Module 14:
Noise Models

GLM

A standard GLM can be written:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \qquad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \mathbf{V})$$

where

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & \cdots & X_{1p} \\ 1 & X_{21} & \cdots & X_{2p} \\ \vdots & \vdots & & \vdots \\ 1 & X_{np} & \cdots & X_{np} \end{bmatrix} \times \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

fMRI Data

Design matrix

Regression coefficients

Noise

V is the covariance matrix whose format depends on the noise model.

The quality of the model depends on our choice of X and V.

Design Matrix

- We has previously discussed various signal and nuisance components that can be included in the design matrix to improve the model.
  - Temporal Basis functions
    - Allows for flexible HRF
  - Parametric modulation
    - Allows for trial-specific variation in amplitude
  - Motion parameters
    - Corrects for 'spin history' artifacts

fMRI Noise

- Functional MRI data typically exhibit significant autocorrelation.
  - Caused by physiological noise and low frequency drift, that has not been appropriately modeled.
  - Typically modeled using either an AR(p) or an ARMA(1,1) process.
  - Single subject statistics are not valid without an accurate model of the noise.

AR(1) model

- Serial correlation can be modeled using a first-order autoregressive model, i.e.

$$\varepsilon_t = \phi \varepsilon_{t-1} + u_t \qquad u_t \sim N(0, \sigma^2)$$

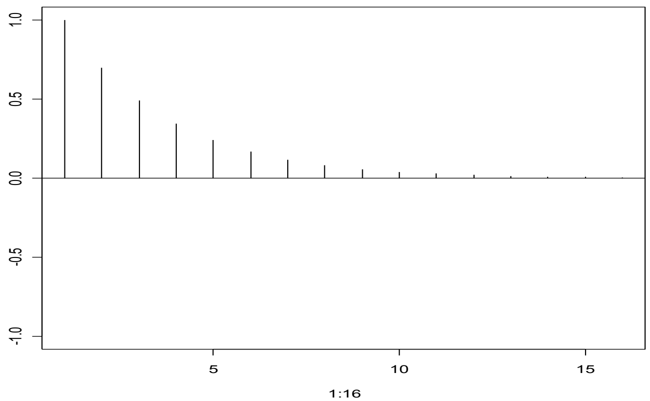
- The error term ε_t depends on the previous error term ε_{t-1} and a new disturbance term u_t.

AR(1) model

- The autocorrelation function (ACF) for an AR(1) process at lag *h*:

$$\rho(h) = \begin{cases} 1, & \text{if } h = 0, \\ \phi^{|h|} & \text{if } h \neq 0 \end{cases}$$

ϕ=0.7



Error Term

- The format of $\mathbf{V}$ will depend on what noise model is used.

IID Case

$$\mathbf{V} \propto \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

AR(1) Case

$$\mathbf{V} \propto \begin{bmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{n-1} \\ \phi & 1 & \phi & \cdots & \phi^{n-2} \\ \phi^2 & \phi & 1 & \cdots & \phi^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} & \cdots & 1 \end{bmatrix}$$

GLM Summary

model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \longrightarrow$$

estimate

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}$$

fitted values

$$\hat{\mathbf{Y}} = \mathbf{X}\hat{\beta}$$

residuals

$$\mathbf{r} = \mathbf{Y} - \hat{\mathbf{Y}}$$

$$= (\mathbf{I} - (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1})\mathbf{Y}$$

$$= \mathbf{R}\mathbf{Y}$$

Estimating V

- In general the form of the covariance matrix is unknown, which means it has to be estimated.
- Estimating $\mathbf{V}$ depends on β's, and estimating β's depends on $\mathbf{V}$. Need iterative procedure.
- Methods for estimating variance components:
  - Method of moments
  - Maximum likelihood
  - Restricted maximum likelihood

Iterative Procedure

1. Assume that $\mathbf{V}=\mathbf{I}$ and calculate the OLS solution.

2. Estimate the parameters of $\mathbf{V}$ using the residuals.

3. Re-estimate the β values using the estimated covariance matrix $\hat{\mathbf{V}}$ from step 2.

4. Iterate until convergence.

Yule-Walker Estimates

- Assume ε_t is an AR(1) process, i.e.

$$\varepsilon_t = \phi \varepsilon_{t-1} + u_t \qquad\qquad t = 0, \pm 1, \ldots$$

where $u_t \sim WN(0, \sigma^2)$

- The Yule-Walker estimates are:

$$\hat{\phi} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} \qquad\qquad \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}\hat{\gamma}(1)$$

Auto Covariance Function

MLE

- Maximum likelihood estimators (MLEs) are obtained by maximizing the log-likelihood:

$$l^*(\lambda) = -\frac{1}{2}\log\left(|\mathbf{V}|\right) - \frac{1}{2}\left(\mathbf{Y} - \mathbf{X}\hat{\beta}\right)^T \mathbf{V}^{-1}\left(\mathbf{Y} - \mathbf{X}\hat{\beta}\right)$$

where λ are parameters associated with $\mathbf{V}$.

ReML

- Restricted maximum likelihood (ReML) requires maximizing the restricted log-likelihood:

$$l^{*}(\lambda) = -\frac{1}{2}\log\left(\left|\mathbf{V}\right|\right) \boxed{-\frac{1}{2}\log\left(\left|\mathbf{X}^{T}\mathbf{V}\mathbf{X}\right|\right)} - \frac{1}{2}\left(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}}\right)^{T}\mathbf{V}^{-1}\left(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}}\right)$$

Extra ReML variance term

where λ are parameters associated with V.

ML vs ReML

- Maximum Likelihood
  - Maximize likelihood of data *y*
  - Used to estimate "mean" parameters β
  - But can produce biased estimates of variance

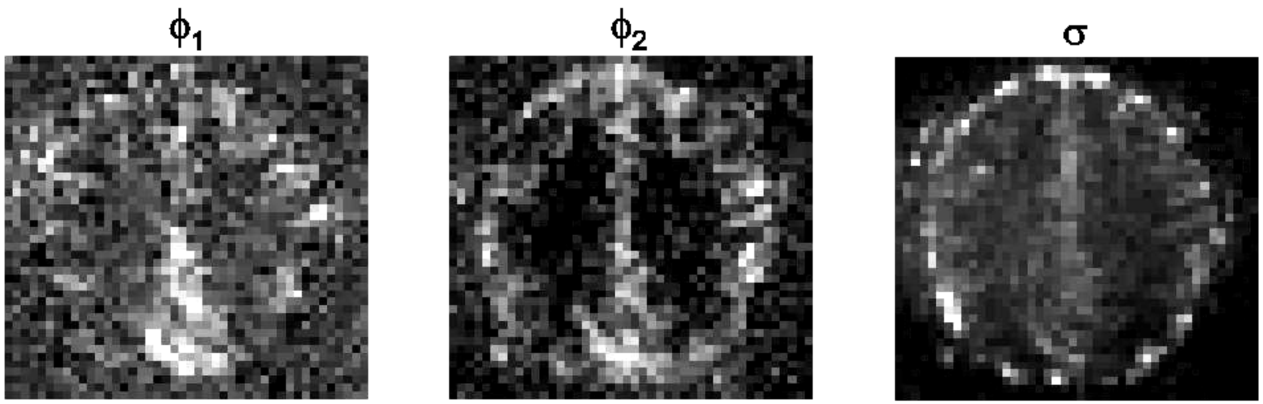
$$\hat{\sigma}^2_{\mathrm{ML}} = \frac{1}{n}\sum (y_i - \bar{y})^2$$

- Restricted Maximum Likelihood
  - Maximize likelihood of residuals $e = y - Xb$
  - Used to estimate variance parameters
  - Provides unbiased estimates

$$\hat{\sigma}^2_{\mathrm{ReML}} = \frac{1}{n-1}\sum (y_i - \bar{y})^2$$

Spatio-temporal Behavior

- The spatiotemporal behavior of these noise processes is complex.



Spatial maps of the model parameters from an AR(2) model estimated for each voxel's noise data.

End of Module