Quantifying Entanglement

Jesse Tobias Rasowsky

Adviser: Kannan Jagannathan

May 8, 2008

Submitted to the
Department of Physics
of Amherst College
in partial fulfillment of the requirements
for the degree of
Bachelor of Arts with honors.

Abstract

Since the revolutionary EPR-Bohm thought experiment, researchers inspired by Bell's theorem have been experimentally generating and employing non-classical correlations in the laboratory. Recently, on the theoretical side, much effort has been placed on precisely defining just how much entanglement there is in a system. The simple case of pure states of two-part systems is rather well understood, since the von Neumann entropy is the only "reasonable" measure in this context. Things get more complicated, however, in the mixed and multi-partite cases. Here, depending on the application, you may be justified in selecting any number of measures. For example, if you want to assemble collection of particles into a given state using an ensemble of maximally entangled states, then you must pay something called the entanglement cost.

There are many open problems in this developing field of quantifying entanglement. In this essay we focus on entanglement distributed amongst many particles; specifically, calculating the largest possible minimum pairwise entanglement between particles in a collection of n subsystems, each of which has d levels. After seeing recent results of Dennison and Wootters on the entanglement of formation, we employ crude numerical methods to estimate the trend of a system's entanglement sharing capacity as a function of the dimension d.

Acknowledgments

First, let me offer sincere thanks to Jagu Jagannathan for his unwavering dedication. Despite his busy schedule, he always made time to meet with me. It's a privilege to be his student. I am also especially grateful to Bill Wootters (Williams College), Mark Williamson, and Vlatko Vedral (University of Leeds) for their selflessness, always offering thoughtful advice and technical support. I also humbly offer my thanks to

- Jonathan Friedman for his thoughtful guidance, and for introducing me to the field of quantum information theory.
- Larry Hunter for his kind advice, and for patiently teaching me formal quantum mechanics.
- Dan Velleman for his insightful mathematical advice.
- the Dean of the Faculty for supporting me last summer.

I am indebted to my fellow students inside and outside the Physics department for their friendly encouragement. Finally, thanks to my dear parents, brothers and sister for providing constant tender love and care. I am unsure how to quantify this "TLC" phenomenon, but I am nevertheless willing to claim it is not one but rather the characteristic trait of harmonious household relations.

Contents

Preface

The central problem we explore in this essay is the quantification of entanglement. For starters (chapter 1), we give some historical background and explain exactly what entanglement is. Next (chapter 2), we move to the main question, namely, how one goes about prescribing and then calculating measures of entanglement. We will consider, in some detail, the special case of pure bipartite quantum states (chapter 3); followed by the richer, general case of mixed, multipartite systems. At this point (chapter 5) the fun begins, since we'll be ready to introduce and discuss some examples of entanglement measures, along with a few applications. After all this preliminary material, we'll move to the focus of my thesis work, namely, the topic of distributed entanglement. To conclude, we will indicate future possible lines of work.

For the sake of clarity and illustration, we will offer examples and applications along the way. One of my primary aims in this essay is to elucidate some of the questions (and answers!) that abound in this exciting field.

The reader will find no figures in this work. We will encounter geometrical reasonings, but I believe that even for these mere prose will suffice.

By the way, we are generally going to be using density matrix formalism to talk about quantum states. Feel free to consult appendix A, in which we review basic density matrix mathematics.

Chapter 1

Introduction to entangled states

Most descriptions of everyday objects are **separable**; that is, they are, in a sense, reducible to the descriptions of the parts. The state of a chess board, for example, is completely specified by the positions of the individual pieces. Thus, in such a case, we know the state of the whole if, and only if, we know the state of the parts. Perhaps prima facie there is nothing noteworthy about this fact of everyday life. However, it so happens that in the strange world of quantum mechanics (QM) a group of objects may as a whole exist in a state that is *not* expressible in terms of the states of the parts. In virtue of its non-separability, such a state is called **entangled**. Erwin Schrödinger, who coined the term *Verschränkung* (German for entanglement), was a key figure among those who laid the theoretical foundation of this novel concept. In 1935, he penned a prescient passage on the topic of entangled states:

> When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one*

> but rather *the* characteristic trait of QM, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled. [43]

Clearly, Schrödinger was among the first to realize the significance of this "entanglement" phenomenon. And, as he might have hoped, in recent decades scientists have discovered that there are many wonderful applications of the phenomenon. Before we get to the applications, however, it will be instructive to put the physics in historical context. Therefore, let me review some of the back story.

1.1 The EPR paradox

The basic idea of entanglement stemmed from a thought experiment published in 1935 by Einstein, Podolsky, and Rosen (EPR) [19]. Eager to demonstrate the incompleteness of QM, they imagined a scenario in which the states of two spatially separated objects generated curiously strong correlations. By a compelling line of reasoning, they concluded that the formalism of QM must have, in some sense, left something out. Their argument, which we'll get to in a minute, relies on two crucial assumptions: **locality** and **realism**. Let us first spell out precisely what is meant by those terms.

Locality

This hypothesis essentially means no action-at-a-distance. Informally, we can express the spirit of locality like this: there is no direct causal link between space-like separated events. That is to say, influences do not propagate immediately. Einstein was famous for expressing [42] the principle of locality (and its importance to him) on a more precise level:

> But on one supposition we should, in my opinion, absolutely hold fast: the real factual situation of system S2 is independent of what is done with the system S1, which is spatially separated from the former.

In other words, objects separated by a space-like interval are not directly causally related. So no influence may propagate instantaneously between two distant objects. Later on, we will make this idea even more precise, particularly in the context of operators. For now, however, the above definition will do.

Realism

In addition to locality, EPR also wanted to preserve the common-sense notion that, in some cases, it is reasonable to believe that there *genuinely are* atoms and spins and so on to speak of. Einstein expressed this when he asked whether we "really believed that the moon exists only if [we] look at it" (see Ref. [35]). Thus, in essence, realism is the common-sense notion that there really is a world "out there" independent of our observations. EPR formulated [19] a more precise principle of realism:

> If, without in any way disturbing a system, one can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

In other words, the realist believes (as most of us would) that if you can accurately (i.e. with probability 1) forecast the outcome of an experiment, then there must have been something real about it in the first place. We make use of this assumption later on.

The argument

EPR aimed to show that QM was incomplete, that is, that additional variables (not supplied by the traditional ψ-function) were needed to completely specify the state of a system. Meanwhile, however, they wished to hold onto the intuitive notions of causal determinism (i.e. realism) and locality. Bell showed that this is untenable, at least insofar as it is not consistent with the statistical predictions of QM.

For the sake of clarity, let's consider the EPR thought experiment in the context suggested by Bohm [11, 10]. Imagine that pairs of spin-1/2 particles in the singlet state $(\uparrow\downarrow - \downarrow\uparrow)$ are periodically emitted from a source situated between Alice and Bob, two lab technicians, with one spin traveling toward each technician. Assume that the technicians can measure any component of the spin-state of their incoming particles: let Alice measure along the direction specified by the unit vector $\vec{a}$; similarly let Bob measure along $\vec{b}$. When a technician performs his or her measurements, there are exactly two possible outcomes for the measured spin components. Either he or she will get "up" (+1) or "down" (-1). We'll denote the bimodal results for each technician as

$$A \equiv \vec{\sigma_1} \cdot \vec{a} = \pm 1 \text{ and } B \equiv \vec{\sigma_2} \cdot \vec{b} = \pm 1, \tag{1.1}$$

where $\vec{\sigma_1}$ and $\vec{\sigma_2}$ denote the spins of the two particles.

Suppose the technicians' measurement events are separated by a space-like interval, so that a measurement Alice performs on her particle in no way influences the outcome of Bob's experiment. (This is the locality assumption.)

Now, consider the special case where Alice and Bob have chosen to measure along the same direction, i.e. $\vec{a}$ and $\vec{b}$ coincide. (If they are operating Stern-Gerlach apparatuses, for example, they will orient their magnets along some common axis.) Then, for the singlet state, we have $A = +1$ whenever $B = -1$, and vice-versa. Notice what this perfect anti-correlation implies: Alice can

predict with certainty the outcome of Bob's measurement B simply by first performing the corresponding measurement on her particle. Moreover, her measurement does not (cannot) influence Bob's experiment, since she is far away. In other words, to predetermine what Bob will get when he measures $B = \vec{\sigma_2} \cdot \vec{b}$ Alice needs only to measure $A = \vec{\sigma_1} \cdot \vec{b}$ on her particle and take the negative. According the realist position, this must mean that the outcome of Bob's experiment was predetermined. After all, Alice can predict Bob's measurement outcome with certainty, and – because of locality – she does so without influencing it.

1.2 Bell's theorem

Hidden variables

Of course, the ψ-function does not of itself determine the outcome of any given experiment, and so the reasoning above suggests that there must exists some *other* parameters that provide a more complete description of the state. Whatever these parameters may be, they are absent from the traditional QM formalism, which is after all merely a mathematical recipe that lets us calculate expectation values; it says nothing whatever regarding the states that a particle is *really* in. EPR believed that a "complete" physical theory ought in its formalism to account for all "elements of reality," and so for them these other parameters were crucial for the sake of quantum theory's completeness.

Some interpretations of QM aim to preserve the notion of causal determinism and so invoke such parameters. They are often called **hidden variable** theories. This term, however, is rather controversial [3], for many argue that the ψ-function itself is a "hidden variable;" after all, whereas we routinely measure positions and velocities, according to orthodox QM, it is *impossible* to determine the ψ-function for any individual unknown system. In any case,

some believe "hidden variables" is a misnomer altogether, and that its use came about by virtue of historical accident. For more on this, please consult Ref. [5].

Bell's inequality

By considering the entangled system above, John S. Bell proved [2] a fantastic result. In essence, he showed that no local realistic theory can reproduce all the predictions of QM. The proof is as follows.

Suppose there were parameters that contain the information that is otherwise "hidden" from the ψ-function. (This is the realism assumption.) We'll denote them by λ, which could represent a number or a list of numbers or even a list of functions. Let $\rho(\lambda)$ denote the probability distribution of λ. Then the expectation value of the product of Alice and Bob's measurements is

$$P(\vec{a}, \vec{b}) = \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda),$$

where A and B, the respective measurement outcomes for Alice and Bob, are defined in (1.1). Notice that A depends on $\vec{a}$ and (possibly) λ only, but *not* on $\vec{b}$; mutatis mutandis, for B. In English, since the two technicians are far away from one another, Alice's measurement outcome does not depend on the orientation of Bob's magnet, and vice-versa. (This is the locality assumption.)

Now, in order to agree with standard QM, this expectation value must equal the QM result

$$P_{QM}(\vec{a}, \vec{b}) = \left\langle (\vec{\sigma_1} \cdot \vec{a})(\vec{\sigma_2} \cdot \vec{b}) \right\rangle = -\vec{a} \cdot \vec{b}. \tag{1.2}$$

Let's see why this is impossible. First, recall that when Alice and Bob choose the same direction, they get perfect anti correlation; that is,

$$A(\vec{a}, \lambda) = -B(\vec{a}, \lambda).$$

Then the integral above may be rewritten as

$$P(\vec{a},\vec{b}) = -\int d\lambda \rho(\lambda) A(\vec{a},\lambda) A(\vec{b},\lambda).$$

Similarly, for another unit vector $\vec{c}$, we have

$$P(\vec{a},\vec{c}) = -\int d\lambda \rho(\lambda) A(\vec{a},\lambda) A(\vec{c},\lambda).$$

Taking the difference, we get

$$\begin{aligned} P(\vec{a},\vec{b}) - P(\vec{a},\vec{c}) &= -\int d\lambda \rho(\lambda) \Big[A(\vec{a},\lambda) A(\vec{b},\lambda) - A(\vec{a},\lambda) A(\vec{c},\lambda) \Big] \\ &= \int d\lambda \rho(\lambda) A(\vec{a},\lambda) A(\vec{b},\lambda) \Big[A(\vec{b},\lambda) A(\vec{c},\lambda) - 1 \Big], \end{aligned}$$

where we have done some algebraic repackaging using the fact that $A(\vec{b},\lambda)^2 = 1$, which follows from (1.1).

Then, taking the absolute value, we can easily see that

$$\Big| P(\vec{a},\vec{b}) - P(\vec{a},\vec{c}) \Big| \leq \int d\lambda \rho(\lambda) \Big[1 - A(\vec{b},\lambda) A(\vec{c},\lambda) \Big].$$

Now look at the right hand side. The first term simplifies via $\int d\lambda \rho(\lambda) = 1$, since ρ is a normalized probability distribution. And the second term is just $P(\vec{b},\vec{c})$. Thus we rewrite the inequality as

$$\Big| P(\vec{a},\vec{b}) - P(\vec{a},\vec{c}) \Big| \leq 1 + P(\vec{b},\vec{c}).$$

This last statement, called **Bell's inequality** after its discoverer, must be obeyed by any local realistic model. The upshot of Bell's theorem is that the inequality is inconsistent with QM. To see this, consider a concrete example: suppose we choose coplanar vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ with respective orientations 0^o, 45^o, and 135^o. Then, if QM expectation values (cf. Eq. (1.2) above) were to satisfy Bell's inequality, then we would have

$$\sqrt{2} = \left| \frac{-1}{\sqrt{2}} - \frac{+1}{\sqrt{2}} \right| \leq 1 + 0 = 1,$$

which is clearly false.

The upshot, then, is that whereas any local causally deterministic theory must obey the inequality derived above, QM clearly violates it. Hence, there is an experimental criterion on which we can rely to discriminate between QM and local realism. Indeed, one of most important consequences of Bell's result is that it is empirically testable.

The above reasoning closely follows the argument in the 1964 paper. It is worth noting, though, that in practice Bell's original inequality is rather inapplicable because of counter inefficiencies, etc. Instead, experimenters will more often rely on the so-called **CHSH inequality** [13], which is more suitable for typical laboratory applications.

Also, note that this argument deals with statistical results: in order to compute the relevant quantities, an experimenter must average his measurements over many trials. More recently, using the so-called **GHZ-state** of three particles, theorists have presented [22] a version of Bell's inequality which is more powerful in the sense that the QM prediction disagrees with that of local realism on every experimental run; hence, in principle, a single reliable test would serve to distinguish quantum theory from its local deterministic competitors.[1]

Not just "either-or"

It is important to distinguish between genuine entanglement ("quantum correlation") and mere classical correlation. To help bring out the difference, consider the following parable.[2] Suppose your mother owns two types of earrings: diamonds and pearls. Now, let's also suppose she always wears earrings

[1]Since this severe test of local realism requires entanglement between at least three particles, experimental confirmation has proved difficult; even so, conclusive results have been obtained [41].

[2]Obviously inspired by Bell's example [4] of Bertlmann's socks.

that mismatch. There's either a pearl on the left and a diamond on the right, or vice-versa.

Well, naturally, just by looking at the earring on her right, you know for sure which type is on her left, namely, the other kind. This anti-correlation among the earrings resembles what we saw with Alice and Bob's entangled electrons. But are the earrings really entangled? Of course not. How, though, do we explain the crucial difference between the earrings and the electrons?

Bell's argument discriminates in the following way. On the one hand, in the case of the earrings, no matter what the outcome of our observations, it is always possible to ascribe a definite prior state to each earring that would have generated those observations. (It was either "pearl-diamond"or "diamond-pearl".) With an entangled pair, on the other hand, certain measurement outcomes show that it is impossible to assign a prior state to each particle separately.

Let's stick with our example of the singlet state $\uparrow\downarrow - \downarrow\uparrow$; each time we measure one particle or the other, it is true that the pair will have collapsed into a separable state (either $\uparrow\downarrow$ or $\downarrow\uparrow$). But – and this is the main point – over many trials there is no separable description of the electron pair that could have generated the statistical results we see. Instead, the most complete description available refers only to correlations between the electrons, not individual particle states. We have no expression of the form "particle A is so-and-so, and particle B is such-and-such" that is consistent with our correlated measurements. In other words, there is no fact of the matter concerning the state of the individual electrons prior to and independent of our observation of them. That is, it is not merely that we don't *know* the individual spin states beforehand; rather, there are no individual spin states to speak of.

Still, you might wonder: how do we know Alice and Bob's electrons didn't just start out mutually opposed, say, in the product state $\downarrow\uparrow$, with one spin up and the other down. Wouldn't that be sufficient to generate the perfect anti-correlation? In general, the answer is no. In this case of $\downarrow\uparrow$, Alice and Bob would see anti-correlation only if they measure in the vertical direction. If instead they measure in, say, the horizontal direction, then they would see any of the four results $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, and $\downarrow\downarrow$, each occurring randomly, 25% of the time.

What, then, is the key feature of the electron spin measurements Alice and Bob do obtain that entails the nonexistence of a separable expression? Loosely speaking, it is the following: no matter which way they measure – sideways, slantways, whatever – so long as they measure in the same direction, their measurement results will always be perfectly anti-correlated. In this case, one can show [30, 29, 31, 1] that there could not have been a fact of the matter concerning the prior state of either particle. With your mother, there is always a separable option that will be able to explain the measurement outcomes; hence, there is no violation of local realism. Thus, the state of your mom's jewelry is not entangled at all. It is only correlations which cannot be explained by local realism that exhibit genuine entanglement.

Comments

Two parts, one state

It is worth pausing for a moment now to clarify what is meant exactly by the holistic nature of entangled states. Suppose Alice and Bob share a pair of entangled particles in, say, the singlet state. Then the novelty is that there is no way of expressing the state of one particle without making reference to the other. It is, however, true to say that each party holds a separate particle; one

particle is indeed over here and the other is over there; there is, after all, a fact of the matter about the particles being distinct entities. (Actually, we shall see later that indistinguishability is a sufficient but not necessary condition for entanglement.)

Particles vs. states

By the way, we will sometimes speak loosely and say something like "particle A is entangled with particle B." In light of the considerations above, however, strictly speaking it is probably better to say that the *state* of particle A is entangled with the *state* of particle B. Therefore, when we do speak about *particles* being entangled, know that it's just a mnemonic for a more precise statement about their *states*.

Hilbert space

Mathematically, we denote the conjunction of subsystems by the tensor product symbol $\otimes$. Thus, for a composite system consisting of both Alice and Bob's parts, the compound Hilbert space is $H_A \otimes H_B$. Now, when talking about this Hilbert space of the particle pair, we do have some freedom in our choice of basis. But be careful; we must respect the identity of the two particles. Specifically, the basis of $H_A \otimes H_B$ must be one that is generated by bases of its parts. That is, although it is true that in a Hilbert space of dimension d, any collection of d linearly independent vectors could in principle serve as a basis, from a physical point of view it is crucial that we first select bases for both parts, say $\{|a\rangle_m\} \subset H_A$ and $\{|b\rangle_n\} \subset H_B$, and then consider the several higher dimensional basis vectors (in $H_A \otimes H_B$) generated by taking their tensor products.

1.3 Physical Examples

In this essay, we are concerned principally with theoretical considerations. But, you may (should!) wonder, "How are entangled states prepared in practice?" So, for the sake of completeness, we will briefly illustrate a couple of techniques that scientists actually use to generate and observe entanglement in the laboratory.

Down conversion

In this popular scheme for generating entanglement [51, 37, 27], a laser beam of high energy ("blue") photons is fired into a nonlinear optical medium along, say, the z-axis. As a consequence, each photon splits into two lower energy ("red") photons, since energy is conserved. The deflection of the red photons from the z-axis are equal and opposite, since momentum is conserved. The resulting optical output from the crystal is a pair of cones, which intersect at precisely two points on the output face. The experimental fact is that there is no individual polarization at these intersections. Rather, upon measurement, the polarizations are found to exhibit correlations indicative of the entangled state $HV - VH$, where H and V denote horizontally and vertically polarized light, respectively.

Cavity Quantum-electrodynamics

Here the aim is to entangle an atom with the optical field inside a cavity [40, 24, 23]. When an atom passes through a radiation box, by adjusting the phases of the electromagnetic pulses, experimenters can generate a correlation between the photon and atom. Roughly speaking, upon measurement it is seen that either the atom is excited and there's no photon in the cavity, or the atom is in the ground state and there is a photon in cavity. In particular, it

is shown that the state of the atom is entangled with the state of the field in the cavity.

Indistinguishable Particles

So much for the fancy machinery physicists use to produce entangled states artificially. It is now worth clarifying something: entangled states are not just artifacts produced by experimentalists in the lab – they naturally occur all the time. Consider, for example, the electrons in a para helium atom; they are in the entangled singlet state $\uparrow\downarrow - \downarrow\uparrow$! In fact, because of the symmetrization requirement of the wave function, any indistinguishable particles are entangled provided there is any overlap of their wave-packets.

1.4 Quantum Teleportation

The purpose of this section is to show you a neat application of entanglement, but it can be skipped without loss of continuity.

As we have seen, entangled states are not just a theoretical curiosity; they are actually observed experimentally. What's more, they can be employed as a valuable resource. In fact, scientists have concocted practical schemes [6] and algorithms [17, 21] that actually make use of entangled states. We will now illustrate one such beautiful scheme, namely, quantum teleportation. In this section we will make use of the so-called Bell states and the density matrix calculus; so, let me gently remind you of the relevant background material in appendix A.

The task and the scheme

Alice holds an unknown state $|\varphi\rangle = a\,|0\rangle + b\,|1\rangle$ that she wishes to send to Bob. Suppose the two share a maximally entangled pair of qubits, say, $|\Phi^+\rangle =$

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Now, the 3-partite state is

$$|\varphi\rangle \otimes \left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(a\,|000\rangle + a\,|011\rangle + b\,|100\rangle + b\,|111\rangle),$$

where the order of the qubits in each ket is: the unknown state ϕ on the left; followed by Alice's qubit in the middle; and finally Bob's qubit on the right.

Using the Bell basis, we can rewrite the 3-part state as

$$\frac{a}{2}(\left|\Phi^+\right\rangle+\left|\Phi^-\right\rangle)\,|0\rangle+\frac{a}{2}(\left|\Psi^+\right\rangle+\left|\Psi^-\right\rangle)\,|1\rangle+\frac{b}{2}(\left|\Psi^+\right\rangle-\left|\Psi^-\right\rangle)\,|0\rangle+\frac{b}{2}(\left|\Phi^+\right\rangle-\left|\Phi^-\right\rangle)\,|1\rangle\,.$$

After a little repackaging, this becomes

$$\frac{1}{2}\left(\left|\Phi^+\right\rangle(a\,|0\rangle + b\,|1\rangle) + \left|\Phi^-\right\rangle(a\,|0\rangle - b\,|1\rangle) + \left|\Psi^+\right\rangle(b\,|0\rangle + a\,|1\rangle) + \left|\Psi^-\right\rangle(-b\,|0\rangle + a\,|1\rangle)\right)$$

Evidently, then, Bob's qubit is in a superposition of four qubit states, any of which can be transformed into the mystery state $|\phi\rangle$ via some unitary operation. If Alice performs a measurement (in the Bell basis) on her two particles, then she will randomly obtain one of the four equiprobable results, each of which corresponds to a term in the expression immediately above. At the moment of measurement, then, the wave function collapses and Bob's particle will have assumed the corresponding qubit state.

Now the work is partly done, but in order to transform it into $|\phi\rangle$, Bob still needs to know which unitary transformation to apply to his "distorted" copy. So, Alice sends Bob a text message broadcasting which result she obtained, thus informing Bob of which "distorted" state he is holding. For instance, the fourth term indicates that if Alice obtains the result $|\Psi^-\rangle$, then Bob's particle must be in the state $-b\,|0\rangle + a\,|1\rangle$. Now, if Alice texts Bob and says, "I got Psi-minus," then Bob knows that he must perform the operation

$$iY = i\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

on his state to obtain

$$iY(-b\,|0\rangle + a\,|1\rangle) = a\,|0\rangle + b\,|1\rangle = |\phi\rangle\,,$$

which is precisely what he wanted.

Comments

Notice that Alice and Bob never find out a or b. That is, neither party ever learns what the mystery state $|\phi\rangle$ is, although it is communicated from one to the other. Hence, in order to communicate the state, Alice cannot just tell Bob the values of a and b, for she does not know them; nor could she determine them from just one instance of $|\phi\rangle$.

According to this protocol, when the state $|\phi\rangle$ is created in Bob's lab, it is simultaneously destroyed in Alice's lab. Otherwise, we would have cloned the qubit, which is impossible [49].

The transmission of $|\phi\rangle$ is *not* superluminal since Alice used a classical channel (i.e. cell phone) to give Bob a crucial piece of information, namely, the result of her measurement.

Chapter 2

Quantifying entanglement

The mere qualitative matter of *whether* a system contains any entanglement is well understood. We can say, for example, that the singlet state is entangled whereas the state of a chessboard is not. It is quite another matter, however, to quantitatively describe *how much* entanglement there is in a given system. If you have some particles in the state σ and I have some particles in the state ρ, then we can ask: is your "system" *more* or *less* entangled than my system? The matter of precisely deciding just how entangled a given system is (i.e., putting a number on it) is known as the **quantification of entanglement**, and it is the central question of this essay.

2.1 Operational approach

From an operational point of view, it is natural to construct some basic measures of entanglement that tell you how efficiently you can prepare an ensemble of particles in a particular state. If you start with m copies of the state ρ and you want to generate n copies of the state σ, then how well can you do it in principle? Historically, it was in the context of these so-called **purification procedures** that some fundamental measures of entanglement were first proposed [8]. Here are a couple:

Distillable entanglement

Suppose you have an ensemble of r particles in the state ρ, and you want to "distill" out as many maximally entangled states as possible. That is to say, by performing basic operations[1] on a given collection of states, you want to generate as many, say, $\uparrow\downarrow - \downarrow\uparrow$ states as possible. Suppose the number of singlet states you can distill is $s = s(r)$. Then, in the limit of many copies, the ratio s/r of ρ "what you can get" over "what you started with" is called the **distillable entanglement**.

Entanglement cost

As a complement to the process above, imagine you are given a collection of s maximally entangled states, and suppose you want to create the largest possible ensemble of particles in some state σ; that is, you want to generate many copies of a particular state σ, given, say, s copies of the singlet state $\uparrow\downarrow - \downarrow\uparrow$ as a resource. By the most efficient procedure, how much will it cost you? Let $n = n(s)$ denote the optimal amount of copies of σ you can generate. In the limit of many copies, this ratio s/n of "what it cost you" over "what you make" is called the **entanglement cost**.

2.2 Axiomatic approach

In order to tackle the problem of prescribing a quantitative **measure of entanglement**, another more abstract method is this: judiciously select a few physically well-motivated desiderata at the outset that any reasonable measure ought to satisfy; that is, write down some well-defined mathematical properties, and stipulate that any "good" measure E of entanglement should

[1] We will make this more precise later on.

satisfy them. The most basic of these properties are specified by the following fundamental axioms:

Axiom 1: Vanishing on separable states

Any kosher measure E should satisfy $E(\sigma) = 0$ is σ is separable. Note that this is a necessary but not sufficient condition for separable states. In particular, it is known that some states, known as "bound" entangled states, may have E measure zero (and cannot be distilled) although they are indeed "entangled" because they violate some Bell inequality. The first example was given by P. Horodecki in 1997 (see Ref. [25]), and since then the states have been studied in greater detail (see, for example, Ref. [7]).

Examples

Because they are separable, the following pure states have no entanglement at all.

1. The simple product state $|10\rangle$.
2. $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$.
3. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.
4. Any (normalized) state that may be written as $a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle = (a_0 |0\rangle + a_1 |1\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle)$

Including the possibility of mixed states, any state of the form $\rho^{AB} = \rho^A \otimes \rho^B$ is separable, and hence satisfies $E(\rho^{AB}) = 0$.

Axiom 2: Invariant under local unitary operations

It is convenient to construct entanglement measures in such a way that they are unaffected by certain "benign" operations. The basic idea is that we want

to specify some set of harmless operations that should not affect the amount entanglement in a system at all. As it happens, those harmless operations amount to **local changes of basis**. In other words, we stipulate that any well-behaved measure of entanglement does not change under local unitary operations. Let me explain what this means more precisely.

Local operations

The most general way to write down an operator O^{AB} on two parties in quantum mechanics is as the sum of tensor products

$$O^{AB} = \sum_i A_i \otimes B_i, \tag{2.1}$$

where the operators A_i act only Alice's part and the B_i's act only on Bob's part. Now, we say an operator O on a bipartite system is **local** if it can be written as a *single* term that is the product of two such operators, to wit: $A \otimes B$. In other words, this condition implies that the total operation genuinely acts *separately* on the respective subsystems. Hence, in the spirit of EPR's ordinary locality principle, when an operator is expressed by a single product term, the action on part A does not depend on that of part B, and vice-versa. I.e., no action-at-a-distance.

(You may have noticed that technically this requirement is more general than conventional locality. For example, A and B may operate on spaces that don't necessarily correspond to the 3-dimensional space that we live in.)

Note that here for simplicity we have focused on the special case $n = 2$. But, as you may have guessed, this notion of locality is completely general and extends naturally to n-part systems: a mathematical operator acting on an n-part system is local if it may be written as a single (tensor) product of n terms.

Unitary operations

You will recall from linear algebra that any unitary operation just amounts to a "change of basis" in your vector space. To be sure, under a unitary transformation, the way a state is represented (viz. its density matrix elements) may well change. But the state itself is only altered in this nominal sense, *not* in the amount of entanglement it contains.

We can illustrate this with a simple example. In the Stern-Gerlach setup, suppose Alice and Bob select the axes for their magnets independently. Then their various measurement outcomes may not correlate, for they are "projecting" along different bases. But of course by merely rotating their magnets they have not really changed the state itself: they have not altered the amount of entanglement in the system.

So, we are insisting in axiom 2 that if you perform only local unitary operations on an ensemble of particles, then the amount of entanglement cannot increase or decrease.

Counterexamples

It is natural to ask [20] whether there are any interesting operators that do *not* meet this condition. That is, are there any interesting "bad" operators, in the sense that they alter the amount of entanglement in a system? If so, then by design they will either be non-local or non-unitary (or both).

Well, we know a whole class of operators that are non-unitary, namely, **general measurements**. After all, when you "measure" the singlet state $\uparrow\downarrow - \downarrow\uparrow$ – mathematically, project it onto the some valid single-particle basis (as opposed to the Bell basis) – then you will end up with either $\uparrow\downarrow$ or $\downarrow\uparrow$ which obviously does not preserve the inner product.

Also, clearly, an operator corresponding to **subselection** could have the

effect of reducing the entanglement. Mathematically, this would corresponding to "tracing out" some subset of particles.[2] For example, suppose Alice, Bob and Carol hold respective particles A, B and C; and assume A and B are entangled, but C is entangled with neither A nor B. Then considering the density matrix of the whole triplet, then there is definitely some entanglement (1 ebit, say) in the system; on the other hand, if you take the partial trace with respect to particle A, then you are left with just two particles, B and C, which are utterly disentangled from one another. Hence, Alice and Bob share two particles that are entangled and Carol has another particle that is entangled with neither. (Exercise for the reader: Can you think of a similar operation that increases the entanglement?)

So much for non-unitary operators. What about non-local ones? Perhaps we can change the amount of entanglement in a system by applying an explicitly non-local operator. Consider the Hamiltonian (or "energy operator")

$$H = a\left(\vec{S}_1 \cdot \vec{S}_2\right).$$

where, in our notation, the spin operator for the jth particle is written as

$$\vec{S}_j = (S_j^x, S_j^y, S_j^z).$$

So S_2^x, for example, denotes the spin-x operator

$$\frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on the second particle.

What is interesting about this operator H? Consider, for instance, a two-part two-level system that begins in the product state $\uparrow\downarrow$. Under the action

[2]In this mathematical context, "tracing out" over a certain part amounts to ignoring that subsystem; please see appendix A for details.

of the unitary operation $U = \exp(iHt/\hbar)$, that state can in fact become maximally entangled! (Please see the relevant calculations and pictures at the very end of appendix B.) The upshot is that, through the action of the operator H, a state which is disentangled initially may gradually become entangled. Actually, it eventually becomes disentangled again, and the cycle repeats.

A couple of comments are in order here. First, this counterexample thus shows that there are operators which are explicitly non-local, under whose action the entanglement of a system *can* in fact change.

Second, this operation is non-local: you can clearly see from the Hamiltonian that the energy of the system, and hence its time-evolution, depends instantaneously on the state of both parts. Another way to see the non-locality is to expand the Hamiltonian into three terms á la Eq. (2.1):

$$H = a\left(S_1^x \otimes S_2^x + S_1^y \otimes S_2^y + S_1^z \otimes S_2^z\right),$$

which is a far cry from "a single product term." So H is manifestly not local.

Finally, note that this Hamiltonian is somewhat realistic; after all, we can imagine that the energy of interaction between two "spinning" objects is a function of their alignment. (Consider, for example, the magnetic interaction between two dipoles.)

Axiom 3: Monotonicity under local operations and classical communication

This property stipulates that if Alice and Bob perform operations on their respective parts and their only means of communication is, say, by telephone, then the entanglement they share cannot be enhanced. (Transfer of quantum particles, though, is *not* allowed under LOCC.) For example, if Alice and Bob

perform measurements followed by correlated sub-selection[3] on their respective ensembles (both completely *local* operations), then they cannot "create" any entanglement in the process. The entanglement of a system can increase, but only through global operations.

Not true classically

Observe, on the other hand, that axiom 3 does not hold for classical correlations, for classical correlations can increase under LOCC. For example, suppose Alice and Bob each have a 100 coins. Then over the phone they can decide to cooperatively manipulate some subset (e.g., they both flip every third bit to heads[4]), and thereby enhance the classical correlations between the collections.

Comments

Sometimes, in the context of ensembles, a stronger version of this monotonicity condition is enforced, namely, that under LOCC even the **expected entanglement** should not exceed the expected entanglement of the original states. In other words, suppose we perform some classically correlated, local operations on the ensemble $\sigma^{\otimes n}$ (n copies of the state σ); and afterward we obtain a weighted ensemble $\{(\sigma_i, p_i)\}$ (various distinct states with respective probabilities). Then, the "strong monotonicity" axiom requires that the weighted sum of the entanglements of the new states not exceed the entanglement of the prior state. Putting that requirement mathematically, $\sum_i p_i E(\sigma_i) \leq E(\sigma)$. Thus, even the (weighted) *average* of the entanglement should not increase under LOCC; that's why it's called *strong* monotonicity.

[3]By this I mean that Alice and Bob selectively ignore corresponding particles, where they have coordinated, say, via telephone.

[4]This operation is not reversible, since some information is clearly lost; hence it is not unitary. But that's no problem, for axiom 3 does not speak of unitarity.

By the way, it follows from axioms 1 and 3 that E must be non-negative. Suppose there were a state with $E < 0$. Then you could measure it in the natural basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, which requires nothing beyond LOCC. But you would end with a product state, for which $E = 0$, by axiom 1. That means E will have *increased* under LOCC, which is impossible, by axiom 3.

Choosing a measure

The basic problem in selecting "good" measures of entanglement is that typically just a few axioms are not enough to uniquely determine an entanglement measure. By this we mean that in some contexts, there exists distinct measures, say E and F, both satisfying the above properties that do not order all quantum states in the same way; that is, you can pick two states ρ and σ such that $E(\rho) > E(\sigma)$ but $F(\rho) < F(\sigma)$. For instance, although for two particles, the idea of a "maximally entangled state" is well-defined, it is not so clear [36] in the case of multiple particles.

This may seem somewhat aggravating, but it means that there is room for exploring various measures of entanglement, and so indicates that plenty of problems and open questions will remain available for study for some time. Generally, however, for a given application, there are other axioms which further restrict the nature of a measure of entanglement.

Chapter 3

Pure bipartite states

We are now going to discuss a special case, namely, that of pure two-part systems. (Not because they are the most interesting, but because they are the best understood.) By the way, in this chapter, we will be using some matrix mathematics, so I'll gently remind you that operations such as taking the log of a matrix or computing the reduced density operator are explained in appendix A.

One system

In QM the state of a pure d-level system is expressed by a **qudit**, that is, a state vector of the form

$$|s\rangle = s_0 |0\rangle + \cdots + s_{d-1} |d-1\rangle ,$$

where the $|i\rangle$'s denote the basis states and the complex coefficients satisfy $\sum_{i=0}^{d-1} |s_i|^2 = 1$ to ensure normalization. For the sake of simplicity, we will focus on the special case where $d = 2$; that is, pure 2-level bipartite systems, or pairs of qubits. Thus, for a qubit $|a\rangle$ we write

$$|a\rangle = a_0 |0\rangle + a_1 |1\rangle$$

where the states $|0\rangle$ and $|1\rangle$ denote "down" and "up" (or $\downarrow$ and $\uparrow$), respectively.[1]

Two systems

In this section, instead of single systems, we wish to consider pairs of systems, for example: a pair of spin-1/2 particles shared by two parties Alice and Bob. Well, if the corresponding state vector is factorisable into a product of two expressions (one for Alice, the other for Bob), then the state is called **separable**. We can write it like this:

$$|\psi\rangle_{sep} = (a_0\,|0\rangle + a_1\,|1\rangle) \otimes (b_0\,|0\rangle + b_1\,|1\rangle) = a_0b_0\,|00\rangle + a_0b_1\,|01\rangle + a_1b_0\,|10\rangle + a_1b_1\,|11\rangle\,,$$

This, however, is not the most general state a pair may exist in. The most general way to write the state of a pair of spin-$\frac{1}{2}$ particles ($d \times d = 4$) is this:

$$|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle\,,$$

where, of course, the state is normalized so

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1.$$

3.1 Qualitative: separable vs. entangled

Consider the qualitative question of whether or not a pure bipartite state $|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$ is entangled (i.e., non-separable). As it happens, in this special case, it is easy to write down the characterization of entangled states:

$$|\psi\rangle \text{ is entangled } \Leftrightarrow \alpha\delta \neq \beta\gamma. \tag{3.1}$$

The condition $\alpha\delta = \beta\gamma$ is obviously necessary for separability (reader, do you see why?). What we are claiming is that it is also sufficient. The proof is easy.

[1] Please be aware of these different notation conventions; from now on we might glide from one to another.

Assuming that $\alpha\delta = \beta\gamma$, we want to show that $|\psi\rangle$ may be written as a tensor product:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}.$$

Our task is to find suitable complex numbers a_0, a_1, b_0, and b_1. This can always be done provided $\alpha\delta = \beta\gamma$. Just consider the four possibilities

$$\begin{pmatrix} 1 \\ \gamma/\alpha \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ , \ \begin{pmatrix} 1 \\ \delta/\beta \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ , \ \begin{pmatrix} \alpha/\gamma \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \ , \text{ and } \begin{pmatrix} \beta/\delta \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix}.$$

Any of these that is well-defined (can't divide by zero) will work. Now, since for any pure state at least one of the complex coefficients will be nonzero, it's always possible to select one of these four options. Thus, given the state vector of any pure bipartite state, we can easily determine whether or not it is entangled: it is separable iff we can "factor" it into a tensor product.

Examples

1. The **product state** $\downarrow\uparrow = |01\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is separable, since $0 \cdot 0 = 0 \cdot 1$. It factors into $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which says of course that particle 1 is "down" and particle 2 is "up."

2. The singlet state $\uparrow\downarrow - \downarrow\uparrow = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ is of course *not* separable; indeed, in this case $\alpha\delta = 0 \neq \beta\gamma$.[2]

[2]For cleanliness' sake, when there is no danger of confusion, we sometimes omit the normalization constants out front.

3. The state $\uparrow\uparrow + \uparrow\downarrow = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is separable, since $0 \cdot 1 = 0 \cdot 1$. It factors into $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which means that particle 1 is in the state $\uparrow = |1\rangle$ and particle 2 is in the **superposed** state $\uparrow + \downarrow = |1\rangle + |0\rangle$.

   Please keep in mind that superposition is *not* the same as entanglement. Rather, entanglement is special kind of superposition in which the states of two or more parts are inextricably linked to one another; that is, they are not separable. Indeed, entangled means intertwined in a way that is **basis-independent**. Superposition, on the other hand – at least the benign type we encountered in this example – is merely a function of what basis vectors Bob chooses. (For instance, he had chosen $|1\rangle$ and $|1\rangle + |0\rangle$, there would not even have been a superposition.)

4. The state $3\,|00\rangle + i\,|01\rangle + 6i\,|10\rangle + -2\,|11\rangle = \begin{pmatrix} 3 \\ i \\ 6i \\ -2 \end{pmatrix}$ is separable, since $3 \cdot -2 = i \cdot 6i$. It factors into $\begin{pmatrix} 1 \\ 2i \end{pmatrix} \otimes \begin{pmatrix} 3 \\ i \end{pmatrix}$, so that particles 1 and 2 are again in superposed separable states.

So, generally we can easily tell *whether* a pure state is entangled. That's nice, but let's get on with the real business, namely, calculating entanglement.

3.2 Quantitative: von Neumann entropy

Uniqueness: an inheritance from information theory

It turns out that just a few intuitive axioms suffice to uniquely determine [46, 33] the only "good" measure of entanglement for pure bipartite systems, namely, the von Neumann entropy.

(As we will see below, however, the story is more complicated for mixed, multipartite systems.)

This quantity is the analogue in QM of the Shannon entropy from classical mechanics [33]. The Shannon entropy, usually written H, provides the unique measure of "accessibility" of states in the sense that a state A may be converted into another state B via adiabatic processes only if $H(A) \leq H(B)$. [3]

Similarly, the von Neumann entropy S provides a unique such measure of accessibility of a quantum states [39], in the sense that ρ may be converted via LOCC operations into σ only if $S(\sigma) \geq S(\rho)$. (Note the discrepancy in sign here: entropy increases under adiabatic processes whereas entanglement decreases.)

Definition

Let the density operator ρ denote a quantum state. Then the **von Neumann entropy** of ρ is defined as

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho),$$

where the log is to base 2 and tr denotes the trace of a matrix.[4]

As an entanglement measure

To construct a suitable measure of entanglement from this function, we will need to make use the reduced density operator (see appendix). The **entropy of entanglement** E_E of a state is defined as the von Neumann entropy of the

[3] Unfortunately, I do not know nearly enough information theory to give you anything but a rough sketch here. In any case, we only bother with these considerations because historically it is where the "axiomatic approach" (see chapter 2) originates. So, please, if the analogy to classical thermodynamics doesn't help you, don't worry about it; we're just including it here for the sake of quasi-completeness.

[4] The base of the log is really arbitrary, but the convention – inherited from information theory, where data is stored in terms of bits – is to use base 2.

reduced density operator of either subsystem (the two are always equal); that is,

$$E_E(\rho^{AB}) \equiv S(\rho^A) = S(\rho^B).$$

Example

All Bell states are maximally entangled. For instance, an example in the appendix shows that the reduced density operator of the singlet state is represented by $\frac{1}{2}I$, which is already in diagonal form. You, the reader, can verify from the definition above that the entropy of entanglement of the entangled pair of qubits is exactly $-(1/2 + 1/2)\log(1/2) = \log(2)$, which is the largest possible value of the entropy of entanglement.

Chapter 4

Other measures of entanglement

In the more generic context of mixed or multiparticle states, unfortunately there is no unique measure of entanglement. There simply are too many suitable functions that fit the bill imposed by the basic axioms above. It is is, however, useful and instructive to construct measures based on supplementary well-motivated considerations. In this chapter we will review two such measures. (For a myriad of other entanglement measures, we encourage you to consult the references [26, 38].)

4.1 Entanglement of Formation

For a pure bipartite state $|\Phi\rangle$ the **entanglement of formation** is defined as

$$E_f(\Phi) = -\sum_i \lambda_i \log \lambda_i,$$

where the λ_is are the eigenvalues of the reduced density matrix of either subsystem.

How, then, is it defined for mixed states? Well, as we know,[1] for any given mixed state there are many possible decompositions into pure states. That is,

[1]For a reminder, see appendix A.

ρ may be written as

$$\rho = \sum_n p_n \left|\psi_n\right\rangle \left\langle\psi_n\right|,$$

for a weighted collection of pure states $\{\psi_n\}$. But the decomposition is not unique; i.e. there are many such collections that work. There is, however, some optimal ensemble, call it (p_i^*, ψ_i^*), for which the weighted average of the entanglements of the pure states is minimal. This is entanglement of formation of the mixed state ρ. That is,

$$E_f(\rho) = \inf \sum_j p_j E_f(\psi_j),$$

where the infimum indicates that we use the optimal ensemble.

By the way, throughout our study, we adopt the convention that $0 \log_2 0 = 0$. On the one hand, this makes sense mathematically since $x \log_2 x$ does indeed approach 0 in the limit as $x \to 0$. It makes sense from a physical point of view, too: an outcome with zero probability should not contribute to the entanglement.

4.2 Relative Entropy of Entanglement (REE)

For some geometric measures of entanglement, it will be useful to define a notion of "distance" between quantum states. We now consider one such measure, whose definition is motivated by analogy with its classical counterpart, the **relative entropy**, which indicates the difference between two probability distributions. The **quantum relative entropy** (QRE) between two matrices ρ and σ is defined as the difference between $-\mathrm{Tr}\rho \log \sigma$ and the von Neumann entropy. That is,

$$\mathrm{QRE}(\rho, \sigma) \equiv \mathrm{Tr}\rho \log \rho - \mathrm{Tr}\rho \log \sigma.$$

An analogue of the classical relative entropy [44] from information theory, this quantity provides a measure of the distinguishability between two states.

For our purposes the important thing about the QRE is that it can serve as a measure of distance between two quantum states, in the sense of distinguishability. In particular, we can use it to define the following important geometric measure of entanglement.

The **relative entropy of entanglement** (REE) [45] of a state ρ is defined as the minimum, taken over all separable states σ, of the QRE between ρ and σ. That is,

$$REE(\rho) := \min_{\sigma \in D} \{\mathrm{QRE}(\rho, \sigma)\}. \tag{4.1}$$

In other words, for a given entangled state, this measure picks out the nearest separable state[2] and spits out the distance (i.e. the QRE) to that closest state. (That's easy for us to say, but beware, computing this analytically is very difficult, since actually proving that a certain state is the closest one is in general quite hard.) The REE is thus a measure of the optimal distance (or distinguishability) from all separable states. Of course, the REE of any disentangled state ρ is just zero.

[2]It natural to wonder [15] whether the nearest separable state to a given entangled one has any further relevance or interesting properties. I don't know the answer to this, but consider that it is not even obvious whether there is a unique nearest state.

Chapter 5

Distributed entanglement

5.1 Introduction

As we know, entanglement, or quantum correlations, are categorically different from classical correlations. One novel manifestation of this distinction is that unlike classical correlations, quantum correlations between multiple parties are in a sense limited.

On the one hand, **classical correlations** amongst multiple systems can in principle be perfect. For instance, the trend of heights and weights of students at Amherst, Williams, and Wesleyan theoretically could be identical. Another example: temperature fluctuations in ten different cities might be exactly parallel [16].

On the other hand, **entanglement** among multiple particles are limited in the following sense. Suppose Alice and Bob share two particles, A and B, in an entangled state. Then there is a limit on the degree to which the state of either particle A or particle B may be entangled with that of some third particle C [14, 12].

For instance, one can show that if particles A and B are each entangled with some third particle, then the density operator of the composite system $A \otimes B$ must be mixed. Therefore, to give a simple example, if Alice and Bob

share singlet state, which is pure and maximally entangled, then their particles A and B cannot be entangled with another particle C.

If, however, you permit *partial* entanglement between systems A and B, then it has been shown [14] there is a trade off in the sense that A and B may share a larger amount of entanglement only at the expense of sacrificing their potential entanglement with other parties.

5.2 The problem

Therefore, a natural question to ask in the context of multiparticle entanglement is this: For a collection of n d-level particles (qudits), what is the largest possible value of the minimum pairwise entanglement between particles, considering all possible states of the system? In other words, for any state $X(n, d)$ there will be some pair of particles $P_{min}(X)$ that is least entangled, say, by the amount $E(X)$.[1] Now, we want to know, over all possible states on our system, what is the greatest value of E? The answer, generally given in ebits (the standard unit of entanglement), is denoted $E_{max}(n, d)$.

5.3 Old results

In past decades, this question has been answered for several special cases, with the **entanglement of formation** being taken as the measure of entanglement. The following results are known:

1. $E_{max}(2, d) = \log_2(d)$.

   This means that if you have just two qudits, the greatest possible entanglement occurs when the entanglement between them is completely

[1] The pair need not be unique; in fact, intuition (as well as serious computational effort [16]) suggest that the optimal state might be symmetric, perhaps in such a way that all pairs are equally entangled.

saturated. This makes sense: there is no third party with which to share, so there is no "trade off" in the above sense.

2. $E_{max}(3,2) = 0.550$.

   It has been shown [18] that the state

   $$\frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

   of three qubits maximizes the pairwise minimum of entanglement. Thus, for a system three spin-1/2 particles, for example, the "trade off" has significant effect.

3. $E_{max}(n,2) \geq f(2/n)$,

   where f is a function called the concurrence, given by Wootters' formula [50]. For n qubits, it has been proven [28] that in the special case where the density matrix is the same for any pair of particles, the maximum pairwise concurrence is $f(2/n)$.

5.4 Recent results

Taking the entanglement of formation as their measure of entanglement, in 2001 Dennison and Wootters achieved [16] a few new results to supplement the list above.

A system of d qudits

For a collection of d qudits ($n = d$), they proved a lower bound: the largest possible value of the minimum entanglement shared by two particles is no less than 1. That is, $E_{max}(d,d) \geq 1$. To prove this, they first consider the special case of three qutrits (i.e. $d = 3$). By invoking certain symmetries and relying on some results [47] about Werner states, they verified that the entanglement

of formation of the singlet state with respect to SU(3) has entanglement of formation equal to unity. Hence they prove 1 is a lower bound on $E_{max}(3,3)$. A completely analogous construction works for arbitrary d.

Three particles: $d = 2$ and $d = 7$

For a threesome of qudits ($n = 3$), they consider the specific cases $d = 2$ and $d = 7$, and prove a lower bound in each case. (The example $d = 3$ is of course included in the first result.) It is natural to wonder how exactly the ability of particles to share entanglement varies with the dimension d. Therefore, it is enlightening to consider the ratio of $E_{max}(n,d)$ to the entanglement capacity $\mathrm{Log}(d)$.

When $n = d = 3$, Dennison and Wootters found that the ratio of E_{max} over the entanglement capacity was (at least) 0.631, which is larger than 0.550, the corresponding ratio for ($n = 3$, $d = 2$). For $n = 3$ and $d = 7$, they estimated[2] the ratio to be .710, which is even larger. Interestingly, then, their three results for the case $n = 3$ suggest that the ability of (pairs of) particles to share entanglement is enhanced by boosting the dimension of the system.

This trend of increasing $E_{max}/\log(d)$, albeit over just a few data points, suggests that proportionally the entanglement capable of being shared by qudits increases with the dimension d. It is not yet clear what this means for "infinite dimensional" systems. Perhaps in a system with very many levels (e.g. molecules), the entanglement sharing capacity will be enormous. Then again, perhaps there will be some definite cap. Either result would be interesting, so it seems worthwhile to investigate the matter further for higher values of d.

[2]Of course, I am using phrases like "at least" and "estimated" because some results are lower bounds, which may or may not be sharp.

5.5 New lines of work

Dennison and Wootters [16], along with Williamson [48], have suggested that it is definitely worthwhile to investigate the same problem (viz. entanglement distributed among qudits) for other measures of entanglement – not just entanglement of formation. We consider the relative entropy of entanglement (REE).

The lazy scheme

Our aim is to calculate the largest possible value of the minimum pairwise REE. In our calculations, we have focused on pure states, completely ignoring mixed states. The approach we adopted is this. First generate a bunch of random pure states, which generally will be entangled. In each state, consider each pair of particles and compute the REE in the following way: Assemble a bunch of random pure separable states. For a given pure (entangled) state, calculate the distance (i.e. the quantum relative entropy) between it and all the (pure) separable states. The smallest such distance will be an approximation of the REE (which, you will recall, is defined as the *actual* minimum distance). Of all the pairs in a given state, write down the minimum value of the pairwise REE. Now, amongst all the states, look at the largest of these values. This largest number is an approximate lower bound on the minimum pairwise relative entropy of entanglement.

This approach is obviously more naive than, say, a formal approach of analytic minimization. After all, we are only looking at a thousand or so random pure separable states, as opposed to the whole universe of states. Furthermore, the calculations are very time-consuming; after all, we need to compute multiple extrema over many states (i.e. density matrices).

Fortunately, we can use some Mathematica software that makes most of the

computation formally very easy. For example, after making slight modifications, we may be able to make use of a partial trace function in Mathematica, courtesy of Mark S. Tame (see appendix B). His program works for qubits, whereas I need to generalize it for qudits. Although my results so far for qubits ($d = 2$) have been consistent with those found in the literature, the more interesting case will be where the value of d varies. Because I have not perfected the code needed to compute the partial trace with respect to qudits ($d > 2$), I cannot declare any new results.

Our immediate goal – still not met yet – is to figure out whether, according to our naive approach, a similar entanglement-sharing-increases-with-dimension trend holds for the relative entropy of entanglement, as it seems to for the entanglement of formation.

Assumptions

This method yields some possibly insightful results; but, to be sure, it is not great because it relies on several assumptions or approximations. Let me list some of these crucial assumptions.

1. Pure separable states fill every nook and cranny of state space. In other words, we are supposing that no crucial region (one that is nearby the state in question) is left untouched by the sample of pure states.

2. Random sampling is efficient. That is, we get a pervasive view of state space by merely examining a few thousand states.

To the extent that these assumptions hold true, our results will be of some value. On the other hand, we are not likely to gain much insight if these assumptions don't hold up.

Improvements

On the one hand, supposing for the moment that this approach is not too naive, it may be fruitful to keep using it after some refinements. Here, then, are some ways to improve the lazy method:

1. Sample over mixed states as well as pure states. Obviously, this will offer us a "fairer" sample of states in the space of density matrices, since until now we have entirely neglected mixed states.

2. Sample over a greater number of random separable states. A brute force approach to improving this method will obviously be to consider a greater number of separable states (e.g. 10,000 as opposed to 1,000).

Cleverer strategies

It is not altogether clear that modifications to our simple approach described above is the best way to proceed. Rather, it may be wiser to abandon the naive approach and take a whole new line of attack. In other words, since the lazy method is essentially naive, it might be most productive to stop toying around and instead employ some heavier artillery. One has to assume that there are superior methods to compute distributed entanglement. Therefore, let me suggest some better approaches:

1. Instead of employing strictly numerical methods, find an insightful analytic approach, perhaps by invoking some kind of symmetry amongst pairs. This of course (one would think) would require more ingenuity than a numerical approach; but one hopes that the payout (viz. a possibly more elegant, insightful result) will be worth the effort!

2. If there is no luck with fancy analytics, then some more sophisticated numerical methods can be employed. Specifically, to improve upon mere

lazy random sampling, we can use the method of steepest descent to determine "closest states." According to this method, we would determine an optimal state by varying parameters one at a time until the distance measure in question was optimized.

Chapter 6

Future work

6.1 Unitary but non-local time-evolution

In the context of the axiom 2 above, we saw that there were examples of operators that were not local and unitary that altered the amount of entanglement in a system. In the corresponding Mathematica notebooks (see appendix B), we took as an example the Hamiltonian $H = \vec{S}_1 \cdot \vec{S}_2$.

It is natural to ask a similar question concerning the analogous operator on 3-part, 2-level system. If there are instead three particles, then we obtain a slightly more complicated version of the preceding counterexample, namely, the time-evolution of a three-electron system under the action of the Hamiltonian

$$H = a_{12}\left(\vec{S}_1 \cdot \vec{S}_2\right) + a_{23}\left(\vec{S}_2 \cdot \vec{S}_3\right) + a_{13}\left(\vec{S}_1 \cdot \vec{S}_3\right),$$

where the factor a_{ij} is just a coefficient that reflects the contribution of the interaction between particles i and j to the total energy.

I don't know if anyone has done this sort of calculation before explicitly. As the two-particle example above suggests, it may yield interesting results.

6.2 Distributed entanglement

As we indicated in the previous chapter, there is more work to be done on the calculation of $E_{max}(n, d)$ for the relative entropy of entanglement. Specifically, as we saw, it will be wise to pursue the problem by first refining the lazy approach (e.g. by more effective sampling), and then possibly implementing new, more sophisticated schemes (e.g. analytics, steepest descent).

Appendix A

Density matrices

In QM the state of a system is frequently represented by a state vector (or "ket") that lives in the associated Hilbert space. That is fine for many applications, but it is really only valid for certain types of states. For arbitrary states, we will need a more comprehensive formalism. The so-called **density matrix** calculus does the job.

A.1 Pure vs. mixed states

A state that is represented by a normalized linear combination of basis vectors is called a **pure state**. That is, for a pure state, the most complete description you could possibly write down is some expression of the form

$$|\psi\rangle = \sum_{i}^{d} c_i \, |e_i\rangle,$$

where the $|e_i\rangle$'s constitute an orthonormal basis spanning the Hilbert space; d is the dimension of the system; and $\sum_i^d |c_i|^2 = 1$ to ensure normalization.

For a pure state, the statistical behavior of any observable O is determined by the state vector $|\psi\rangle$: if the hermitian operator A represents the observable O, then, in Dirac's notation, the expectation value of O is given by the **inner**

product

$$\langle O \rangle = \langle \psi | \, A \, | \psi \rangle \, .$$

The family of pure states thus defined, however, is only a proper subset of all possible of states. There exist so-called **mixed states** for which there is no corresponding ket vector in Hilbert space that generates all possible expectation values in the above sense. Imagine, for example, that you have a bunch of particles in a probabilistic admixture of two pure states: $|\psi\rangle$, with probability p; and $|\phi\rangle$, with probability q. This is indeed a mixed state, so you cannot rely on a single ket to properly represent the state. Instead, consider the expression

$$p \, |\psi\rangle \, \langle \psi| + q \, |\phi\rangle \, \langle \phi| \, .$$

This $d \times d$ matrix (d is the dimension of the Hilbert space) is called the **density matrix** (or **density operator**) of the system. It suffices to represent the state of the whole admixture in the sense that it generates the expectation values via

$$\langle O \rangle = \mathrm{Tr}(A\rho).$$

In the above example, we assumed there were just two possible states. In general, though, we could have many. The extension is straight forward: a system with m "component" states $|\psi_i\rangle$, each with probability (or "relative weight") p_i, is represented by the density matrix

$$\rho = \sum_{i}^{m} p_i \, |\psi_i\rangle \, \langle \psi_i|. \tag{A.1}$$

Thus, we say a mixed state is the weighted sum of **outer products** of pure states.

Not only does this formalism work for mixed states, but also we can use it to recover the standard results for pure states, i.e. $\langle \psi | \, A \, | \psi \rangle = \mathrm{Tr}(A\rho_\psi)$.

Pure example

Consider the simple case in which all the particles are in the pure state $|\psi\rangle$. Then just ascribe the matrix $\rho_\psi \equiv |\psi\rangle \langle\psi|$ to the ensemble. To be a little more explicit, let's suppose the ket vector in question is the 4-level pure state

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Then it has the 4×4 density matrix

$$\rho_\psi = |\psi\rangle \langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* & \alpha\gamma^* & \alpha\delta^* \\ \alpha^*\beta & |\beta|^2 & \beta\gamma^* & \beta\delta^* \\ \alpha^*\gamma & \beta^*\gamma & |\gamma|^2 & \gamma\delta^* \\ \alpha^*\delta & \beta^*\delta & \gamma^*\delta & |\delta|^2 \end{pmatrix}.$$

Mixed example

Consider the following so-called **Bell states**:

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

These constitute an orthonormal basis on the composite Hilbert space of our two-particle system; the reader may wish to verify this as a simple exercise.

For two 2-level systems, consider the mixed state that is the equally weighted sum of the two pure states Φ^- and Φ^+:

$$\rho = \frac{1}{2}\left|\Phi^-\right\rangle\left\langle\Phi^-\right| + \frac{1}{2}\left|\Phi^+\right\rangle\left\langle\Phi^+\right| = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

Notice that we'd get the very same density matrix from 50/50 admixture of the two pure states $|00\rangle$ and $|11\rangle$. (The reader may wish to check this as an exercise.) Notice that $|\Phi^-\rangle$ and $|\Phi^+\rangle$ are entangled, whereas $|00\rangle$ and $|11\rangle$ are not; and yet once the admixtures above are formed, they generate the same measurement statistics. Thus this example shows that two ensembles prepared in different ways can share the same density operator. From an information theory point of view, this means that some of the information about an ensemble's history is erased when the mixture is created.

Comments

If ρ is a density matrix, then $\mathrm{Tr}(\rho) = 1$. Also, $\mathrm{Tr}(\rho^2) \leq 1$ with equality if ρ is a pure state (Dear reader, can you see why this is true from the definition (A.1)?). We have attempted to clearly discriminate between mixed states and pure states. From a mathematical point of view, the definitions stated above are all there is to it. What, though, is the distinction from a physical point of view? The answer is that in a pure state there is no statistical indeterminacy as there is for a mixed state.[1] In a pure state the system is completely specified, but in a mixed state we do not have maximal knowledge of the system. For example, as we just saw, two systems could have different histories, and yet still have the same density matrix.[2] What's more, it is impossible to discriminate between the two states experimentally; after all they have the same statistics.

In pure states, then, there is no statistical indeterminacy in the state of part of the system. The state as a whole is entirely specified. For a mixed state, on the other hand, this is not so. To build the intuition, consider a

[1] Of course, even for a pure state the various possible outcomes of a measurement are indeed *probabilistically* determined by the complex coefficients of the eigenstates. That is of course not what we are talking about.

[2] It is possible to characterize [34] precisely which statistical admixtures share the same density matrix. See Mermin's discussion [32] of this in the context of the EPR paradox.

simple example. Suppose you pass a bunch of silver atoms through a Stern-Gerlach magnet. Then, the spin state describing the beam of particles is in a pure state if, and only if, there is some axis along which you can orient the magnet such that all the electrons are deflected the same way.[3]

The distinctions between **pure vs. mixed** and **separable vs. entangled** is sometimes confusing. To help clarify this, let's consider an example of each possibility:

1. The state $\uparrow\uparrow + \uparrow\downarrow$ is **pure and separable**. It can be written as the product of two states: Alice's particle is up and Bob's particle is in a superposition of up and down.

2. The state $\uparrow\uparrow + \downarrow\downarrow$ is **pure and entangled**. It cannot be written as a separable product of singletons, since $1 \cdot 1 \neq 0 \cdot 0$ (cf. (3.1)). This is a bell state, and it is maximally entangled, with entropy of entanglement equal to $\text{Log}(2) \approx .693$.

3. The 50%/50% admixture of the two states

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   is represented by the density matrix

$$\rho = \frac{1}{2}\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + \frac{1}{2}|00\rangle\langle 00| = \frac{1}{4}\begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

[3]The Stern-Gerlach system is deceptively simple because it has dimension $d = 2$. We can thus identify a "change of basis" in Hilbert space with a mere "rotation of the magnets." But, bear in mind, that is only because the group $SU(2)$ is locally isomorphic to the group of rotations in 3-dimensional space. We are not so lucky in systems of higher dimension ($d \geq 3$), since then the relevant special unitary group, $SU(d)$, becomes quite large.

It is **mixed and entangled**. Indeed, its entropy of entanglement is $(3\text{Log}(4/3) + \text{Log}(4))/4 \approx .562$

4. The 50%/50% admixture consisting of the two states $\uparrow\uparrow$ and $\downarrow\downarrow$ is **mixed and separable**. The diagonal density matrix for this state has the entries 1/2 at the top-left and 1/2 at the bottom-right, zero otherwise.

   How do we know whether it's mixed? Well, if it were pure, then it could be written as the outer-product of a single ket (i.e., column state vector) with itself. But then since the top-left and bottom-right entries are nonzero, the extremes of the ket would both have to be nonzero. But in that case there would some nonzero off-diagonal elements (e.g. top-right and bottom-left); but there are not. Hence the state is not pure, so it must be mixed. Also, there is a quick mathematical check: ρ^2 is not equal to ρ. (Also $\text{Tr}(\rho^2) < 1$.)

   Now, is it entangled? Of course not. After all, it is prepared completely without recourse to entangled systems. To generate the admixture, all you need is a fair coin. But why you may ask, are there perfect correlations on every measurement? There are correlations only if you measure in this basis! If you measure in the y-basis, then you get random results!

A.2 The reduced density matrix

The so-called reduced density matrix (also called the reduced density operator) becomes especially useful in the context of subsystems. Suppose, for example, that we want to focus on Alice's part of the state ρ^{AB} she shares with Bob. Alice's **reduced density operator** is defined as

$$\rho^A \equiv \text{Tr}_B(\rho^{AB}).$$

where by tr_B we mean the **partial trace** with respect to Bob's part. The partial trace is the appropriate way of "ignoring" Bob's part, and is defined as follows. Let $|a\rangle$ and $|a'\rangle$ be any two states in the vector space H_A; likewise let $|b\rangle$ and $|b'\rangle$ be members of H_B. Then the partial trace with respect to B is

$$\mathrm{Tr}_B\left(|a\rangle\langle a'|\otimes|b\rangle\langle b'|\right)\equiv|a\rangle\langle a'|\,\mathrm{Tr}(|b\rangle\langle b'|)=|a\rangle\langle a'|\langle b'|b\rangle\,.$$

Of course this only defines the partial trace for a certain type of operator; however, the further requirement that the partial trace be linear ensures that it is well defined for arbitrary operators.

Though it is by no means obvious at first glance, it turns out that the reduced density operator ρ^A is the unique matrix that will generate all the appropriate statistics for measurements performed on Alice's part of the system.

Example

For the pure state $|\Psi^-\rangle=\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)$ the density matrix is

$$\rho^{AB}=\left|\Psi^-\right\rangle\left\langle\Psi^-\right|=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&+1&-1&0\\0&-1&+1&0\\0&0&0&0\end{pmatrix}.$$

The reduced density operators for this (and *any*) Bell state are

$$\rho^A=\rho^B=\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}.$$

Notice that these are mixed states. Although we began with a pure (Bell) state about which we had maximal information, when we consider the state of just one particle or the other, we see that it is a probabilistic admixture the "up" and "down" states; i.e., we have lost all information. You can see, therefore, why we say that an entangled state, as a whole, contains more information than either of its parts.

If you feel the material here is lacking, feel free to consult the references [9, 34] for more details.

A.3 The log of a matrix

How do you take the $\log_2$ of a positive semi-definite matrix M? Well, M is diagonalizable, so first find a suitable matrix P such that M may be rewritten in diagonal form D via the similarity transformation $D = PMP^{-1}$. Now, the elements of D are all zero except on the diagonal, where we find the eigenvalues of M. Now, write down $\log(D)$, which is the diagonal matrix whose jth element is the $\log_2$ of the jth element of D. Finally, reverse the diagonalization procedure via the inverse similarity transformation (using the same matrix P); that is, write $\log(M) = P^{-1}\log(D)P$, where $\log(D) = \log(PMP^{-1})$ is diagonal.

Appendix B

Mathematica calculations

For our computations, we used the following Mathematica notebooks, which contain calculations on:

1. Basic matrix computations
2. Entanglement measures
3. Distributed entanglement
4. Non-local but unitary time-evolution

Bibliography

[1] P. K. Aravind. Quantum mysteries revisited again. *American Journal of Physics*, 72(10):1303–1307, 2004.

[2] John S. Bell. On the Einstein-Podolsky-Rosen paradox. *Physics*, 1:195–200, 1964.

[3] John S. Bell. On the problem of hidden variables in quantum mechanics. *Rev. Mod. Phys.*, 38(3):447–452, Jul 1966.

[4] John S. Bell. Bertlmanns socks and the nature of reality. *Journal de Physique*, Colloque C2, suppl. au numero 3, Tome 42:pp C2 41–61, 1981.

[5] John S. Bell. *Speakable and Unspeakable in Quantum Mechanics*. Cambridge University Press, 1988.

[6] C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters. Teleporting an unknown quantum state via dual classical and EPR channels. *Phys Rev Lett*, pages 1895–1899, 1993.

[7] Charles H. Bennett, David P. DiVincenzo, Tal Mor, Peter W. Shor, John A. Smolin, and Barbara M. Terhal. Unextendible product bases and bound entanglement. *Phys. Rev. Lett.*, 82(26):5385–5388, Jun 1999.

[8] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. *Phys. Rev. A*, 54(5):3824–3851, Nov 1996.

[9] Karl Blum. *Density Matrix Theory and Applications*. Plenum Press, New York, NY, 1981.

[10] D. Bohm and Y. Aharonov. Discussion of experimental proof for the paradox of einstein, rosen, and podolsky. *Phys. Rev.*, 108(4):1070–1076, Nov 1957.

[11] David Bohm. *Quantum Theory*. Dover, New York, 1989.

[12] Dagmar Bruß. Entanglement splitting of pure bipartite quantum states. *Phys. Rev. A*, 60(6):4344–4348, Dec 1999.

[13] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.*, 23(15):880–884, Oct 1969.

[14] Valerie Coffman, Joydip Kundu, and William K. Wootters. Distributed entanglement. *Physical Review A*, 61:052306, 2000.

[15] N. Darnton. Question during my defense of this thesis, May 2008.

[16] Kenneth A. Dennison and William K. Wootters. Entanglement sharing among quantum particles with more than two orthogonal states. *Phys. Rev. A*, 65(1):010301, Dec 2001.

[17] David P. DiVincenzo. Quantum Computation. *Science*, 270(5234):255–261, 1995.

[18] W. Dür, G. Vidal, and J. I. Cirac. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A*, 62(6):062314, Nov 2000.

[19] Albert Einstein, Boris Podolsky, and Nathan Rosen. Can Quantum-Mechanical Description of Physical Reality be Considered Complete? *Physical Review*, 47:777, 1935.

[20] Jonathan R. Friedman. Private communication, Decmeber 2007.

[21] Christopher A. Fuchs, Nicolas Gisin, Robert B. Griffiths, Chi-Sheng Niu, and Asher Peres. Optimal eavesdropping in quantum cryptography. i. information bound and optimal strategy. *Phys. Rev. A*, 56(2):1163–1172, Aug 1997.

[22] Daniel M. Greenberger, Michael A. Horne, Abner Shimony, and Anton Zeilinger. Bell's theorem without inequalities. *American Journal of Physics*, 58(12):1131–1143, 1990.

[23] S. Haroche. Quantum information in cavity quantum electrodynamics: Logical gates, entanglement engineering and 'schrodinger-cat states'. *Philosophical Transactions: Mathematical, Physical and Engineering Sciences*, 361(1808):1339–1347, 2003.

[24] S. Haroche, M. Brune, and J. M. Raimond. Experiments with single atoms in a cavity: Entanglement, schrodinger's cats and decoherence. *Philosophical Transactions: Mathematical, Physical and Engineering Sciences*, 355(1733):2367–2380, 1997.

[25] Pawel Horodecki. Separability criterion and inseparable mixed states with positive partial transposition. *Physics Letters A*, 232:333, 1997.

[26] Ryszard Horodecki, Pawel Horodecki, Michal Horodecki, and Karol Horodecki. Quantum entanglement, 2007. eprint arXiv:quant-ph/0702225.

[27] Bradley Jost, Alexander Sergienko, Ayman Abouraddy, Bahaa Saleh, and Malvin Teich. Spatial correlations of spontaneously down-converted photon pairs detected with asingle-photon-sensitive ccd camera. *Opt. Express*, 3(2):81–88, 1998.

[28] Masato Koashi, Vladimír Bužek, and Nobuyuki Imoto. Entangled webs: Tight bound for symmetric sharing of entanglement. *Phys. Rev. A*, 62(5):050302, Oct 2000.

[29] N. D. Mermin. Quantum mysteries refined. *American Journal of Physics*, 62:880–887, October 1994.

[30] N. David Mermin. Is the moon there when nobody looks? reality and the quantum theory. *Physics Today*, 38(4):38–47, 1985.

[31] N. David Mermin. Quantum mysteries revisited. *American Journal of Physics*, 58(8):731–734, 1990.

[32] N. David Mermin. What do these correlations know about reality? nonlocality and the absurd. *Found. Phys.*, 29:571, 1999.

[33] Fumiaki Morikoshi, Marcelo França Santos, and Vlatko Vedral. Accessibility of physical states and non-uniqueness of entanglement measure. *Journal of Physics A: Mathematical and General*, 37(22):5887–5893, 2004.

[34] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Oxford University Press, New York, NY, 2005.

[35] Abraham Pais. *Subtle Is the Lord: The Science and the Life of Albert Einstein*. Plenum Press, New York, NY, 1981.

[36] A. K. Pati. Quantifying entanglement in "experimental entanglement of four particles". *Physics Letters A*, 322:301, 2004.

[37] T. B. Pittman, Y. H. Shih, D. V. Strekalov, and A. V. Sergienko. Optical imaging by means of two-photon quantum entanglement. *Phys. Rev. A*, 52(5):R3429–R3432, Nov 1995.

[38] Martin B. Plenio and S. Virmani. An introduction to entanglement measures. *Quant. Inf. Comp.*, 7:1, 2007.

[39] Sandu Popescu and Daniel Rohrlich. Thermodynamics and the measure of entanglement. *Phys. Rev. A*, 56(5):R3319–R3321, Nov 1997.

[40] J. M. Raimond, M. Brune, and S. Haroche. Manipulating quantum entanglement with atoms and photons in a cavity. *Rev. Mod. Phys.*, 73(3):565–582, Aug 2001.

[41] C. A. Sackett, D. Kielpinski, B. E. King, C. Langer, V. Meyer, C. J. Myatt, M. Rowe, Q. A. Turchette, W. M. Itano, D. J. Wineland, , and C. Monroe. Experimental entanglement of four particles. *Nature*, 404:256–259, 2000.

[42] P. A. Schilp, editor. *Albert Einstein, Philosopher Scientist*. Library of Living Philosophers, Evanston, Illinois, 1949.

[43] Erwin Schrodinger. Discussion of Probability Relations Between Separated Systems. *Proceedings of the Cambridge Philosophical Society*, 31:555–563 and 446–451, 1935 and 1936.

[44] V. Vedral. The role of relative entropy in quantum information theory. *Rev. Mod. Phys.*, 74(1):197–234, Mar 2002.

[45] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight. Quantifying Entanglement. *Physical Review Letters*, 78:2275–2279, 1997.

[46] Vlatko Vedral and Elham Kashefi. Uniqueness of the entanglement measure for bipartite pure states and thermodynamics. *Phys. Rev. Lett.*, 89(3):037903, Jun 2002.

[47] K. G. H. Vollbrecht and R. F. Werner. Entanglement measures under symmetry. *Phys. Rev. A*, 64(6):062307, Nov 2001.

[48] Mark Williamson. Private communication, September 2007.

[49] W. K. Wootters and W. H. Zurek. A single quantum cannot be cloned. *Nature*, 299:802–803, October 1982.

[50] William K. Wootters. Entanglement of formation of an arbitrary state of two qubits. *Phys. Rev. Lett.*, 80(10):2245–2248, Mar 1998.

[51] Anton Zeilinger. Experiment and the foundations of quantum physics. *Rev. Mod. Phys.*, 71(2):S288–S297, Mar 1999.