Discrete Differential Operators

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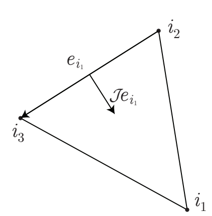
The idea is to construct the operators *gradient, divergence* and *Laplacian* (i.e., $\mathrm{grad}, \mathrm{div}$ and L, respectively) based on the area matrices and a *single* operator E.

Notation. M denotes our surface and we use f for a function and v for a vector field. In the discrete case, $\mathcal{V}$ and $\mathcal{F}$ denote the vertex set and face set of M, respectively. Moreover, we use piecewise-linear functions, i.e., $f \in \mathbb{R}^{|\mathcal{V}| \times 1}$ and piecewise-constant vector fields, i.e., $v \in \mathbb{R}^{3\,|\mathcal{F}| \times 1}$. One way to interpret the discrete v is to assume that the first $|\mathcal{F}|$ elements represent the x-coordinate of v over all of the faces, the next $|\mathcal{F}|$ entries are the y-coordinate, and the last $|\mathcal{F}|$ elements are the z-coordinate. Then, the dimensions of the required operators are as follows, $\mathrm{grad} \in \mathbb{R}^{3\,|\mathcal{F}| \times |\mathcal{V}|}$, $\mathrm{div} \in \mathbb{R}^{|\mathcal{V}| \times 3\,|\mathcal{F}|}$, and $L \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$. Finally, $\mathrm{G}_{\mathcal{V}}$ is a diagonal matrix with the areas of the vertices along the diagonal. Similarly, $\mathrm{G}_{\mathcal{F}}$ is a diagonal matrix with the areas of the faces along the diagonal, replicated 3 times, thus $\mathrm{G}_{\mathcal{F}} \in \mathbb{R}^{3\,|\mathcal{F}| \times 3\,|\mathcal{F}|}$.

Discrete gradient. The only operator which we need to construct "manually" is the gradient. The rest of the operators will result through multiplication of formerly built operators. A common discretization of the gradient at a face j is given by

$$(\mathrm{grad}\, f)(j) = \frac{1}{2\,\mathrm{G}_{\mathcal{F}}(j,j)} \sum_{i=1}^{3} f_i\, \mathcal{J}\, e_{ji},$$

where the sum runs over the three vertices of the face, and $\mathcal{J}$ is the in-plane $\pi/2$ rotation of the edge e_{ji} opposite vertex i in the face j (see the inset figure). Therefore, we can define the gradient as $\mathrm{grad} = \frac{1}{2}\,\mathrm{G}_{\mathcal{F}}{}^{-1}\,E$, where E encodes the rotated edges. Specifically, if we consider face j, then only three rows of E hold information related to this face, namely $j, j+|\mathcal{F}|$ and $j+2\,|\mathcal{F}|$. In these rows, only 3 columns are relevant, those are exactly the columns which correspond to the vertices i of the face. For instance, on row j, you will encode the x-coordinate of the vectors $\mathcal{J}\, e_{ji_1}, \mathcal{J}\, e_{ji_2}$ and $\mathcal{J}\, e_{ji_3}$ of vertices i_1, i_2 and i_3, respectively.



Integration by parts. Given a discrete gradient, one possible way to construct a discrete Laplacian is to derive a discrete divergence through integration by parts. Recall that for f and v satisfying some conditions, we have:

$$\int_M v \cdot \nabla f \,\mathrm{d}a + \int_M f\, \nabla\cdot v \,\mathrm{d}a = \int_{\partial M} f(v\cdot n)\,\mathrm{d}l = 0, \quad (1)$$

where the rightmost equality holds when ∂M is empty or when $v \cdot n = 0$. We can rewrite Eq. (1) in a discrete form by defining inner products on functions f_1, f_2 and on vector fields v_1, v_2. Namely,

$$\int_M f_1 f_2 \,\mathrm{d}a = f_1^T\, \mathrm{G}_{\mathcal{V}}\, f_2, \qquad \int_M v_1 \cdot v_2 \,\mathrm{d}a = v_1^T\, \mathrm{G}_{\mathcal{F}}\, v_2.$$

Then, Eq. (1) becomes

$$v^T\, \mathrm{G}_{\mathcal{F}}\, \mathrm{grad}\, f + v^T\, \mathrm{div}^T\, \mathrm{G}_{\mathcal{V}}\, f = 0. \quad (2)$$

Discrete divergence. Eq. (2) holds for any f and v thus we can extract the involved operators and arrive at

$$\begin{aligned} &\mathrm{G}_{\mathcal{F}}\,\mathrm{grad} + \mathrm{div}^T\,\mathrm{G}_{\mathcal{V}} = 0 \\ \Rightarrow\ &\mathrm{div}^T = -\,\mathrm{G}_{\mathcal{F}}\,\mathrm{grad}\,\mathrm{G}_{\mathcal{V}}{}^{-1} \\ \Rightarrow\ &\mathrm{div} = -\,\mathrm{G}_{\mathcal{V}}{}^{-1}\,\mathrm{grad}^T\,\mathrm{G}_{\mathcal{F}}\,. \end{aligned}$$

Discrete Laplacian. Finally, we can construct our discrete Laplacian using our notions of gradient and divergence. Namely,

$$\begin{aligned} L &= -\,\mathrm{div}\,\mathrm{grad} \\ &= \mathrm{G}_{\mathcal{V}}{}^{-1}\,\mathrm{grad}^T\,\mathrm{G}_{\mathcal{F}}\,\mathrm{grad} \\ &= \frac{1}{4}\,\mathrm{G}_{\mathcal{V}}{}^{-1}\,E^T\,\mathrm{G}_{\mathcal{F}}{}^{-1}\,\mathrm{G}_{\mathcal{F}}\,\mathrm{G}_{\mathcal{F}}{}^{-1}\,E \\ &= \frac{1}{4}\,\mathrm{G}_{\mathcal{V}}{}^{-1}\,E^T\,\mathrm{G}_{\mathcal{F}}{}^{-1}\,E. \end{aligned}$$

Interestingly, L corresponds to the commonly used Laplace–Beltrami operator and specifically, the matrix $\frac{1}{4}E^T\,\mathrm{G}_{\mathcal{F}}{}^{-1}\,E$ encodes the well-known cotangent weights.