this problem.
Firstly, condition on R, we have

$$\begin{aligned} EX &= E[E[X|R]] \\ &= \int_0^\infty E[X|R=t]dF(t) \end{aligned}$$

To compute $E[X|R=t]$, we condition on T

$$\begin{aligned} E[X|R=t] &= E[E[X|R=t,T]] \\ &= \int_0^\infty E[X|R=t,T=s]f_T(s)ds \\ &= \int_0^\infty E[X|R=t,T=s]\lambda e^{-\lambda s}ds \end{aligned}$$

For a given $R=t$, there are two cases

$$E[X|R=t,T=s] = \begin{cases} t, \text{ if } s<t \\ s+EX, \text{ otherwise} \end{cases}$$

Therefore, we can separate the integration into two parts

$$\begin{aligned} \int_0^\infty E[X|R=t,T=s]\lambda e^{-\lambda s}ds &= \int_t^\infty t\lambda e^{-\lambda s}ds \int_0^t (s+EX)\lambda e^{-\lambda s}ds \\ &= \int_t^\infty t\lambda e^{-\lambda s}ds + \int_0^t s\lambda e^{-\lambda s}ds + \int_0^t EX\lambda e^{-\lambda s}ds \\ &= EX(1-e^{-\lambda t}) + \frac{1}{\lambda} - te^{-\lambda t} - \frac{1}{\lambda}e^{-\lambda t} + te^{-\lambda t} \end{aligned}$$

Then, back to the previous integration

$$
\begin{aligned}
EX &= \int_0^\infty (E[X][1 - e^{-\lambda t}] + \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda t})dF(t) \\
&= (EX + \frac{1}{\lambda})[1 - \int_0^\infty e^{-\lambda t} f(t) ft]
\end{aligned}
$$

Note that this integration is the Laplace transformation of F: $\eta(\lambda)$. Finally, we have

$$
EX = \frac{1 - \eta(\lambda)}{\lambda\eta(\lambda)}
$$

8 Question8

Consider a renewal process with mean interarrival time μ. Suppose that each event of this process is independently "counted" with probability p. Let $N_C(t)$ denote the number of counted events by time $t, t > 0$.
(a) Is $N_C(t), t \geq 0$ a renewal process?
(b) What is $\lim_{t\to\infty} N_C(t)/t$?

8.1 Answer of Q8(a)

Yes, it is. Firstly, let's define

$$Y := \text{inter-arrival time of } N_C(t) = \sum^{N_i} X_i$$
$$N_i := \text{counts of consecutive 'counted' events}$$

Note that each Y are iid. By definition of renewal process, it is a renewal process.

8.2 Answer of Q8(b)

Note that if the current event is not counted, the inter-arrival time for $N_C(t)$ will be accumulated until an event is counted. It's clear that

$$N_i \sim geom(p)$$

Therefore, the expectation of Y is given by

$$\mu_c = EY_i = EN_iEX = \frac{\mu}{p}$$

By SLRP,

$$\lim_{t\to\infty} \frac{N_c(t)}{t} \longrightarrow \frac{1}{\mu_c} = \frac{p}{\mu}$$

9 Question9

Events occur according to a Poisson process with rate λ. Any event that occurs within a time d of the event that immediately preceded it is called a d-event. For instance, if $d = 1$ and events occur at times $2, 2.8, 4, 6, 6.6, \ldots$, then the events at times 2.8 and 6.6 would be d-events.
(a) At what rate do d-events occur?
(b) What proportion of all events are d-events?

9.1 Answer of Q9(a)

Firstly, let's define

$$\begin{aligned} T_i &:= \text{inter-arrival time of two groups of d- events} = \sum^{N_i} X_i \\ N_i &:= \text{counts of consecutive non d-events} \\ X_i &:= inter - arrivaltimeohthisPoissonProcess \sim exp(\lambda) \end{aligned}$$

Similar to the last question, for the consecutive non d-events, the failure rate is given by

$$Pr\{X_i < d\} = 1 - e^{-\lambda d}$$

And we have

$$N \sim geom(1 - e^{-\lambda d})$$

Then,

$$ET = ENEX_i = \frac{1}{\lambda}\frac{1}{1 - e^{-\lambda d}}$$

Therefore,

$$\text{rate} = \lambda(1 - e^{-\lambda d})$$

9.2 Answer of Q9(b)

In the long run, the proportion of d-events is given by

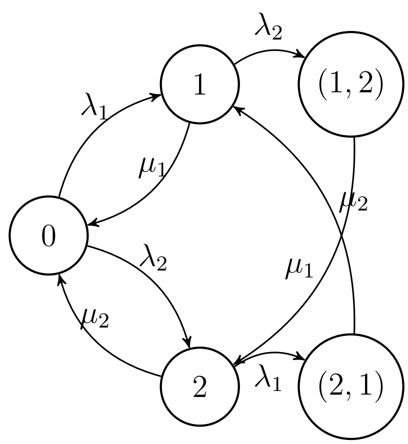
$$Pr\{X_i < d\} = 1 - e^{-\lambda d}$$

10 Question10: Additional question1

A single technician is responsible for two machines. For machine i, the time to fail $\sim exp(\lambda_i)$ and the time to repair$\sim exp($
$mu_i)$, and all such times are independent. Suppose preemption is permitted without penalty, i.e., the technician can stop repairing one machine and switch to the other at any time. For $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = 1$, $\mu_2 = 3$, find the optimal preemptive repair policy (which machine should have priority when both machines are down), when the objective is
(a) Maximize the time both machines are working (a series system)
(b) Minimize the time neither machine is working (a parallel system)
(c) Maximize long-run average production rates, if machine i produces widgets at rate γ_i, $\gamma_1 = 2$, $\gamma_2 = 3$, and they operate independently.

10.1 Answer of Q10(a)

Note that without any priority to repair, this is what we learned in class. Follow the same notation in the class, the CTMC of this model is given by



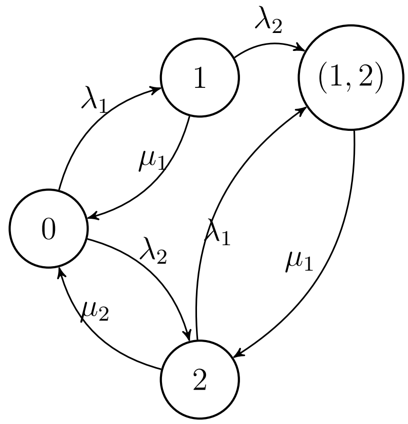
The limiting probabilities of each state can be solved by the linear system below

$$\begin{cases} (\lambda_1 + \lambda 2)P_0 = \mu_1 P_1 + \mu_2 P_2 \\ (\mu_1 + \lambda 2)P_1 = \lambda_1 P_0 + \mu_2 P_4 \\ (\mu_2 + \lambda 1)P_2 = \lambda_2 P_0 + \mu_1 P_3 \\ \mu 1 P_3 = \lambda_2 P_1 \\ \mu 2 P_4 = \lambda_1 P_2 \\ \sum P_j = 1 \end{cases}$$

The solution is given by

$$\vec{P_1} = \begin{bmatrix} 0.2586 & 0.1552 & 0.2069 & 0.3103 & 0.069 \end{bmatrix}$$

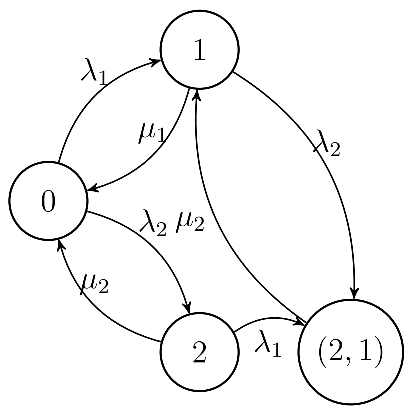
If the priority is to fix one, the CTMC is given by



The limiting probabilities of each state can be solved by the similar method. The solution is given by

$$\vec{P}_2 = \begin{bmatrix} 0.2647 & 0.0882 & 0.2353 & 0.4118 \end{bmatrix}$$

If the priority is to fix two, the CTMC is given by



The limiting probabilities of each state can be solved by the similar method. The solution is given by

$$\vec{P}_3 = \begin{bmatrix} 0.24 & 0.36 & 0.12 & 0.28 \end{bmatrix}$$

To max the time stays in state 1, since 0.2647 is the biggest one, then the priority is to fix 1.

10.2 Answer of Q10(b)

To min the time that there's broken machine, compare $P_{(1,2)} + P_{(2,1)}$ in a, $P_{(1,2)}$ in b, and $P_{(2,1)}$ in c. Note that 0.28 is the min, then the priority is to fix 2.

10.3 Answer of Q10(c)

To max production rate, let's compare

$$\begin{cases} (0.2586 + 0.2069)2 + (0.2586 + 0.1552)3 = 2.1724 \\ (0.2647 + 0.2353)2 + (0.647 + 0.0882)3 = 2.0587 \\ (0.24 + 0.12)2 + (0.24 + 0.36)3 = 2.52 \end{cases}$$

Therefore, the priority is to fix 2.

11 Question11: Additional question2

If the mean value function of a renewal process $\{N(t)\}$ is given by $m(t) = \frac{t}{2}$, what is $Pr\{N(5) = 0\}$?

11.1 Answer of Q11

Note that $m(t)$ could characterize a renewal process, and we have known that the Poisson process has linear mean value function. Therefore, we have

$$N(t) \sim P.P.(\frac{1}{2})$$
$$N(5) \sim Poisson(\frac{5}{2})$$

Thus, we have

$$Pr\{N(5) = 0\} = e^{-\frac{5}{2}} \approx 0.082$$