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by A. N. Guz', G. N. Savin, and I. A. Tsurpal

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Translation of "Kontsentratsiya napryazheniy okolo krivolineynykh otverstiy v fizicheski nelineynoy uprugoy plastinke."
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STRESS CONCENTRATION ABOUT CURVILINEAR HOLES IN PHYSICALLY NONLINEAR ELASTIC PLATES

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A.N.Guz', G.N.Savin, I.A.Tsurpal (Kiev)

Start

An approximate solution method of plane physical nonlinear problems of stress concentration about curvilinear holes in thin plates made of a material subject to a nonlinear law of elasticity is given. The solutions are represented in the form of expansions in the small parameter μ and ϵ. The determination of the stress function F for a physically nonlinear elastic plate with a hole reduces for each approximation to the integration of nonlinear differential equations. Stress concentration about an elliptic hole is considered in zero, first, and second approximation. The coefficient of stress concentration k is found on the contour of the hole, depending nonlinearly on the tensile forces P, the ellipticity of the hole, and a parameter λ characterizing the mechanical properties of the material. Tables represent the values of the coefficient of stress concentration for various values of the parameters P, λ, and ϵ.

end

1. The problem considered is that of stress concentrations in the neighborhood of curvilinear orifices without sharp corners in a thin plate consisting of a material for which the stress - strain ratio is nonlinear even in the presence of comparatively small strains. Given the deformation magnitude considered, all geometric relations of elasticity remain linear, i.e., we are dealing with a variant of the physically nonlinear theory of elasticity, with a specified nonlinear law of elasticity.

A previous study by the author (Bibl.1) examined this problem for the nonlinear law of elasticity* (Bibl.2), using conformal mapping of the region in question, outside the curvilinear orifice, onto the exterior of a unit circle and introducing the Kolosov-Muskhelishvili complex potentials. For the sought stress function, represented in the form of expansions in a small parameter, differential equations and boundary conditions for successive approximations in curvilinear coordinates, given by a mapping function, have been derived. However, in view of the cumbersomeness of the right-hand sides of the equations, this method has led to extremely complex calculations with respect to orifices of noncircular shape.

This paper, utilizing the same nonlinear law of elasticity (Bibl.2), pro-

* The stress concentration in the neighborhood of a circular orifice for this law of elasticity has been investigated elsewhere (Bibl.2, 7, 8, and 9).

** Numbers in the margin indicate pagination in the original foreign text.

poses another approximate method for the solution of the above problem, which makes it possible to complete this solution with respect to certain noncircular orifices. The new approach is based on the approximate method of "perturbation of the boundary form" (Bibl.3), as successfully used by the authors (Bibl.4, 5, 6) in investigating stress concentrations in the neighborhood of analogous curvilinear orifices in shells.

2. The approximate method described here requires the representation of all basic equations in a polar coordinate system; hence we will use, in the form given elsewhere (Bibl.2), the nonlinear law of elasticity for a generalized plane stress state:

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$$\begin{aligned}\epsilon_r &= \frac{1}{3K}k(s_0)\sigma_0 + \frac{1}{2G}g(t_0^2)(\sigma_r - \sigma_0),\\ \epsilon_\varphi &= \frac{1}{3K}k(s_0)\sigma_0 + \frac{1}{2G}g(t_0^2)(\sigma_\varphi - \sigma_0),\\ \epsilon_{r\varphi} &= \frac{1}{G}g(t_0^2)\tau_{r\varphi},\end{aligned} \tag{2.1}$$

where ϵ_r, ϵ_φ, and $\epsilon_{r\varphi}$, as well as σ_r, σ_φ, and $\tau_{r\varphi}$, correspondingly, are the mean stress and strain components over the plate thickness in the polar coordinate system (r, φ); K and G are constant moduli of volume deformation and shear, respectively, for the physically nonlinear material of the plate in the presence of vanishingly small deformations; $k(s_0)$ and $g(t_0^2)$ are the pressure and shear-stress functions which characterize, respectively, the change in volume and shape at any point of the body during its deformation. The dimensionless quantities s_0 and t_0 are expressed in the form of invariants:

$$\begin{aligned}s_0 &= \frac{\sigma_0}{3K} = \frac{1}{9K}(\sigma_r + \sigma_\varphi),\\ t_0^2 &= \frac{\tau_0^2}{G^2} = \frac{2}{9G^2}(\sigma_r^2 + \sigma_\varphi^2 - \sigma_r\sigma_\varphi + 3\tau_{r\varphi}^2).\end{aligned} \tag{2.2}$$

For many materials, volume deformation over a wide range obeys Hooke's law so that, in eq.(2.1), it may be assumed with a high degree of accuracy that $k(s_0) \equiv 1$. The slight deviations from linear dependence between stress and strain in the elasticity relations (2.1) can be, with sufficient accuracy, mapped by the function $g(t_0^2) = 1 + g_2 t_0^2$.

Hereafter, we will assume that, in the elasticity relations (2.1),

$$k(s_0) \equiv 1, \quad g(t_0^2) = 1 + g_2 t_0^2, \tag{2.3}$$

where g_2 is a dimensionless constant.

At such a choice of the nonlinear law of elasticity (2.1) under the conditions (2.3), the problem of the stressed state of a thin plate reduces to finding the stress function $F(r, \varphi)$ from the fourth-order nonlinear differential equation (Bibl.2, 8):

$$\Delta\Delta F+\frac{2\lambda}{R^4}\left[\frac{1}{r^4}F_\varphi T_\varphi-\frac{1}{r^3}(F_{r\varphi}T_\varphi+F_\varphi T_{r\varphi})\right.$$
$$\left.-\frac{1}{r^2}\left(\frac{1}{2}F_{rr}T_{\varphi\varphi}-F_{r\varphi}T_{r\varphi}+\frac{1}{2}F_{\varphi\varphi}T_{rr}\right)-\frac{1}{2r}(F_rT_{rr}+F_{rr}T_r)-\frac{1}{3}\Delta(T\Delta F)\right]=0 \tag{2.4}$$

in the presence of corresponding boundary conditions over the orifice contour and at "infinity". If the function $F(r, \varphi)$ is known, the stress components σ_r, σ_φ, and $\tau_{r\varphi}$ may be directly found with the aid of $F(r, \varphi)$ from the formulas

$$\sigma_r=\frac{1}{R^2}\left[\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\varphi^2}\right]\equiv\frac{1}{R^2}\left[\frac{1}{r}F_r+\frac{1}{r^2}F_{\varphi\varphi}\right];$$
$$\sigma_\varphi=\frac{1}{R^2}F_{rr},\qquad \tau_{r\varphi}=\frac{1}{R^2}\left[\frac{1}{r^2}F_\varphi-\frac{1}{r}F_{r\varphi}\right]. \tag{2.5}$$

In eqs.(2.4) and (2.5), r is a dimensionless coordinate referring to the /1011
quantity R which characterizes the absolute dimensions of the orifice; Δ is the Laplace operator which, in dimensionless coordinates, has the form

$$\Delta\equiv\frac{1}{r}F_r+\frac{1}{r^2}F_{\varphi\varphi}+F_{rr}. \tag{2.6}$$

The material constant λ and the function $T(r, \varphi)$ have the form

$$\lambda=\frac{Kg_2}{(3K+G)G^2}=\mu\beta^2; \tag{2.7}$$

where $\mu = \dfrac{1}{g_2}$ is a small dimensionless quantity

$$\beta^2=\frac{Kg_2^2}{(3K+G)G^2},\qquad T(r,\varphi)=\frac{9}{2}G^2t_0^2,$$

where t_0^2 is given by the expression (2.2) and the components σ_r, σ_φ, and $\tau_{r\varphi}$ are associated with the stress function $F(r, \varphi)$ by the relations (2.5). The small parameter λ, which characterizes the deviation of the nonlinear law of elasticity from Hooke's law, has the dimension 1/bar^2 and a magnitude of the order of 10^{-5} to 10^{-6}, while the dimensionless constant g_2, for certain nonferrous metals and their alloys, is of the order of 10^5 to 10^6 (Bibl.2).

The components of the displacements $u(r, \varphi)$ and $v(r, \varphi)$ for the nonlinear law of elasticity (2.1) under the conditions (2.3) are determined from the system of equations

$$
\begin{aligned}
\frac{\partial u}{\partial r} = {} & \frac{1}{9KR}\left(F_{rr}+\frac{1}{r}F_r+\frac{1}{r^2}F_{\varphi\varphi}\right)+\frac{1}{6GR}\left(-F_{rr}+2\frac{1}{r}F_r+2\frac{1}{r^2}F_{\varphi\varphi}\right) \\
& +\frac{\lambda}{R^5}\left(\frac{1}{27K}+\frac{1}{9G}\right)\Big[F_{rr}^2-\frac{1}{r}F_rF_{rr}+\frac{1}{r^2}(F_r^2-F_{\varphi\varphi}F_{rr}+3F_{r\varphi}^2) \\
& +\frac{2}{r^3}(F_rF_{\varphi\varphi}-3F_\varphi F_{r\varphi})+\frac{1}{r^4}(F_{\varphi\varphi}^2+3F_\varphi^2)\Big]\left(-F_{rr}+2\frac{1}{r}F_r+2\frac{1}{r^2}F_{\varphi\varphi}\right), \\
\frac{1}{r}\frac{\partial v}{\partial \varphi}+\frac{u}{r} = {} & \frac{1}{9KR}\left(F_{rr}+\frac{1}{r}F_r+\frac{1}{r^2}F_{\varphi\varphi}\right)+\frac{1}{6GR}\left(2F_{rr}-\frac{1}{r}F_r-\frac{1}{r^2}F_{\varphi\varphi}\right) \\
& +\frac{\lambda}{R^5}\left(\frac{1}{27K}+\frac{1}{9G}\right)\Big[F_{rr}^2-\frac{1}{r}F_rF_{rr}+\frac{1}{r^2}(F_r^2-F_{\varphi\varphi}F_{rr}+3F_{r\varphi}^2) \\
& +2\frac{1}{r^3}(F_rF_{\varphi\varphi}-3F_\varphi F_{r\varphi})+\frac{1}{r^4}(F_{\varphi\varphi}^2+3F_\varphi^2)\Big]\left(2F_{rr}-\frac{1}{r}F_r-\frac{1}{r^2}F_{\varphi\varphi}\right).
\end{aligned}
\tag{2.8}
$$

Thus, the solution of the problem reduces to an integration of the complex fourth-order nonlinear equation (2.4) or of the system of nonlinear equations (2.8) under definite boundary conditions over the orifice contour as well as under conditions of the behavior of these functions "at infinity" (Bibl.1).

3. Consider orifices of a shape such that the function /1012

$$
Z^* = R[\zeta+\varepsilon f(\zeta)], \quad (z^* = r^*e^{i\varphi};\ r^* = Rr;\ z = re^{i\varphi};\quad \zeta = \varrho e^{i\theta}) \tag{3.1}
$$

realizes the conformal mapping of an infinite plane with a circular orifice of unit radius onto an infinite plane with an orifice having the shape considered. In the function (3.1), R is the true constant characterizing the dimensions of the orifice; the function $f(\zeta)$ depends on the shape of the orifice; ϵ is a small parameter; the true quantity satisfying the condition $\epsilon \ll 1$ and the roots of the equation $1 + \epsilon f'(\zeta) = 0$ should lie within the unit circle in the plane ζ. We will present the solution of eq.(2.4) and of the system of equations (2.8) in the form of expansions in the small parameters μ and ϵ (2.7):

$$
F(r,\varphi;\mu;\varepsilon) = H_0\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j F^{(k,j)}(r,\varphi); \tag{3.2}
$$

$$
\begin{aligned}
u(r,\varphi;\mu;\varepsilon) &= H_0\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j u^{(k,j)}(r,\varphi), \\
v(r,\varphi;\mu;\varepsilon) &= H_0\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j v^{(k,j)}(r,\varphi),
\end{aligned}
\tag{3.3}
$$

where H_0 is selected from the condition $H_0^2\beta^2/R^4 = 1$. Hence,

$$H_0 = \frac{R^2}{\beta} = \frac{GR^2}{g_2}\sqrt{3 + \frac{G}{K}}. \tag{3.4}$$

The stress and strain components in the coordinate system (ρ, θ) also will be presented in the form of series in μ and ϵ:

$$\sigma_\varrho = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j\sigma_\varrho^{(k,j)}, \quad \sigma_\theta = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j\sigma_\theta^{(k,j)}, \tag{3.5}$$

$$\tau_{\varrho\theta} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j\tau_{\varrho\theta}^{(k,j)};$$

$$u_\varrho = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j u_\varrho^{(k,j)}, \quad u_\theta = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j u_\theta^{(k,j)}, \tag{3.6}$$

Substituting the function $F(r, \varphi; \mu; \epsilon)$ of eq.(3.2) into the fundamental equation (2.4) and equating to zero the coefficients in the presence of identical exponents of μ^k, ϵ^j, will yield the equation for determining the function $F^{(k,j)}$ in the form of

$$\Delta\Delta F^{(k,j)}(r, \varphi) = L_{k,j}(F^{(0,0)}, \ldots, F^{(k-1,j-1)}). \tag{3.7}$$

We will present explicit expressions for the right-hand sides of eq.(3.7), for certain values of k and j.

For $k = 0$, $j = 0, 1, \ldots$

$$L_{0,j}(F^{(0,0)}, \ldots, F^{(k-1,j-1)}) \equiv 0. \tag{3.8}$$

For $k = 1$, $j = 0$ /1013

$$L_{1,0}(F^{(0,0)}) \equiv L_0(F^{(0,0)}). \tag{3.9}$$

The developed form of the operator $L_0(\Gamma^{(0,0)})$ was given by Tsurpal (Bibl.8).

For $k = 2$, $j = 0$

$$L_{2,0}(F^{(0,0)}, F^{(1,0)}) \equiv L_1(F^{(0,0)}, F^{(1,0)}). \tag{3.10}$$

The developed form of the operator $L_1(F^{(0,0)}, F^{(1,0)})$ was given in another report (Bibl.9).

For $k = 1$, $j = 1$, the developed form of the operator $L_{1,1}$ will be

$$
\begin{aligned}
L_{1,1}(F^{(0,0)}, F^{(0,1)}) = {} & T_{rr}^{(0,0)}\left(\frac{1}{r^2} F_{\varphi\varphi}^{(0,1)} + \frac{1}{r} F_r^{(0,1)}\right) + T_{rr}^{(0,1)}\left(\frac{1}{r^2} F_{\varphi\varphi}^{(0,0)} + \frac{1}{r} F_r^{(0,0)}\right) \\
& + F_{rr}^{(0,1)}\left(\frac{1}{r^2} T_{\varphi\varphi}^{(0,0)} + \frac{1}{r} T_r^{(0,0)}\right) + F_{rr}^{(0,0)}\left(\frac{1}{r^2} T_{\varphi\varphi}^{(0,1)} + \frac{1}{r} T_r^{(0,1)}\right) - 2\Big[\Big(\frac{1}{r} T_{r\varphi}^{(0,0)} \\
& - \frac{1}{r^2} T_\varphi^{(0,0)}\Big)\left(\frac{1}{r} F_{r\varphi}^{(0,1)} - \frac{1}{r^2} F_\varphi^{(0,1)}\right) + \left(\frac{1}{r} F_{r\varphi}^{(0,0)} - \frac{1}{r^2} F_\varphi^{(0,0)}\right)\left(\frac{1}{r} T_{r\varphi}^{(0,1)} - \frac{1}{r^2} T_\varphi^{(0,1)}\right)\Big] \\
& - \frac{2}{3} \Delta (T^{(0,1)} \Delta F^{(0,0)} + T^{(0,0)} \Delta F^{(0,1)}),
\end{aligned}
\tag{3.11}
$$

where

$$
\begin{aligned}
T_{(r,\varphi)}^{(0,0)} = {} & (F_{rr}^{(0,0)})^2 - \frac{1}{r} F_{rr}^{(0,0)} F_r^{(0,0)} + \frac{1}{r^2}[(F_r^{(0,0)})^2 - F_{rr}^{(0,0)} F_{\varphi\varphi}^{(0,0)} + 3(F_{r\varphi}^{(0,0)})^2] \\
& + \frac{2}{r^3}(F_r^{(0,0)} F_{\varphi\varphi}^{(0,0)} - 3F_{r\varphi}^{(0,0)} F_\varphi^{(0,0)}) + \frac{1}{r^4}[(F_{\varphi\varphi}^{(0,0)})^2 + 3(F_\varphi^{(0,0)})^2];
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
T_{(r,\varphi)}^{(0,1)} = {} & 2F_{rr}^{(0,0)} F_{rr}^{(0,1)} - \frac{1}{r}(F_{rr}^{(0,0)} F_r^{(0,1)} + F_{rr}^{(0,1)} F_r^{(0,0)}) + \frac{1}{r^2}(2F_r^{(0,0)} F_r^{(0,1)} \\
& - F_{rr}^{(0,0)} F_{\varphi\varphi}^{(0,1)} - F_{rr}^{(0,1)} F_{\varphi\varphi}^{(0,0)} + 6F_{r\varphi}^{(0,0)} F_{r\varphi}^{(0,1)}) + \frac{2}{r^3}(F_r^{(0,0)} F_{\varphi\varphi}^{(0,1)} + F_r^{(0,1)} F_{\varphi\varphi}^{(0,0)} \\
& - 3F_{r\varphi}^{(0,0)} P_\varphi^{(0,1)} - 3F_{r\varphi}^{(0,1)} F_\varphi^{(0,0)}) + \frac{1}{r^4}(2F_{\varphi\varphi}^{(0,0)} F_{\varphi\varphi}^{(0,1)} + 6F_\varphi^{(0,0)} F_\varphi^{(0,1)}).
\end{aligned}
\tag{3.13}
$$

Proceeding analogously, we can write explicit expressions for the operators $L_{k,j}$ at any ($k > 1$, $j > 1$) values of k and j. The solution of eq.(3.7) is sought in the form of a Fourier series

$$
F_{(r,\varphi)}^{(k,j)}(r, \varphi) = \sum_{m=0}^{\infty} \{f_{k,j}^{(m)}(r) \cos m\varphi + g_{k,j}^{(m)}(r) \sin m\varphi\}.
\tag{3.14}
$$

To find the stress components σ_ρ, σ_θ, and $\tau_{\rho\theta}$ and the displacement components u_ρ and u_θ in the curvilinear orthogonal coordinate system* (ρ, θ) given by the function (3.1), we utilize, by analogy with another paper (Bibl.6), the corresponding formulas for conversion from the polar coordinates (r, φ) to the curvilinear orthogonal coordinate system (ρ, θ).

Expanding the obtained expressions for the stress components σ_ρ, σ_θ, and $\tau_{\rho\theta}$ and for the displacement components u_ρ and u_θ in series in μ and ϵ and taking into account the form of the function (3.1), we have /1014

$$
\sigma_\varrho^{(k,j)} = H_0\left(\frac{1}{\varrho}\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}\right)F^{(k,j)}(\varrho, \theta) + H_0 \sum_{m=0}^{j-1}\left[L_1^{(j-m)}\left(\frac{1}{\varrho}\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}\right)\right.
$$

* The linear coordinate $\rho = 1$ coincides with the contour of the orifice under study.

$$+L_2^{(j-m)}\left(\frac{\partial^2}{\partial\varrho^2}-\frac{1}{\varrho}\frac{\partial}{\partial\varrho}-\frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}\right)-L_3^{(j-m)}\frac{\partial^2}{\partial\varrho\partial\theta}\frac{1}{\varrho}\Bigg]F^{(k,m)}(\varrho,\theta);$$

$$\sigma_\theta^{(k,j)}=H_0\frac{\partial^2}{\partial\varrho^2}F^{(k,j)}(\theta,\varrho)+H_0\sum_{m=0}^{j-1}\Bigg[L_1^{(j-m)}\frac{\partial^2}{\partial\varrho^2}+L_2^{(j-m)}\Bigg(\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+$$

$$+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}-\frac{\partial^2}{\partial\varrho^2}\Bigg)+L_3^{(j-m)}\frac{\partial^2}{\partial\varrho\partial\theta}\frac{1}{\varrho}\Bigg]F^{(k,m)}(\varrho,\theta); \qquad (3.15)$$

$$\tau_{\varrho\theta}^{(k,j)}=-H_0\frac{\partial^2}{\partial\varrho\partial\theta}\frac{1}{\varrho}F^{(k,j)}(\varrho,\theta)-H_0\sum_{m=0}^{j-1}\Bigg[(L_1^{(j-m)}-2L_2^{(j-m)})\frac{\partial^2}{\partial\varrho\partial\theta}\frac{1}{\varrho}$$

$$+\frac{1}{2}L_3^{(j-m)}\left(\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}-\frac{\partial^2}{\partial\varrho^2}\right)\Bigg]F^{(k,m)}(\varrho,\theta;$$

$$u_\varrho^{(k,j)}=u^{(k,j)}+\sum_{m=0}^{j-1}[L_5^{(j-m)}u^{(k,m)}+L_6^{(j-m)}v^{(k,m)}],$$

$$u_\theta^{(k,j)}=v^{(k,j)}+\sum_{m=0}^{j-1}[L_5^{(j-m)}v^{(k,m)}-L_6^{(j-m)}u^{(k,m)}]. \qquad (3.16)$$

Substituting the functions $F(r, \varphi; \mu; \epsilon)$ [eq.(3.2)] and $u(r, \varphi)$, $v(r, \varphi)$ [eq.(3.3)] into eq.(2.8) and equating to zero the coefficients in the presence of identical exponents of μ^k, ϵ^j, we will find the system of equations for determining $u^{(k,j)}(r, \varphi)$, $v^{(k,j)}(r, \varphi)$ which enter into eq.(3.16).

The functions $F^{(k,j)}(\rho, \theta)$ entering into eq.(3.15) are solutions of eqs.(3.7) $F^{(k,j)}(r, \varphi)$ in the form of eq.(3.14), in which the variables r and φ are replaced by ρ and θ, respectively. Then, the arbitrary constants entering into $f_{k,j}^{(n)}(r)$ and $g_{k,j}^{(n)}(r)$ [eq.(3.14)] are determined from the corresponding boundary conditions for $F^{(k,j)}(\rho, \theta)$; these conditions are derived from the expansions analogous to eqs.(3.2) and (3.3), for the orifice contour under study, on the basis of an expansion in a double series in μ and ϵ.

The stress and strain coefficients over the contour of the orifice in question are found from eqs.(3.4), (3.5), and (3.15) at $\rho = 1$.

Consider the problem in which the stresses

$$\sigma_\varrho|_\Gamma=\psi_1(r,\varphi;\mu),\qquad \tau_{\varrho\theta}|_\Gamma=\psi_2(r,\varphi;\mu) \qquad (3.17)$$

are specified over the orifice contour. For the sake of universality, let us assume that, in eq.(3.17), the quantities ψ_1 and ψ_2 depend on μ. The equation of the orifice contour in parametric form is determined by the function (3.1) /1015 and may be written as

$$r=r(\varrho,\theta),\qquad \varphi=\varphi(\varrho,\theta) \qquad (3.18)$$

for $\rho = 1$.

Having utilized eqs.(3.1) and (3.18), we will present the right-hand sides of eq.(3.17) as double series in μ and ϵ

$$\sigma_\varrho|_\Gamma = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j\psi_1^{(k,j)}(\theta), \qquad \tau_{\varrho\theta}|_\Gamma = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mu^k\varepsilon^j\psi_2^{(k,j)}(\theta). \tag{3.19}$$

Substituting expressions (3.4) into eq.(3.19), assuming that $\rho = 1$, and comparing the coefficients with identical exponents, we obtain the relations

$$\sigma_\varrho^{(k,j)}|_\Gamma = \psi_1^{(k,j)}(\theta), \qquad \tau_{\varrho\theta}^{(k,j)}|_\Gamma = \psi_2^{(k,j)}(\theta). \tag{3.20}$$

From eq.(3.15), taking eq.(3.20) into account, we obtain the boundary conditions for determining the $(k, j)^{th}$ function $F^{(k,j)}(r, \varphi)$ in the form of

$$\begin{aligned}
&\left(\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}\right)F^{(k,j)}(\varrho,\theta)\bigg|_{\varrho=1} = \frac{1}{H_0}\psi_1^{(k,j)}(\theta)\bigg|_{\varrho=1}\\
&\quad -\left\{\sum_{m=0}^{j-1}\left[L_1^{(j-m)}\left(\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}\right)+L_2^{(j-m)}\left(\frac{\partial^2}{\partial\varrho^2}-\frac{1}{\varrho}\frac{\partial}{\partial\varrho}-\frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}\right)\right.\right.\\
&\qquad\qquad \left.\left. -L_3^{(j-m)}\frac{\partial^2}{\partial\varrho\partial\theta}\frac{1}{\varrho}\right]F^{(k,m)}(\varrho,\theta)\right\}\bigg|_{\varrho=1},\\
&\frac{\partial^2}{\partial\varrho\partial\theta}\frac{1}{\varrho}F^{(k,j)}(\varrho,\theta)\bigg|_{\varrho=1} = -\frac{1}{H_0}\psi_2^{(k,j)}(\theta)\bigg|_{\varrho=1} - \left\{\sum_{m=0}^{j-1}\left[\left(L_1^{(j-m)}-2L_2^{(j-m)}\right)\frac{\partial^2}{\partial\varrho\partial\theta}\frac{1}{\varrho}\right.\right.\\
&\qquad\qquad \left.\left. +\frac{1}{2}L_3^{(j-m)}\left(\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}-\frac{\partial^2}{\partial\varrho^2}\right)\right]F^{(k,m)}(\varrho,\theta)\right\}\bigg|_{\varrho=1}.
\end{aligned} \tag{3.21}$$

In accordance with eq.(3.2), the solution obtained to the n^{th} approximation will be construed as the function

$$F_n(r,\varphi;\mu;\varepsilon = H_0\sum_{k,j=0}^{k+j=n-1}\mu^k\varepsilon^j F^{(k,j)}(r,\varphi). \tag{3.22}$$

From eqs.(3.15) and (3.21) we can see that, for each function $F^{(k,j)}$, we obtain the boundary problem for a circular orifice. This also explains why all the basic equations and the relations (2.1), (2.4), (2.6), and (2.8) were written in polar coordinates.

In eqs.(3.15) and (3.16), $L_1^{(j-m)}, \ldots, L_6^{(j-m)}$ are differential operators whose form depends on the function $f(\zeta)$ [eq.(3.1)]. The series expressions for these operators for the zeroth, first, and second approximations are given by Savin and Guz' (Bibl.6).

4. Consider by way of an example the most elementary case, namely, /1016
the case of omnilateral uniform tension produced by stresses P applied to an infinite physically nonlinear isotropic plate with an elliptical orifice (Fig.1) obeying the elasticity law (2.1) under the conditions (2.3).

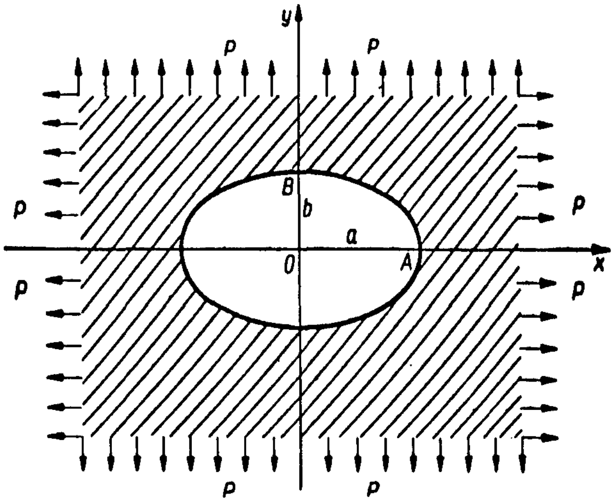

Fig.1

The function (3.1) which maps the exterior of the elliptical orifice onto the exterior of a circle of unit radius for this case, as is known (Bibl.10), has the form

$$Z^* = R\left[\zeta + \frac{\varepsilon}{\zeta}\right], \tag{4.1}$$

where

$$R = \frac{a+b}{2}, \quad \varepsilon = \frac{a-b}{a+b}, \quad \zeta = \varrho e^{i\theta}, \quad z^* = Rre^{i\varphi} \tag{4.2}$$

and where a and b are the semiaxes of the ellipse (Fig.1). Obviously, the function* $f(\zeta)$ in eq.(3.1) will, in this case, be $f(\zeta) = 1/\zeta$.

The approximate solution of this problem, taking into account the three approximations, reduces to the successive integration of eq.(3.7) allowing for the form of the operators (3.8) - (3.13).

The stress functions with the zeroth $F^{(0,0)}$, first $F^{(1,0)}$, and second $F^{(2,0)}$ approximations for the mentioned omnilaterally stressed physically nonlinear elastic plate, with a circular orifice, are known (Bibl.9) and have the form

$$F^{(0,0)} = \frac{P}{2H_0}(r^2 - 2\ln r), \tag{4.3}$$

* For an orifice with rounded angles, the function $f(\zeta)$ will be (Bibl.10), for a square orifice, $f(\zeta) \equiv 1/\zeta^3$; for a triangular orifice, $f(\zeta) = 1/\zeta^2$, etc.

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$$F^{(1,0)} = -\frac{P^3}{H_0^3}\left[\frac{1}{4}\left(\frac{1}{r^2}+\frac{1}{2}\frac{1}{r^4}\right)+\ln r\right], \tag{4.4}$$

$$F^{(2,0)} = \frac{P^5}{H_0^5}\left[\frac{13}{5}\ln r+\frac{1}{4}\left(-\frac{1}{r^2}-\frac{1}{3}\frac{1}{r^4}+\frac{47}{36}\frac{1}{r^6}+\frac{51}{80}\frac{1}{r^8}\right)\right]. \tag{4.5}$$

The stress function for a linearly elastic plate with an elliptic orifice has the form (Bibl.5):

$$\begin{aligned} F^{(0,1)}(r,\varphi) &= \frac{P}{H_0}\left(\frac{1}{r^2}-1\right)\cos 2\varphi, \\ F^{(0,2)}(r,\varphi) &= \frac{P}{H_0}\left[\left(\frac{3}{2}\frac{1}{r^4}-\frac{1}{r^2}\right)\cos 4\varphi-\ln r\right]. \end{aligned} \tag{4.6}$$

Substituting the function (4.3) into eq.(3.12), we find the functions $T^{(0,0)}(r, \varphi)$ in the form of

$$T^{(0,0)} = \frac{P^2}{H_0^2}\left(1+3\frac{1}{r^4}\right). \tag{4.7}$$

Knowing the functions (4.3) and (4.6) we find from eq.(3.13) the function $T^{(0,1)}(r, \varphi)$:

$$T^{(0,1)}(r,\varphi) = 4\frac{P^2}{H_0^2}\frac{1}{r^2}\left(1-3\frac{1}{r^2}+9\frac{1}{r^4}\right)\cos 2\varphi. \tag{4.8}$$

Substituting the functions (4.3), (4.6), (4.7), (4.8) and their derivatives into eq.(3.11), we find the series expression for the operator $L_{1,1}$.

The differential equation (3.7) for the function $F^{(1,1)}(r, \varphi)$ will be

$$\Delta\Delta F^{(1,1)}+64\frac{P^3}{H_0^3}\left(\frac{1}{r^8}+36\frac{1}{r^{10}}\right)\cos 2\varphi = 0. \tag{4.9}$$

The specific integral of eq.(4.9) becomes

$$F_{rac}^{(1,1)}(r,\varphi) = -\frac{P^3}{H_0^3}\left(\frac{1}{6}\frac{1}{r^4}+\frac{6}{5}\frac{1}{r^6}\right)\cos 2\varphi. \tag{4.10}$$

The general integral of the homogeneous equation (4.9) will be taken as

$$F_{homog}^{(1,1)}(r,\varphi) = \sum_{m=2}^{\infty}(C_{m3}r^{-m+2}+C_{m4}r^{-m})\cos m\varphi. \tag{4.11}$$

The integration constants C_{m3} and C_{m4} in eq.(4.11) are determined from the boundary conditions (3.21):

$$\left(\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}\right)F^{(1,1)}(\varrho,\theta)\Big|_{\varrho=1}+R\left[\cos 2\theta\frac{\partial}{\partial\varrho}\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\right]F^{(1,0)}(\varrho,\theta)\Big|_{\varrho=1}=0,$$
$$\frac{\partial^2}{\partial\varrho\partial\theta}\frac{1}{\varrho}F^{(1,1)}(\varrho,\theta)\Big|_{\varrho=1}-R\left[2\sin 2\theta\left(\frac{\partial^2}{\partial\varrho^2}-\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\right)\right]F^{(1,0)}(\varrho,\theta)\Big|_{\varrho=1}=0. \qquad (4.12)$$

Omitting the intermediary calculations, we will present the final expressions for the function $F^{(1,1)}(r, \varphi)$: /1018

$$F^{(1,1)}(r,\varphi)=-\frac{1}{30}\frac{P^3}{H_0^3}\left(32-73\frac{1}{r^2}+5\frac{1}{r^4}+36\frac{1}{r^6}\right)\cos 2\varphi. \qquad (4.13)$$

Let us consider the second approximation in greater detail. The stress function $F_2(r, \varphi; \mu; \epsilon)$ [eq.(3.22)] will be

$$F_2(r,\varphi;\mu;\varepsilon)=H_0(F^{(0,0)}+\mu F^{(1,0)}+\mu^2F^{(2,0)}+\varepsilon F^{(0,1)}+\varepsilon^2F^{(0,2)}+\mu\varepsilon F^{(1,1)}). \qquad (4.14)$$

From eqs.(3.5), taking into account the values of the components $\sigma_\rho^{(k,j)}$, $\sigma_\theta^{(k,j)}$, and $\tau_{\rho\theta}^{(k,j)}$ of eq.(3.15) as well as the values of the functions $F^{(k,j)}$ ($k = 0, 1, 2$, $j = 0, 1, 2$) in eqs.(4.3) - (4.7), (4.13), and (4.14), we will determine the stress state in a physically nonlinear thin plate, weakened by an elliptical orifice, to a second approximation. Over the orifice contour, the coefficient of stress concentration $k^{(2)} = \sigma_\vartheta/P$ will be*

$$k^{(2)}=\left(\frac{\sigma_\theta}{P}\right)_{\varrho=1}=2[1-1.500\lambda p^2+10.605\lambda^2p^4+2\varepsilon\cos 2\theta$$
$$+2\varepsilon^2\cos 4\theta-10,660\lambda\varepsilon p^2\cos 2\theta]. \qquad (4.15)$$

5. It can be seen from eq.(4.15) that, if allowance is made for the physical nonlinearity of the materials satisfying the elasticity relations (2.1) under the conditions (2.3), the coefficient of stress concentration will nonlinearly depend not only on the magnitude of the tensile stresses P (Fig.1) and on the parameter λ (characterizing the strength properties of the plate material) but also on the ellipticity of the orifice, characterized by the parameter ϵ [eq.(4.2)]. Setting $\epsilon = 0$ in eq.(4.15), we obtain the values of k for the case of a circular orifice (Bibl.9). Setting $\lambda = 0$ in eq.(4.15), we obtain the values of k found by Guz' (Bibl.5) when using the mentioned approximate method for the case of an elliptical orifice where the plate material obeys Hooke's law. For this last case, there exists an exact solution (Bibl.10) of the problem. A comparison of the corresponding values of k given by the exact solution (Bibl.10) with the approximate k (Bibl.5) will yield a clear idea on the rate

* The superscript of the coefficient of concentration in eq.(4.15) gives the number of the approximation.

TABLE 1

	a/b	1.00	1.05	1.10	1.20	1.30	1.50	1.60
Linear Theory	Exact Solution	2 / 2	2.101 / 1.904	2.212 / 1.818	2.444 / 1.666	2.616 / 1.538	3.000 / 1.333	3.200 / 1.250
$\lambda = 0$	Approximate Solution	2 / 2	2.097 / 1.904	2.198 / 1.818	2.435 / 1.669	2.587 / 1.546	2.960 / 1.360	3.136 / 1.289
	P	In the linear theory, k is independent of P						
Nonlinear Theory	60	1.920 / 1.920	2.002 / 1.843	2.084 / 1.775	2.248 / 1.658	2.412 / 1.565	2.730 / 1.431	2.709 / 1.557
	70	1.904 / 1.904	1.980 / 1.834	2.055 / 1.772	2.208 / 1.667	2.361 / 1.584	2.660 / 1.469	2.568 / 1.667
λ_1	80	1.895 / 1.895	1.962 / 1.832	2.301 / 1.777	2.170 / 1.686	2.310 / 1.616	2.587 / 1.522	2.414 / 1.802
	90	1.868 / 1.868	1.954 / 1.841	2.014 / 1.794	2.138 / 1.718	2.264 / 1.662	2.517 / 1.593	2.250 / 1.966

TABLE 2

		a/b	1.00	1.05	1.10	1.20	1.30	1.50	1.60
Linear Theory		Exact Solution	2.0000 / 2.0000	2.1010 / 1.9047	2.2120 / 1.8182	2.4440 / 1.6667	2.6160 / 1.5387	3.0000 / 1.3333	3.2006 / 1.2509
		Approximated Solution	2.0000 / 2.0000	2.0970 / 1.9048	2.1980 / 1.8188	2.4350 / 1.6696	2.5870 / 1.5460	2.9600 / 1.3600	3.1360 / 1.2896
	P		In the linear theory, k is independent of k						
Nonlinear Theory	600	λ_2	1.9606 / 1.9606	2.0406 / 1.8654	2.1308 / 1.7886	2.3102 / 1.6570	2.4874 / 1.5500	2.8298 / 1.3914	2.8987 / 1.4278
		λ_3	1.9682 / 1.9682	2.0622 / 1.8790	2.1562 / 1.7984	2.3424 / 1.6600	2.5260 / 1.5466	2.8790 / 1.3774	2.9909 / 1.3712
	800	λ_2	1.9226 / 1.9226	2.0050 / 1.8450	2.0878 / 1.7756	2.2534 / 1.6578	2.4196 / 1.5616	2.7380 / 1.4272	2.7248 / 1.5460
		λ_3	1.9474 / 1.9474	2.0368 / 1.8628	2.1262 / 1.7868	2.3044 / 1.6564	2.4914 / 1.5504	2.8200 / 1.3948	2.8810 / 1.4386
	1000	λ_2	1.9006 / 1.9006	1.9730 / 1.8330	2.0466 / 1.7730	2.1944 / 1.6728	2.3430 / 1.5944	2.6346 / 1.4866	2.5150 / 1.7118
		λ_3	1.9258 / 1.9258	2.0092 / 1.8472	2.0930 / 1.7768	2.2604 / 1.6572	2.4266 / 1.5612	2.7494 / 1.4222	2.7468 / 1.5302
	1200	λ_2	1.8946 / 1.8946	1.9550 / 1.8390	2.0168 / 1.7906	2.1432 / 1.7120	2.2722 / 1.6532	2.5292 / 1.5800	2.2279 / 1.9353
		λ_3	1.9068 / 1.9068	1.9828 / 1.8356	2.0596 / 1.7722	2.2142 / 1.6654	2.3684 / 1.5814	2.6704 / 1.4632	2.5892 / 1.6496

of convergence of the approximate solution of the problems of stress concentration about curvilinear orifices for which exact solutions are lacking, as proposed above. Such a comparison is presented in the first two rows of Tables 1 and 2. The values of k in these Tables indicate that even for the greatly extended ellipse $a/b = 1.6$, the third approximation [eq.(4.15)] gives for k a very good agreement with the exact value (the difference does not exceed 2.5 - 3.0%). Tables 1 and 2 present values of k [eq.(4.15)] calculated for two points A and B on the orifice contour (see Fig.1). The values of k at the point $A(\theta = 0)$ are substituted into the numerator and at the point $B(\theta = \pi/2)$, into the denominator, for different values of a/b, P, and λ. The values of λ were taken from another report (Bibl.7): $\lambda_1 = 1.02 \times 10^{-6} \frac{1}{\text{bar}^2}$ (copper); $\lambda_2 = 0.055 \times 10^{-6} \frac{1}{\text{bar}^2}$ (aluminum bronze); $\lambda_3 = 0.033 \times 10^{-6} \frac{1}{\text{bar}^2}$ (open- /1020
hearth steel). The numerical data presented in Tables 1 and 2 indicate that: a) the ellipticity of the orifice - as in the classical case, i.e., when the plate material obeys Hooke's law - greatly affects the coefficient k of stress concentration; b) as the applied tensile stresses P (Fig.1) are increased, k will decrease at the point A and will increase at the point B. It follows that, as the numerical values of the parameters P and λ increase, a consideration of the physical nonlinearity generally yields a more uniform stress distribution over the orifice contour. The approximate method for the solution of the problems, formulated in Point 1 above, was based on a formal expansion of the required functions in double series over the small parameters μ and ϵ, without evaluating at all the convergence of the series. An idea as to the rapidity of convergence of the proposed method in the general case (Bibl.5, 9) can be obtained by calculating the concentration coefficients $k^{(n-2)}$, $k^{(n-1)} k^{(n)}$ from formulas analogous to eq.(4.15) and corresponding to the stress functions F_{n-2}, F_{n-1}, F_n for the preceding approximations.

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