Chapter 2. Vectors

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Outline

- 2.1 Scalars and Vectors
- 2.2 Vector Addition
- 2.3 Components and Unit Vectors
- 2.4 The Scalar (Dot) Product
- 2.5 The Vector (Cross) Product

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2.1 Scalars and Vectors

- **Scalars (純量)**: a scalar is a quantity that is completely specified by a number and its unit. It has a magnitude but no direction. Scalars obey the rules of ordinary algebra.

- **Vector (向量):** a vector is a quantity that is specified by both a magnitude and a direction in space. Vectors obey the laws of vector algebra.

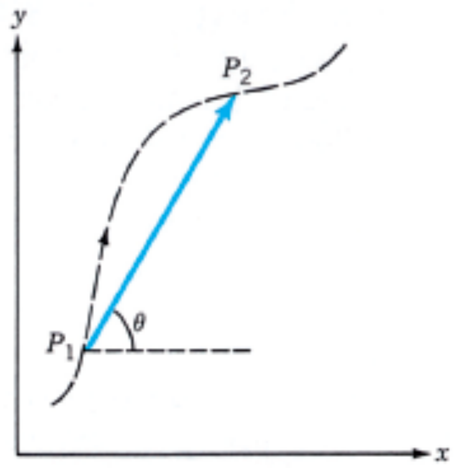


FIGURE 2.1 When a particle moves along the dashed path, the displacement from P_1 to P_2 is represented by the arrow.

- Notations:
  - **Vector**: $\mathbf{A}$ or $\vec{A}$
  - **Magnitude**: $|\mathbf{A}|$ or A

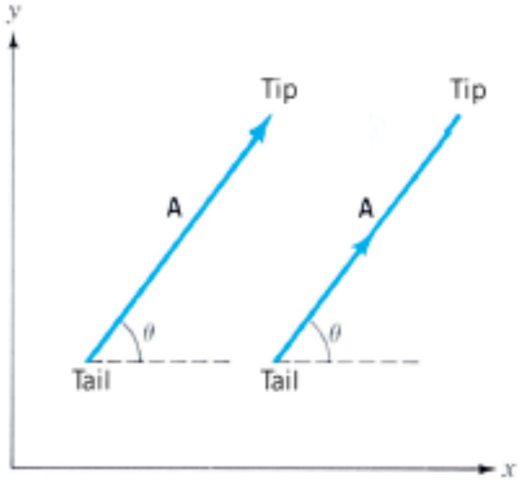


FIGURE 2.2 The geometrical representation of a vector. The tail of a vector may be placed at any point relative to the coordinate system.

- When we say $\mathbf{A} = \mathbf{B}$, it means
  1. $A = B$
  2. $\theta_A = \theta_B$

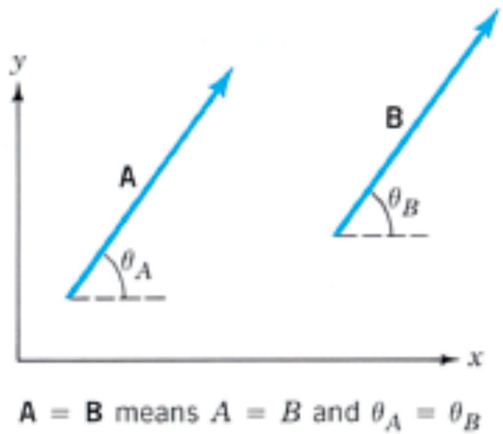


FIGURE 2.3 The vector equality $\mathbf{A} = \mathbf{B}$ means that $A = B$ and $\theta_A = \theta_B$.

- Important concepts:
  - When adding, subtracting or equating physical quantities, they must have the same unit.

    Ex: 1 m + 3 m = 4 m ✔

    1 m + 3 kg = ? ✖
  - Similarly, all the terms on both sides of equation must be either scalar or vector.

    Ex: $\mathbf{A} + \mathbf{B} = \mathbf{C}$ ✔

    $\mathbf{A} + B = ?$ ✖
  - A vector can be multiplied by a pure number or a scalar:
    - By a pure number: only change the magnitude
    - By a scalar: the new vector will have a different meaning (physical quantity)

    Ex: linear momentum $m\mathbf{v} = \mathbf{p}$

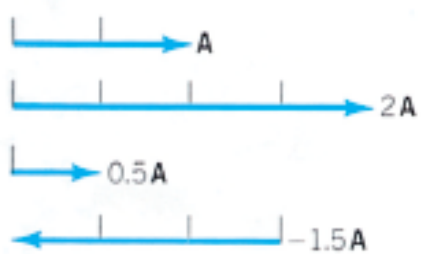


FIGURE 2.4 When a vector is multiplied by a scalar its magnitude and/or its direction changes.

2.2 Vector Addition

- Addition rules:
  - Vector + vector = vector: $\mathbf{A}+\mathbf{B}=\mathbf{C}$
  - $|\mathbf{A}+\mathbf{B}| \neq A+B$
  - Commutative: $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
  - Associative: $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$
  - Subtraction: $\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B})$

    $\mathbf{A}=\mathbf{B}+(\mathbf{A}-\mathbf{B})$

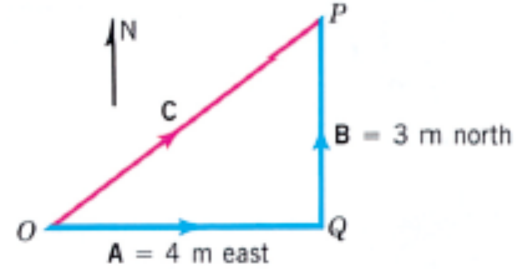


FIGURE 2.5 The net effect of two displacements **A** and **B** is equivalent to a single displacement **C** called the *sum* or *resultant* of **A** and **B**.

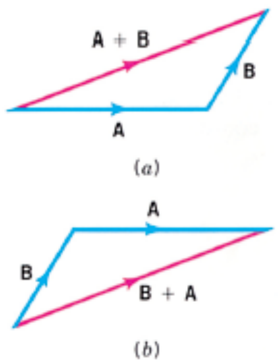


FIGURE 2.6 To add the vectors **A** and **B** by the tail-to-tip method, place the tail of the second vector at the tip of the first. The sum is drawn from the tail of the first to the tip of the second. Comparison of (*a*) and (*b*) shows that the vectors may be added in any order.

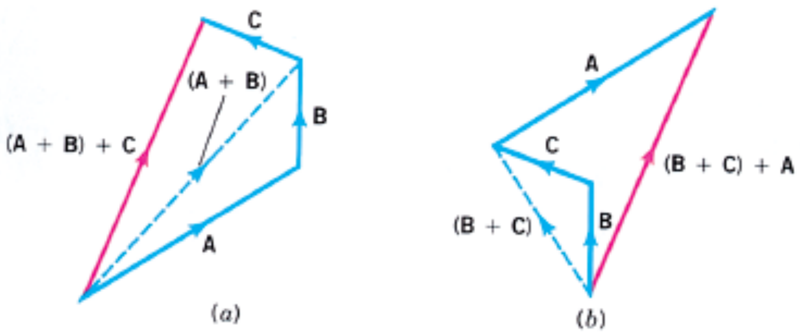


FIGURE 2.7 To add several vectors, the tail of each vector is placed at the tip of the preceding one. The resultant is drawn from the tail of the first vector to the tip of the last.

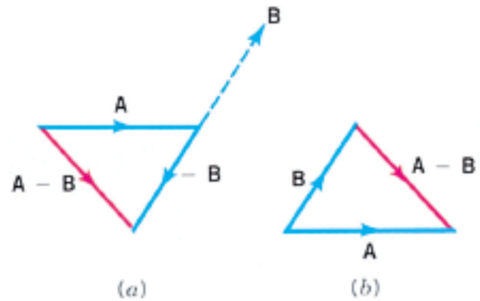


FIGURE 2.8 Two methods of finding the difference of two vectors. In (*a*), the subtraction is treated as a special case of addition. In (*b*), **A** is the resultant.

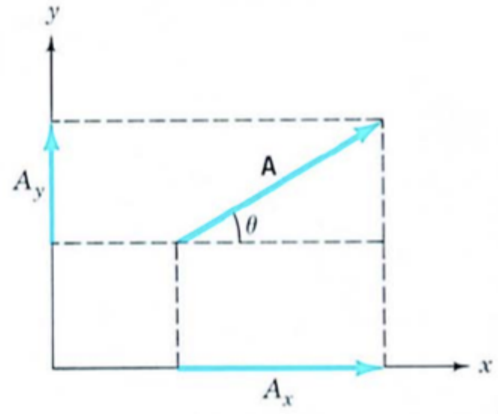
2.3 Components and Unit Vectors

- A vector can be represented by the rectangular components
  - To define a vector, you need magnitude and direction

$$\cos\theta = A_x/A \qquad \sin\theta = A_y/A \quad \Rightarrow \quad A_x = A\cos\theta \qquad A_y = A\sin\theta$$

magnitude $A = \sqrt{A_x^2 + A_y^2}$

direction $\tan\theta = \dfrac{A_y}{A_x}$



  - How to write down a vector: **component** + **unit vector**

$$\begin{aligned}\mathbf{A} &= A_x\hat{x} + A_y\hat{y} \\ &= A_x\mathbf{x} + A_y\mathbf{y} \\ &= A_x\hat{i} + A_y\hat{j} \\ &= A_x\mathbf{i} + A_y\mathbf{j}\end{aligned}$$

2.3 Components and Unit Vectors

- Unit vectors: a unit vector is a dimensionless quantity that serves only to specify a direction in space.

  It has a unit length: $|\hat{x}| = |\hat{y}| = |\hat{z}| = 1$

$$|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = 1$$

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$

$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$$

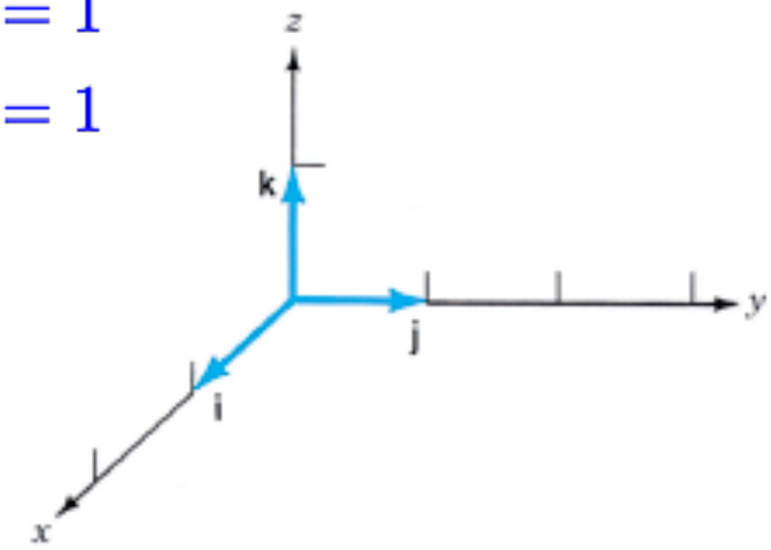


FIGURE 2.15 The unit vectors **i**, **j**, and **k** are directed along the x, y, and z axes, respectively.

2.3 Components and Unit Vectors

- Example 2.3:

Given the vectors **A** = 2**i** – 3**j** + 6**k** m and **B** = **i** + 2**j** – 3**k** m, find:

(a) A + B; (b) |**A**+**B**|; (c) 2**A** – 3**B**

Solution:

(a)

$$A = \sqrt{2^2 + 3^2 + 6^2} = 7$$

$$B = \sqrt{1^2 + 2^2 + 3^2} = 3.74$$

$$\begin{aligned} A + B &= 7 + 3.74 \\ &= 10.74m \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) + (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \\ &= 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}m \end{aligned}$$

$$|\mathbf{A} + \mathbf{B}| = \sqrt{3^2 + 1^2 + 3^2} = 4.36m$$

(c)

$$\begin{aligned} 2\mathbf{A} - 3\mathbf{B} &= 2 \times (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) - 3 \times (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \\ &= (4 - 3)\mathbf{i} + (-6 - 6)\mathbf{j} + (12 + 9)\mathbf{k}m \\ &= \mathbf{i} - 12\mathbf{j} + 21\mathbf{k}m \end{aligned}$$

2.4 The Scalar (Dot) Product

- The scalar product is defined as: $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$
  - **Commutative:** $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$
  - **Distributive:** $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$

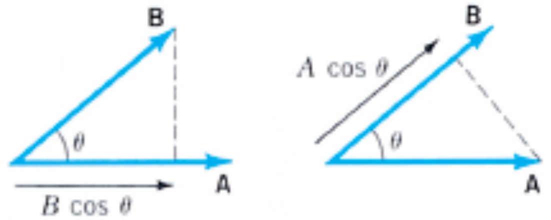


FIGURE 2.18 The scalar product $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$ of two vectors is the product of one vector and the component of the second vector along the direction of the first.

- Because of the properties of the unit vectors:

➔ $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$ (Hints: $\cos 0 = 1$

$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0$ $\cos 90 = 0$)

➔
$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

- Example 2.4:

  Find the scalar product of **A** = 8**i** +2**j** – 3**k** and **B** = 3**i** – 6**j** + 4**k**

  Solution:

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= A_xB_x + A_yB_y + A_zB_z \\
&= (8\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})\cdot(3\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}) \\
&= 8\times 3 + 2\times(-6) + (-3)\times 4 \\
&= 24 - 12 - 12 \\
&= 0
\end{aligned}$$

2.4 The Scalar (Dot) Product

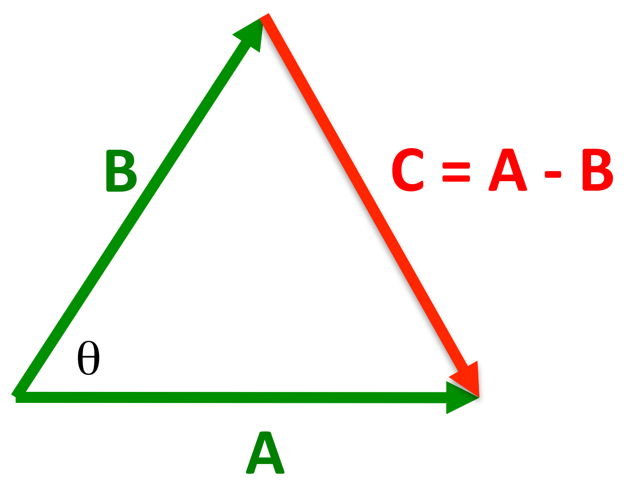
- Example 2.6:

  Drive the law of cosines using the scalar product.

  Solution:

  Consider **C** = **A** - **B**

$$\mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})$$
$$= A^2 + B^2 - 2\mathbf{A}\cdot\mathbf{B}$$
$$C^2 = A^2 + B^2 - 2AB\cos\theta$$

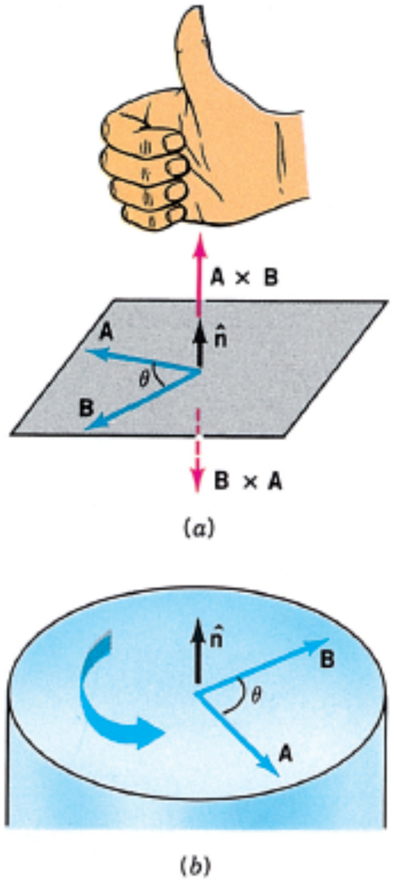


2.5 The Vector (Cross) Product

- The vector (cross) product of two vectors A and B is defined as

  $$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{n}$$

  - A vector (unlike the scalar product) with magnitude *ABsinθ* and direction **n** (follow the right-hand rule)
  - Non-commutative: $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$
  - Distributive: $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$



- The vector product of unit vectors:

$$\mathbf{i}\times\mathbf{i} = 0 \qquad \mathbf{j}\times\mathbf{j} = 0 \qquad \mathbf{k}\times\mathbf{k} = 0$$

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\times\mathbf{i} = -\mathbf{k} \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j}$$

$$\mathbf{i}\times\mathbf{k} = -\mathbf{j} \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\times\mathbf{j} = \mathbf{i}$$

2.5 The Vector (Cross) Product

- The vector product in terms of the components (complicated!)

$$
\begin{aligned}
\mathbf{C} &= \mathbf{A} \times \mathbf{B} \\
&= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\
&= (A_xB_y\mathbf{k} - A_xB_z\mathbf{j}) + (-A_yB_x\mathbf{k} + A_yB_z\mathbf{i}) + (A_zB_x\mathbf{j} - A_zB_y\mathbf{i}) \\
&= \underbrace{(A_yB_z - A_zB_y)}_{C_x}\mathbf{i} + \underbrace{(A_zB_x - A_xB_z)}_{C_y}\mathbf{i} + \underbrace{(A_xB_y - A_yB_x)}_{C_z}\mathbf{k}
\end{aligned}
$$

- The vector product can be expressed as (determinant)

$$
\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}
$$

2.5 The Vector (Cross) Product

- Example 2.7:

  Find the vector product of **A** = 3**i** – 2**j** + **k** and **B** = **i** + 4**j** – 2**k**.

  Solution:

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{i} + (A_xB_y - A_yB_x)\mathbf{k}\\ &= ((-2)\times(-2) - 1\times 4)\mathbf{i} + (1\times 1 - 3\times(-2))\mathbf{j} + (3\times 4 - (-2)\times 1)\mathbf{k}\\ &= (4-4)\mathbf{i} + (1+6)\mathbf{j} + (12+2)\mathbf{k}\\ &= 0\mathbf{i} + 7\mathbf{j} + 14\mathbf{k}\\ &= 7\mathbf{j} + 14\mathbf{k}\end{aligned}$$

- Example 2.8:

Derive the law of sines using the cross product.

Solution:

Consider **C** = **A** - **B**

$$\mathbf{C} \times \mathbf{C} = (\mathbf{A} - \mathbf{B}) \times C$$
$$= \mathbf{A} \times \mathbf{C} - \mathbf{B} \times \mathbf{C}$$
$$0 = AC \sin\beta \mathbf{n} - BC \sin\alpha \mathbf{n}$$

$$\Rightarrow \frac{\sin\alpha}{A} = \frac{\sin\beta}{B}$$