COMPACTIFICATIONS OF MODULI SPACE OF (QUASI-)TRIELLIPTIC K3 SURFACES

YITAO CHEN, HAOYU WU, HANYU YAO

Abstract. We study the moduli space $\mathcal{F}_{T_1}$ of quasi-trielliptic K3 surfaces of type I, whose general member is a smooth bidegree $(2,3)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^2$. Such moduli space plays an important role in the study of Hassett-Keel-Looijenga program of the moduli space of degree 8 quasi-polarized K3 surfaces.

In this paper, we consider several natural compactifications of $\mathcal{F}_{T_1}$, such as the GIT compactification and arithmetic compactifications. We give a complete analysis of GIT stability of $(2,3)$-hypersurfaces and provide a concrete description of boundary of the GIT compactification. We also compute the configurations of the boundary of the Baily-Borel compactification of the quasi-trielliptic K3 surfaces by classifying certain lattice embeddings. As an application, we show that $(\mathbb{P}^1 \times \mathbb{P}^2, \epsilon S)$ with small ϵ is K-stable if S is a K3 surface with at worst ADE singularities. This gives a concrete description of boundary of the K-stability compactification via the identification of the GIT stability and the K-stability. We also discuss the connection between the GIT, Baily-Borel compactification and Looijenga's compactifications by studying the projective models of quasi-trielliptic K3 surfaces.

1. Introduction

Let $\mathcal{F}_{2\ell}$ be the moduli space of primitively quasi-polarized K3 surfaces of degree 2ℓ, i.e., a K3 surface S and a nef line bundle L such that $L^2 = 2\ell$ and $c_1(L) \in H^2(S, \mathbb{Z})$ is a primitive class. There are natural geometric (partial) compactifications of $\mathcal{F}_{2\ell}$ constructed by geometric invariant theory. One can construct the moduli of polarized K3's as the GIT quotient of an open subset of the Hilbert scheme or Chow variety parametrizing (S, L) in $|mL|$ for sufficiently large m. In the case of low degree, Mukai showed that general members in $\mathcal{F}_{2\ell}$ are complete intersections in some homogeneous spaces for $2\ell \leq 22$ and he provided natural GIT compactification of moduli spaces of such K3 surfaces (cf. [Muk88]). On the other hand, by the Global Torelli theorem, $\mathcal{F}_{2\ell}$ is a locally symmetric variety and hence admits natural arithmetic compactifications such as the Baily-Borel compactification $\mathcal{F}^*_{2\ell}$ [BB66], Mumford's toroidal compactifications [AMRT10] and Looijenga's semitoric compactifications [Loo03]. In recent years, there are a series of study of the compactifications of moduli of K3 surfaces via various methods (cf. [Bru15, AB19, ABE20, Laz16, HLL20, AET19, AE21]).

A natural question is to investigate the connection between various compactifications. When $2\ell = 2$, the birational map between $\overline{\mathcal{F}}_2^{\mathrm{GIT}}$ and $\mathcal{F}_2^*$ is described by Shah and Looijenga [Sha80, Loo86]. For quartic K3 surfaces, this is so-called Hassett-Keel-Looijenga program, proposed by Laza and O'Grady [LO19] for $\mathcal{F}_4$. They conjectured that the birational map from the natural GIT compactification to the Baily-Borel compactification can be factorized into a series of elementary birational transformations whose center is the proper transformation of Shimura subvarieties. Recently, this was generalized to larger degree by Greer-Laza-Li-Si-Tian in [GLL$^+$]. When $2\ell = 4$, the program was confirmed by Ascher-DeVleming-Liu in [ADL22] by using the HKL program on moduli space of hyperelliptic K3 of degree 4 (See also [LO19, LO21]).

In this paper, we investigate the compactification of moduli space $\mathcal{F}_{T_n}$ of (quasi)-trielliptic K3 surfaces, of which the general member admits an elliptic fibration with a degree three multisection. The Picard group of a trielliptic K3 surface contains a sublattice T_n whose Gram matrix is

$$\begin{array}{c|c|c} & C & E \\ \hline C & 2n & 3 \\ \hline E & 3 & 0 \end{array}.$$

The moduli space of such K3 surfaces have recently been studied by Beauville in [Bea21]. When $n=1$, the general K3 surface in $\mathcal{F}_{T_n}$ is a smooth bidgree $(2,3)$-hypersurface of $\mathbb{P}^1\times\mathbb{P}^2$. There is a natural GIT construction of moduli space of these hypersurfaces

$$\overline{\mathcal{M}}_2^{\mathrm{GIT}} := |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,3)| /\!\!/ \mathrm{SL}_2\times\mathrm{SL}_3 .$$

Let $\mathcal{M}_2$ be the domain of the natural birational map

$$\overline{\mathcal{M}}_2^{\mathrm{GIT}} \dashrightarrow \mathcal{F}_{T_1}.$$

Moreover, the moduli space $\mathcal{F}_{T_1}$ admits a natural morphism to $\mathcal{F}_8$, whose image is an irreducible Noether-Lefschetz divisor. Let $\mathcal{F}_{T_1}^*$ be the Baily-Borel compactification of $\mathcal{F}_{T_1}$. According to the spirit of HKL program (see [GLL$^+$, ADL22]), the birational map $\overline{\mathcal{M}}_2^{\mathrm{GIT}} \dashrightarrow \mathcal{F}_{T_1}^*$ has a close relation to the HKL program of $\mathcal{F}_8$. For instance, there is a one-to-one correspondence between their wall-crossing phenomenon. As a start, we first give a detailed description of the GIT compactification and $\mathcal{F}_{T_n}^*$. Let π,π' be the two projections of $(2,3)$-hypersurface to $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively. Our first main result is

Theorem 1.1. *The locus $\mathcal{M}_2$ is contained in the stable locus and the boundary $\overline{\mathcal{M}}_2^{\mathrm{GIT}}\backslash\mathcal{M}_2$ consists of 11 irreducible components whose general member S is described as follows:*

(i) *The **strictly semistable** components consist of*

(α): *S is singular along a vertical line L. And S has a corank 3 singularity p such that $p\notin\pi(L)$ and $\pi'(p)\notin\pi'(L)$.*

(β): *S has two isolated $\widetilde{E}_8$-type singularities p and q such that $\pi(p)\neq\pi(q)$ and $\pi'(p)\neq\pi'(q)$. The fiber over p and q are both triple lines with different directions.*

(γ): *$S=(\mathbb{P}^1\times\mathbb{P}^1)\cup S'$ for some $S'\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,2)|$ and S is singular along a horizontal line C such that the intersection of $\mathbb{P}^1\times\mathbb{P}^1$ and C is empty.*

(η): *S has two isolated $\widetilde{E}_7$-type singularities p and q such that $\pi(p)\neq\pi(q)$ and $\pi'(p)\neq\pi'(q)$. The fiber over $\pi(p)$ and $\pi(q)$ contain a line L_1 and L_2 respectively with $L_1\neq L_2$. And $\mathrm{mult}_{\pi'(p)}((\pi'(F)),L_1)\geq 2$, $\mathrm{mult}_{\pi'(q)}((\pi'(F)),L_2)\geq 2$ for any fiber F.*

(δ): *S is the union of two $(1,0)$ hypersurfaces and one $(0,3)$ hypersurface.*

(ii) *The **stable** components consist of*

(ζ): *S has an isolated $\widetilde{E}_7$-type singularities p and there exists a fiber F such that $\pi'(F)$ does not contain $\pi'(p)$.*

(ξ): *S has an isolated $\widetilde{E}_8$-type singularities p and the fiber containing p is not a triple line.*

(θ): *S is singular along a section of degree 1.*

(ϕ): *S is singular along a section of degree 2.*

$(r1)$: *S is a union of a $(1,1)$ hypersurface and a $(1,3)$ hypersurface.*

$(r2)$: *S is a union of a $(0,2)$ hypersurface and a $(2,1)$ hypersurface.*

The dimension of the stratum is given by

Strata	α	β	γ	η	δ	ζ	ξ	θ	ϕ	$r1$	$r2$
dimension	4	1	4	3	1	10	7	8	5	13	2

And the complement of the image of $\mathcal{M}_2$ in $\mathcal{F}_{T_1}$ is the union of two irreducible Noether-Lefschetz divisors $\mathbf{H}_u \cup \mathbf{H}_h$.

As proved in [Zho21], an open subset of $\mathcal{F}_{T_1}$ also admits a K-stability compactification via the identification of the GIT stability and the K-stability. More precisely, by [Zho21, Theorem 1.1], there exists some rational number $0 < c < 1$ such that one can identify the K-stability of the log Fano pair $(\mathbb{P}^1 \times \mathbb{P}^2, \epsilon S)$ for $\epsilon \in (0, c)$ with the GIT-stability of S under the action of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^2)$ for any $(2,3)$-hypersurface S. Moreover, there is a isomorphism

$$\overline{\mathcal{M}}_2^K := M^K_{\mathbb{P}^1\times\mathbb{P}^2,\epsilon} \xrightarrow{\cong} \overline{\mathcal{M}}_2^{\mathrm{GIT}}$$

where $M^K_{\mathbb{P}^1\times\mathbb{P}^2,\epsilon}$ is the good moduli space parametrized K-polystable pairs of the form $(\mathbb{P}^1 \times \mathbb{P}^2, \epsilon S)$ where S is a anticanonical section of $\mathbb{P}^1 \times \mathbb{P}^2$. And we have

Corollary 1.2. *There exists a rational number $0 < c < 1$ such that the log Fano pair $(\mathbb{P}^1 \times \mathbb{P}^2, \epsilon S)$ is K-stable for $0 < \epsilon < c$ where $S \in |-K_{\mathbb{P}^1\times\mathbb{P}^2}|$ is a $(2,3)$-hypersurface with at worst simple singularities (i.e., isolated ADE singularities) or of type $(\zeta), (\xi), (\theta), (\phi)$ or $(r_1), (r_2)$.*

Following [Fri84], we also compute the boundary of the Baily-Borel compactification of $\mathcal{F}_{T_n}$. More precisely, we describe the type II and III boundary components by classifying the primitive lattice embedding of $(\Sigma_n)^{\perp}_{\mathbb{I}_{2,26}} \hookrightarrow \mathbb{I}_{0,24}$.

Theorem 1.3 (Theorem 7.5)**.** *Let $\mathcal{F}^*_{T_n}$ be the Baily-Borel compactification of $\mathcal{F}_{T_n}$. Then the boundary $\partial\mathcal{F}_{T_n} := \mathcal{F}^*_{T_n} - \mathcal{F}_{T_n}$ is given as follows:*

(i) *$n = 1$, $\partial\mathcal{F}_{T_1}$ consists of 14 modular curves and 2 points. All curves meet at one point. There are 2 curves meet at another point.*
(ii) *$n = 2$, $\partial\mathcal{F}_{T_2}$ consists of 11 modular curves and 2 points. All curves meet at one point. There are 2 curves meet at another point.*
(iii) *$n = 3$, $\partial\mathcal{F}_{T_3}$ consists of 10 modular curves and 2 points. All curves meet at one point. There are 2 curves meet at another point.*

In a sequel to this paper, we would like to study the birational transformations between $\overline{\mathcal{M}}_2^{\mathrm{GIT}}$ and $\mathcal{F}^*_{T_1}$.

Organization of the paper. We start in Section 2 by constructing $\mathcal{F}_{T_n}$ the moduli space of the trielliptic K3 surface of given type. We also compute the Picard group of $\mathcal{F}_{T_n}$ using Noether-Lefschetz theory. We finish this section by finding out the projective models of trielliptic K3 surfaces. For $n = 1$, generic quasi-trielliptic surfaces are $(2,3)$-hypersurfaces of $\mathbb{P}^1 \times \mathbb{P}^2$ which gives rise to natural GIT compactification we considered below. The Sections 3-6 are devoted to the standard GIT analysis of $(2,3)$-hypersurface. Section 3 consists of the combinatorics of the unstable and not-properly stable surfaces and we characterize their geometry in Section 4. In section 5, using Luna's criterion, we study the strictly semistable locus of the GIT compactification while in Section 6, the stable locus with non-simple singularities are discussed. Together this gives the complete deceptions of the boundary of $\mathcal{M}_2$ in $\overline{\mathcal{M}}_2^{\mathrm{GIT}}$ and proves the main result Theorem 1.1. Finally, Baily-Borel compactification and Looijenga's compactification of $\mathcal{F}_{T_n}$ are discussed in section 7. In particular, we show that the GIT quotient $\overline{\mathcal{M}}_2^{\mathrm{GIT}}$ is not isomorphic to the Looijenga's compactification.

Acknowledgments. This project was initiated in the Individual study of the Talented Program in Fudan University. The authors are grateful for the helpful discussions and comments from Zhiyuan Li and Ruxuan Zhang. The authors are supported by NKRD Program of China (No. 2020YFA0713200) and NSFC General Program (No. 12171090).

Notations & Conventions.

- $\overline{\mathcal{M}}_2^{\mathrm{GIT}}$: the GIT quotient of $(2,3)$-hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^2$.
- $\mathcal{F}_{T_n}$: the moduli space of quasi-trielliptic K3 surfaces of type I,II,III (n=1,2,3) respectively.
- $\mathcal{M}_2$: the domain of the birational map $\overline{\mathcal{M}}_2^{\mathrm{GIT}} \dashrightarrow \mathcal{F}_{T_1}$.
- $\mathcal{F}^*_{T_n}$: the Baily-Borel compactification of $\mathcal{F}_{T_n}$.
- Let A_n, D_n, E_n be the negative definite root lattices.

We work over $\mathbb{C}$.

2. Moduli space of (quasi)-trielliptic K3 surfaces

2.1. **(Quasi)-Trielliptic K3 surface.** A K3 surface S is **trielliptic** if it admits an elliptic fibration $\pi\colon S \to \mathbb{P}^1$ together with a multisection C of degree 3. We may say S is trielliptic of type n if $C^2 = 2n$. In this case, the Picard lattice $\mathrm{Pic}(S)$ contains a primitive sublattice T_n given by

$$\begin{array}{c|c|c} & C & E \\ \hline C & 2n & 3 \\ \hline E & 3 & 0 \end{array}\,. \tag{2.1}$$

Up to an isometry, we may assume $n \in \{1,2,3\}$. Then S admits a T_n-polarization in the sense of [Dol96]. Hence we may say a K3 surface is **quasi-trielliptic** of type I (II, III respectively) if it admits a T_1-polarization (respectively T_2, T_3-polarization).

2.2. **Moduli space of (quasi)-trielliptic K3 surface.** Let $\mathcal{F}_{\mathrm{T}_n}$ be the coarse moduli space of T_n-polarized K3 surfaces which parameteries pairs (S,ϕ) where S is a smooth K3 surface and $\phi : T_n \hookrightarrow \mathrm{Pic}(S) \subset \mathrm{H}^2(S,\mathbb{Z})$ is a primitive embedding and $\phi(T_n)$ contains a quasi-polarization. Here two T_n-polarized K3 surfaces (S,ϕ) and (S',ϕ') are equivalent if there exits isomorphism $h : S \to S'$ such that $\phi = h^* \circ \phi'$ that takes the quasi-polarization to a quasi-polarization. Here the middle cohomology $\mathrm{H}^2(S,\mathbb{Z})$ is a unimodular even lattice of signature $(3,19)$ under the intersection form $\langle , \rangle$ by cup product. For every S, we choose a identification of $\mathrm{H}^2(S,\mathbb{Z}) \xrightarrow{\cong} \Lambda = U^2 \oplus E_8^3$. According to [Nik79], there is a unique primitive embedding up to the automorphism of Λ

$$\mathrm{T}_n \hookrightarrow \Lambda. \tag{2.2}$$

We denote by Σ_n the orthogonal complement of T_n in Λ, which is an even lattice of signature $(2,18)$. Let $\Sigma_n^{\mathbb{C}} = \Sigma_n \otimes \mathbb{C}$. The period domain $\mathbb{D}$ associated to Σ_n can be realized as a connected component of

$$\mathbb{D}^{\pm} := \{v \in \mathbb{P}(\Sigma_n^{\mathbb{C}}) |\, \langle v, v\rangle = 0, \langle v, \bar{v}\rangle > 0\}.$$

The monodromy group

$$\Gamma_n = \{g \in O^+(\Sigma_n) |\; g \text{ acts trivially on } \Sigma_n^\vee/\Sigma_n\}$$

naturally acts on $\mathbb{D}$, where $O^+(\Sigma_n)$ is the identity component of $O(\Sigma_n)$. According to the Global Torelli theorem of K3 surfaces (See [Dol96, Remark 3.4]), there is an isomorphism

$$\mathcal{F}_{\mathrm{T}_n} \cong \Gamma_n \backslash \mathbb{D}$$

via the period map. Then $\mathcal{F}_{\mathrm{T}_n}$ is a locally Hermitian symmetric variety with only quotient singularities, and hence $\mathbb{Q}$-factorial.

Note that by the definition of lattice polarization, we may assume the generators C, E of T_n are both effective. Moreover, up to an automorphism of Λ, we may assume that C is big and nef (see [Huy16, Chapter 8, Corollary 2.9]).

2.3. **Picard group of $\mathcal{F}_{\mathrm{T}_n}$.** The Noether-Lefschetz (NL) divisors on $\mathcal{F}_{\mathrm{T}_n}$ parameterizes the K3 surfaces in $\mathcal{F}_{\mathrm{T}_n}$ containing additional curve classes. According to [BLMM17, Theorem 1], the Picard group of $\mathcal{F}_{\mathrm{T}_n}$ is generated by NL-divisors. Let us recall the construction of some irreducible NL-divisors on $\mathcal{F}_{\mathrm{T}_n}$.

Definition 2.1. Let $\beta \in A_{\Sigma_n}, m \in \mathbf{Q}_{<0}$. The Heegner divisor is given by

$$\mathbf{H}_{\beta,m} := \widetilde{O}^+(\Sigma_n) \setminus \bigcup_{\substack{v \in \Sigma_n^*,\, v^2 = 2m \\ v \in \beta + \Sigma_n}} v^{\perp}.$$

Note that $\mathbf{H}_{\beta,m} = \mathbf{H}_{-\beta,m}$.

In general, the Heegner divisor $\mathbf{H}_{\beta,m}$ can be non-reduced and reducible. In the case that Λ is a transcendental lattice of a K3 surface, it can be written as the sum of some irreducible NL divisors which we will introduce below.

Definition 2.2. We define $\mathbf{H}_u$, $\mathbf{H}_h$ and $\mathbf{H}_t$ to be the locus of K3 surfaces $(S, C, E) \in \mathcal{F}_{\mathrm{T}_n}$ such that $\mathrm{Pic}(S)$ contains a divisor class E' satisfying

- $\mathbf{H}_u$: ${E'}^2 = 0$, $C \cdot E' = 1$ and $E \cdot E' = 1$.
- $\mathbf{H}_h$: ${E'}^2 = 0$, $C \cdot E' = 2$ and $E \cdot E' = 1$.
- $\mathbf{H}_t$: ${E'}^2 = 0$, $C \cdot E' = 3$ and $E \cdot E' = 1$.

such that the rank 3 sublattice of Picard group generated by $\{C, E, E'\}$ is primitive. This is the generalization of primitive NL divisors of moduli space of (quasi)-polarized K3 surfaces. Following the same proof of [O'G86, Proposition 1.3], one can show that these primitive NL divisors are irreducible.

Now we give the computation of $\mathrm{Pic}(\mathcal{F}_{T_n})$. let L be an even lattice of signature $(2, k)$ containing two hyperbolic lattices. Let

$$\rho_L \colon \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[A_L])$$

be the dual Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$. Due to [Bru02] and [BLMM17], there is an isomorphism

$$\mathrm{Pic}_{\mathbb{Q}}(\widetilde{O}^+(L) \setminus \mathcal{D}_L) \cong \mathrm{Acusp}(\frac{k+2}{2}, \rho_L)^{\vee} \tag{2.3}$$

sending Heegner divisor $\mathbf{H}_{\beta,m}$ to the coefficient function $c_{-m,\beta}$

$$\sum_{\tau \in A_L} \sum_{d \in \mathbb{Q}} c_{d,\tau} q^d e_\tau \mapsto c_{-m,\beta}.$$

Here $\mathrm{Acusp}(\frac{k+2}{2}, \rho_L)$ is the space of almost cusp form of weight $\frac{k+2}{2}$ and type ρ_L. So one can read the relation of Heegner divisors from the the relation between modular forms. The space of vector-valued modular form can be computed by Raum's method using lattice level Jacobi forms. For details, we refer to [Pet15, Section 3, 4].

For the dimension of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{T_n})$, one can use the dimension formula of Bruinier [Bru02]:

$$\begin{aligned}\rho(\mathcal{F}_{T_n}) := \dim_{\mathbb{Q}} \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{T_n}) = \frac{29}{4} - \frac{1}{12}\,\mathrm{Re}\, G(2,\Sigma_n) - \alpha_3(n) - \alpha_4(n) \\ - \frac{1}{9\sqrt{3}}\,\mathrm{Re}\left[\sqrt{-1}\big(G(1,\Sigma_n) + G(-3,\Sigma_n)\big)\right]\end{aligned} \tag{2.4}$$

where $G(m,L)$ is the generalized quadratic Gauss sum

$$G(m,L) = \sum_{\gamma\in A_L} e^{2\pi\sqrt{-1}\frac{m\gamma^2}{2}}$$

and $\alpha_3(n) = \sum_{\gamma\in A_{\Sigma_n}/\pm 1}\{-\frac{\gamma^2}{2}\}$, $\alpha_4(n) = |A_{\Sigma_n}/\pm 1|$.

Since in our case, the discriminant groups are as follows:

(i) $n=1$, $A_{\Sigma_1} = \mathbb{Z}\langle\xi_1\rangle \cong \mathbb{Z}/9\mathbb{Z}$, $\xi_1^2 = \frac{2}{9}$,
(ii) $n=2$, $A_{\Sigma_2} = \mathbb{Z}\langle\xi_2\rangle \cong \mathbb{Z}/9\mathbb{Z}$, $\xi_2^2 = \frac{4}{9}$,
(iii) $n=3$, $A_{\Sigma_3} = \mathbb{Z}\langle\eta_1\rangle \times \mathbb{Z}\langle\eta_2\rangle \cong \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$, $\eta_1^2 = -\frac{2}{3}$, $\eta_2^2 = -\frac{4}{3}$, $\eta_1\cdot\eta_2 = 0$,

One can compute

$$\alpha_3(n) = \frac{6-n}{3}, \quad \alpha_4(n) = \begin{cases} 2, & n=1,2 \\ 3, & n=3 \end{cases}$$

and

$$G(m,\Sigma_n) = \begin{cases} 3\left(\frac{nm}{9}\right), & n=1,2,\ m=1,2 \\ 3\sqrt{-3}\left(\frac{-n}{3}\right), & n=1,2,\ m=-3 \\ 3\gcd(n,|m|), & n=3,\ m=1,2,-3 \end{cases}.$$

where $\left(\frac{a}{b}\right)$ is the Jacobi symbol. Therefore we have

Proposition 2.3. *The Picard number of $\mathcal{F}_{T_n}$ is given by*

n	1	2	3
$\rho(\mathcal{F}_{T_n})$	3	4	3

and the corresponding Hodge relations are given by

$$\begin{aligned} 27\lambda_1 &= 20\mathbf{H}_{4\xi_1,-\frac{2}{9}} - 8\mathbf{H}_{2\xi_1,-\frac{5}{9}} + \mathbf{H}_{\xi_1,-\frac{8}{9}}, \\ 108\lambda_2 &= \mathbf{H}_{\bar{0},-1} + 130\mathbf{H}_{2\xi_2,-\frac{1}{9}} + 28\mathbf{H}_{4\xi_2,-\frac{4}{9}} + 2\mathbf{H}_{\xi_2,-\frac{7}{9}}, \\ 102\lambda_3 &= \mathbf{H}_{\bar{0},-1} + 54\mathbf{H}_{\eta_1,-\frac{1}{3}} + 6\mathbf{H}_{\eta_2,-\frac{2}{3}}. \end{aligned}$$

Proof. The relation follows from the identification (2.3) and the explicit computation of basis of the almost cusp forms $\mathrm{Acusp}(10,\rho_{\Sigma_n})$ which can be easily computed by `Sage` using computer. For examples of quasi-polarized K3 surfaces, see [Pet15, Section 4.4]. ♣

2.4. **Projective models of triple-elliptic K3 surfaces.** Let us recall the Saint-Donat's classical result of projective models of K3 surface.

Proposition 2.4 (see [SD74])**.** *Let (S,L) be a smooth K3 surface with a primitive quasi-polarization L of degree 2ℓ and let φ_L be the map defined by $|L|$. Then there are the following possibilities:*

(i) (Generic case) *φ_L is birational to a degree 2ℓ surface in $\mathbb{P}^{\ell+1}$. In particular, φ_L is a closed embedding when L is ample.*

(ii) (Hyperelliptic case) φ_L *is a generically* $2:1$ *map and* $\varphi_L(S)$ *is a smooth rational normal scroll of degree* ℓ*, or a cone over a rational normal curve of degree* ℓ*. Moreover, in this case,* $\mathrm{Pic}(S)$ *contains a curve class* E' *satisfying* ${E'}^2 = 0$ *and* $L\cdot E' = 2$ *for* $L^2 \geq 4$*.*

(iii) (Unigonal case) $|L|$ *has a fixed component* D*, which is a smooth rational curve. In this case,* $\mathrm{Pic}(S)$ *contains a curve class* E' *satisfying* ${E'}^2 = 0$ *and* $L\cdot E' = 1$*.*

The projective models of T_n-polarized K3 surfaces are given as below.

Proposition 2.5. *Let* (S, C, E) *be a* T_n*-quasipolarized K3 surface. Consider the rational map* $\varphi : S \dashrightarrow |\mathcal{O}_S(E)| \times |\mathcal{O}_S(C)|$ *and* $\overline{\varphi(S)}$ *is*

(i) *a bidegree* $(2,3)$ *hypersurface of* $\mathbb{P}^1 \times \mathbb{P}^2$ *if and only if* $(S, C, E) \notin \mathbf{H}_u \cup \mathbf{H}_h$ *when* $n = 1$*.*

(ii) *the complete intersection of two hypersurfaces of bidegree* $(1,3), (1,1)$ *in* $\mathbb{P}^1 \times \mathbb{P}^3$ *if* $(S, C, E) \notin \mathbf{H}_u \cup \mathbf{H}_h \cup \mathbf{H}_t$ *when* $n = 2$*.*

(iii) *the intersection of three hypersurfaces of bidegree* $(0,3), (1,1), (1,1)$ *in* $\mathbb{P}^1 \times \mathbb{P}^4$ *if* $(S, C, E) \notin \mathbf{H}_h \cup \mathbf{H}_t$ *when* $n = 3$*.*

up to some higher codimension locus.

Proof. According to Saint-Donat's result, the map φ_L defined by a primitive quasi-polarization L is a closed embedding after contracting all exceptional (-2) curves if L is base point free and (S, L) is not hyperelliptic. In our case, we always assume that C is big and nef and E is effective as acknowledged in Subsection 2.2. Moreover, one can deduce that C is not unigonal iff $(S, C, E) \notin \mathbf{H}_u$ and when $n \geq 2$, (S, C) is not hyperelliptic iff $(S, C, E) \notin \mathbf{H}_h$ due to the Hodge Index Theorem. In the following, we denote by $\widetilde{C}, \widetilde{E}$ the movable part of C and E respectively. Hence one can identify $\overline{\varphi(S)}$ with $\tilde{\varphi}(S)$ where $\tilde{\varphi}$ is defined by $|\widetilde{E}| \times |\widetilde{C}|$.

For $n = 1$, we first show that the intersection matrix given by $(\widetilde{C}, \widetilde{E})$ still equals to $\left(\begin{smallmatrix} 2 & 3 \\ 3 & 0 \end{smallmatrix}\right)$ unless $(S, C, E) \in \mathbf{H}_u \cup \mathbf{H}_h$ or in some higher codimension locus. We claim that in this case E is necessarily nef hence base point free by [Huy16, Chapter 2, Proposition 3.10]. Assume on the contrary, there exists some irreducible (-2) curve Δ, such that the corresponding intersection matrix is

	C	E	Δ
C	2	3	x
E	3	0	y
Δ	x	y	-2

with $x \geq 0$ and $y < 0$. One can see that the only possibilities are $(x, y) = (0, -1)$, $(0, -2)$, $(1, -1)$ or $(2, -1)$ according to the Hodge index theorem. It is clearly that $(S, C, E) \in \mathbf{H}_h$ if $(x, y) = (1, -1)$ and $(S, C, E) \in \mathbf{H}_u$ if $(x, y) = (2, -1)$. If $(x, y) = (0, -2)$, then the Gram matrix given by basis $\{C, E, C + \Delta\}$ shows $(S, C, E) \in \mathbf{H}_h$. If $(x, y) = (0, -1)$, one see that $(E - \Delta)$ is nef unless S lies in some higher codimension locus. Thus $(E - \Delta)$ is base point free as $(E - \Delta)^2 = 0$ and we have $\widetilde{E} = E - \Delta$. Thus $(S, C, E) \notin \mathbf{H}_u \cup \mathbf{H}_h$, the Gram matrix given by $(\widetilde{C} = C, \widetilde{E})$ still equals to $\left(\begin{smallmatrix} 2 & 3 \\ 3 & 0 \end{smallmatrix}\right)$ Set $L = C + \widetilde{E}$. Moreover, since the sublattice spanned by $\{C, \widetilde{E}\}$ is obtained from T_1 by a reflection respect to Δ, we see L is not hyperelliptic or unigonal by assumption. One can see that $s \circ \tilde{\varphi} = \tilde{\varphi}_{|L|}$ is a closed embedding after contracting all exceptional (-2) curves, where s is the Segre embedding. Then $\tilde{\varphi}$ is birational to its image which is a $(2,3)$-hypersurface by the adjunction formula.

Conversely, for $(S, C, E) \in \mathbf{H}_u$, one find that $\Delta := E - E'$ is effective and makes E not nef. And it's easy to find that $\widetilde{E} = E'$ and $\widetilde{C} = 2E'$ with $h^0(S, \mathcal{O}_S(2E')) = 3$ by [SD74, Proposition

2.6 and 2.7.2]. Then the image of the projection $\overline{\varphi(S)} \to |\widetilde{C}|$ is a rational normal curve. Hence $\overline{\varphi(S)}$ can not be a $(2,3)$-hypersurface. For $(S, C, E) \in \mathbf{H}_h$, like the above case we have $\widetilde{E} = E'$ and $\overline{\varphi(S)}$ is equal to the image induced by $|C| \times |E'|$ given by intersection matrix $\left(\begin{smallmatrix} 2 & 2 \\ 2 & 0 \end{smallmatrix}\right)$. Notice that the projection map $\overline{\varphi(S)} \to |C| \cong \mathbb{P}^2$ restricted to $\varphi(\widetilde{E})$ has degree 2. So the projection $\overline{\varphi(S)} \to \mathbb{P}^2$ is generically injective, hence φ is a generically 2 to 1 map and $\overline{\varphi(S)}$ is not a $(2,3)$-hypersurface.

For $n = 2$ and 3, one can show that E is nef unless $(S, C, E) \in \mathbf{H}_u \cup \mathbf{H}_h \cup \mathbf{H}_t$ using the similar analysis as $n = 1$ case. Note that $H_u = \emptyset$ for $n = 3$. Hence when S does not lies in the union of NL divisors, we have E is base point free and φ is a morphism. Note that $\varphi_{|C|}$ is a closed embedding after contracting all exceptional (-2) curves. Then $\varphi = \varphi_{|C| \times |E|}$ is also a closed embedding.

Next, we analyse the projective model $\varphi(S)$ for $n = 2$ and 3. For $n = 2$, note that the composition with the Segre embedding $s \circ \varphi \colon S \to \mathbb{P}^7$ is induced by the line bundle $\mathcal{O}_S(L)$ and the dimension of linear system $|C + E|$ is 6. Then $s \circ \overline{\varphi(S)}$ is contained in a hyperplane and so $\overline{\varphi(S)}$ is contained in a bidegree $(1,1)$ hypersurface $X_{1,1}$ of $\mathbb{P}^1 \times \mathbb{P}^3$. Using the Lefschetz hyperplane theorem and adjunction formula, one can see that the divisor class of S in $X_{1,1}$ is $(1,3)$. Note that

$$H^1(I_{X_{1,1}}(1,3)) = H^1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(0,2)) = 0,$$

where $I_{X_{1,1}}$ is the ideal sheaf of $X_{1,1}$. We get a surjection $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1,3)) \twoheadrightarrow H^0(\mathcal{O}_{X_{1,1}}(1,3))$. Thus $\overline{\varphi(S)}$ is the complete intersection of two hypersurfaces of bidegree $(1,3), (1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^3$.

For $n = 3$, similarly, $s \circ \varphi \colon S \to \mathbb{P}^1 \times \mathbb{P}^4 \xrightarrow{s} \mathbb{P}^9$ is induced by the line bundle $\mathcal{O}_S(L)$ where s is the Segre embedding and the dimension of linear system $|C + E|$ is 7, one can see that $s \circ \overline{\varphi(S)}$ is contained in the intersection of two hyperplanes. So $\overline{\varphi(S)}$ is contained in the intersection of two bidegree $(1,1)$ hypersurfaces X_1 and X_2 of $\mathbb{P}^1 \times \mathbb{P}^4$. We denote $X_1 \cap X_2$ by Y. One can see that if a $(1,1)$ hypersurface is singular, the only possibility is that it is reducible. Thus X_1 and X_2 are smooth. Otherwise, S is contained in a $\{\mathrm{pt}\} \times \mathbb{P}^4$ or a $\mathbb{P}^1 \times \mathbb{P}^3$, which is impossible. By checking the Jacobian of Y, one can conclude that Y is also smooth. Note that the divisor class of S in Y is $(0,3)$ by adjunction. We deduce that the divisor class $(0,3)$ of Y comes from a bidegree $(0,3)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^4$ since $H^1(\mathcal{O}_{X_1}(-Y)(0,3))$ and $H^1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(-X_1)(0,3))$ vanish. Therefore, $\varphi(S)$ is the complete intersection of three hypersurfaces of bidegree $(0,3), (1,1), (1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^4$. ♣

3. Stability of bidegree $(2,3)$ hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^2$

3.1. **Numerical criteria.** Using the Hilbert-Mumford's numerical criteria [MFK94, Thm. 2.1], we have: A bidegree $(2,3)$ hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^2$ is stable (resp. semistable) if and only if $\mu(f, \lambda) > 0$ (resp. ≥ 0) for all one parameter subgroups λ of $\mathrm{SL}_2 \times \mathrm{SL}_3$, where $\mu(f, \lambda)$ is the numerical weight introduced by Hilbert and Mumford.

As is customary, a one parameter subgroup (1-PS) of $\mathrm{SL}_2 \times \mathrm{SL}_3$ can be diagonalized as

$$\lambda \colon t \in \mathbb{C}^* \to \mathrm{diag}(t^a, t^{-a}, t^b, t^c, t^{-b-c})$$

for some $a, b, c \in \mathbb{Z}$. We call such λ a normalized 1-PS of $\mathrm{SL}_2 \times \mathrm{SL}_3$ if $a \geq 0$ and $b \geq c \geq -b - c$.

Let λ be a normalized 1-PS. Then the weight of a monomial $x_0^u x_1^{2-u} y_0^v y_1^w y_2^{3-v-w}$ with respect to λ is

$$au - a(2-u) + bv + cw + (-b-c)(3-v-w).$$

If we denote by $M^{\ominus}(\lambda)$ (resp. $M^{-}(\lambda)$) the set of monomials of bidegree $(2,3)$ which have non-positive (resp. negative) weight with respect to λ, one can easily compute the maximal subsets $M^{\ominus}(\lambda)$ (resp. $M^{-}(\lambda)$) listed in the next subsection.

Cases	1-PS	Maximal monomials	Invariant
N1	$\lambda'_1 = (3,-3,2,2,-4)$	$x_1^2y_0^3, x_0x_1y_0^2y_2, x_0^2y_0y_2^2$	(α)
N2	$\lambda'_2 = (3,-3,2,0,-2)$	$x_1^2y_0^3, x_0x_1y_0y_1y_2, x_0^2y_2^3$	(β)
N3	$\lambda'_3 = (3,-3,4,-2,-2)$	$x_1^2y_0^2y_1, x_0x_1y_0y_1^2, x_0^2y_1^3$	(α)
N4	$\lambda'_4 = (0,0,2,-1,-1)$	$x_0^2y_0y_1^2$	(γ)
N5	$\lambda'_5 = (1,-1,2,0,-2)$	$x_1^2y_0^2y_2, x_0x_1y_0y_1y_2, x_0^2y_0y_2^2, x_0^2y_1^2y_2$	(η)
N6	$\lambda'_6 = (0,0,1,1,-2)$	$x_0^2y_0^2y_2$	(γ)
N7	$\lambda'_7 = (1,-1,0,0,0)$	$x_0x_1y_0^3$	(δ)

Table 1. Not properly stable

Cases	1-PS	Description (roughly)	inclusion
U1	$\lambda_1 = (5,-5,3,-1,-2)$	reducible	S1
U2	$\lambda_2 = (4,-4,2,1,-3)$	singular along a vertical line	S2, S3
U3	$\lambda_3 = (4,-4,4,-1,-3)$	corank 3 isolated	S2, S3 $(\star)$
U4	$\lambda_4 = (3,-3,4,-1,-3)$	corank 3 isolated	S3
U5	$\lambda_5 = (1,-1,3,-1,-2)$	singular along a horizontal line	S3
U6	$\lambda_6 = (1,-1,5,-1,-4)$	singular along a horizontal line	S4
U7	$\lambda_7 = (2,-2,3,1,-4)$	singular along a vertical line	S5

Table 2. Unstable

3.2. Maximal subsets for not properly stable points.

Lemma 3.1. *For any normalized 1-PS λ, $M^{\ominus}(\lambda)$ is contained in one of $M^{\ominus}(\lambda'_i)$ in table 1. The surface S is not properly stable if its defining polynomial is one of the following:*

(N1) $f(x_0,x_1,y_0,y_1,y_2) = x_1^2c(y_0,y_1,y_2) + x_0x_1y_2q(y_0,y_1) + x_0^2y_2^2l(y_0,y_1,y_2)$.

(N2) $f(x_0,x_1,y_0,y_1,y_2) = x_1^2c_0(y_0,y_1,y_2) + x_0x_1[c_1(y_1,y_2) + y_0y_2l(y_1,y_2)] + \mu x_0^2y_2^3$.

(N3) $f(x_0,x_1,y_0,y_1,y_2) = x_1^2[c_1(y_1,y_2) + y_0q_1(y_1,y_2) + y_0^2l(y_1,y_2)]$
$+ x_0x_1[c_2(y_1,y_2) + y_0q_2(y_1,y_2)] + x_0^2c_0(y_1,y_2)$.

(N4) $f(x_0,x_1,y_0,y_1,y_2) = x_1^2[c_0(y_1,y_2) + y_0q_0(y_1,y_2)] + x_0x_1[c_1(y_1,y_2) + y_0q_1(y_1,y_2)]$
$+ x_0^2[c_2(y_1,y_2) + y_0q_2(y_1,y_2)]$.

(N5) $f(x_0,x_1,y_0,y_1,y_2) = x_1^2[c_0(y_1,y_2) + y_0q(y_1,y_2) + \mu y_0^2y_2] + x_0x_1[c_1(y_1,y_2) + y_0y_2l(y_1,y_2)]$
$+ x_0^2y_2(q(y_1,y_2) + \nu y_0y_2)$.

(N6) $f(x_0,x_1,y_0,y_1,y_2) = y_2 \cdot q(x_0,x_1,y_0,y_1,y_2)$.

(N7) $f(x_0,x_1,y_0,y_1,y_2) = x_1[x_0c_0(y_0,y_1,y_2) + x_1c_1(y_0,y_1,y_2)]$.

Using the destabilizing 1-PS, the invariant part of equation (N1)-(N7) are given as follows:

$$(\alpha): f = x_1^2 c(y_0, y_1) + x_0x_1 q(y_0, y_1)y_2 + x_0^2 l(y_0, y_1)y_2^2. \tag{3.1}$$

$$(\beta): f = ax_1^2 y_0^3 + x_0x_1(by_1^3 + cy_0y_1y_2) + dx_0^2 y_2^3. \tag{3.2}$$

$$(\gamma): f = x_1^2 q_1(y_0, y_1)y_2 + x_0x_1 q_2(y_0, y_1)y_2 + x_0^2 q_3(y_0, y_1)y_2. \tag{3.3}$$

$$(\eta): f = x_1^2(a_1y_1^2y_0 + b_1y_0^2y_2) + x_0x_1(a_2y_0y_1y_2 + b_2y_1^3) + x_0^2(a_3y_0y_2^2 + b_3y_1^2y_2). \tag{3.4}$$

$$(\delta): f = x_0x_1c(y_0, y_1, y_2). \tag{3.5}$$

Similarly, we can get maximal subsets for unstable points.

Lemma 3.2. *For any normalized 1-PS λ, $M^-(\lambda)$ is contained in one of $M^-(\lambda_i)$ in table 2. The surface S is unstable if its defining polynomial is one of the following:*

(U1) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2 c(y_0, y_1, y_2) + x_0x_1[c_0(y_1, y_2) + \mu y_0y_2^2]$.

(U2) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2 c_0(y_0, y_1, y_2) + x_0x_1(y_2^2 l(y_0, y_1, y_2) + \mu y_2y_1^2) + \nu x_0^2 y_2^3$.

(U3) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0 q(y_1, y_2) + y_0^2 l(y_1, y_2)] + x_0x_1[c_1(y_1, y_2) + \mu y_0y_2^2] + \nu x_0^2 y_2^3$.

(U4) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0 q(y_1, y_2) + \mu y_0^2 y_2] + x_0x_1[c_1(y_1, y_2) + \nu y_0y_2^2] + x_0^2 y_2^2 l(y_1, y_2)$.

(U5) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0 q_1(y_1, y_2))] + x_0x_1[c_1(y_1, y_2) + \mu y_0y_2^2] + x_0^2 c_2(y_1, y_2)$.

(U6) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0y_2 l(y_1, y_2)] + x_0x_1[c_1(y_1, y_2) + \mu y_0y_2^2] + x_0^2[c_2(y_1, y_2) + \nu y_0y_2^2]$.

(U7) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c(y_1, y_2) + y_0y_2 l_0(y_1, y_2) + \mu y_0^2 y_2] + x_0x_1(y_2^2 l_1(y_0, y_1, y_2) + \nu y_2y_1^2) + x_0^2 y_2^2 l_2(y_0, y_1, y_2)$.

4. Geometric interpretation of stability

Let us first give some notations and conventions.

4.1. **Additional Conventions.** Given a bidegree $(2,3)$ hypersurface $S \subseteq \mathbb{P}^1 \times \mathbb{P}^2$, it admits an elliptic fibration

$$\pi : S \to \mathbb{P}^1$$

via the first projection and all the fibers we consider are respect to π. On the other hand, the second projection

$$\pi' : S \to \mathbb{P}^2$$

is a double cover branching along a sextic curve denoted by $\mathbf{B}(S)$. In particular, we call a curve $C \subseteq S$

- *a vertical line*, if C has the form $\{\mathrm{pt}\} \times \mathbb{P}^1$.
- *a horizontal line* if C has the form $\mathbb{P}^1 \times \{\mathrm{pt}\}$.
- *a section of degree d* if $\pi(C) = \mathbb{P}^1$ and $\pi^{-1}(p) \cdot C = d$ for any $p \in \mathbb{P}^1$.

In the rest of this paper, We use the terminology of the corank of the hypersurface singularities as in [AGZV12] and [Laz09].

Definition 4.1. Let $0 \in \mathbb{C}^n$ be a hypersurface singularity given by an equation $f(z_1, \ldots, z_n) = 0$. The corank of 0 is n minus the rank of the Hessian of $f(z_1, \ldots, z_n)$ at 0.

Moreover, let $f(u, v, w)$ be an analytic function in $\mathbb{C}[[u, v, w]]$ whose leading term defines an isolated singularity at the origin. We are concerned with the following analytic types of isolated hypersurface singularities:

- Simple singularities: $A_n (n \geq 1)$, $D_n (n \geq 4)$ and $E_r (r = 6, 7, 8)$.
- Simple elliptic singularities $\widetilde{E}_r (r = 6, 7, 8)$:
  - $\widetilde{E}_6$: $f = u^3 + v^3 + w^3 + uvw$.
  - $\widetilde{E}_7$: $f = u^2 + v^4 + w^4 + uvw$.
  - $\widetilde{E}_8$: $f = u^2 + v^3 + w^6 + uvw$.

4.2. **Geometry of not properly stable surface.**

Theorem 4.2. *A bidegree-$(2, 3)$ hypersurface S in $\mathbb{P}^1 \times \mathbb{P}^2$ is not properly stable if and only if one of the following conditions holds*

(N1) *S is singular along a vertical line;*
(N2) *S contains a singularity p of at least $\widetilde{E}_8$-type, and the fiber over $\pi(p)$ is a triple line.*
(N3) *S contains at least a singularity of corank 3;*
(N4) *S is singular along a horizontal line;*
(N5) *S contains a corank 2 singularity p of at least $\widetilde{E}_7$-type, such that the fiber over $\pi(p)$ contains a line L, and $\mathrm{mult}_{\pi'(p)}((\pi'(F)), L) \geq 2$ for any fiber F.*
(N6) *$S = (\mathbb{P}^1 \times \mathbb{P}^1) \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2, 2)|$;*
(N7) *$S = \mathbb{P}^2 \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 3)|$.*

Proof. According to Lemma 3.1, it suffices to find the geometric characterizations of type (N1)-(N7). We give detailed proof for types (N2) and (N5) here. The other cases are similar and relatively simple so we omit them.

If S is of type (N2), then the equation of S is given by

$$x_1^2 c_0(y_0, y_1, y_2) + x_0 x_1 [c_1(y_1, y_2) + y_0 y_2 l(y_1, y_2)] + \mu x_0^2 y_2^3 = 0.$$

One can assume that the coefficient of $x_1^2 y_0^3$ and $x_0^2 y_2^3$ is nonzero. Otherwise, the equation will degenerate to type (N3) and (N6). Set $p = (1, 0, 1, 0, 0)$, then the fiber over $(1, 0)$ is the triple line $3L$: $\{y_2^3 = 0\}$. Letting $x_0 = y_0 = 1$, the affine equation near p is

$$x_1^2 + y_2^3 + x_1^2 l(y_1, y_2) + x_1 y_2 l(y_1, y_2) + x_1^2 q(y_1, y_2) + x_1 c_1(y_1, y_2) = 0.$$

It is clear that the weight on variables x_1, y_2, y_1 is $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$. Thus p is at least $\widetilde{E}_8$-type.

Conversely, we take $p = (1, 0, 1, 0, 0)$ to be the isolated singularity of at least $\widetilde{E}_8$-type. Up to coordinate change, one can assume the fiber over $\pi(p)$ is the triple line $3L$: $\{y_2^3 = 0\}$, then the equation of S can be written as

$$x_1^2 c_0(y_0, y_1, y_2) + x_0 x_1 [c_1(y_1, y_2) + y_0 q_1(y_1, y_2)] + \mu x_0^2 y_2^3 = 0.$$

If the coefficient of $x_0 x_1 y_0 y_1^2$ is nonzero, one can see that the weight of p has weight no worse than $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, which contradicts the condition that p is at least $\widetilde{E}_8$-type.

If S is of type (N5), then the equation of S is given by

$$\begin{aligned} x_1^2[c_0(y_1, y_2) + y_0 q(y_1, y_2) + \mu y_0^2 y_2] + x_0 x_1 [c_1(y_1, y_2) + y_0 y_2 l(y_1, y_2)] \\ + x_0^2 y_2 (q(y_1, y_2) + \nu y_0 y_2) = 0. \end{aligned}$$

One can assume that the coefficients of $x_1^2 y_0^2 y_2$ and $x_0^2 y_0 y_2^2$ are nonzero. Otherwise the equation will degenerate to type (N3) and (N4). Set $p = (1, 0, 1, 0, 0)$, then the fiber over $(1, 0)$ contains the line $\{y_2 = 0\}$. And $\mathrm{mult}_{(1,0,0)}((\pi'(F)), y_2) \geq 2$ for any fiber F since there are no y_0^3, $y_0^2 y_1$ terms. One can check that p is at least $\widetilde{E}_7$-type.

Conversely, we take $p = (1, 0, 1, 0, 0)$ to be the isolated corank 2 singularity of at least $\widetilde{E}_7$-type. Up to coordinate change, one can assume that the fiber over $(1, 0)$ contains the line $\{y_2 = 0\}$, then the equation of S can be written as

$$x_1^2 c_0(y_0, y_1, y_2) + x_0x_1[c_1(y_1, y_2) + y_0q_1(y_1, y_2) + y_0^2y_2] + x_0^2y_2(q(y_1, y_2) + \nu y_0y_2) = 0.$$

One can deduce that the coefficients of $x_1^2y_0^3$ and $x_1^2y_0^2y_1$ are zero since $\mathrm{mult}_{\pi'(p)}((\pi'(F)), y_2) \geq 2$ for any fiber F. And the coefficient of $x_0x_1y_0^2y_2$ is also zero due to the corank 2 condition. From now on, the equation of S can be written as

$$\begin{aligned} x_1^2[c_0(y_1, y_2) + y_0q(y_1, y_2) + \mu y_0^2y_2] + x_0x_1[c_1(y_1, y_2) + y_0q_1(y_1, y_2)] \\ + x_0^2y_2(q(y_1, y_2) + \nu y_0y_2) = 0. \end{aligned}$$

Moreover, if the coefficient of $x_0x_1y_0y_1^2$ is nonzero, then the singularity of p has weight no worse than $(\frac{1}{2}, \frac{1}{4}, \frac{3}{8})$, which is impossible.

♣

4.3. **Geometry of unstable surface.**

Theorem 4.3. *A bidegree-$(2, 3)$ hypersurface $S \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ is unstable if and only if one of the following conditions holds*

(U1) *$S = \mathbb{P}^2 \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1, 3)|$, meeting along a cuspidal cubic curve.*

(U2) *S is singular along a line in the fiber whose projection under π' is a triple line $3L$. And $\mathbf{B}(S) = 2\ell \cup B'$ with $L \cap B'$ is a quartic point $\{o\}$.*

(U3) *S contains at least a corank 3 isolated singularity p and the fiber over $\pi(p)$ is a triple line $3L$. And the projective tangent cone at $\pi'(p)$ of branching locus $\mathbb{P}CT_{\pi'(p)}\mathbf{B}(S)$ contains $3L$.*

(U4) *S contains at least a corank 3 isolated singularity p and the fiber over $\pi(p)$ is the union of a double line $2L_1$ and a line L_2 meeting at $\pi'(p)$. And $\mathrm{mult}_{\pi'(p)}((\pi'(F)), L_1) \geq 2$ for any fiber F. The projective tangent cone at $\pi'(p)$ of the branching locus $\mathbb{P}CT_{\pi'(p)}\mathbf{B}(S)$ is at least $3L_1 \cup L'$.*

(U5) *S is singular alone a type horizontal line C with $\pi'(C)$ is a point $o \in \mathbb{P}^2$. S has a fiber which is the union of three lines intersecting at o. The projective tangent cone at o of branching locus $\mathbb{P}CT_o\mathbf{B}(S)$ is at least a quadruple line.*

(U6) *S is singular along a type horizontal line C with $\pi'(C)$ is a point $o \in \mathbb{P}^2$. S has a fiber F_0 such that the projective tangent cone of $\pi'(F_0)$ at o is a double line $\mathbb{P}CT_o(\pi'(F_0)) = 2\ell$. And $\mathrm{mult}_o((\pi'(F)), L) \geq 3$ for any fiber F. The projective cone $\mathbb{P}CT_o\mathbf{B}(S)$ is at least $3L \cup L'$.*

(U7) *S is singular along a type vertical line whose projection under π' is a line L . For any F with $L \not\subseteq \pi'(F)$, $\{o\} = L \cap \pi'(F)$ is at least a triple point. The branching locus $\mathbf{B}(S) = 2\ell \cup B'$ with $L \cap B'$ containing at least a triple point which is exactly o when such F exists.*

Proof. As in the proof of Theorem 4.2, we present explicit proof for three slightly complicated cases.

(1) If S is of type (U4), then the equation of S is given by

$$x_1^2[c_0(y_1, y_2) + y_0q(y_1, y_2) + \mu y_0^2y_2] + x_0x_1[c_1(y_1, y_2) + \nu y_0y_2^2] + x_0^2y_2^2l(y_1, y_2) = 0.$$

One observe that S contains a corank 3 isolated singularity $p\colon (1,0,1,0,0)$, and the fiber over $\pi(p)$ is the union of a double line $2L_1\colon \{y_2^2=0\}$ and a line $L_2\colon \{l(y_1,y_2)=0\}$ meeting at $\pi'(p)$. One can assume that the coefficient of $x_1^2y_0^2y_2$ is nonzero, and the coefficients of $x_0^2y_2^3$, $x_0^2y_1y_2^2$ are not simultaneously zero. Otherwise, the equation will degenerate to type (U5) and (U1). Then $\mathrm{mult}_{(1,0,0)}((\pi'(F)),y_2)\geq 2$ for any fiber F and

$$\mathbb{P}CT_{\pi'(p)}\mathbf{B}(S)=\nu^2y_2^4-\mu y_2^3l(y_1,y_2)=3L_1\cup L'.$$

Conversely, suppose that S contains at least a corank 3 isolated singularity p and the fiber containing p is a union of a double line $2L_1$ and a line L_2 meeting at p. One can assume that $p=(1,0,1,0,0)$ and L_1 is given by $\{y_2=0\}$. It's easy to show that there are no $x_ix_jy_0^2y_1$ terms in the equation of S since $\mathrm{mult}_{(1,0,0)}((\pi'(F)),y_2)\geq 2$ for any fiber F. Then the equation of S can be written as

$$x_1^2[c_0(y_1,y_2)+y_0q(y_1,y_2)+\mu y_0^2y_2]+x_0x_1[c_1(y_1,y_2)+y_0q_1(y_1,y_2)]+x_0^2y_2^2l_0(y_1,y_2)=0.$$

The $\mathbf{B}(S)$ is given by

$$(c_1(y_1,y_2)+y_0q_1(y_1,y_2))^2-y_2^2l_0(y_1,y_2)(c_0(y_1,y_2)+y_0q_0(y_1,y_2)+\mu y_0^2y_2)=0.$$

Thus one can deduce that the $q_1=\nu y_2^2$ since $\mathbb{P}CT_{(1,0,0)}\mathbf{B}(S)=3L_1\cup L'$.

(2) If S is of type (U5), then the equation of S is given by

$$x_1^2[c_0(y_1,y_2)+y_0q_1(y_1,y_2)]+x_0x_1[c_1(y_1,y_2)+\mu y_0y_2^2]+x_0^2c_2(y_1,y_2)=0.$$

We see at once that S is singular along a horizontal line $C\colon \mathbb{P}^1\times(1,0,0)$ with $\pi'(C)$ is a point $o\colon (1,0,0)\in\mathbb{P}^2$. And the fiber over $(1,0)\in\mathbb{P}^1$ is given by $\{c_2(y_1,y_2)=0\}$, which is the union of three lines intersecting at o. In addition, $\mathbb{P}CT_o\mathbf{B}(S)$ is a quadruple line $\{y_2^4=0\}$ if μ is nonzero. Otherwise, the degree of $\mathbb{P}CT_o\mathbf{B}(S)$ is at least 5.

Conversely, up to a coordinate change, one can assume that S is singular along a horizontal line $C\colon \mathbb{P}^1\times(1,0,0)$ with $\pi'(C)$ is a point $o\colon (1,0,0)\in\mathbb{P}^2$. In addition, the fiber over $(1,0)\in\mathbb{P}^1$ is the union of three lines intersecting at o. So the equation of S can be written as

$$x_1^2[c_0(y_1,y_2)+y_0q_1(y_1,y_2)]+x_0x_1[c_1(y_1,y_2)+y_0q_2(y_1,y_2)]+x_0^2c_2(y_1,y_2)=0.$$

If $q_2=0$, then obviously S is of type (U5). If $q_2\neq 0$, then $\mathbb{P}CT_o\mathbf{B}(S)$ is given by $q_2(y_1,y_2)^2$. Thus one can assume $q_2(y_1,y_2)=\mu y_2^2$ up to a coordinate change of y_1 and y_2 since $\mathbb{P}CT_o\mathbf{B}(S)$ is a quadruple line.

(3) If S is of type (U7), then the equation of S is given by

$$\begin{aligned}x_1^2[c(y_1,y_2)+y_0y_2l(y_1,y_2)+\mu y_0^2y_2]+x_0x_1(y_2^2l_0(y_0,y_1,y_2)+\nu y_1^2y_2)&\\+x_0^2y_2^2l_1(y_0,y_1,y_2)=0&\end{aligned}$$

with $\mathbf{B}(S)$ given by

$$y_2^2(y_2l_0(y_0,y_1,y_2)+\nu y_1^2)^2-y_2^2l_1(y_0,y_1,y_2)(c(y_1,y_2)+y_0y_2l(y_1,y_2)+\mu y_0^2y_2)=0.$$

One observes that S is singular along a vertical line $\{x_1=y_2=0\}$ with $L\colon \{y_2=0\}$ the projection to $\mathbb{P}^2$. Also one can notice that $\mathbf{B}(S)=2L\cup B'$. If the coefficient of $x_1^2y_1^3$ is zero, then the projection of every fiber under π' contains L. One can see that $L\cap B'$ is a quadruple point $(1,0,0)$. If the coefficient of $x_1^2y_1^3$ is nonzero, it is clear that $L\cap\pi'(F)$ is a triple point $o\colon (1,0,0)$ for any fiber F. Finally, $L\cap B'$ is given by $\{\nu^2y_1^4-y_1^3l_1(y_0,y_1)=y_2=0\}$, which contains a triple point $(1,0,0)$.

Conversely, up to a coordinate change one may assume that S is singular along a horizontal line $\{x_1 = y_2 = 0\}$ with $L\colon \{y_2 = 0\}$ the projection to $\mathbb{P}^2$. One can then write the equation of S as

$$x_1^2[c(y_0, y_1) + y_2 q_0(y_0, y_1) + y_2^2 l_0(y_0, y_1) + y_2^3] + x_0 x_1 (y_2^2 l_1(y_0, y_1, y_2) + y_2 q(y_0, y_1)) \\ + x_0^2 y_2^2 l_2(y_0, y_1, y_2) = 0.$$

If the projection of every fiber under π' contains L, then there are no $x_1^2 c(y_0, y_1)$ terms. One can compute $\mathbf{B}(S) = 2L \cup B'$ with $B' \cap L = \{q(y_0, y_1)^2 = 0\} \subset \mathbb{P}^1$. By our assumption, we see $q(y_0, y_1) = l(y_0, y_1)^2$ and we may set $l = y_1$ by making coordinate change. This completes the proof.

From here we assume that there is a fiber F whose projection under π' does not contain L. One can assume that F is over $(0, 1)$. Then $L \cap \pi'(F)$ is given by $\{c(y_0, y_1) = 0\}$. By our assumption, we see $c(y_0, y_1) = y_1^3$ and the triple point o is $(1, 0, 0)$ after applying coordinate change. Still $\mathbf{B}(S) = 2L \cup B'$ and $L \cap B'$ is given by $\{q(y_0, y_1)^2 - y_1^3 l_2(y_0, y_1, 0) = 0\}$. It follows that $q = y_1^2$ since $L \cap B'$ contains the triple point o.

♣

5. Minimal orbits

In this section, we study the strictly semistable locus of the GIT compactification . According to Subsection 3.2, it suffices to discuss the points of type $(\alpha) - (\delta)$. The results are obtained by using the following criterion of Luna.

Lemma 5.1 (Luna's criterion [Lun75])**.** *Let G be a reductive group acting on an affine variety X. If H is a reductive subgroup of G and $x \in X$ is stabilized by H, then the orbit $G \cdot x$ is closed if and only if $C_G(H) \cdot x$ is closed.*

As an immediate application, one can calculate the dimensions of the boundary strata (see subsection 6.3).

Lemma 5.2. *A generic cubic fourfold of type $(\alpha) - (\delta)$ gives a closed orbit. Therefore, each of the types $(\alpha) - (\delta)$ gives an irreducible boundary component for the GIT compactification. The dimensions of these boundary strata are 4 for (α) and (γ), 3 for (η), and 1 for (β) and (δ).*

Before starting the detailed analysis of the strata $(\alpha) - (\delta)$, we describe the relations of their common degeneration. One observes that the types (α), (γ) and (η) have a common specialization

$$(\tau)\colon a_1 x_1^2 y_0^2 y_1 + a_2 x_0 x_1 y_0 y_1 y_2 + a_3 x_0^2 y_1 y_2^2 = 0.$$

The stabilizer of (τ) contains a 1-PS $(4, -4, 3, 2, -5)$. Its equation further degenerates (for $a_1 a_3 = 0$) to

$$(\tau')\colon x_0 x_1 y_0 y_1 y_2 = 0.$$

In addition, (τ') is also a specialization of the cases (β), (γ) and (δ). The resulting incidence diagram is given in the figure 1.

Lemma 5.3. *A generic hypersurface of type (τ) and (τ') is semi-stable with closed orbit.*

Proof. This follows from Luna's criterion cited above. The stabilizer of (τ) and (τ') both contain a 1-PS H of distinct weights. Thus it suffices to check the semi-stability with respect to the standard maximal torus $T = C_G(H)$ in G. The proposition follows easily. For example, the fact

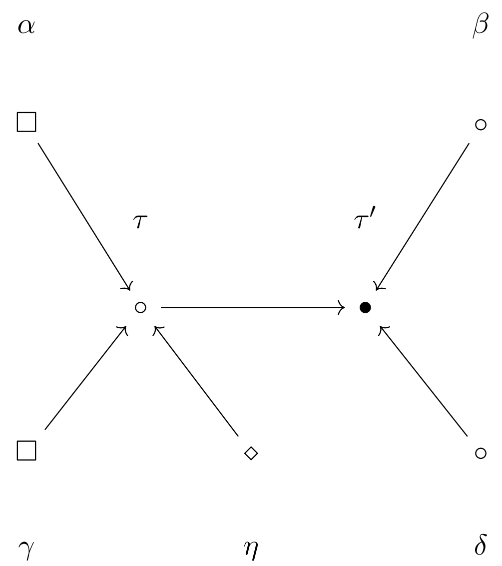

FIGURE 1. Incidence of the boundary components

that (τ) is semi-stable follows from the simple observation that $(-2a+2b+c)+(2a+c+2(-b-c))=0$ implies either $-2a+2b+c \geq 0$ or $2a+c+2(-b-c) \geq 0$, where $(a,-a,b,c,-(b+c))$ are the weights of a 1-PS of T. ♣

We now do the case-by-case analysis of the minimal orbits of type $(\alpha)-(\delta)$. The common feature of all these cases is that the analysis reduces to some lower dimensional GIT problem. Generally speaking, as the dimension of stratum increases, the analysis gets more involved. We first consider several cases with higher dimension.

Proposition 5.4. *Let S be a hypersurface of type (α). Then S is singular along a vertical line L in fiber F . And S has a corank 3 singularity p such that $p \notin F$ and $\pi'(p) \notin \pi'(L)$. Moreover, we have*

(1) S is unstable if one of the following conditions hold
  (i) *$S=\mathbb{P}^2 \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,3)|$, where $\mathbb{P}^2=\pi^{-1}(\pi(F))$. And $\mathbb{P}^2 \cap S' = l_1^2\pi'(L)$, where l_1 is a line in $\mathbb{P}^2$.*
  (ii) *S has an another triple point o such that $o \in L$. And S is singular along a horizontal line containing o. In addition, the branch locus $\mathbf{B}(S)$ contains a triple line.*
  (iii) *The fiber over $\pi(p)$ is a triple line $3L_1$ and the branch locus $\mathbf{B}(S)$ also contains $3L_1$.*

(2) The orbit of S is not closed if one of the following conditions hold
  (i) *S is singular along a vertical line in the fiber over $\pi(p)$. It degenerates to type (τ).*
  (ii) *$S=(\mathbb{P}^1\times\mathbb{P}^1)\cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,2)|$. It degenerates to type (τ).*
  (iii) *$S=\mathbb{P}^2 \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,3)|$. It degenerates to type (τ').*

Otherwise, S is semistable with closed orbit.

Proof. By inspecting the equation of (α), it is easy to see that S is singular along a vertical line $\{x_1=y_2=0\}$, whose projection under π' is given by $L\colon \{y_2=0\}$. And S has a corank 3 singularity at $(0,1,0,0,1)$. The converse is trivial.

Since the stabilizer of type (α) contains a 1-PS:

$$H_1=\{\mathrm{diag}(t^3,t^{-3},t^2,t^2,t^{-4}) \mid t \in \mathbb{C}^*\}$$

and the center $C_G(H_1) \cong \mathbb{C}^* \times \mathbb{C}^* \times SL_2(\mathbb{C}) \times \mathbb{C}^*$, we can reduce our problem to a simpler GIT problem $V^{H_1} /\!\!/ C_G(H_1)$ by Luna's criterion where $V^{H_1} = \langle x_0^u x_1^{2-u} y_0^v y_1^{3-u-v} y_2^u \rangle$. Then following the Hilbert-Mumford criterion, by diagonalizing 1-PS λ into the form

$$\lambda(t) = \mathrm{diag}(t^a, t^{-a}, t^b, t^c, t^{-b-c}) \quad (b \geq c)$$

we can deduce that S is unstable if and only if its defining equation is one of the following forms up to a coordinate change of y_0 and y_1

- $x_1^2 c(y_0, y_1) + \mu x_0 x_1 y_1^2 y_2 = 0$.
- $x_1^2 y_1^2 l(y_0, y_1) + \mu x_0 x_1 y_1^2 y_2 + \nu x_0^2 y_1 y_2^2 = 0$.
- $\mu x_1^2 y_1^3 + \nu x_0 x_1 y_1^2 y_2 + x_0^2 y_2^2 l(y_0, y_1) = 0$.

And the orbit of S is not closed if its equation has one of the following forms

- $x_1(x_1 c(y_0, y_1) + x_0 q(y_0, y_1) y_2) = 0$.
- $x_0(x_1 q(y_0, y_1) y_2 + x_0 l(y_0, y_1) y_2^2) = 0$.
- $y_1(x_1^2 q_1(y_0, y_1) + x_0 x_1 l_1(y_0, y_1) y_2 + \mu x_0^2 y_2^2) = 0$.
- $x_1^2 y_1^2 l_0(y_0, y_1) + x_0 x_1 y_1 y_2 l_1(y_0, y_1) + x_0^2 l_2(y_0, y_1) y_2^2 = 0$.

Then the first two cases degenerate to type (τ'), the last two cases degenerates to type (τ).

The proof of the geometric description is similar to Theorem 4.2 and Theorem 4.3. To give an example, we show the third unstable case. If the equation of S is given by $\mu x_1^2 y_1^3 + \nu x_0 x_1 y_1^2 y_2 + x_0^2 y_2^2 l(y_0, y_1) = 0$. Then it is clear that the fiber over $(0, 1)$ is a triple line $\{y_1^3 = 0\}$ and the branch locus also contains $\{y_1^3 = 0\}$.

Conversely, if S is of type (α) whose fiber over $\pi(p)$ is a triple line $3L_1$. Then one can assume that $L_1 \colon \{y_1 = 0\}$ up to a coordinate change. Its equation has the form

$$x_1^2 y_1^3 + x_0 x_1 q(y_0, y_1) y_2 + x_0^2 l(y_0, y_1) y_2^2 = 0$$

with $\mathbf{B}(S)$ given by $y_2^2(q(y_0, y_1)^2 - y_1^3 l(y_0, y_1)) = 0$. Thus one can deduce that $q = y_1^2$ by our assumption. In conclusion, the equation of S can be written as

$$x_1^2 y_1^3 + x_0 x_1 y_1^2 y_2 + x_0^2 y_2^2 l(y_0, y_1) = 0.$$

♣

Proposition 5.5. *Let S be a hypersurface of type (γ), then $S = (\mathbb{P}^1 \times \mathbb{P}^1) \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2, 2)|$ and S is singular along a horizontal line C such that the intersection of $\mathbb{P}^1 \times \mathbb{P}^1$ and C is empty. Moreover, we have*

(i) *S is unstable if the intersection $S' \cap (\mathbb{P}^1 \times \mathbb{P}^1)$ is of type $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 0) \cup \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 2)$ or $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0, 1) \cup \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2, 1)$, where the two components meet at a double point.*

(ii) *The orbit of S is not closed if the intersection $S' \cap (\mathbb{P}^1 \times \mathbb{P}^1)$ is a singular curve. It degenerates to type (τ).*

Otherwise, S is semistable with closed orbit.

Proof. The stabilizer of type (γ) contains a 1-PS:

$$H_2 = \{\mathrm{diag}(1, 1, t^1, t^1, t^{-2}) \mid t \in \mathbb{C}^*\}.$$

The center $C_G(H_2) \cong SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times \mathbb{C}^*$. By Luna's criterion, we can reduce our problem to a simpler GIT analysis $V^{H_2} /\!\!/ C_G(H_2)$ where $V^{H_2} = \langle x_0^u x_1^{2-u} y_0^v y_1^{2-v} y_2 \rangle$. Any 1-PS λ can be diagonalized in the form

$$\lambda(t) = \mathrm{diag}(t^a, t^{-a}, t^b, t^c, t^{-b-c}).$$

where $a \geq 0$ and $b \geq c$. Then following the Hilbert-Mumford criterion, we deduce that S is unstable if and only if its defining equation is one of the following forms after we make a coordinate change.

- $y_1y_2(x_1^2l(y_0,y_1)+\mu x_0x_1y_1+\nu x_0^2y_1)=0$.
- $x_1y_2(x_1q(y_0,y_1)+\mu x_0y_1^2)=0$.

And the orbit of S is not closed if its equation has the form

- $y_2(x_1^2q(y_0,y_1)+x_0x_1y_1l(y_0,y_1)+\mu x_0^2y_1^2)=0$.
- $y_1y_2(x_1^2l_0(y_0,y_1)+x_0x_1l_1(y_0,y_1)+x_0^2l_2(y_0,y_1))=0$.
- $x_1y_2(x_1q_1(y_0,y_1)+x_0q_0(y_0,y_1))=0$.

Then first case degenerates to type (τ), and the other two cases are unstable.

The remaining part follows by an analysis of the geometric description of the possible degenerations similar to before. ♣

The remaining cases are quite similar, we omit the details.

Proposition 5.6. *Let S be a hypersurface of type (η) given by the equation (3.4). Then S has two isolated $\widetilde{E}_7$-type singularities p and q such that $\pi(p)\neq\pi(q)$ and $\pi'(p)\neq\pi'(q)$. The fiber over $\pi(p)$ and $\pi(q)$ contain a line L_1 and L_2 respectively with $L_1\neq L_2$. And $\mathrm{mult}_{\pi'(p)}((\pi'(F)),L_1)\geq 2$, $\mathrm{mult}_{\pi'(q)}((\pi'(F)),L_2)\geq 2$ for any fiber F. Moreover, we have*

(1) S is unstable if one of the following conditions holds
  - (i) *$S=\mathbb{P}^2\cup S'$ for some $S'\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,3)|$, meeting along a triple line.*
  - (ii) *The fiber over p and q contain double line $2L_1$ and $2L_2$ respectively, with $\mathbf{B}(S)=3L_1\cup 3L_2$.*
  - (iii) *The fiber over p and q are l^2L_1 and l^2L_2 respectively, where l is a line. And the branch locus $\mathbf{B}(S)$ contains $4l$.*

(2) The orbit of S is not closed if the fiber over p or q contains a double line $2l$ and the branch locus also contains the double line $2l$. It degenerates to type (τ).

Otherwise, it is semistable with closed orbit.

Proposition 5.7. *Let S be a hypersurface of type (β), then S has two isolated $\widetilde{E}_8$-type singularities p and q such that $\pi(p)\neq\pi(q)$ and $\pi'(p)\neq\pi'(q)$. The fiber over p and q are $3L_1$ and $3L_2$ respectively with $L_1\neq L_2$. Moreover, we have*

- (i) *S is unstable if the branch locus contains a triple line.*
- (ii) *The orbit of S is not closed if the branch locus contains a double line. It degenerates to type (τ').*

Otherwise, S is semistable with closed orbit.

Proposition 5.8. *Let S be a hypersurface of type (δ), then $S=\mathbb{P}^2\cup\mathbb{P}^2\cup(\mathbb{P}^1\times C)$ for some cubic curves C in $\mathbb{P}^2$. Moreover, we have*

- (i) *S is unstable if if C has cusps or triple points, or is reducible with two components tangent at a point.*
- (ii) *The orbit of S is not closed if C is the union of a conic and a transversal line.*

Otherwise, it's semistable with closed orbit.

Proof. One can deduce this from the well-known GIT analysis of plane cubic curves, for example see [Mum77, Section 1.11].

♣

6. Boundary of $\overline{\mathcal{M}}_2^{\mathrm{GIT}}$

6.1. **Stable surface with isolated singularity.** The following is a direct consequence of Theorem 4.2.

Proposition 6.1. *Let $S \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,3)|$ with only isolated singularity. Then S with non ADE singularities is stable if and only if one of the following holds*

(ζ) *S has a $\widetilde{E}_7$-type singular point p such that there exists a fiber F with $\pi'(p) \notin \pi'(F)$.*
(η) *S has a $\widetilde{E}_8$-type singular point p such that the fiber containing p is not a triple line.*

Proof. Suppose S has only at worst isolated singularities of type (ζ) or (η). By Theorem 4.2, we immediately know that S is stable. Conversely, suppose S is stable and it has a non-ADE corank 2 isolated singularity at p: $(1,0,1,0,0)$, then p is an isolated $\widetilde{E}_7$ or $\widetilde{E}_8$ singularity. Up to a change of coordinates preserving p, the local defining equation near p has three possibilities :

- $\mu x_1^2 + 3$ jets $+4$ jets $+5$ jets,
- $\mu y_1^2 + 3$ jets $+4$ jets $+5$ jets,
- $\mu(x_1+y_1)^2 + 3$ jets $+4$ jets $+5$ jets.

Direct computation shows the only possible case is the third one.

If p is $\widetilde{E}_7$ type, take the change of coordinate: $x_1 + y_1 \mapsto z$. Then there are no terms $y_1^k y_2^{3-k}$ in the third jets. The equation of S is of the form

$$\begin{aligned} f = {} & x_1^2(c_0(y_1,y_2) + y_0q_0(y_1,y_2) + y_0^2 l_0(y_1,y_2)) + x_0x_1(c_1(y_1,y_2) + y_0q_1(y_1,y_2)) \\ & + x_0^2(y_1^2 l_1(y_1,y_2) + y_1y_2^2) + \mu(x_1^2y_0^3 + 2x_0x_1y_0^2y_1 + x_0^2y_0y_1^2), \end{aligned} \tag{6.1}$$

satisfying $a_{210} - a_{120} + a_{030} = 0$, $a_{201} - a_{111} + a_{021} = 0$ and $a_{102} = a_{012}$. Where a_{ijk} represents the coefficient of $x_0^{2-i}x_1^i y_0^{3-j-k} y_1^j y_2^k$. Conversely, for the general equations of the form (6.1), take the change of coordinate: $x_1 + y_1 \to z$. Then the weights on variables z, y_1 and y_2 are $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Thus p is $\widetilde{E}_7$ type. And one can see that $\pi'(F)$ does not contain $\pi'(p) = (1,0,0)$, where F is the fiber over $(0,1)$.

If p is $\widetilde{E}_8$ type, take the change of coordinate: $x_1 + y_1 \to z$. Suppose that the weight of y_1 is less than $\frac{1}{6}$, then there are no terms y_1^3, $y_2^2y_1$, $y_2y_1^2$, zy_1^2 in the third jets. No term y_1^4, $y_1^3y_2$, and y_1^5 in the fourth and fifth jets. Thus the equation of S is of the form

$$\begin{aligned} f = {} & x_1^2(y_2^3 + y_2^2y_1 + y_2y_1^2 + y_0q_0(y_1,y_2) + y_0^2l_0(y_1,y_2)) \\ & + x_0x_1(c_0(y_1,y_2) + y_0q_1(y_1,y_2)) + x_0^2c_1(y_1,y_2) \\ & + \mu(x_1^2y_0^3 + 2x_0x_1y_0^2y_1 + x_0^2y_0y_1^2), \end{aligned} \tag{6.2}$$

satisfying $a_{201} - a_{111} + a_{021} = 0$, $a_{220} = a_{130}$, $a_{211} = a_{121}$, $a_{120} = 2a_{210} = 2a_{030}$ and $a_{102} = a_{012}$. Conversely, for the general equations of the form (6.2), take the change of coordinate: $x_1 + y_1 \to z$. Then the weights on variables z, y_1 and y_2 are $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$. And one can see that the fiber containing p is not a triple line.

Similarly, if the weight of y_2 is less than $\frac{1}{6}$, then equation of S is of the form

$$\begin{aligned} f = {} & x_1^2(c_0(y_1,y_2) + y_0q_0(y_1,y_2) + y_0^2l_0(y_1,y_2)) + x_0^2y_1^2l_1(y_1,y_2) \\ & + x_0x_1(y_1^3 + y_1^2y_2 + y_1y_2^2 + y_0y_1l_2(y_1,y_2)) \\ & + \mu(x_1^2y_0^3 + 2x_0x_1y_0^2y_1 + x_0^2y_0y_1^2). \end{aligned}$$

satisfying $a_{201} - a_{111} + a_{021} = 0$. One can easily see that S is singular along $\{x_1 = y_1 = 0\}$. Thus p is not an isolated singularity. ♣

6.2. Stable surface with non-isolated singularity.

Proposition 6.2. *Let S be a bidegree $(2,3)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^2$ with non-isolated singularities. Then one of the following situations holds:*

(i) $\operatorname{Sing}(S)$ *contains a vertical line and S is not properly stable.*
(ii) $\operatorname{Sing}(S)$ *contains a horizontal line and S is not properly stable.*
(iii) $\operatorname{Sing}(S)$ *contains an elliptic curve and S is not properly stable unless S is of type $(1,1)+(1,2)$ or $(0,2)+(2,1)$.*
(iv) $\operatorname{Sing}(S)$ *contains a section of degree 1 and S is stable.*
(v) $\operatorname{Sing}(S)$ *contains a section of degree 2 and S is stable.*

Proof. Let $C \subseteq \operatorname{Sing}(S)$ be an irreducible curve. If $S = S_1 \cup S_2$ is reducible, one can assume the bidegree of S_1 is $(1,0)$, $(0,1)$, $(0,2)$ or $(1,1)$. This gives the elliptic curve cases. For the first two cases, S is not properly stable by Theorem 4.2.

If S is irreducible, then C is either a section or contained in some fiber as the intersection of C with any fiber is at most 1 by genus calculation. Suppose that C is contained in some fiber. Then the only possible case is that S is singular along a vertical line by some simple computation. Next, we assume that C is a section. Let d be the degree of C as a section. Take a general hypersurface H of type $(0,1)$, then $H \cap S$ is an irreducible curve and it is singular along $H \cap C$. Since the arithmetic genus of $H \cap S$ is at most 2, then $H \cap C$ has at most 2 points. It follows that d is at most 2.

For $d = 0$, the surface is just (N4) type, which is not properly stable.

For $d = 1$, We claim that up to change of coordinate, the equation of S is of the form

$$(\theta)\colon f = y_2^2 c_{2,1}(x_0, x_1|y_0, y_1, y_2) + y_2(x_0y_1 - x_1y_0)q_{1,1}(x_0, x_1|y_0, y_1) \\ +(x_0y_1 - x_1y_0)^2 l(y_0, y_1),$$

where $c_{2,1}(x_0, x_1|y_0, y_1, y_2)$ represents the bidegree $(2,1)$ homogeneous polynomial of x_0, x_1 and y_0, y_1, y_2. And $q_{1,1}(x_0, x_1 \,|\, y_0, y_1)$ is similar. After changing coordinates, we may assume that S is singular along a curve C with parametric form $\{(x_0, x_1, x_0, x_1, 0)|(x_0, x_1) \in \mathbb{P}^1\}$. Its equation is given by $C\colon \{y_2 = x_0y_1 - x_1y_0 = 0\}$. Since S contains C, the equation of S is of the form $f = y_2 g + (x_0y_1 - x_1y_0)h$ where $g \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,2)|$ and $h \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$. Via computing $\frac{\partial f}{\partial y_2}$ and $\frac{\partial f}{\partial y_1}$ on C, one can deduce that both $\{g = 0\}$ and $\{x_0 h = 0\}$ contain C. This gives the equation of type (θ). Conversely, if the equation of S is of type (θ), then the Jacobi of S is zero along C.

For $d = 2$, we claim that the general equation is of the form

$$(\phi)\colon f = (x_0y_1 - x_1y_0)^2 l_0(y_0, y_1, y_2) + (x_0y_1 - x_1y_0)(x_0y_2 - x_1y_1)l_1(y_0, y_1, y_2) \\ +(x_0y_2 - x_1y_1)^2 l_2(y_0, y_1, y_2).$$

As above, one can assume that S is singular along the curve $C\colon \{x_0y_2 - x_1y_1 = x_0y_1 - x_1y_0 = 0\}$. Since C is contained in S, the equation of S is of the form $f = (x_0y_1 - x_1y_0)g + (x_0y_2 - x_1y_1)h$, where g, $h \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,2)|$. One can deduce that both $\{x_1 g = 0\}$ and $\{x_0 h = 0\}$ contain C because $\frac{\partial f}{\partial y_2}$ and $\frac{\partial f}{\partial y_0}$ are zero on C. Furthermore, using the parametric equation of C, it turns out the only possibility is that both $\{g = 0\}$ and $\{h = 0\}$ contain C. Thus the equation of S is of type (ϕ). Conversely, if the equation of S is of type (ϕ), the Jacobi of S is zero along C. Finally, the assertion of stability follows directly from Theorem 4.2. ♣

6.3. **Proof of Theorem 1.1.** Note that the only normal surface singularity that admits crepant resolution is of ADE type. So $\mathcal{M}_2$ consists of $(2,3)$-hypersurfaces with only simple singularities and by Theorem 4.2, we know that $\mathcal{M}_2$ is lying in the stable locus of $\overline{\mathcal{M}_2}^{\mathrm{GIT}}$. The description of the boundary components simply follows from the combination of Proposition 5.4-5.8, 6.1 and 6.2. The dimension of the strictly semistable boundary $(\alpha)-(\delta)$ can be computed via Luna's criterion. For example,

$$\dim(\alpha) = \dim \mathbb{P}\left(V^{H_1}\right) /\!\!/ C_G(H_1) = 4,$$

where $V^{H_1} = \langle x_0^u x_1^{2-u} y_0^v y_1^{3-u-v} y_2^u \rangle$ and $C_G(H_1) \cong \mathbb{C}^* \times \mathbb{C}^* \times SL_2(\mathbb{C}) \times \mathbb{C}^*$. For stable components, we can also compute the dimension as follows:

- For (ζ), it can be viewed as the quotient space $\mathbb{P}(V_1)/G_1$, where V_1 is the vector space spanned by monomials in the equation (ζ) and G_1 is the group fixing the singular point p: $(1,0,1,0,0)$ and the 2- jets $(x_1+y_1)^2$. As $\dim \mathbb{P}(V_1) = 16$ and $\dim G = 6$, we get $\dim(\zeta) = 10$. Similarly, (ξ) is the quotient space $\mathbb{P}(V_2)/G_2$ with $\dim \mathbb{P}(V_2) = 13$ and $\dim G_2 = 6$. It follows that $\dim(\xi) = 7$.
- Similar as above, (θ) is the quotient space $\mathbb{P}(V_3)/G_3$, where V_3 is the vector space spanned by monomials in the equation (θ) and G_3 is the group fixing the section of degree 1 with the parametric form $\{(x_0,x_1,x_0,x_1,0)|(x_0,x_1) \in \mathbb{P}^1\}$. One can calculate that $\dim \mathbb{P}(V_3) = 14$ and $\dim G_3 = 6$. It follows that $\dim(\theta) = 8$. Similarly, one can see that $\dim(\phi) = 5$.
- (r_1) is a union of a $(1,1)$ hypersurface and a $(1,3)$ hypersurface. The elements of (r_1) are parametrized by the product of two projective spaces $\mathbb{P}(V) \times \mathbb{P}(V')$, where $V = H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,1))$ and $V' = H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,3))$. Thus

  $$\dim(r_1) = \dim(V) + \dim(V') - \dim \mathrm{PGL}_2 \times \mathrm{PGL}_3 = 13.$$

  With the same method, one can get $\dim(r_2) = 2$.

The last statement follows from Proposition 2.5. ♣

7. Arithmetic compactifications

7.1. **Baily-Borel compactifications of $\mathcal{F}_{T_n}$.** Recall that the type II and III boundary components of the Baily-Borel compactifications of $\mathcal{F}_{T_n}$ correspond one-to-one to the classes of rank 2 and 1 isotropic sublattices of Σ_n modulo Γ_n.

Let $I_2(\Sigma_n) = \bigcup_e I_{2,e}(\Sigma_n)$ be the set-theoretic partition of rank 2 isotropic sublattices of Σ_n where

$$I_{2,e}(\Sigma_n) = \left\{ J \in I_2(\Sigma_n) \,\middle|\, |H_J| = e \right\}.$$

Here $H_J = (J^{\perp})^{\perp}_{\Sigma_n^*}$ with both the orthogonals taken inside Σ_n^*, is a isotropic subgroup of A_{Σ_n} with the induced finite quadratic form. Therefore $e^2 \,\big|\, (9 = |A_{\Sigma_n}|)$ and $e = 1, 3$.

Lemma 7.1. *For $n = 1,2,3$ and $e = 1,3$ fixed, all $J^{\perp}/J$ have the signature $(0,16)$ and the same discriminant form $(A_{n,e}, q_{n,e})$ for $J \in I_{2,e}(\Sigma_n)$, i.e., are in the same genus which we denote by $\mathcal{G}(n,e)$. Moreover, for $J \in I_{2,e}(\Sigma_n)$ and $e = 1$ or $e = 3$, there exists $\{v_1, \ldots, v_{20}\}$ a basis of the lattice Σ_n such that $J = \langle v_1, v_2 \rangle$, $J^{\perp} = \langle v_1, \ldots, v_{18} \rangle$ and the quadratic form in this basis is of the form*

$$Q = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ {}^tA & 0 & D \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ e & 0 \end{pmatrix}, D = \begin{pmatrix} 2t & 0 \\ 0 & 0 \end{pmatrix},$$

$0 \le t < e$ *is uniquely determined by* n *and* B *is any matrix representing the quadratic form on* $J^\perp/J$.

Proof. It's well-known that we have the isomorphism of finite quadratic form

$$A_{J^\perp/J} \cong H_J^\perp/H_J$$

(c.f. [Cam18, Proposition 6.5] and [Sca87, Proof of Lemma 5.1.3]). For $e=1$, we have $A_{J^\perp/J} \cong H_J^\perp/H_J \cong A_{\Sigma_n}$. For $e=3$, one can find that $H_J^\perp = H_J$ for $n=1,2,3$ and we have $A_{J^\perp/J} = \{0\}$. So $\mathcal{G}(n,e)$ is well-defined.

For the second statement, the proof of [Sca87, lemma 5.2.1] works and we sketch the proof here for the convenience of readers. Since $J \subset J^\perp$ are primitive sublattices of Σ_n, we can choose a basis of Σ_n in which $J = \langle v_1, v_2\rangle$, $J^\perp = \langle v_1, \dots, v_{20}\rangle$ The matrix $Q(v_i, v_j)$ will then have the form

$$Q = \begin{pmatrix} 0 & 0 & A_0 \\ 0 & B & C_0 \\ A_0^t & C_0^t & D_0 \end{pmatrix}$$

where B represents the bilinear form of $J^\perp/J$. Recalling that $J \in I_{2,e}(\Sigma_n)$, $H_J \cong Z/eZ$, a direct application of elementary divisors produces matrices U,$Z \in \mathrm{GL}_2(\mathbb{Z})$ such that $U^t A_0 Z = \left(\begin{smallmatrix} 0 & 1 \\ e & 0 \end{smallmatrix}\right)$. Therefore, the change of basis described by the matrix $g = \mathrm{diag}(U, \mathrm{Id}_{16}, Z) \in \mathrm{GL}_{20}(\mathbb{Z})$ transform A_0 into A, C_0 into C_1 and preserves B.

Next Scattone showed that there exist integral matrices V and Y such that $BY + VA + C_1 = 0$ if $\gcd(e, \det B) = 1$, which is satisfied since $e^2 \cdot \det B = 9$ and we always have e or $\det B$ equals 1. By choosing V and Y as above, and applying change of basis $g = \begin{pmatrix} I & V^t & \\ & I & Y \\ & & I \end{pmatrix}$ we put Q into the form $\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ A^t & 0 & D_2 \end{pmatrix}$. Finally, by applying $g = \begin{pmatrix} I & 0 & W \\ & I & 0 \\ & & I \end{pmatrix}$ and choose a appropriate W, we can put D_2 into the required form. For $e=1$, t can only be 1. ♣

Let $N_H(J) := \mathrm{Im}\left(\mathrm{Stab}_H(J) \xrightarrow{r} \mathrm{GL}(J) \cong \mathrm{GL}_2(\mathbb{Z})\right)$ be image of the stabilizer of J under the action of the group $H < O(\Sigma_n)$ in $\mathrm{GL}_2(\mathbb{Z})$.

Lemma 7.2. *For* $J \in I_{2,e}(\Sigma_n)$, *we have:*

$$N_{O(\Sigma_n)}(J) \cong \left\{ \begin{pmatrix} a & be \\ c & d \end{pmatrix} \in \mathrm{GL}_2(Z) \;\middle|\; a^2 \equiv 1 \pmod{e} \right\}$$

and

$$N_{O^+(\Sigma_n)}(J) \cong \left\{ \begin{pmatrix} a & be \\ c & d \end{pmatrix} \in \mathrm{SL}_2(Z) \;\middle|\; a^2 \equiv 1 \pmod{e} \right\}.$$

Proof. For the case of polarized K3 surface, this is [Sca87, Lemma 5.6.3, 5.6.6]). The proof for our case is basically the same. Let g be a general element of $\mathrm{Stab}_{O(\Sigma_n)}(J)$ of the form

$$g = \begin{pmatrix} U & V & UW \\ & X & Y \\ & & Z \end{pmatrix}.$$

Take Q as in the previous lemma. The condition $g^t Q g = Q$ gives

$$A = U^t AZ, B = X^t BX, C = X^t BY + V^t AZ, D = Z^t DZ + Y^t BY + W^t U^t AZ + Z^t A^t UW$$

which implies $U \in N_{O(\Sigma_n)}(J)$ is of the form $\begin{pmatrix} a & be \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ and $D \equiv \begin{pmatrix} 2a^2t & * \\ * & * \end{pmatrix} \pmod{2e}$. Note that in our case we have $\gcd(t, e) = 1$, which implies $a^2 \equiv 1 \pmod{e}$ is a necessary condition.

Conversely, if $U \in \mathrm{GL}_2(\mathbb{Z})$ is an arbitrary matrix of the form $U = \begin{pmatrix} a & be \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ with $a^2 \equiv 1 \pmod{e}$, let $Z = A^{-1} \left(U^{-1}\right)^t A$ and then

$$g = \begin{pmatrix} U & 0 & UW \\ & I & 0 \\ & & Z \end{pmatrix}$$

gives a lift of U in $\mathrm{Stab}_{O(\Sigma_n)}(J)$ with an appropriately chosen W.

The second statement holds naturally by considering the sign of the determinant. ♣

Proposition 7.3. *There is a bijection between*

$$I_{2,e}(\Sigma_n)/\Gamma_n \xleftrightarrow{1:1} \mathcal{G}(n, e).$$

Proof. We divide the proof into three steps.

Firstly we show that $I_{2,e}(\Sigma_n)/O(\Sigma_n) \xleftrightarrow{1:1} \mathcal{G}(n, e)$. Let $L(e, t)$ be the lattice given by the Gram matrix $\left(\begin{smallmatrix} 0 & e \\ e & 2t \end{smallmatrix}\right)$. By Lemma 7.1 above, given $J \in I_{2,e}(\Sigma_n)$, one get $\Sigma_n \cong U \oplus L(e,t) \oplus \left(J^\perp/J\right)$ where $J^\perp/J$ belongs to $\mathcal{G}(n, e)$. Conversely, for any $M \in \mathcal{G}(n, e)$ since $U \oplus L(e, t) \oplus M$ and Σ_n belong to the same genus, we have $\Sigma_n \cong U \oplus L(e, t) \oplus M$ so by the uniqueness of indefinite even lattice (see [Nik79, Corollary 1.13.3]). With this isomorphism one can find $J \in I_{2,e}(\Sigma_n)$ such that $J^\perp/J \cong M$. Therefore,

$$\mathcal{G}(n, e) \to I_{2,e}(\Sigma_n)/O(\Sigma_n)$$

is a bijection.

Next we show that

$$I_{2,e}(\Sigma_n)/O^+(\Sigma_n) \to I_{2,e}(\Sigma_n)/O(\Sigma_n)$$

is a bijection. Let $H_1 < H_2$ be a finite index subgroup acting on set $\mathscr{S}$. Note that the fiber of $\mathscr{S}/H_1 \to \mathscr{S}/H_2$ over $[a]/H_2$ has cardinality $\frac{[H_2:H_1]}{[\mathrm{Stab}_{H_2}(a):\mathrm{Stab}_{H_1}(a)]}$ for $a \in \mathscr{S}$. In our case, $[O(\Sigma_n) : O^+(\Sigma_n)] = 2$. By lemma 7.2 we have $N_{O^+(\Sigma_n)}(J) \neq N_{O(\Sigma_n)}(J)$, hence $\mathrm{Stab}_{O^+(\Sigma_n)}(J) \neq \mathrm{Stab}_{O(\Sigma_n)}(J)$. Then one can conclude that $I_2(\Sigma_n)/O(\Sigma_n) \cong I_2(\Sigma_n)/O^+(\Sigma_n)$.

Finally we consider the map

$$I_{2,e}(\Sigma_n)/\Gamma_n \to I_{2,e}(\Sigma_n)/O^+(\Sigma_n).$$

For $n = 1$ and 2, one have $O^+(\Sigma_n)/\Gamma_n \cong O(A_{\Sigma_n}) = \{\mathrm{id}, -\mathrm{id}\}$. Since $-\mathrm{id}$ doesn't change the Γ_n-class of J, the conclusion follows.

For $n = 3$, we have $O^+(\Sigma_3)/\Gamma_n \cong O(\Sigma_3)/\widetilde{O}(\Sigma_3) \cong O(A_{\Sigma_3}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by [Nik79, Theorem 1.14.2]. To determine $\mathrm{Stab}_{O^+}(J)/\mathrm{Stab}_{\Gamma_n}(J)$, we consider the exact sequence

$$1 \to \mathrm{Fix}_{O^+(\Sigma_n)}(J)/\mathrm{Fix}_{\Gamma_n}(J) \xrightarrow{i} \mathrm{Stab}_{O^+}(J)/\mathrm{Stab}_{\Gamma_n}(J) \xrightarrow{r} N_{O^+}(J)/N_{\Gamma_n}(J) \to 1.$$

Note that

$$\begin{tikzcd} 1 \arrow[r] & \ker\tilde{\beta} \arrow[r] \arrow[d, hook] & \mathrm{Fix}_{\Gamma_n}(J) \arrow[r, "\tilde{\beta}"] \arrow[d, hook] & \widetilde{O}^+(W_e) \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \ker\beta \arrow[r] & \mathrm{Fix}_{O^+(\Sigma_n)}(J) \arrow[r, "\beta"] & O^+(W_e) \arrow[r] & 1 \end{tikzcd}.$$

We have

$$1 \to \ker\beta/\ker\tilde{\beta} \to \mathrm{Fix}_{O^+(\Sigma_n)}(J)/\mathrm{Fix}_{\Gamma_n}(J) \to O^+(W_e)/\widetilde{O}^+(W_e) \to 1$$

for $W_e = J^\perp/J$ where $J \in I_{2,e}(\Sigma_n)$ and $\beta, \tilde{\beta}$ are the corresponding restriction maps.

For the case $e = 1$, note that $\Sigma_3 \cong U^2 \oplus W_1$, we have $\ker\beta = \ker\tilde{\beta}$. Hence

$$\mathrm{Fix}_{O^+(\Sigma_n)}(J)/\mathrm{Fix}_{\Gamma_n}(J) \cong O^+(W_e)/\widetilde{O}^+(W_e).$$

Since there are only two types of even (negative) definite unimodular lattice of rank 16, $D_{16} \subset D_{16}^+$ and $E_8 \oplus E_8$, Since the only two isotropic subgroups of A_{Σ_3} are differed by $\mathrm{id} \times (-\mathrm{id}) \in O(A_{W_1})$, one see that $\mathrm{id} \times (-\mathrm{id})$ is in the image of $O(W_1) \to O(A_{W_1})$ by Lemma 7.4 below, which generates $O(A_{W_1})$. Hence $O(W_1) \to O(A_{W_1})$ is surjective and $O^+(W_1)/\widetilde{O}^+(W_1) \cong O(A_{\Sigma_3})$. Form the exact sequence, the cardinality of the fiber over $[J]_{O^+(\Sigma_n)}$ is less or equal than 1 which gives the conclusion.

For $e = 3$, the lattice W_3 is unimodular and we have $\Sigma_3 \cong U \oplus U(3) \oplus W_3$. Using normalized basis, one can easily show that

$$N_{\Gamma_3}(J) \cong \left\{ \begin{pmatrix} a & 3b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(Z) \,\middle|\, a \equiv 1 \pmod 3 \right\}.$$

and $\left|N_{O^+}(J)/N_{\Gamma_3}(J)\right| = 2$ for $J \in I_{2,3}(\Sigma_3)$. One the other hand we have

$$\mathrm{Fix}_{O^+(\Sigma_n)}(J)/\mathrm{Fix}_{\Gamma_n}(J) \cong \left(\ker\beta/\ker\tilde{\beta}\right) \neq 1$$

So $\left[\mathrm{Stab}_{O^+}(J) : \mathrm{Stab}_{\Gamma_3}(J)\right] \geq 4$. The conclusion follows from the cardinality formula of the fiber.

♣

Lemma 7.4. [Ebe13, Proposition 3.6] *Let $\Lambda \subset \mathbb{R}^n$ be a lattice. There is a natural one-to-one correspondence between isomorphism classes of even overlattices $\Lambda \hookrightarrow \Gamma$ and orbits of isotropic subgroups $H \subset A_\Lambda$ under the image of the natural homomorphism $O(\Lambda) \to O(q_\Lambda)$. Unimodular lattices correspond to subgroups H with $|H|^2 = |A_\Lambda|$.*

In other words, there is a bijection between isotropic subgroups of discriminant group G_Λ and overlattices of Λ.

With Proposition 7.3, we have

Theorem 7.5. *Let $\mathcal{F}_{T_n}^*$ be the Baily-Borel compactification of $\mathcal{F}_{T_n}$. Then the boundary $\partial\mathcal{F}_{T_n} := \mathcal{F}_{T_n}^* - \mathcal{F}_{T_n}$ is given as follows:*

(i) *$n = 1$, $\partial\mathcal{F}_{T_1}$ consists of 14 modular curves and 2 points. All curves meet at one point. There are 2 curves meet at another point.*

(ii) $n = 2$, $\partial\mathcal{F}_{T_2}$ *consists of* 11 *modular curves and* 2 *points. All curves meet at one point. There are* 2 *curves meet at another point.*

(iii) $n = 3$, $\partial\mathcal{F}_{T_3}$ *consists of* 10 *modular curves and* 2 *points. All curves meet at one point. There are* 2 *curves meet at another point.*

Proof. By Proposition 7.3, there is a one-to-one correspondence between isomorphic classes of lattices in the genus $\mathcal{G}(n,e)$ and modular curves. To determine the classes L in $\mathcal{G}(n,e)$, by Nikulin's result [Nik79, Proposition 1.6.1], it's sufficient to classify all the the primitive embeddings of F of signature $(0,8)$ with $A_L \cong A_F$ and $q_L \cong -q_F$, into some $H \in \mathbb{I}_{0,24}$ unimodular signature $(0,24)$ lattices up to $O(H)$-equivalence.

(i). For $n = 1$, we have $(A_{\Sigma_1}, q_{\Sigma_1}) \cong (A_{A_8}, -q_{A_8})$. So we consider the all possible primitive embeddings of A_8 for $e = 1$ and the primitive embedding of E_8 for $e = 3$, into some $H \in \mathbb{I}_{0,24}$. By simple facts about root lattice embeddings (see, for example [Nis96, Section 4.1]), it turns out that the modular curves are given by the following 14 isomorphic classes of lattices $L = J^{\perp}/J \in \mathcal{G}(1,e)$ labeled by its root system:

- $e = 3$: E_8^2, D_{16},
- $e = 1$: $E_8 \oplus D_7$, $E_7^2 \oplus A_1$, $E_7 \oplus A_8$, D_{15}, $D_{12} \oplus A_3$, $D_9 \oplus A_6\,(A_8 \hookrightarrow A_{15} \subset D_9 \oplus A_{15})$, $A_{15}\,(A_8 \hookrightarrow D_9 \subset D_9 \oplus A_{15})$, $A_9 \oplus D_6$, $E_6 \oplus D_7 \oplus A_2$, A_8^2, $A_{12} \oplus A_3$, $A_{15}\,(A_8 \hookrightarrow A_{24})$.

Note that there are two isomorphic classes with the same root system A_{15}. Since there is only one orbit of isotropic subgroup of $A_{A_8} \oplus A_{1,1}$ giving primitive embedding of A_8 into $H \in \mathbb{I}_{0,24}$ and their direct sum with A_8 are contained in the different overlattice, by Lemma 7.4, the two classes are different.

(ii). For $n = 3$, we have $(A_{\Sigma_3}, q_{\Sigma_3}) \cong (A_{E_6 \oplus A_2}, -q_{E_6 \oplus A_2})$. So we consider the primitive embedding of $E_6 \oplus A_2$ and E_8 respectively. There are actually 10 isomorphic classes of lattices $L = J^{\perp}/J \in \mathcal{G}(3,e)$ labeled by its root system:

- $e = 3$: E_8^2, D_{16}.
- $e = 1$: $E_8 \oplus E_6 \oplus A_2$, $D_{13} \oplus A_2$, $D_{10} \oplus A_5$, $E_7 \oplus D_7$, A_{14}, $E_6^2 \oplus A_2^2$, $D_4 \oplus A_{11}$, $D_7 \oplus A_8$.

(iii). Before giving the proof for case $n = 2$, we first fix the following notations.

Denote $A_n = \langle a_1, a_2, \ldots, a_n\rangle$ Let $\alpha_k := a_k^*$ be the dual basis in A_n^* with $\alpha_k^2 = \frac{k}{k+1}$. Let ϵ_6 be the dual basis in E_6^* with $\epsilon_6^2 = -\frac{4}{3}$ Let ϵ_7 be the dual basis in E_7^* with $\epsilon_6^2 = -\frac{3}{2}$. Denote $D_n = \langle d_1, d_2, \ldots, d_n\rangle$ where $\{d_i\}_{i=1}^{24}$ is the standard basis of D_{24} with $d_1 \cdot d_3 = d_i \cdot d_{i+1} = 1$ for $2 \le i \le n-1$ and $d_1 \cdot d_2 = 0$. Let $\delta_k := d_k^*$ be the dual basis in D_n^*. We know δ_1, δ_n are the generators for A_{D_n} when n is even, δ_1 is the generator for A_{D_n} when n is odd with $\delta_1^2 = -\frac{n}{4}$, $\delta_n^2 = -1, \delta_1 \cdot \delta_n = -\frac{1}{2}$.

For $n = 2$, we have $(A_{\Sigma_2}, q_{\Sigma_2}) \cong (A_{\mathrm{L}_8}, -q_{\mathrm{L}_8}) \cong (A_{A_8}, q_{A_8})$ where L_8 is the even lattice of rank 8 with the following Gram matrix under the basis $\{d_1, \cdots d_7, w\}$:

$$\begin{pmatrix} -2 & 1 & & & & & & 1 \\ 1 & -2 & 1 & & & & & 0 \\ & 1 & -2 & 1 & & & & 0 \\ & & 1 & -2 & 1 & & & 0 \\ & & & 1 & -2 & 1 & 1 & -1 \\ & & & & 1 & -2 & & 1 \\ & & & & 1 & & -2 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & -4 \end{pmatrix}.$$

where $\{d_1, \cdots, d_7\}$ are the bases of D_7.

Consider the primitive embedding of L_8 and E_8 and we claim that the modular curves are given by the following isomorphic classes of lattices labeled by its root system:

- $e = 3$: E_8^2, D_{16}.

- $e = 1$:
  - $E_7 \oplus E_8$ $(\mathrm{L}_8 \hookrightarrow M(E_8^{\oplus 3}))$
  - $A_1 \oplus D_{14}$ $(\mathrm{L}_8 \hookrightarrow M(E_8 \oplus D_{16}))$
  - $E_8 \oplus A_8$ $(\mathrm{L}_8 \hookrightarrow M(D_{16} \oplus E_8))$
  - $A_4 \oplus D_{11}$ $(\mathrm{L}_8 \hookrightarrow M(D_{12}^{\oplus 2}))$
  - $E_7 \oplus E_6 \oplus A_2$ $(\mathrm{L}_8 \hookrightarrow M(D_{10} \oplus E_7^{\oplus 2}))$
  - $A_1 \oplus A_1 \oplus A_{13}$ $(\mathrm{L}_8 \hookrightarrow M(D_9 \oplus A_{15}))$
  - $D_7 \oplus D_7$ $(\mathrm{L}_8 \hookrightarrow M(D_8^{\oplus 3}))$
  - $D_8 \oplus A_6$ $(\mathrm{L}_8 \hookrightarrow M(D_8^{\oplus 3}))$
  - $D_5 \oplus A_{10}$ $(\mathrm{L}_8 \hookrightarrow M(D_7 \oplus E_6 \oplus A_{11}))$

The $e = 3$ case is as the same as above.

For $e = 1$, note that we have $D_7 \subset \mathrm{L}_8$ as a primitive sublattice. It's sufficient to consider the embedding of L_8 into lattices in $\mathbb{I}_{0,24}$ with root system E_8^3, $E_8 \oplus D_{16}$, D_{24}, D_{12}^2, $D_{10} \oplus E_7^2$, $D_9 \oplus A_{15}$, D_8^3 and $D_7 \oplus E_6 \oplus A_{11}$. We denote the corresponding unique unimodular lattice by $M(R)$ for R the root lattice in the above list. We give the proof in the next lemma 7.6.

(iv). For type III boundary components, by [Sca87, Lemma 4.1.2, Proposition 4.1.3], they are bijective to

$$\{\text{isotropic elements of } A_{\Sigma_n}\}/\{\pm 1\}.$$

For every n, there are only 2 such elements class, one is 0. Any isotropic rank 2 sublattice contains the vector corresponding to 0, and only those with $J^{\perp}/J \in \mathcal{G}(n,3)$ contain the vector corresponding to the nontrivial discriminant class. This completes the proof.

♣

Lemma 7.6. *Consider the primitive embedding of* L_8 *in a unimodular lattice of rank* 24*, we claim that the orthogonal complement of* L_8 *are classified by the following isomorphic classes of lattices labeled by its root system:*

- $E_7 \oplus E_8$ $(\mathrm{L}_8 \hookrightarrow M(E_8^{\oplus 3}))$
- $A_1 \oplus D_{14}$ $(\mathrm{L}_8 \hookrightarrow M(E_8 \oplus D_{16}))$
- $E_8 \oplus A_8$ $(\mathrm{L}_8 \hookrightarrow M(D_{16} \oplus E_8))$
- $A_4 \oplus D_{11}$ $(\mathrm{L}_8 \hookrightarrow M(D_{12}^{\oplus 2}))$
- $E_7 \oplus E_6 \oplus A_2$ $(\mathrm{L}_8 \hookrightarrow M(D_{10} \oplus E_7^{\oplus 2}))$
- $A_1 \oplus A_1 \oplus A_{13}$ $(\mathrm{L}_8 \hookrightarrow M(D_9 \oplus A_{15}))$
- $D_7 \oplus D_7$ $(\mathrm{L}_8 \hookrightarrow M(D_8^{\oplus 3}))$
- $D_8 \oplus A_6$ $(\mathrm{L}_8 \hookrightarrow M(D_8^{\oplus 3}))$
- $D_5 \oplus A_{10}$ $(\mathrm{L}_8 \hookrightarrow M(D_7 \oplus E_6 \oplus A_{11}))$

Proof. First we deal with the cases where the root systen contains a direct summand of D_n type with $n \neq 16$ in a unified way. Notations are the same as in 7.5. In these cases, one can recover $M(R)$ as the sublattice $D_n^* \oplus T$. For a embedding τ, $\tau(D_7) \subseteq D_n$, $p\ (\tau(w)) = \sum_{i=1}^{n} x_i d_i + k\delta_1 + l\delta_n$ where p is the projection to $D_n \supseteq D_7$

By solving equations given by inner product on L_8, one can assume $k = -1$.

we write $(p(\tau(w)), p(\tau(w))) = g_1 + g_2$ where $x_i \in \mathbb{Z}$, $k, l \in \mathbb{Z}/2\mathbb{Z}$

$$g_1 = (z,z) - 2x_1 - 2\left(x_1^2 + \dots x_7^2\right) - x_8^2 + 2(x_1x_3 + x_2x_3 + x_3x_4 \dots + x_7x_8)$$

$$g_2 = -(x_8 - x_9)^2 - (x_9 - x_{10})^2 - \dots - (x_{n-1} - x_n)^2 - x_n^2 + l - l^2 + 2x_n l$$

$1 \leq i \leq n$

Thus

$$y=\tau(w)=-\delta_1+cd_1+cd_2+(1+2c)\ (d_3+d_4+d_5+d_6+d_7)+(2+2c)d_8+x_9d_9\ldots+x_nd_n$$

$g_1=-2+(z,z)-2c,\quad x_8=2+2c.\quad$ Besides, if $l=0$ then $g_2\leq 0$.

- $M(D_{12}\oplus D_{12})$

  In this case, we will show l has only one possible value 0 and there is only one kind of embedding. $M(D_{12}\oplus D_{12})$ can be recovered as the sublattice $D_{12}^*\oplus D_{12}^*$ spanned by $D_{12}^{\oplus 2}$, (δ_1,δ_{12}) and (δ_{12},δ_1)

  By solving equations ,we know we can't find a vector of form $\delta_1-\delta_{12}+x$ where $x\in D_{12}$ with norm -1 Therefore, in this case $l=0$

  By considering the function we introduced before, we know

  $-5-2c=g_1\geq -3\ \ \&\ \ x_8\ \geq -4\qquad$ then $\ c=-1,\ -2\ ,\ -3$

  $y=\tau(w)=-\delta_1+cd_1+cd_2+(1+2c)\ (d_3+d_4+d_5+d_6+d_7)+(2+2c)d_8+x_9d_9+x_{10}d_{10}+....+x_nd_{12})$ we call the embedding is of the type $(d_8,d_9,d_{10},d_{11},d_{12})$

  Firstly, for each fixed c, the corresponding orthogonal complement is independent of the choice of $(d_8,d_9,d_{10},d_{11},d_{12})$, and we denote it by $Q(c)$. This is because one can relate any two embeddings by composition of a series of reflections. For instance, one see the embedding of type (-4,-3,-2,-1,0) can be identified to the embedding of type (-4,-3,-2,-1,-1) by the reflection of d_{12}.

  Secondly, we show $Q(-1)\cong Q(-2)\cong Q(-3)$

  In D_{12}, $D_7^{\perp}$=$\langle x_8,d_9,\ldots,d_{12}\rangle$ where $x_8=d_1+d_2+2(d_3+\ldots+d_8)+d_9$

  For $c=-1$, we embed w to $y_1=-\delta_1-d_1-d_2\ldots-d_7$

  For $c=-3$, we embed w to $y_3=-\delta_1-d_1-d_2\ldots-d_7-4d_8-3d_9-2d_{10}-d_{11}$ Notice that $(x_8,y_3)=1=-(x_8,y_1)$ Then one can construct a isomorphism ψ which induces a isomorphism between $Q(-1)$ and $Q(-3)$

$$\begin{aligned}\psi:\quad M\left(D_{12}\oplus D_{12}\right)\ &\longrightarrow M\left(D_{12}\oplus D_{12}\right)\\ (x,y)&\mapsto(-x,y)\end{aligned}$$

  For $c=-2$, we embed w to $y_2=-\delta_1-d_1-d_2\ldots-d_7-2d_8-d_9$ The same argument indicates that $Q(-1)\cong Q(-2)$ And one easily sees that the unique orthogonal complement is $Q(-1)=A_4\oplus D_{10}$.

- $M(D_9\oplus A_{15})$

  In this case, l has only one possible value 0 corresponding to the unique embedding. We recover $M(D_9\oplus A_{15})$ as a sublattice of $(D_9\oplus A_{15})^*$ generated by $\delta_1+2\alpha_1+a_1$ In this case, $z=\delta_1\ \ ,n=12,(z,z)=-\frac{9}{4}$ Similarly from $g_1=-\frac{17}{4}-2c\geq -4\ \ \&\ \ x_8\geq -4$ we get $-3\leq c\leq -1$By simple calculation one sees $-2\leq c\leq -1$ The same method in $D_{12}\oplus D_{12}$ implies $Q(-1)\cong Q(-2)$. By embedding w to $y_1=-\delta_1-d_1-d_2\ldots-d_7$ The unique orthogonal complement is $A_1\oplus A_1\oplus A_{13}$

- $M(D_{10}\oplus E_7^{\oplus 2})$

  In this case, l has only one possible value 0 corresponding to the unique embedding. By the same argument, we get $-3\leq c\leq -1$. And we map w to $y_1=-\delta_1-d_1-d_2\ldots-d_7$, which corresponds to the unique orthogonal complement $E_7\oplus E_6\oplus A_2$

- $M(D_{24})$

  In this distinguished case, the root system only has one direct sum of D_n type, and we show that L_8 can not be embedded into $M(D_{24})$.

  Suppose on the contrary we have primitive embedding $\tau\colon \mathrm{L}_8 \hookrightarrow L := M(D_{24})$. Then

$$L \cong \mathrm{Span}\{\delta_1\} + D_{24} \subset (D_{24})^*.$$

  We have

$$-4 \leq \tau(w)^2 = -2c - 8 - \sum_{i=8}^{23}(m_i - m_{i+1})^2 - m_{24}^2 \leq -2c - 8 - |m_8| = -6$$

  a contradiction.

- $M(D_8^{\oplus 3})$

  In this case, there are two different kinds of embeddings corresponding to two possible values of l. We recover $M(D_8 \oplus D_8 \oplus D_8)$, as the sublattice $D_8^* \oplus D_8^* \oplus D_8^*$ spanned by $D_8^{\oplus 3}$, $(\delta_1, \delta_8, \delta_8)$, $(\delta_8, \delta_1, \delta_8)$ and $(\delta_8, \delta_8, \delta_1)$

  If $l = 0$

  $(w, w) = -2c - 4 - (2 + 2c)^2 = -4c^2 - 10c - 8$ then $c = -1$,which corresponds to one kind of orthogonal complement $D_7 \oplus D_7$

  If $l = 1$ by calculation we know $p\ (\tau(w)) = -\sum_{i=1}^7 d_i - \delta_1 + \delta_8$ $(w, w) = -2c - 4 - (2 + 2c)^2 + 2(2 + 2c) = -4c^2 - 6c - 4$ again we get $c = -1$ In this case, w is mapped to $(-\sum_{i=1}^7 d_i - \delta_1 + \delta_8, -\sum_{i=1}^7 d_i - \delta_1 + \delta_8, 0)$ which corresponds to the other kind of orthogonal complement $D_8 \oplus A_6$

- $M(D_7 \oplus E_6 \oplus A_{11})$

  In this case, l has only one possible value 0 corresponding to the unique embedding. We recover $M(D_7 \oplus E_6 \oplus A_{11})$ as the sublattice of $(D_7 \oplus E_6 \oplus A_{11})^*$ generated by $\epsilon_6 + \delta_1 + \alpha_{11}$ and $D_7 \oplus E_6 \oplus A_{11}$ We write $p\ (\tau(w)) = -\sum_{i=1}^7 x_i d_i - \delta_1$ as before. By the inner product on L_8, we get a group of integral coefficient equations of m_i, k containing 7 linear ones. We get a unique solution $(x_1, x_2, ..x_7) = (-1, -1, ... - 1)$ Therefore, the unique embedding is given by mapping w to $(\epsilon_6, -\sum_{i=1}^7 x_i d_i - \delta_1, 2\alpha_1)$ with orthogonal complement $D_5 \oplus A_{10}$

For the case of $M(E_8 \oplus D_{16}^+)$, there are two different kinds of embeddings. Firstly, a similar calculation shows that L_8 can be primitively embedded into D_{16}^+, hence also $M(D_{16} \oplus E_8) = E_8 \oplus D_{16}^+$. Note that the genus of A_8 contains two classes. The other one contains the root sublattice E_7 hence it cannot lie in D_{16}^+. This shows that the orthogonal complement of L_8 in D_{16}^+ is unique, hence also the complement in $E_8 \oplus D_{16}^+$. Another kind of embedding can be given as follows. Notice that $E_8 \cong D_8 + \mathbb{Z}\langle\delta_1$, here we specify $\delta_1 \in D_8^*$. Then one can embed $\mathrm{L}_8 \hookrightarrow E_8 \oplus D_{16}^+$ by letting $w = (-\delta_1 - \sum_{i=1}^7 d_i, v)$ where $v \in D_{16}^+$ is a (-2)-vector.

For the case of E_8^3, there is only one embedding given as follows. We notice that if one changes the element of Gram matrix (8,8) element of L_8 to -2, the corresponding lattice is isomorphism to E_8. Therefore, we can find $w_1 \in E_8$ satisfying $w_1 \cdot w_1 = -2$ and $(d_1, d_2, \ldots d_7, w_1)$ is a basis of E_8. We map D_7 to the sublattice $(d_1, d_2, \ldots d_7)$ and map w to $(w_1, w_2, 0)$ where w_2 is a (-2) vector in the second copy of E_8

♣

7.2. **Looijenga's compactifications.** Let D be the type IV domain associated to an integral lattice (L, q) of signature $(2, n)$. Let Γ be a congruence arithmetic subgroup of the stable orthogonal group $\widetilde{O}(L)$. Following [Loo03], let $\mathcal{H}$ be a Γ-invariant hyperplane arrangement, we set

$$\mathrm{D}^{\circ} = \mathrm{D} - \bigcup_{H \in \mathcal{H}} H$$

and define

- $\mathrm{PO}(\mathcal{H})$: the collection of subspaces $M \subseteq L$ which are intersection of members in $\mathcal{H}$ meeting D. Denote by
$$\pi_M : \mathbb{P}(L_{\mathbb{C}}) - \mathbb{P}(M) \longrightarrow \mathbb{P}(L_{\mathbb{C}}/M)$$
the natural projection. The projection also defines a natural subdomain $\pi_M \mathrm{D}^{\circ} \subseteq \mathrm{D}^{\circ}$ (*cf.* [Loo03, §7]).
- $\Sigma(\mathcal{H})$: the collection of the common intersection of $I^{\perp}$ and members in $\mathcal{H}$ containing I, where I is a $\mathbb{Q}$-isotropic line or plane of L.

Then Looijenga's compactification of $\Gamma \backslash \mathrm{D}^{\circ}$ can be interpreted as below: let

$$\widehat{\mathrm{D}} = \mathrm{D}^{\circ} \cup \coprod_{M \in \mathrm{PO}(\mathcal{H})} \pi_M \mathrm{D}^{\circ} \cup \coprod_{V \in \Sigma(\mathcal{H})} \pi_V \mathrm{D}^{\circ}, \tag{7.1}$$

we define $\overline{\Gamma \backslash \mathrm{D}}^{\mathcal{H}}$ to be the quotient $\Gamma \backslash \widehat{\mathrm{D}}$, which compactifies $\Gamma \backslash \mathrm{D}^{\circ}$ and boundary decomposes into finitely many stratas. The birational map $\overline{\Gamma \backslash \mathrm{D}}^{\mathcal{H}} \dashrightarrow (\Gamma \backslash \mathrm{D})^*$ can be resolved by the following diagram

$$\begin{tikzcd} \widetilde{\Gamma \backslash \mathrm{D}}^{\mathcal{H}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\widetilde{\pi}"] & (\Gamma \backslash \mathrm{D})^{\Sigma(\mathcal{H})} \arrow[d, "\pi_{\mathcal{H}}"] \\ \overline{\Gamma \backslash \mathrm{D}}^{\mathcal{H}} \arrow[r, dotted] & \Gamma \backslash \mathrm{D}^* \end{tikzcd} \tag{7.2}$$

where $\pi_{\mathcal{H}} : (\Gamma \backslash \mathrm{D})^{\Sigma(\mathcal{H})} \to (\Gamma \backslash \mathrm{D})^*$ is the $\mathbb{Q}$-Cartierization of the hyperplane arrangement in $\mathcal{H}$ and π_i are the blow up and blow down respectively. Note that when $\mathcal{H}^{(r)} \neq \emptyset$, i.e. there exists non-empty common intersection of D with r linearly independent hyperplanes in $\mathcal{H}$, the dimension of the boundary

$$\dim(\overline{\Gamma \backslash D}^{\mathcal{H}} - \Gamma \backslash D^{\circ}) \geq r - 1 \tag{7.3}$$

.

For $L = \Sigma_1$, $\Gamma = \widetilde{O}(\Sigma_1)$, let $\mathcal{H}_1$ be the collection of hyperplanes such that $\Gamma \backslash D^{\circ}$ is the complement of the Heegner divisors $\mathbf{H}_u$ and $\mathbf{H}_h$. To be specific, we have $\mathcal{H}_1 = \mathcal{H}_u \cup \mathcal{H}_h$ where

$$\mathcal{H}_u = \bigcup_{\substack{v \in \Lambda,\, v^2 = 0 \\ vC = 1,\, vE = 1}} v^{\perp}, \qquad \mathcal{H}_h = \bigcup_{\substack{v \in \Lambda,\, v^2 = 0 \\ vC = 2,\, vE = 1}} v^{\perp}$$

Then $\overline{\mathcal{F}}_{T_1}^{\mathcal{H}_1}$ is the Looijenga's compactification of $\mathcal{F}_{T_1} - \mathbf{H}_u \cup \mathbf{H}_h$.

Lemma 7.7. *When $n = 1$, the boundary $\overline{\mathcal{F}}_{T_1}^{\mathcal{H}_1} - \Gamma \backslash D^{\circ}$ has codimension 1.*

Proof. Consider the signature $(1,19)$ lattice N given by the span of $C, E, e_1, \cdots, e_{18}$ where $e_i \cdot C = 2$, $e_i \cdot E = 1$ and $e_i \cdot e_j = \delta_{ij}$. Since $A_N \cong \mathbb{Z}/27\mathbb{Z}$, it's easy to show that N can be primitively embedded into the K3 lattice Λ by Nikulin 's results (see [Nik79, Corollary 1.12.3]). Then N can represent the common intersection of 18 hyperplanes in $\mathcal{H}_h$. This proves the assertion by (7.3). ♣

Corollary 7.8. *For $n = 1$, the GIT quotient $\overline{\mathcal{M}_2}^{\mathrm{GIT}}$ is not isomorphic to the Looijenga's compactification $\overline{\mathcal{F}}_{T_1}^{\mathcal{H}_1}$.*

Proof. Since $\mathcal{M}_2$ is isomorphic to $\Gamma\backslash D^\circ \cong \mathcal{F}_{T_1} - \mathbf{H}_u \cup \mathbf{H}_h$, this is obtained by comparing the dimension of the boundary of $\overline{\mathcal{M}_2}^{\mathrm{GIT}} - \mathcal{M}_2$ and $\overline{\mathcal{F}}_{T_1}^{\mathcal{H}_1} - \Gamma\backslash D^\circ$. ♣

References

[AB19] Kenneth Ascher and Dori Bejleri. Compact moduli of elliptic K3 surfaces. *arXiv preprint arXiv:1902.10686*, 2019.

[ABE20] Valery Alexeev, Adrian Brunyate, and Philip Engel. Compactifications of moduli of elliptic K3 surfaces: stable pair and toroidal. *arXiv preprint arXiv:2002.07127*, 2020.

[ADL22] Kenneth Ascher, Kristin DeVleming, and Yuchen Liu. K-stability and birational models of moduli of quartic K3 surfaces. *Inventiones mathematicae*, pages 1–82, 2022.

[AE21] Valery Alexeev and Philip Engel. Compact moduli of K3 surfaces. *arXiv preprint arXiv:2101.12186*, 2021.

[AET19] Valery Alexeev, Philip Engel, and Alan Thompson. Stable pair compactification of moduli of K3 surfaces of degree 2. *arXiv preprint arXiv:1903.09742*, 2019.

[AGZV12] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Volume 1.* Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2012. Classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds, Reprint of the 1985 edition.

[AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-sheng Tai. *Smooth Compactifications of Locally Symmetric Varieties.* Cambridge Mathematical Library. Cambridge University Press, 2 edition, 2010.

[BB66] W. L. Baily and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *The Annals of Mathematics*, 84(3):442, nov 1966.

[Bea21] Arnaud Beauville. A remark on the generalized Franchetta conjecture for K3 surfaces. *Mathematische Zeitschrift*, 300(4):3337–3340, apr 2021.

[BLMM17] Nicolas Bergeron, Zhiyuan Li, John Millson, and Colette Moeglin. The Noether-Lefschetz conjecture and generalizations. *Invent. Math.*, 208(2):501–552, 2017.

[Bru02] Jan Hendrik Bruinier. On the rank of Picard groups of modular varieties attached to orthogonal groups. *Compositio Math.*, 133(1):49–63, 2002.

[Bru15] Adrian Brunyate. *A modular compactification of the space of elliptic K3 surfaces.* PhD thesis, University of Georgia, 2015.

[Cam18] Chiara Camere. Some remarks on moduli spaces of lattice polarized holomorphic symplectic manifolds. *Communications in Contemporary Mathematics*, 20(04):1750044, 2018.

[Dol96] I. V. Dolgachev. Mirror symmetry for lattice polarized K3 surfaces. *Journal of Mathematical Sciences*, 81(3):2599–2630, sep 1996.

[Ebe13] Wolfgang Ebeling. Lattices and codes. In *Lattices and Codes*, pages 1–32. Springer, 2013.

[Fri84] Robert Friedman. The period map at the boundary of moduli. *Topics in transcendental algebraic geometry (Princeton, NJ, 1981/1982)*, 106:183–208, 1984.

[GLL+] François Greer, Radu Laza, Zhiyuan Li, Fei Si, and Zhiyu Tian. Compactifications of the moduli space of quasi-polarized K3 surfaces of degree 6. *In preparation.*

[HLL20] Klaus Hulek, Christian Lehn, and Carsten Liese. On the GHKS compactification of the moduli space of K3 surfaces of degree two. *arXiv preprint arXiv:2010.06922*, 2020.

[Huy16] Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[Laz09] Radu Laza. The moduli space of cubic fourfolds. *J. Algebraic Geom.*, 18(3):511–545, 2009.

[Laz16] Radu Laza. The KSBA compactification for the moduli space of degree two $k3$ pairs. *Journal of the European Mathematical Society*, 18(2):225–279, 2016.

[LO19] Radu Laza and Kieran O'Grady. Birational geometry of the moduli space of quartic surfaces. *Compositio Mathematica*, 155(9):1655–1710, 2019.

[LO21] Radu Laza and Kieran O'Grady. GIT versus Baily-Borel compactification for K3's which are double covers of $\mathbb{P}^1 \times \mathbb{P}^1$. *Advances in Mathematics*, 383:107680, 2021.

[Loo86] Eduard Looijenga. New compactifications of locally symmetric varieties, in" proc. of the 1984 vancouver conf. in algebraic geometry". In *CMS Conf. Proc.*, volume 6, pages 341–364. AMS, Providence, 1986.

[Loo03] Eduard Looijenga. Compactifications defined by arrangements. II. Locally symmetric varieties of type IV. *Duke Math. J.*, 119(3):527–588, 2003.

[Lun75] D. Luna. Adhérences d'orbite et invariants. *Invent. Math.*, 29(3):231–238, 1975.

[MFK94] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.

[Muk88] Shigeru Mukai. Curves, K3 surfaces and Fano 3-folds of genus ≤ 10. In *Algebraic geometry and commutative algebra*, pages 357–377. Elsevier, 1988.

[Mum77] David Mumford. Stability of projective varieties. *Enseign. Math. (2)*, 23(1-2):39–110, 1977.

[Nik79] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979.

[Nis96] Ken-ichi Nishiyama. The jacobian fibrations on some k3 surfaces and their mordell-weil groups. *Japanese journal of mathematics. New series*, 22(2):293–347, 1996.

[O'G86] KIERAN GREGORY O'GRADY. *Moduli of abelian and K3 surfaces*. PhD thesis, Brown University, 1986.

[Pet15] Arie Peterson. Modular forms on the moduli space of polarised K3 surfaces. *https://hdl.handle.net/11245/1.478095, PhD thesis, Korteweg-de Vries Institute for Mathematics (KdVI)*, 2015.

[Sca87] Francesco Scattone. On the compactification of moduli spaces for algebraic $K3$ surfaces. *Mem. Amer. Math. Soc.*, 70(374):x+86, 1987.

[SD74] B. Saint-Donat. Projective models of $K-3$ surfaces. *Amer. J. Math.*, 96:602–639, 1974.

[Sha80] Jayant Shah. A complete moduli space for K3 surfaces of degree 2. *Annals of Mathematics*, 112(3):485–510, 1980.

[Zho21] Chuyu Zhou. Log K-stability of GIT-stable divisors on Fano manifolds, 2021.