Mathematics

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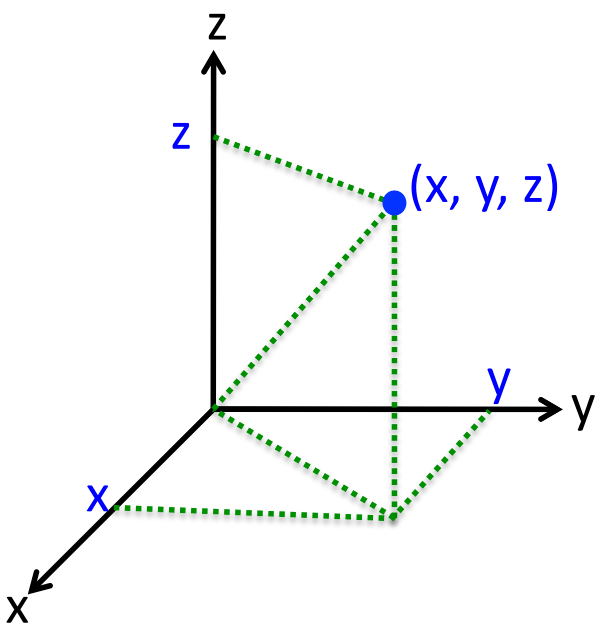
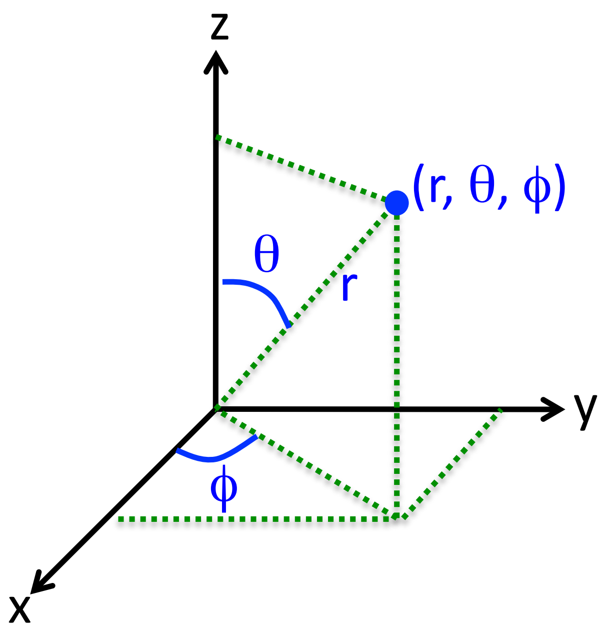
Outline

- Coordinate System
- Functions
- Differentiation
- Integration

Coordinate System



Cartesian
(直角坐標)

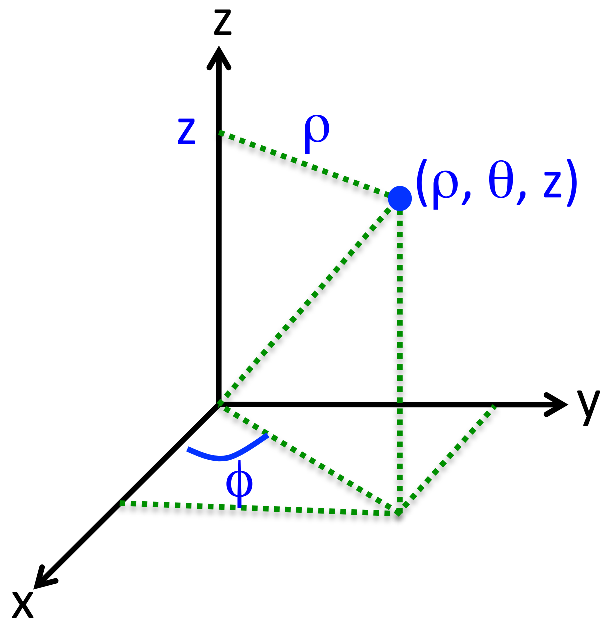


Polar
(球坐標)

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$



Cylindrical
(圓柱坐標)

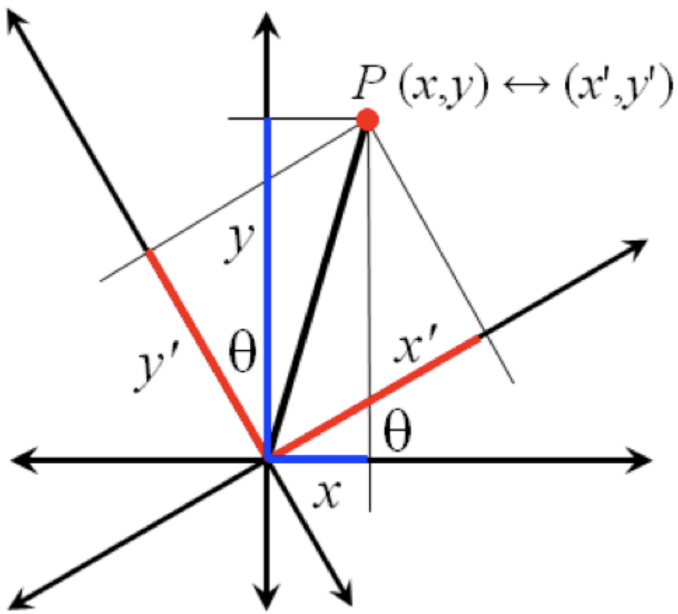
$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

Coordinate System

Coordinate transformation



$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Functions

- A function is a mechanism to transfer a number (or a set of numbers) to another number

  Ex: $f(x) = x + 2, \quad f(5) = 5 + 2 = 7$

  $f(x, y) = xy^2, \quad f(3, 5) = 3 \times 5^2 = 75$

- **Polynomial functions**

$$
\begin{aligned}
f(x) &= A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots \\
&= \sum_{i=0}^{n} A_i x^i
\end{aligned}
$$

Ex: $f(x) = 4 + 10x + 4x^4 + x^8$

Functions

Trigonometric functions

$$\sin\theta = \frac{c}{a}$$

$$\cos\theta = \frac{c}{b}$$

$$\tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta}$$

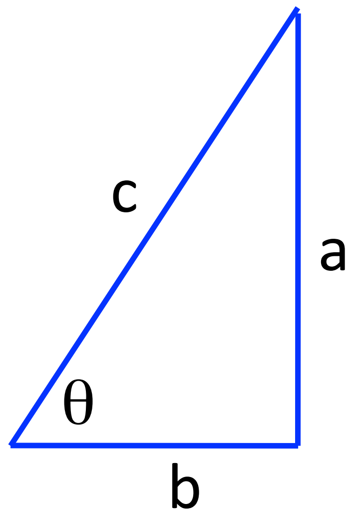
$$\cot\theta = \frac{b}{a} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{b}{c} = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{a}{c} = \frac{1}{\sin\theta}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$
$$= x - \frac{x^3}{3\times2\times1} + \frac{x^5}{5\times4\times3\times2\times1} + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$



Functions

❑ Trigonometric functions: useful relations

$$\sin(-\theta) = -\sin\theta$$

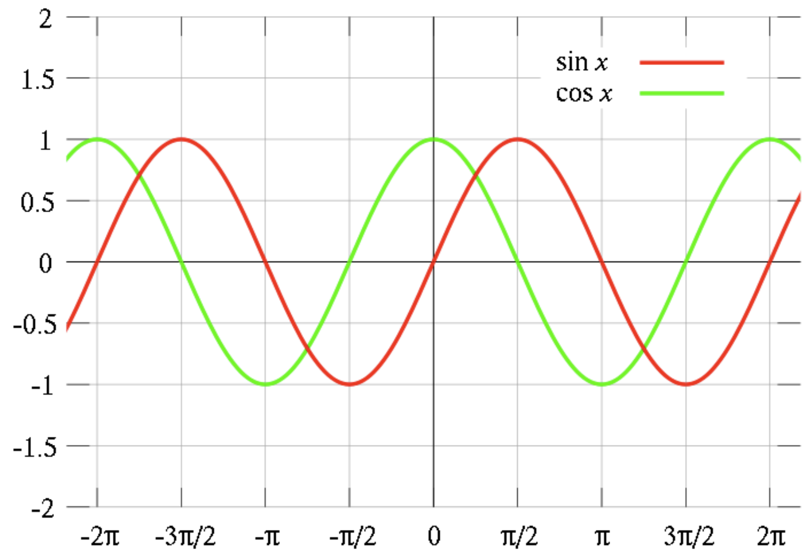
$$\cos(-\theta) = \cos\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$



Exercise: $\cos(\alpha + \beta) = ?$

$\sin(\alpha - \beta) = ?$

Exponential function

Definition: (a) $f(x) = exp(x) = e^x, where\ e = 2.718281...$

(b) $$e^x = \sum_{n=0}^{\infty} \frac{x^2}{n!}$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$
$$= 1 + x + \frac{x^2}{2 \times 1} + \frac{x^3}{3 \times 2 \times 1} + \frac{x^4}{4 \times 3 \times 2 \times 1} + ...$$

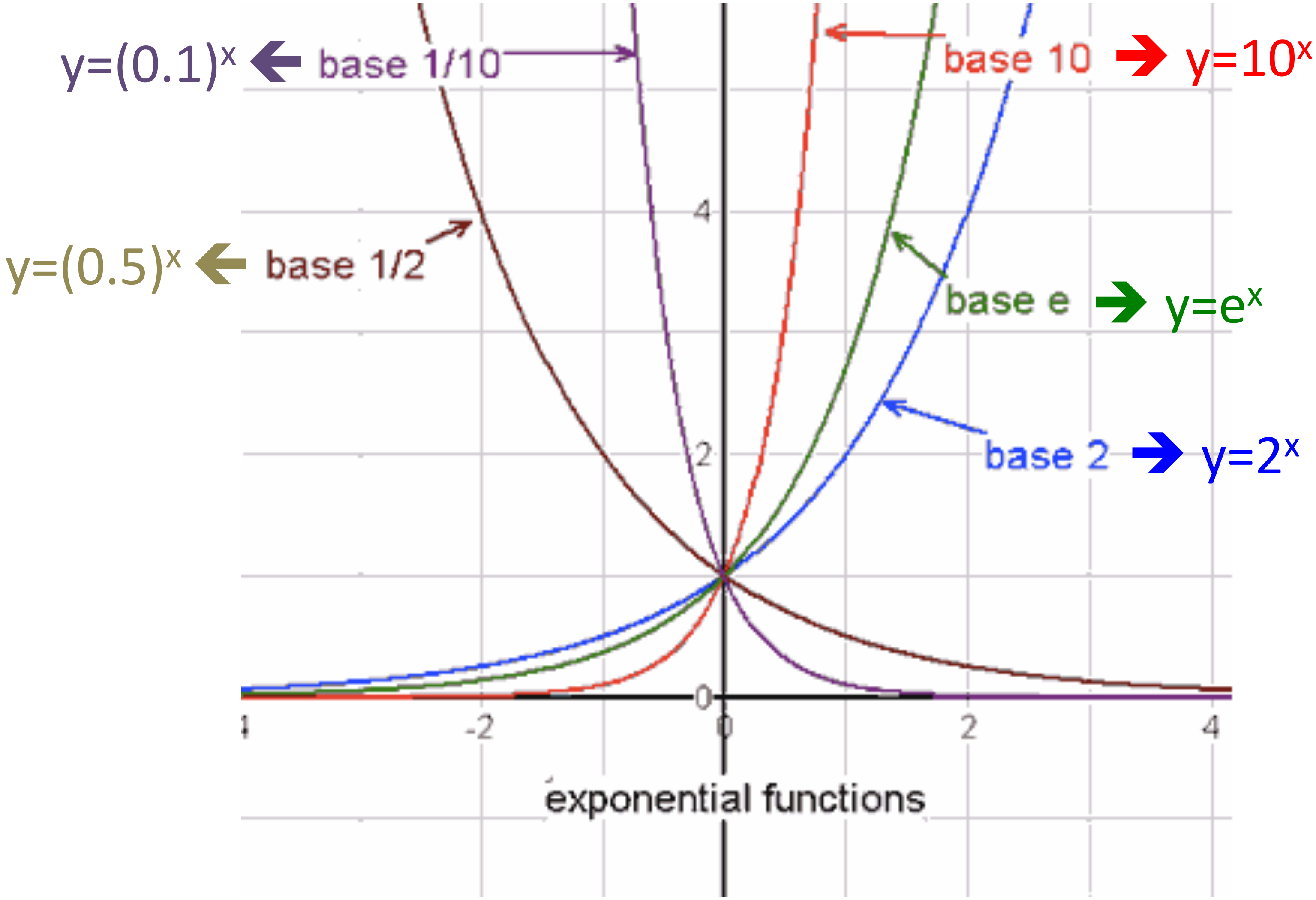
(c) $$e^x = \lim_{n \to \infty} (1 + \frac{x}{n})^2$$

As the unique solution of the equation

$$\frac{df(x)}{dx} = f(x)\ with\ f(0) = 1$$

Functions

❑ Exponential function



Functions

❑ Exponential function: useful relations

$$e^x e^y = e^{x+y}$$

Euler formula: $e^{i\theta} = \cos\theta + i\sin\theta, \text{ where } i = \sqrt{-1}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

In complex analysis: $\cos(ix) = \frac{e^x + e^{-x}}{2} \equiv \cosh(x)$

$$\sin(ix) = \frac{e^x - e^{-x}}{2i} \equiv \sinh(x)$$

Logarithm function

The logarithm of a number to a given *base* is the power or exponent to which the base must be raised in order to produce the number.

Example: $\log_{10} 100 = 2 \quad \because \log_{10} 10^2$

Some properties:

Natural logarithm: $\ln x = \log_e x$

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b(\frac{x}{y}) = \log_b x - \log_b y$$

$$\log_b x^y = y \log_b x$$

$$\log_b \sqrt[y]{x} = \frac{1}{y} \log_b x$$

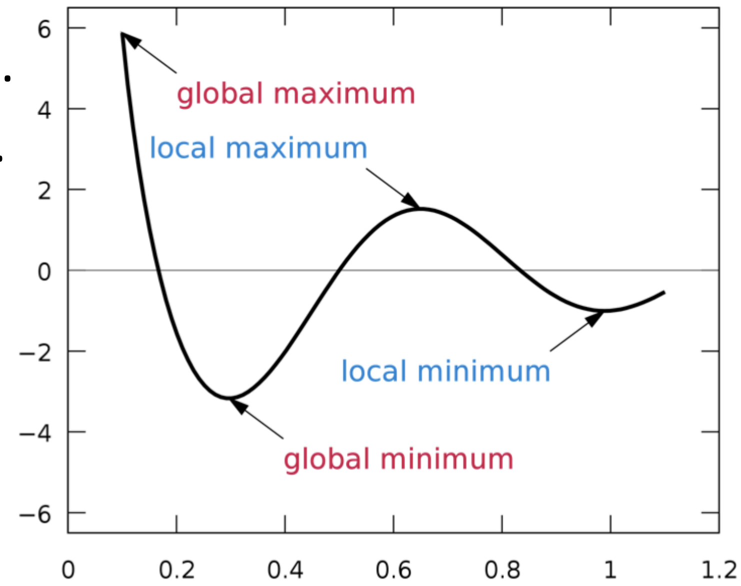
Differentiation

- Definition of the derivative of a function f(x) with respect x:

$$f'(x) \equiv \frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- Physics: $v = \dfrac{dx}{dt}$

- Extrema: maxima or minima
  - If f'(a) = 0, there is an extreme point
  - If f''(a) < 0, then f(x) has a local maximum at a.
  - If f''(a) > 0, then f(x) has a local minimum at a.
  - If f''(a) = 0, then the second derivative test says nothing about the point a, which can possibly be an inflection point.



Differentiation

- **Rules:**
  - **Constant rule**: if f(x) = a, then f'(x) = 0.
  - **Sum rule**: $$\frac{d(af(x)+bg(x))}{dx} = a\frac{df(x)}{dx} + b\frac{dg(x)}{dx}$$
    for all functions f(x) and g(x) and real numbers a and b
  - **Product rule**: $$\frac{d(f(x)g(x))}{dx} = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$
    for all functions f(x) and g(x)
  - **Quotient rule**: $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'g - g'f}{g^2}$$
    for all functions f(x) and g(x) where g(x) ≠ 0
  - **Chain rue**: $$\text{if } f(x) = h(g(x)),\ \frac{df(x)}{dx} = \frac{dg(x)}{dx}\left(\frac{h(g(x))}{dx}\right)$$

Differentiation

- Derivatives of elementary functions

$$f(x) = x^n, \;\; f'(x) = nx^{n-1}$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} a^x = ln(a) e^x$$

$$\frac{d}{dx} log_a(x) = \frac{1}{x ln(a)}$$

$$\frac{d}{dx} ln(x) = \frac{1}{x}, \;\; \text{for } x > 0$$

$$\frac{d}{dx} \sin(x) = \cos(x)$$

$$\frac{d}{dx} \cos(x) = -\sin(x)$$

$$\frac{d}{dx} \tan(x) = \sec^2(x)$$

$$\frac{d}{dx} \cot(x) = -\csc^2(x)$$

$$\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx} \csc(x) = \csc(x)\cot(x)$$

Differentiation

- Example:

Given $f(x) = 3x^5 - 4x^2 + 10x + 1$

Find $f'(x) = ?$

Solution:

$$
\begin{aligned}
f'(x) &= 3 \times 5x^{5-1} - 4 \times 2x^{2-1} + 10 \times 1x^{1-1} + 0 \\
&= 15x^4 - 8x + 10
\end{aligned}
$$

Differentiation

- Example:

Given $f(x) = 5\sin(x) - 4\cos(x)$

Find $f'(x) = \,?$

Solution:

Use $\dfrac{d(af(x)+bg(x))}{dx} = a\dfrac{df(x)}{dx} + b\dfrac{dg(x)}{dx}$

$$\begin{aligned} f'(x) &= 5\cos(x) - 4(-1)\sin(x) \\ &= 5\cos(x) + 4\sin(x) \end{aligned}$$

Differentiation

- Example:

  Given $f(x) = 3\sin^2(x) - 6\cos(2x^2)$

  Find $f'(x) = ?$

  Solution:

  Use if $f(x) = h(g(x))$, $\dfrac{df(x)}{dx} = \dfrac{dg(x)}{dx}(\dfrac{h(g(x))}{dx})$

$$f'(x) = 3 \times 2\cos(x) + 6 \times (2 \times 2x)\sin(2x^2)$$
$$= 6\cos(x) + 24x\sin(2x^2)$$

- Example:

Given $f(x) = 5x^2e^x$

Find $f'(x) = ?$

Solution:

Use $\frac{d(f(x)g(x))}{dx} = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$ and $\frac{d}{dx}e^x = e^x$

$$f'(x) = 5 \times 2x^{2-1}e^x + 5x^2e^x$$
$$= (10x + 5x^2)e^x$$

Differentiation

- Example:

  Given $f(x) = e^{x^3+x+1}$

  Find $f'(x) = ?$

  Solution:

  Use if $f(x) = h(g(x)),\ \dfrac{df(x)}{dx} = \dfrac{dg(x)}{dx}(\dfrac{h(g(x))}{dx})$

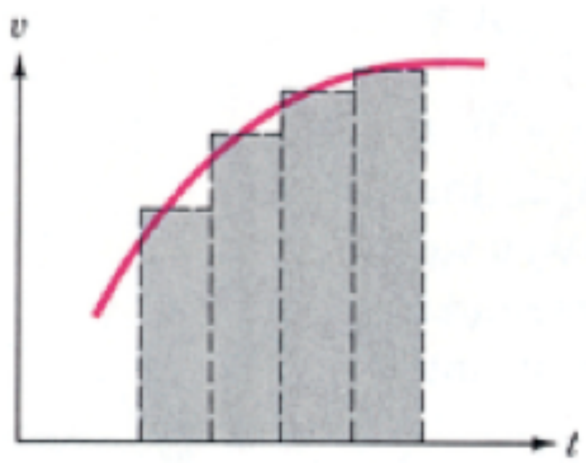
  $$f'(x) = (3x^2+1)e^{x^3+x+1}$$

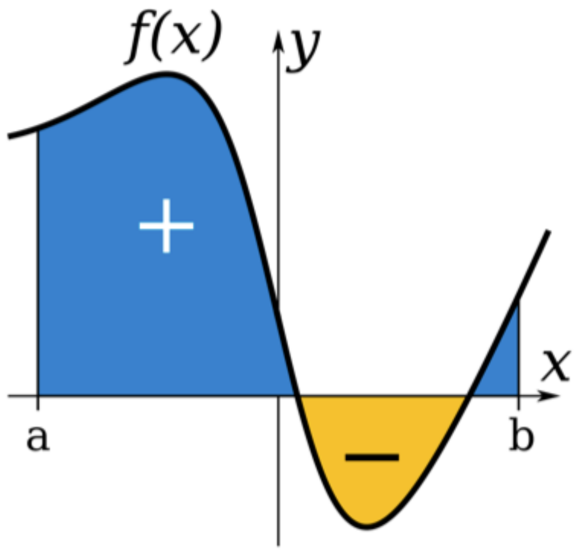
Integration

- Integration is a mathematic method to calculate "area" under a function (*not an approximation*)

$$A = \int_a^b f(x)dx$$

In the region where f(x) is positive (negative), the signed area defined as positive (negative).





Integration

- How to calculate an integration?

$$\text{if } f(x) = g'(x)$$

$$\text{then } \int f(x)dx = g(x) + C$$

$$\text{or } \int_a^b f(x)dx = g(b) - g(a)$$

- Some rules:

$$\int af(x)dx = a\int f(x)dx$$

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

(Note: please check the details from Calculus class / textbook)

Integration

- Example:

Given $f(x) = 3x^5 + 5x^2 + x + 5$

Find $\int f(x)dx = ?$

Solution:

Use $f(x) = x^n, \ f'(x) = nx^{n-1}$

$$\because \quad g(x) = \frac{3}{6}x^6 + \frac{5}{3}x^3 + \frac{1}{2}x^2 + 5x$$

$$g'(x) = f(x)$$

$$\therefore \quad \int f(x)dx = g(x) + C$$

$$= \frac{1}{2}x^6 + \frac{5}{3}x^3 + \frac{1}{2}x^2 + 5x + C$$

- Example:

Given $f(x) = \sin(5x) + \cos(2x)$

Find $\int f(x)dx = ?$

Solution:

Use $\frac{d}{dx}\cos(x) = -\sin(x)$ and $\frac{d}{dx}\sin(x) = \cos(x)$

$$\because \quad g(x) = -\frac{1}{5}\cos(5x) + \frac{1}{2}\sin(2x)$$

$$g'(x) = f(x)$$

$$\therefore \quad \int f(x)dx = g(x) + C$$

$$= -\frac{1}{5}\cos(5x) + \frac{1}{2}\sin(2x) + C$$

Integration

- Example:

  Given $f(x) = 6x^2 e^{5x}$, find $\int f(x)dx = ?$

  Solution:

  Use $\int g(x)h'(x)dx = g(x)h(x) - \int g'(x)h(x)dx$

  $g(x) = 6x^2 \rightarrow g'(x) = 12x$

  $h(x) = \frac{1}{5}e^{5x} \rightarrow h'(x) = e^{5x}$

$$\int f(x)dx = \int g(x)h'(x)dx$$

$$= 6x^2(\frac{1}{5}e^{5x}) - \int 12x(\frac{1}{5}e^{5x})dx + C$$

$$= \frac{6}{5}x^2 e^{5x} - \frac{12}{5}\int xe^{5x}dx + C$$

$$\because \ k(x) = x \rightarrow k'(x) = 1$$

$$= \frac{6}{5}x^2 e^{5x} - \frac{12}{5}(x\frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x}dx) + C'$$

$$= \frac{6}{5}x^2 e^{5x} - \frac{12}{25}xe^{5x} + \frac{12}{125}e^{5x} + C''$$

Integration

- Example:

Given $f(x) = \cos(2x)e^{3x}$, find $\int f(x)dx = ?$

Solution:

Use

$$\int g(x)h'(x)dx = g(x)h(x) - \int g'(x)h(x)dx$$

$$g(x) = \cos(2x) \rightarrow g'(x) = -2\sin(2x)$$

$$h(x) = \frac{1}{3}e^{3x} \rightarrow h'(x) = e^{3x}$$

$$\int f(x)dx = \int g(x)h'(x)dx$$

$$= \cos(2x)(\frac{1}{3}e^{3x}) - \int(-2\sin(2x)(\frac{1}{3}e^{3x})dx + C$$

$$= \frac{1}{3}\cos(2x)e^{3x} + \frac{2}{3}\int \sin(2x)e^{3x}dx + C$$

$$\because \; k(x) = \sin(2x) \rightarrow k'(x) = 2\cos(2x)$$

$$= \frac{1}{3}\cos(2x)e^{3x} + \frac{2}{3}(\sin(2x)\frac{1}{3}e^{3x} - \int 2\cos(2x)\frac{1}{3}e^{3x}dx) + C'$$

$$= \frac{1}{3}\cos(2x)e^{3x} + \frac{2}{9}\sin(2x)e^{3x} - \frac{4}{9}\underbrace{\int \cos(2x)e^{3x}dx}_{f(x)} + C'$$

$$= \frac{9}{13}(\frac{1}{3}\cos(2x)e^{3x} + \frac{2}{9}\sin(2x)e^{3x}) + C''$$

$$= (\frac{3}{13}\cos(2x) + \frac{2}{13}\sin(2x))e^{3x} + C''$$