COMPACTIFICATIONS OF MODULI SPACE OF (QUASI-)TRIELLIPTIC K3 SURFACES

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Abstract. We study the moduli space $\mathcal{F}_{T_1}$ of quasi-trielliptic K3 surfaces of type I, whose general member is a smooth bidegree $(2,3)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^2$. Such moduli space plays an important role in the study of Hassett-Keel-Looijenga program of the moduli space of degree 8 quasi-polarized K3 surfaces.

In this paper, we consider several natural compactifications of $\mathcal{F}_{T_1}$, such as the GIT compactification and arithmetic compactification. We give a complete GIT analysis of the $(2,3)$-hypersurfaces and provide the description of the corresponding geometry. We also compute the configurations of the boundary of the Baily-Borel compactification of the quasi-trielliptic K3 surfaces by classifying certain lattice embeddings.

As an application, we show that $(\mathbb{P}^1 \times \mathbb{P}^2, \epsilon S)$ is K-stable if S is a K3 surface with at worst ADE singularities. This gives a concrete description of boundary of the K-stability compactification via the identification of the GIT stability and the K-stability. We also discuss the connection between GIT moduli, Baily-Borel compactifications and Looijenga's compactifications by studying the projective models of quasi-trielliptic K3 in terms of NL-divisors.

1. Introduction

It has been an important problem to construct geometric compactifications of moduli space of various types of projective K3 surfaces. There are several approaches developed during the study, including Hodge theory, birational geometry, Kähler-Einstein geometry and mirror symmetry. Let us first recall the mostly investigated case of the primitively quasi-polarized K3 surfaces of degree 2ℓ. The data consists of a K3 surface S and a nef line bundle L such that $L^2 = 2\ell$ and $c_1(L) \in H^2(S, \mathbb{Z})$ is a primitive class. Let $\mathcal{F}_{2\ell}$ be the coarse moduli space of such surfaces. By the Global Torelli theorem, $\mathcal{F}_{2\ell}$ is a locally symmetric variety and hence admits natural arithmetic compactifications such as the Baily-Borel compactification $\mathcal{F}^*_{2\ell}$ [BB66], Mumford's toroidal compactifications [AMRT10] and more generally Looijenga's semitoric compactifications [Loo03]. For example, the union of $\mathcal{F}_2$ with all possible type II degeneration introduced by Kulikov-Persson-Pinkham [Kul77, PP81] is a toroidal compactification corresponding to certain limiting Hodge structure (cf. [Fri84]). Another example is the GHKS compactification of $\mathcal{F}_{2\ell}$ introduced by Hulek, Lehn and Lisese [HLL20], which is a semitoric compactification obtained by the birational geometry of the mirror family. Especially, it's a toroidal compactification when $\ell = 1$.

On the other hand, there are other geometric (partial) compactifications constructed, (i.e., the geometric compactification of a Zariski open subset of $\mathcal{F}_{2\ell}$), for example, using geometric invariant theory, KSBA stability, K-stability, etc. Certainly, one can construct the moduli of polarized K3's as the GIT quotient of the Hilbert scheme or Chow variety parametrizing (S, L) in $|mL|$ for sufficiently large m (for instance, see [Vie12, Don01]). In the case of low degree, Mukai showed that general members in $\mathcal{F}_{2\ell}$ are complete intersections in some homogeneous spaces for $2\ell \leq 22$ and he provided a natural GIT compactification of moduli spaces of such K3 surfaces (cf. [Muk88]). Moreover, the work in [GLT15] shows that Mukai's GIT model actually compactifies the complement of a union of extremal Noether-Lefschetz divisors. As for KSBA stability, Laza constructed a 21-dimensional KSBA compactification of degree 2 surfaces in [Laz16] with no restriction on boundary divisors. More recently, Alexeev and Engel compactified $\mathcal{F}_{2\ell}$ in [AE21] using slc stable pairs with a canonically chosen divisor, the so-called recognizable divisor. Moreover, the normalization of this compactification is a semitoric compactification associated with the unique semifan.

With various compactifications constructed, it's a natural question to investigate their relations. In 1980, J. Shah [Sha80] compared the GIT moduli space $\mathfrak{M}_{ps}$ of plane sextic curves to the Baily-Borel compactification $\mathcal{F}_2^*$ of the period space of degree 2 polarized K3 surfaces. The birational period map is resolved by blowing up the triple conic point. Later Looijenga noticed that the period map lifted to the blow up is actually the Cartierization of the unigonal divisor H_u. And $\mathfrak{M}_{ps}$ is isomorphic to the Looijenga's compactification $\overline{\mathcal{F}}_2^{H_u}$ (cf. [Loo86]).

For moduli of degree 4 polarized K3 surface, inspired by Looijenga's framework, Laza and O'Grady proposed the so-called Hassett-Keel-Looijenga program for $\mathcal{F}_4$ in a series of papers [LO19, LO18, LO21] to resolve such period maps from GIT to the Baily-Borel compactification systematically. The key observation is the birational map between two models may be factored into a series of birational modifications centered at Shimura subvarieties. In [LO21], Laza and O'Grady verified their proposal for the the moduli of hyperelliptic quartic K3 surfaces, which is an 18 dimensional divisor in $\mathcal{F}_4^*$ and provided prediction and partial verification for $\mathcal{F}_4$. The HKL program for $\mathcal{F}_4$ was fully confirmed by Ascher-DeVleming-Liu [ADL22] using the theory of K-stable moduli space.

In [GLL$^+$], the authors formulate the HKL program for K3 surfaces with Mukai models in higher degrees, and they analyze the case of degree 6 K3 surfaces in detail. In this case, with the tool of variation of GIT quotient $\mathscr{M}(t)$, they show that the wall-crossing behavior of $\mathscr{M}(t)$ is controlled by the arithmetic of a certain hyperplane arrangement.

So far most of the results are obtained for (quasi)-polarized K3 surfaces and it's natural to consider extending the above results to the case of lattice polarized cases. However, it becomes more involved as the rank of the polarized lattice growing. Recently Brunyate, Ascher and Bejleri, Alexeev-Brunyate-Engel [Bru15, AB19, ABE20] compactified the moduli space of elliptic K3 surface with a section using KSBA stable pairs with different choice of the boundary divisors. It also suggests that when the lattice has a geometric meaning, the corresponding moduli of polarized K3 surfaces may admit interesting compactifications. We remark that in the sequel work of Alexeev and Engel [AE21], they developed a general way for KSBA compactification of lattice polarized K3 surfaces.

In this paper, we investigate the moduli space of (quasi-)trielliptic K3 surfaces, which are K3 surfaces admitting elliptic fibration and multisection. This can be viewed as a parallel case of elliptic K3 and also in its own interests but we adopt a different approach. By finding the projective model of quasi trielliptic K3 surfaces, we give a partial compactification of the trielliptic K3 surfaces by identifying the K-stability and GIT stability, and give the explicit description of the boundary. The relation with the Looijenga's compactification from the arithmetic perspective is considered via the period map, which provides certain evidence for Hassett-Keel-Looijenga program for general lattice polarized K3 surfaces.

One of our main results is in the following. Let X be a K-polystable Fano manifold and $D \in |-K_X|$. Denote the $M^K_{X,\epsilon}$ be the good moduli space parametrized K-polystable pairs of the form $(Y, \epsilon\widetilde{D})$ where $(Y, \widetilde{D})$ is the degeneration of (X, D). Let $\mathcal{M}_2$ be the moduli space of

$(2,3)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$ with at worst simple singularities where $\mathcal{M}_2^\circ \subset \mathcal{M}_2$ be the open part of smooth hypersurfaces.

Theorem 1.1. *There exists a rational number $0 < c < 1$ such that the log Fano pair $(\mathbb{P}^1 \times \mathbb{P}^2, \epsilon S)$ is K-stable for $0 < \epsilon < c$ where $S \in |-K_{\mathbb{P}^1\times\mathbb{P}^2}|$ is a $(2,3)$-hypersurface with at worst simple singularities (i.e., isolated ADE singularities). Hence $\mathcal{M}_2$ admits a natural K-stability compactification $\overline{\mathcal{M}}_2^K = M^K_{\mathbb{P}^1\times\mathbb{P}^2,\epsilon}$. The period map $\mathcal{M}_2^\circ \to \mathcal{F}_{T_1}$ to the moduli space of quasi-trielliptic K3 surfaces extends to $\mathcal{M}_2$, and its image in $\mathcal{F}_{T_1}$ is the complement of $\mathbf{H}_u, \mathbf{H}_h$.*

Moreover, the K-moduli compactification $M^K_{\mathbb{P}^1\times\mathbb{P}^2,\epsilon}$ is not isomorphic to the Looijenga's compactification of the complement $\mathcal{F}_{T_1}\backslash \mathbf{H}_u \cup \mathbf{H}_h$.

Secondly, we have classified the boundary components of $\mathcal{M}_2$ in its K-stability compactification. The main result is as follows.

Theorem 1.2. *The boundary of $\overline{\mathcal{M}}_2^K \backslash \mathcal{M}_2$ consists of 11 irreducible components whose general member S is described as follows:*

*(α): (**Semistable**) S is singular along a vertical line L. And S has a corank 3 singularity p such that $p \notin \pi(L)$ and $\pi'(p) \notin \pi'(L)$.*

*(β): (**Semistable**) S has two isolated $\widetilde{E}_8$-type singularities p and q such that $\pi(p) \neq \pi(q)$ and $\pi'(p) \neq \pi'(q)$. The fiber over p and q are both triple lines with different directions.*

*(γ): (**Semistable**) $S = (\mathbb{P}^1 \times \mathbb{P}^1) \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,2)|$ and S is singular along a horizontal line C such that the intersection of $\mathbb{P}^1 \times \mathbb{P}^1$ and C is empty.*

*(η): (**Semistable**) S has two isolated $\widetilde{E}_7$-type singularities p and q such that $\pi(p) \neq \pi(q)$ and $\pi'(p) \neq \pi'(q)$. The fiber over $\pi(p)$ and $\pi(q)$ contain a line L_1 and L_2 respectively with $L_1 \neq L_2$. And $\mathrm{mult}_{\pi'(p)}((\pi'(F)), L_1) \geq 2$, $\mathrm{mult}_{\pi'(q)}((\pi'(F)), L_2) \geq 2$ for any fiber F.*

*(δ): (**Semistable**) S is the union of two $(1,0)$ hypersurfaces and one $(0,3)$ hypersurface.*

*(ζ): (**Stable**) S has an isolated $\widetilde{E}_7$-type singularities p and there exists a fiber F such that $\pi'(F)$ does not contain $\pi'(p)$.*

*(ξ): (**Stable**) S has an isolated $\widetilde{E}_8$-type singularities p and the fiber containing p is not a triple line.*

*(θ): (**Stable**) S is singular along a section of degree 1.*

*(ϕ): (**Stable**) S is singular along a section of degree 2.*

*$(r1)$: (**Stable**) S is a union of a $(1,1)$ hypersurface and a $(1,3)$ hypersurface.*

*$(r2)$: (**Stable**) S is a union of a $(0,2)$ hypersurface and a $(2,1)$ hypersurface.*

The dimension of the stratum is given by

Strata	α	β	γ	η	δ	ζ	ξ	θ	ϕ	$r1$	$r2$
dimension	4	1	4	3	1	10	7	8	5	13	2

As an application, we find the generators of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{T_1})$ in terms of primitive NL divisors thorough the geometric approach.

Corollary 1.3. *The Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{T_1})$ with rational coefficients is spanned by primitive NL-divisors $\mathbf{H}_u, \mathbf{H}_h$ and $\mathbf{H}_t$. Moreover,*

$$\dim_{\mathbb{Q}} \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{T_1}) = 3.$$

Organization of the paper. We start in section 2 by discussing trielliptic K3 surface for motivation. The fact that trielliptic K3 surface is lattice polarized K3 surface for a certain rank

2 lattice T_n lead us to study the moduli space of K3 surface which admits a T_n-polarization as in 2.1. For $n = 1$, generically the projective model of such K3 surface is bidegree $(2, 3)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^2$. Therefore, GIT compactification of $\mathcal{M}_2$ gives a projective birational model of moduli space of quasi-trielliptic K3 surface. In the following three sections, we mainly study this compactification constructed by GIT. In section 3, we find out the semistable and properly stable $(2, 3)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^2$. In section 4, we give a description of the unstable and non-properly stable $(2, 3)$-hypersurface in terms of geometry such as singularities. Moreover, with the identification of GIT stability and K stability, we prove Theorem 1.1 which compares moduli space of quasi-trielliptic K3 surface to the moduli space $\mathcal{M}_2$ of $(2, 3)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$ with at worst simple singularities via period map. In section 5, by using Luna slice theorem, we study the strictly semistable locus of the GIT compactification. In section 6, we discuss the boundary of $\mathcal{M}_2^\circ$ in $\overline{\mathcal{M}_2}^{GIT}$ and prove Theorem 1.2. Finally, Baily-Borel compactification and Looijenga's compactification of $\mathcal{F}_{T_n}$ are discussed in section 7. In particular, we show that the GIT quotient $\overline{\mathcal{M}_2}^{GIT}$ is not isomorphic to the Looijenga's compactification.

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Notation & Conventions.

- We work over $\mathbb{C}$.
- Let A_n, D_n, E_n be the negative definite root lattices.

2. Moduli space of (quasi)-trielliptic K3 surfaces

2.1. **(Quasi)-Trielliptic K3 surface.** A K3 surface S is **trielliptic** if it admits an elliptic fibration $\pi : S \to \mathbb{P}^1$ together with a multisection C of degree 3. We may say S is trielliptic of type n if $C^2 = 2n$. In this case, the Picard lattice $\mathrm{Pic}(S)$ contains a primitive sublattice T_n given by

$$\begin{array}{c|c|c} & C & E \\ \hline C & 2n & 3 \\ \hline E & 3 & 0 \end{array} . \tag{2.1}$$

Up to an isometry, we may assume $n \in \{1, 2, 3\}$. Then S admits a T_n-polarization in the sense of [Dol96]. Hence we may say a K3 surface is **quasi-trielliptic** of type I (II, III respectively) if it admits a T_1 -polarization (respectively T_2, T_3-polarization).

2.2. **Moduli space of (quasi)-trielliptic K3 surface.** Let $\mathcal{F}_{\mathrm{T}_n}$ be the coarse moduli space of T_n-polarized K3 surfaces which parameteries pairs (S, ϕ) where S is a smooth $K3$ surface and $\phi : T_n \hookrightarrow \mathrm{Pic}(S) \subset \Lambda$ is a primitive embedding and $\phi(T_n)$ contains a quasi-polarization. Here the middle cohomology $\Lambda := \mathrm{H}^2(S, \mathbb{Z})$ is a unimodular even lattice of signature $(3, 19)$ under the intersection form $\langle , \rangle$. According to [Nik79], there is a unique primitive embedding

$$\mathrm{T}_n \hookrightarrow \Lambda. \tag{2.2}$$

We denote by Σ_n the orthogonal complement of T_n in Λ, which is an even lattice of signature $(2, 18)$. Let $\Sigma_n^{\mathbb{C}} = \Sigma_n \otimes \mathbb{C}$. The period domain $\mathbb{D}$ associated to Σ_n can be realized as a connected component of

$$\mathbb{D}^{\pm} := \{v \in \mathbb{P}(\Sigma_n^{\mathbb{C}}) | \, \langle v, v\rangle = 0, \langle v, \bar{v}\rangle > 0\}.$$

The monodromy group

$$\Gamma_n = \{g \in O^+(\Sigma_n) | \ g \text{ acts trivially on } \Sigma_n^\vee/\Sigma_n\}$$

naturally acts on $\mathbb{D}$, where $O^+(\Sigma_n)$ is the identity component of $O(\Sigma_n)$. According to the Global Torelli theorem of K3 surfaces (See [Dol96, Remark 3.4]), there is an isomorphism

$$\mathcal{F}_{\mathrm{T}_n} \cong \Gamma_n \backslash \mathbb{D}$$

via the period map. Then $\mathcal{F}_{\mathrm{T}_n}$ is a locally Hermitian symmetric variety with only quotient singularities, and hence $\mathbb{Q}$-factorial.

2.3. **Picard group of $\mathcal{F}_{\mathrm{T}_n}$.** The Noether-Lefschetz (NL) divisors on $\mathcal{F}_{\mathrm{T}_n}$ parameterizes the K3 surfaces in $\mathcal{F}_{\mathrm{T}_n}$ containing additional curve classes. According to [BLMM17, Theorem 1], the Picard group of $\mathcal{F}_{\mathrm{T}_n}$ is generated by NL-divisors. Let us recall the construction of some irreducible NL-divisors on $\mathcal{F}_{\mathrm{T}_n}$.

Definition 2.1. Let $\beta \in A_{\Sigma_n}, m \in \mathbf{Q}_{<0}$. The Heegner divisor is given by

$$\mathbf{H}_{\beta,m} := \widetilde{O}^+(\Sigma_n) \setminus \bigcup_{\substack{v \in \Sigma_n^*,\, v^2 = 2m \\ v \in \beta + \Sigma_n}} v^\perp.$$

Note that $\mathbf{H}_{\beta,m} = \mathbf{H}_{-\beta,m}$.

In general, the Heegner divisor $\mathbf{H}_{\beta,m}$ can be non-reduced and reducible. In the case that Λ is a transcendental lattice of a K3 surface, it can be written as the sum of some irreducible NL divisors we will introduce below.

Definition 2.2. We define $\mathbf{H}_u$, $\mathbf{H}_h$ and $\mathbf{H}_t$ to be the locus of K3 surfaces $(S, C, E) \in \mathcal{F}_{\mathrm{T}_n}$ such that $\mathrm{Pic}(S)$ contains a divisor class E' satisfying

- $\mathbf{H}_u$: ${E'}^2 = 0$, $C \cdot E' = 1$ and $E \cdot E' = 1$.
- $\mathbf{H}_h$: ${E'}^2 = 0$, $C \cdot E' = 2$ and $E \cdot E' = 1$.
- $\mathbf{H}_t$: ${E'}^2 = 0$, $C \cdot E' = 3$ and $E \cdot E' = 1$.

such that the rank 3 sublattice of Picard group generated by $\{C, E, E'\}$ is primitive. This is the generalization of primitive NL divisors of moduli space of (quasi)-polarized K3 surfaces. Follow the same proof of [O'G86, Proposition 1.3], one can show that these primitive NL divisors are irreducible.

Now we give the computation of $\mathrm{Pic}(\mathcal{F}_{T_n})$. let L be a even lattice of signature $(2, k)$ containing two hyperbolic lattices. Let

$$\rho_L \colon \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[A_L])$$

be the dual Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$. Due to [Bru02] and [BLMM17], there is an isomorphism

$$\mathrm{Pic}_{\mathbb{Q}}(\widetilde{O}^+(L) \setminus \mathcal{D}_L) \cong \mathrm{Acusp}(\frac{k+2}{2}, \rho_L)^\vee \tag{2.3}$$

sending Heegner divisor $\mathbf{H}_{\beta,m}$ to the coefficient function $c_{-m,\beta}$

$$\sum_{\tau \in A_L} \sum_{d \in \mathbb{Q}} c_{d,\tau} q^d e_\tau \mapsto c_{-m,\beta}.$$

Here $\mathrm{Acusp}(\frac{k+2}{2}, \rho_L)$ is the space of almost cusp form of weight $\frac{k+2}{2}$ and type ρ_L. So one can read the relation of Heegner divisors from the the relation between modular forms. The space of vector-valued modular form can be computed by Raum's method using lattice level Jacobi forms. For details, we refer to [Pet15, Section 3, 4].

For the dimension of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{T_n})$, one can use the dimension formula of Bruinier [Bru02]:

$$\begin{aligned}\rho(\mathcal{F}_{T_n}) := \dim_{\mathbb{Q}} \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{T_n}) = \frac{29}{4} - \frac{1}{12} \operatorname{Re} G(2, \Sigma_n) - \alpha_3(n) - \alpha_4(n) \\ - \frac{1}{9\sqrt{3}} \operatorname{Re}\left[\sqrt{-1}\big(G(1, \Sigma_n) + G(-3, \Sigma_n)\big)\right]\end{aligned} \tag{2.4}$$

where $G(m, L)$ is the generalized quadratic Gauss sum

$$G(m, L) = \sum_{\gamma \in A_L} e^{2\pi\sqrt{-1}\frac{m\gamma^2}{2}}$$

and $\alpha_3(n) = \sum_{\gamma \in A_{\Sigma_n}/\pm 1} \{-\frac{\gamma^2}{2}\}$, $\alpha_4(n) = |A_{\Sigma_n}/\pm 1|$.

Since in our case, the discriminant groups are as following:

(i) $n = 1$, $A_{\Sigma_1} = \mathbb{Z}\langle \xi_1 \rangle \cong \mathbb{Z}/9\mathbb{Z}$, $\xi_1^2 = \frac{2}{9}$,
(ii) $n = 2$, $A_{\Sigma_2} = \mathbb{Z}\langle \xi_2 \rangle \cong \mathbb{Z}/9\mathbb{Z}$, $\xi_2^2 = \frac{4}{9}$,
(iii) $n = 3$, $A_{\Sigma_3} = \mathbb{Z}\langle \eta_1 \rangle \times \mathbb{Z}\langle \eta_2 \rangle \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\eta_1^2 = -\frac{2}{3}$, $\eta_2^2 = -\frac{4}{3}$, $\eta_1 \cdot \eta_2 = 0$,

One can compute

$$\alpha_3(n) = \frac{6-n}{3}, \quad \alpha_4(n) = \begin{cases} 2, & n = 1, 2 \\ 3, & n = 3 \end{cases}$$

and

$$G(m, \Sigma_n) = \begin{cases} 3\left(\frac{nm}{9}\right), & n = 1, 2,\ m = 1, 2 \\ 3\sqrt{-3}\left(\frac{-n}{3}\right), & n = 1, 2,\ m = -3 \\ 3\gcd(n, |m|), & n = 3,\ m = 1, 2, -3 \end{cases}.$$

where $\left(\frac{a}{b}\right)$ is the Jacobi symbol. Therefore we have

Proposition 2.3. *The Picard number of $\mathcal{F}_{T_n}$ is given by*

n	1	2	3
$\rho(\mathcal{F}_{T_n})$	3	4	3

and the corresponding Hodge relations are given by

$$\begin{aligned} 27\lambda_1 &= 20\mathbf{H}_{4\xi_1, -\frac{2}{9}} - 8\mathbf{H}_{2\xi_1, -\frac{5}{9}} + \mathbf{H}_{\xi_1, -\frac{8}{9}} \\ 108\lambda_2 &= \mathbf{H}_{\bar{0}, -1} + 130\mathbf{H}_{2\xi_2, -\frac{1}{9}} + 28\mathbf{H}_{4\xi_2, -\frac{4}{9}} + 2\mathbf{H}_{\xi_2, -\frac{7}{9}} \\ 102\lambda_3 &= \mathbf{H}_{\bar{0}, -1} + 54\mathbf{H}_{\eta_1, -\frac{1}{3}} + 6\mathbf{H}_{\eta_2, -\frac{2}{3}} \end{aligned}$$

Proof. The relation follows from the identification (2.3) and the explicit computation of basis of the almost cusp forms $\mathrm{Acusp}(10, \rho_{\Sigma_n})$ which can be easily computed by `Sgae` using computer. For examples of quasi-polarized K3 surfaces, see [Pet15, Section 4.4]. ♣

2.4. **Projective models of triple-elliptic K3 surfaces.** Let us recall the Saint-Donat's classical result of projective models of K3 surface.

Proposition 2.4 (see [SD74]). *Let (S, L) be a smooth K3 surface with a primitive quasi-polarization L of degree 2ℓ and let φ_L be the map defined by $|L|$. Then there are the following possibilities:*

(i) *(Generic case) φ_L is birational to a degree 2ℓ surface in $\mathbb{P}^{l+1}$. In particular, φ_L is a closed embedding when L is ample.*

(ii) *(Hyperelliptic case)* φ_L *is a generically* $2:1$ *map and* $\varphi_L(S)$ *is a smooth rational normal scroll of degree* l*, or a cone over a rational normal curve of degree* l*. Moreover, in this case,* $\mathrm{Pic}(S)$ *contains a curve class* E' *satisfying* ${E'}^2=0$ *and* $L\cdot E'=2$ *for* $L^2\geq 4$*.*

(iii) *(Unigonal case)* $|L|$ *has a fixed component* D*, which is a smooth rational curve. In this case,* $\mathrm{Pic}(S)$ *contains a curve class* E' *satisfying* ${E'}^2=0$ *and* $L\cdot E'=1$*.*

The projective models of T_n-polarized K3 surfaces are given as below.

Proposition 2.5. *Let* (S,C,E) *be a* T_n*-quasipolarized K3 surface. Then up to some high codimension locus, the rational map* $\varphi:S\dashrightarrow\mathbb{P}^1\times\mathbb{P}^{n+1}=|\mathcal{O}_S(E)|\times|\mathcal{O}_S(C)|$ *is well-defined and* $\overline{\varphi(S)}$ *is*

(i) *a bidegree* $(2,3)$ *hypersurface of* $\mathbb{P}^1\times\mathbb{P}^2$ *if and only if* $S\notin\mathbf{H}_u\cup\mathbf{H}_h$ *when* $n=1$*.*

(ii) *the complete intersection of two hypersurfaces of bidegree* $(1,3),(1,1)$ *in* $\mathbb{P}^1\times\mathbb{P}^3$ *if* $S\notin\mathbf{H}_u\cup\mathbf{H}_h\cup\mathbf{H}_t$ *when* $n=2$ *.*

(iii) *the intersection of three hypersurfaces of bidegree* $(0,3),(1,1),(1,1)$ *in* $\mathbb{P}^1\times\mathbb{P}^4$ *if* $S\notin\mathbf{H}_h\cup\mathbf{H}_t$ *when* $n=3$*.*

Proof. According to Saint-Donat's result, the map φ_L defined by $|L|$ is a closed embedding after contracting all exceptional (-2) curves if the quasi-polarization L is base point free and (S,L) is not hyperelliptic.

In the following, we always assume that C is effective. When C,E both effective, we denote by $\widetilde{C},\widetilde{E}$ the free part of C and E respectively. For $n=1,2,3$, one first observe that φ is well-defined if $S\notin\mathbf{H}_u\cup\mathbf{H}_h$ by Hodge index theorem. Indeed, assume $-E$ is effective, one see either $S\in\mathbf{H}_u\cup\mathbf{H}_h$ or S or lies in the locus of codimension greater than 1. For $n=1$, when $S\notin\mathbf{H}_u\cup\mathbf{H}_h$, the intersection matrix given by $(\widetilde{C},\widetilde{E})$ still equals to $\left(\begin{smallmatrix}2&3\\3&0\end{smallmatrix}\right)$. Then $\overline{\varphi(S)}$ is equal to the image of the morphism induced by $|\widetilde{C}|\times|\widetilde{E}|$ which is birational, hence a $(2,3)$-hypersurface by the adjunction formula. Conversely, for $S\in\mathbf{H}_u$, if φ is well-defined, the image of the projection $\overline{\varphi(S)}\to\mathbb{P}^2$ is a rational curve. Hence $\overline{\varphi(S)}$ can not be a $(2,3)$-hypersurface. For $S\in\mathbf{H}_h$ and φ is well-defined, $\overline{\varphi(S)}$ is equal to the image induced by $|\widetilde{C}|\times|\widetilde{E}|$ given by intersection matrix $\left(\begin{smallmatrix}2&2\\2&0\end{smallmatrix}\right)$. Notice that the projection map $\overline{\varphi(S)}\to\mathbb{P}^2$ restricted $\varphi(\widetilde{E})$ has degree 2. So the projection $\overline{\varphi(S)}\to\mathbb{P}^2$ is generically injective, hence φ is a generically 2 to 1 map and $\overline{\varphi(S)}$ is not a $(2,3)$-hypersurface.

Note that $H_u=\emptyset$ for $n=3$. For $n=2$ and 3, if C and E are both nef, then C is base point free and non-hyperelliptic due to the Hodge index theorem and Proposition 2.4. Thus $\varphi_{|C|}$ is a closed embedding after contracting all exceptional (-2) curves. Then $\varphi=\varphi_{|C|\times|E|}$ is also a closed embedding since E is base point free. For the general case, the analysis is similar as $n=1$.

Next, we analyse the projective model $\overline{\varphi(S)}$. For $n=1$, $\overline{\varphi(S)}$ is a bidegree $(2,3)$ hypersurface of $\mathbb{P}^1\times\mathbb{P}^2$ due to the adjunction. For $n=2$, note that the composition with the Segre embedding $s\circ\phi\colon S\to\mathbb{P}^7$ is induced the line bundle $\mathcal{O}_S(L)$ and the dimension of linear system $|C+E|$ is 6, one can see that $s\circ\overline{\varphi(S)}$ is contained in a hyperplane. So $\overline{\varphi(S)}$ is contained in a bidegree $(1,1)$ hypersurface $X_{1,1}$ of $\mathbb{P}^1\times\mathbb{P}^3$. Using the Lefschetz hyperplane theorem and adjunction formula, one can see that the divisor class of S in $X_{1,1}$ is $(1,3)$. Note that

$$H^1(I_{X_{1,1}}(1,3))=H^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(0,2))=0.$$

We get a surjection $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,3))\twoheadrightarrow H^0(\mathcal{O}_{X_{1,1}}(1,3))$. Thus $\overline{\varphi(S)}$ is the complete intersection of two hypersurfaces of bidegree $(1,3),(1,1)$ in $\mathbb{P}^1\times\mathbb{P}^3$.

For $n=3$, similarly, $s\circ\phi\colon S\to\mathbb{P}^1\times\mathbb{P}^4\xrightarrow{s}\mathbb{P}^9$ is induced the line bundle $\mathcal{O}_S(L)$ where s is the Segre embedding and the dimension of linear system $|C+E|$ is 7, one can see that $s\circ\overline{\varphi(S)}$ is contained in the intersection of two hyperplanes. So $\overline{\varphi(S)}$ is contained in the intersection of two bidegree $(1,1)$ hypersurfaces X_1 and X_2 of $\mathbb{P}^1\times\mathbb{P}^4$. We denote $X_1\cap X_2$ by Y. One can see that if a $(1,1)$ hypersurface is singular, the only possibility is that it is reducible. Thus X_1 and X_2 are smooth. Otherwise, S is contained in a $\{\mathrm{pt}\}\times\mathbb{P}^4$ or a $\mathbb{P}^1\times\mathbb{P}^3$, which is impossible. By checking the Jacobian of Y, one can conclude that Y is also smooth. Note that the divisor class of S in Y is $(0,3)$ by adjunction. We deduce that the divisor class $(0,3)$ of Y comes from a bidegree $(0,3)$ hypersurface of $\mathbb{P}^1\times\mathbb{P}^4$ since $H^1(\mathcal{O}_{X_1}(-Y)(0,3))$ and $H^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(-X_1)(0,3))$ vanish. Therefore, $\varphi(S)$ is the complete intersection of three hypersurfaces of bidegree $(0,3),(1,1),(1,1)$ in $\mathbb{P}^1\times\mathbb{P}^4$. ♣

3. Stability of bidegree $(2,3)$ hypersurfaces in $\mathbb{P}^1\times\mathbb{P}^2$

3.1. **Numerical criteria.** Using the Hilbert-Mumford's numerical criteria [MFK94, Thm. 2.1], we have: A bidegree $(2,3)$ hypersurfaces in $\mathbb{P}^1\times\mathbb{P}^2$ is stable (resp. semistable) if and only if $\mu(f,\lambda)>0$ (resp. ≥ 0) for all λ one parameter subgroups of $\mathrm{SL}_2\times\mathrm{SL}_3$, where $\mu(f,\lambda)$ is the numerical weight of Mumford.

As is customary, a one parameter subgroup (1-PS) of $\mathrm{SL}_2\times\mathrm{SL}_3$ can be diagonalized as

$$\lambda\colon t\in\mathbb{C}^*\to\mathrm{diag}(t^a,t^{-a},t^b,t^c,t^{-b-c})$$

for some $a,b,c\in\mathbb{Z}$. We call such λ a normalized 1-PS of $\mathrm{SL}_2\times\mathrm{SL}_3$ if $a\geq 0$ and $b\geq c\geq -b-c$.

Let λ be a normalized 1-PS. Then the weight of a monomial $x_0^u x_1^{2-u} y_0^v y_1^w y_2^{3-v-w}$ with respect to λ is

$$au-a(2-u)+bv+cw+(-b-c)(3-v-w).$$

If we denote by $M^{\ominus}(\lambda)$ (resp. $M^-(\lambda)$) the set of monomials of bidegree $(2,3)$ which have non-positive (resp. negative) weight with respect to λ, one can easily compute the maximal subsets $M^{\ominus}(\lambda)$ (resp. $M^-(\lambda)$), as listed in the next subsection.

Cases	1-PS	Maximal monomials	Invariant
N1	$\lambda_1'=(3,-3,2,2,-4)$	$x_1^2y_0^3, x_0x_1y_0^2y_2, x_0^2y_0y_2^2$	(α)
N2	$\lambda_2'=(3,-3,2,0,-2)$	$x_1^2y_0^3, x_0x_1y_0y_1y_2, x_0^2y_2^3$	(β)
N3	$\lambda_3'=(3,-3,4,-2,-2)$	$x_1^2y_0^2y_1, x_0x_1y_0y_1^2, x_0^2y_1^3$	(α)
N4	$\lambda_4'=(0,0,2,-1,-1)$	$x_0^2y_0y_1^2$	(γ)
N5	$\lambda_5'=(1,-1,2,0,-2)$	$x_1^2y_0^2y_2, x_0x_1y_0y_1y_2, x_0^2y_0y_2^2, x_0^2y_1^2y_2$	(η)
N6	$\lambda_6'=(0,0,1,1,-2)$	$x_0^2y_0^2y_2$	(γ)
N7	$\lambda_7'=(1,-1,0,0,0)$	$x_0x_1y_0^3$	(δ)

Table 1. Not properly stable

3.2. **Maximal subsets for not properly stable points.**

Lemma 3.1. *For any normalized 1-PS λ, $M^{\ominus}(\lambda)$ is contained in one of $M^{\ominus}(\lambda_i')$ in table 1. The surface S is not properly stable if it defining polynomial is one of the following:*

(N1) $f(x_0,x_1,y_0,y_1,y_2)=x_1^2c(y_0,y_1,y_2)+x_0x_1y_2q(y_0,y_1)+x_0^2y_2^2l(y_0,y_1,y_2)$.
(N2) $f(x_0,x_1,y_0,y_1,y_2)=x_1^2c_0(y_0,y_1,y_2)+x_0x_1[c_1(y_1,y_2)+y_0y_2l(y_1,y_2)]+\mu x_0^2y_2^3$.

Cases	1-PS	Description (roughly)	inclusion
U1	$\lambda_1 = (5, -5, 3, -1, -2)$	reducible	S1
U2	$\lambda_2 = (4, -4, 2, 1, -3)$	singular along a vertical line	S2, S3
U3	$\lambda_3 = (4, -4, 4, -1, -3)$	corank 3 isolated	S2, S3 ($\star$)
U4	$\lambda_4 = (3, -3, 4, -1, -3)$	corank 3 isolated	S3
U5	$\lambda_5 = (1, -1, 3, -1, -2)$	singular along a horizontal line	S3
U6	$\lambda_6 = (1, -1, 5, -1, -4)$	singular along a horizontal line	S4
U7	$\lambda_7 = (2, -2, 3, 1, -4)$	singular along a vertical line	S5

TABLE 2. Unstable

(N3) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_1(y_1, y_2) + y_0q_1(y_1, y_2) + y_0^2 l(y_1, y_2)]$
$+ x_0x_1[c_2(y_1, y_2) + y_0q_2(y_1, y_2)] + x_0^2 c_0(y_1, y_2)$.

(N4) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0q_0(y_1, y_2)] + x_0x_1[c_1(y_1, y_2) + y_0q_1(y_1, y_2)]$
$+ x_0^2[c_2(y_1, y_2) + y_0q_2(y_1, y_2)]$.

(N5) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0q(y_1, y_2) + \mu y_0^2 y_2] + x_0x_1[c_1(y_1, y_2) + y_0y_2l(y_1, y_2)]$
$+ x_0^2 y_2(q(y_1, y_2) + \nu y_0y_2)$.

(N6) $f(x_0, x_1, y_0, y_1, y_2) = y_2 \cdot q(x_0, x_1, y_0, y_1, y_2)$.

(N7) $f(x_0, x_1, y_0, y_1, y_2) = x_1[x_0c_0(y_0, y_1, y_2) + x_1c_1(y_0, y_1, y_2)]$.

Using the destabilizing 1-PS, the invariant part of equation (N1)-(N7) are given as follows:

$$(\alpha) : x_1^2 c(y_0, y_1) + x_0x_1q(y_0, y_1)y_2 + x_0^2 l(y_0, y_1)y_2^2 = 0. \tag{3.1}$$

$$(\beta) : ax_1^2y_0^3 + x_0x_1(by_1^3 + cy_0y_1y_2) + dx_0^2y_2^3 = 0. \tag{3.2}$$

$$(\gamma) : x_1^2q_1(y_0, y_1)y_2 + x_0x_1q_2(y_0, y_1)y_2 + x_0^2q_3(y_0, y_1)y_2 = 0. \tag{3.3}$$

$$(\eta) : x_1^2(a_1y_1^2y_0 + b_1y_0^2y_2) + x_0x_1(a_2y_0y_1y_2 + b_2y_1^3) + x_0^2(a_3y_0y_2^2 + b_3y_1^2y_2) = 0. \tag{3.4}$$

$$(\delta) : x_0x_1c(y_0, y_1, y_2) = 0. \tag{3.5}$$

Similarly, we can get maximal subsets for unstable points.

Lemma 3.2. *For any normalized 1-PS λ, $M^-(\lambda)$ is contained in one of $M^-(\lambda_i)$ in table 2. The surface S is unstable if its defining polynomial is one of the following:*

(U1) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2 c(y_0, y_1, y_2) + x_0x_1[c_0(y_1, y_2) + \mu y_0y_2^2]$.

(U2) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2 c_0(y_0, y_1, y_2) + x_0x_1(y_2^2 l(y_0, y_1, y_2) + \mu y_2y_1^2) + \nu x_0^2y_2^3$.

(U3) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0q(y_1, y_2) + y_0^2 l(y_1, y_2)] + x_0x_1[c_1(y_1, y_2) + \mu y_0y_2^2]$
$+ \nu x_0^2y_2^3$.

(U4) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0q(y_1, y_2) + \mu y_0^2y_2] + x_0x_1[c_1(y_1, y_2) + \nu y_0y_2^2]$
$+ x_0^2y_2^2 l(y_1, y_2)$.

(U5) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0q_1(y_1, y_2))] + x_0x_1[c_1(y_1, y_2) + \mu y_0y_2^2] + x_0^2 c_2(y_1, y_2)$.

(U6) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c_0(y_1, y_2) + y_0y_2l(y_1, y_2)] + x_0x_1[c_1(y_1, y_2) + \mu y_0y_2^2]$
$+ x_0^2[c_2(y_1, y_2) + \nu y_0y_2^2]$.

(U7) $f(x_0, x_1, y_0, y_1, y_2) = x_1^2[c(y_1, y_2) + y_0y_2l_0(y_1, y_2) + \mu y_0^2y_2] + x_0x_1(y_2^2 l_1(y_0, y_1, y_2) + \nu y_2y_1^2)$
$+ x_0^2y_2^2 l_2(y_0, y_1, y_2)$.

4. GEOMETRIC INTERPRETATION OF STABILITY

Let us first give some notations and conventions that are used in the following of the paper.

4.1. **Notations and Conventions.** Given a bidegree $(2,3)$ hypersurface $S \subseteq \mathbb{P}^1 \times \mathbb{P}^2$, it admits an elliptic fibration

$$\pi : S \to \mathbb{P}^1$$

via the first projection and all the fibers we consider are respect to π. On the other hand, the second projection

$$\pi' : S \to \mathbb{P}^2$$

is a double cover branching along a sextic curve, denoted by $\mathrm{B}(S)$. In particular, we call a curve $C \subseteq S$

- a line in the fiber or equivalently of type $(0,1)$, if C has the form $\{\mathrm{pt}\} \times \mathbb{P}^1$.
- of type $(1,0)$ if C has the form $\mathbb{P}^1 \times \{\mathrm{pt}\}$.
- a section of degree d if $\pi(C) = \mathbb{P}^1$ and $|\pi^{-1}(p) \cap C| = d$ for any $p \in \mathbb{P}^1$.

We use the terminology of the corank of the hypersurface singularities as in [AGZV12] and [Laz09].

Definition 4.1. Let $0 \in \mathbb{C}^n$ be a hypersurface singularity given by an equation $f(z_1, \dots, z_n) = 0$. The corank of 0 is n minus the rank of the Hessian of $f(z_1, \dots, z_n)$ at 0.

4.2. **Geometry of non-properly stable surface.**

Theorem 4.2. *A bidegree-$(2,3)$ hypersurface S in $\mathbb{P}^1 \times \mathbb{P}^2$ is not properly stable if and only if one of the following conditions holds*

(N1) *S is singular along a vertical line;*
(N2) *S contains a singularity p of at least $\widetilde{E}_8$-type, and the fiber over $\pi(p)$ is a triple line.*
(N3) *S contains at least a singularity of corank 3;*
(N4) *S is singular along a type horizontal line;*
(N5) *S contains a corank 2 singularity p of at least $\widetilde{E}_7$-type, such that the fiber over $\pi(p)$ contains a line L, and $\mathrm{mult}_{\pi'(p)}((\pi'(F)), L) \geq 2$ for any fiber F.*
(N6) *$S = (\mathbb{P}^1 \times \mathbb{P}^1) \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,2)|$;*
(N7) *$S = \mathbb{P}^2 \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,3)|$.*

Proof. According to Lemma 3.1, it suffices to find the geometric characterizations of type (N1)-(N7). We give detailed proof for types (N2) and (N5) here. The other cases are similar and relatively simple and we omit them.

If S is of type (N2), then the equation of S is given by

$$x_1^2 c_0(y_0, y_1, y_2) + x_0 x_1 [c_1(y_1, y_2) + y_0 y_2 l(y_1, y_2)] + \mu x_0^2 y_2^3 = 0.$$

One can assume that the coefficient of $x_1^2 y_0^3$ and $x_0^2 y_2^3$ is nonzero. Otherwise, the equation will degenerate to type (N3) and (N6). Set $p = (1,0,1,0,0)$, then the fiber over $(1,0)$ is the triple line $3L$: $\{y_2^3 = 0\}$. Letting $x_0 = y_0 = 1$, the affine equation near p is

$$x_1^2 + y_2^3 + x_1^2 l(y_1, y_2) + x_1 y_2 l(y_1, y_2) + x_1^2 q(y_1, y_2) + x_1 c_1(y_1, y_2) = 0.$$

It is clear that the weight on variables x_1, y_2, y_1 is $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$. Thus p is at least $\widetilde{E}_8$-type.

Conversely, we take $p = (1,0,1,0,0)$ to be the isolated singularity of at least $\widetilde{E}_8$-type. Up to coordinate change, one can assume the fiber over $\pi(p)$ is the triple line $3L$: $\{y_2^3 = 0\}$, then the equation of S can be written as

$$x_1^2 c_0(y_0, y_1, y_2) + x_0 x_1 [c_1(y_1, y_2) + y_0 q_1(y_1, y_2)] + \mu x_0^2 y_2^3 = 0.$$

If the coefficient of $x_0x_1y_0y_1^2$ is nonzero, one can see that the weight of p has weight worse than $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, which contradicts the condition that p is at least $\widetilde{E}_8$-type.

If S is of type (N5), then the equation of S is given by

$$x_1^2[c_0(y_1, y_2) + y_0q(y_1, y_2) + \mu y_0^2y_2] + x_0x_1[c_1(y_1, y_2) + y_0y_2l(y_1, y_2)] \\ +x_0^2y_2(q(y_1, y_2) + \nu y_0y_2) = 0.$$

One can assume that the coefficient of $x_1^2y_0^2y_2$ and $x_0^2y_0y_2^2$ are nonzero. Otherwise the equation will degenerate to type (N3) and (N4). Set $p = (1, 0, 1, 0, 0)$, then the fiber over $(1, 0)$ contains the line $\{y_2 = 0\}$. And $\mathrm{mult}_{(1,0,0)}((\pi'(F)), y_2) \geq 2$ for any fiber F since there are no y_0^3, $y_0^2y_1$ terms. One can check that p is at least $\widetilde{E}_7$-type.

Conversely, we take $p = (1, 0, 1, 0, 0)$ to be the isolated corank 2 singularity of at least $\widetilde{E}_7$-type. Up to coordinate change, one can assume that the fiber over $(1, 0)$ contains the line $\{y_2 = 0\}$, then the equation of S can be written as

$$x_1^2c_0(y_0, y_1, y_2) + x_0x_1[c_1(y_1, y_2) + y_0q_1(y_1, y_2) + y_0^2y_2] + x_0^2y_2(q(y_1, y_2) + \nu y_0y_2) = 0.$$

One can deduce that the coefficient of $x_1^2y_0^3$ and $x_1^2y_0^2y_1$ is zero since $\mathrm{mult}_{\pi'(p)}((\pi'(F)), y_2) \geq 2$ for any fiber F. And the coefficient of $x_0x_1y_0^2y_2$ is also zero due to the corank being 2. From now on, the equation of S can be written as

$$x_1^2[c_0(y_1, y_2) + y_0q(y_1, y_2) + \mu y_0^2y_2] + x_0x_1[c_1(y_1, y_2) + y_0q_1(y_1, y_2)] \\ +x_0^2y_2(q(y_1, y_2) + \nu y_0y_2) = 0.$$

Moreover, if the coefficient of $x_0x_1y_0y_1^2$ is nonzero, then the singularity of p has weight no worse than $(\frac{1}{2}, \frac{1}{4}, \frac{3}{8})$, which is impossible.

♣

4.3. Geometry of unstable surface.

Theorem 4.3. *A bidegree-$(2, 3)$ hypersurface $S \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ is unstable if and only if one of the following conditions holds*

- (U1) *$S = \mathbb{P}^2 \cup S'$ for some $S' \in |\mathcal{O}(1, 3)|$, meeting along a cuspidal cubic curve.*
- (U2) *S is singular along a line in the fiber whose projection under π' is a triple line $3L$. And $\mathbf{B}(S) = 2\ell \cup B'$ with $L \cap B'$ is a quartic point $\{o\}$.*
- (U3) *S contains at least a corank 3 isolated singularity p and the fiber over $\pi(p)$ is a triple line $3L$. And the projective tangent cone at $\pi'(p)$ of branching locus $\overline{\mathbf{C}}_{\pi'(p)}\mathbf{B}(S)$ contains $3L$.*
- (U4) *S contains at least a corank 3 isolated singularity p and the fiber over $\pi(p)$ is the union of a double line $2L_1$ and a line L_2 meeting at $\pi'(p)$. And $\mathrm{mult}_{\pi'(p)}((\pi'(F)), L_1) \geq 2$ for any fiber F. The projective tangent cone at $\pi'(p)$ of the branching locus $\overline{\mathbf{C}}_{\pi'(p)}\mathbf{B}(S)$ is at least $3L_1 \cup L'$*
- (U5) *S is singular alone a type horizontal line C with $\pi'(C)$ is a point $o \in \mathbb{P}^2$. S has a fiber which is the union of three lines intersecting at o. The projective tangent cone at o of branching locus $\overline{\mathbf{C}}_o\mathbf{B}(S)$ is at least a quadruple line.*
- (U6) *S is singular along a type horizontal line C with $\pi'(C)$ is a point $o \in \mathbb{P}^2$. S has a fiber F_0 such that the projective tangent cone of $\pi'(F_0)$ at o is a double line $\overline{\mathbf{C}}_o(\pi'(F_0)) = 2\ell$. And $\mathrm{mult}_o((\pi'(F)), L) \geq 3$ for any fiber F. $\overline{\mathbf{C}}_o\mathbf{B}(S)$ is at least $3L \cup L'$.*

(U7) *S is singular along a type vertical line whose projection under π' is a line L . For any F with $L \nsubseteq \pi'(F)$, $\{o\} = L \cap \pi'(F)$ is at least a triple point. $\mathbf{B}(S) = 2\ell \cup B'$ with $L \cap B'$ containing at least a triple point which is exactly o when such F exists.*

Proof. As in the proof of Theorem 4.2, we present explicit proof for three slightly complicated cases.

(1) If S is of type (U4), then the equation of S is given by

$$x_1^2[c_0(y_1,y_2)+y_0q(y_1,y_2)+\mu y_0^2y_2]+x_0x_1[c_1(y_1,y_2)+\nu y_0y_2^2]+x_0^2y_2^2l(y_1,y_2)=0.$$

S contains a corank 3 isolated singularity p: $(1,0,1,0,0)$, and the fiber over $\pi(p)$ is the union of a double line $2L_1$: $\{y_2^2=0\}$ and a line L_2: $\{\ell(y_1,y_2)=0\}$ meeting at $\pi'(p)$. One can assume that the coefficient of $x_1^2y_0^2y_2$ is nonzero, and the coefficients of $x_0^2y_2^3$, $x_0^2y_1y_2^2$ are not simultaneously zero. Otherwise, the equation will degenerate to type (U5) and (U1). Then $\mathrm{mult}_{(1,0,0)}((\pi'(F)),y_2)\geq 2$ for any fiber F and

$$\overline{\mathbf{C}}_{\pi'(p)}\mathbf{B}(S)=\nu^2y_2^4-\mu y_2^3\ell(y_1,y_2)=3L_1\cup L'.$$

Conversely, suppose that S contains at least a corank 3 isolated singularity p and the fiber containing p is a union of a double line $2L_1$ and a line L_2 meeting at p. One can assume that $p=(1,0,1,0,0)$ and L_1 is given by $\{y_2=0\}$. It's easy to show that there is no $x_ix_jy_0^2y_1$ terms in the equation of S since $\mathrm{mult}_{(1,0,0)}((\pi'(F)),y_2)\geq 2$ for any fiber F. Then the equation of S can be written as

$$x_1^2[c_0(y_1,y_2)+y_0q(y_1,y_2)+\mu y_0^2y_2]+x_0x_1[c_1(y_1,y_2)+y_0q_1(y_1,y_2)]+x_0^2y_2^2\ell_0(y_1,y_2)=0.$$

The $\mathbf{B}(S)$ is given by

$$(c_1(y_1,y_2)+y_0q_1(y_1,y_2))^2-y_2^2l_0(y_1,y_2)(c_0(y_1,y_2)+y_0q_0(y_1,y_2)+\mu y_0^2y_2)=0.$$

Thus one can deduce that the $q_1=\nu y_2^2$ since $\overline{\mathbf{C}}_{(1,0,0)}\mathbf{B}(S)=3L_1\cup L'$.

(2) If S is of type (U5), then the equation of S is given by

$$x_1^2[c_0(y_1,y_2)+y_0q_1(y_1,y_2)]+x_0x_1[c_1(y_1,y_2)+\mu y_0y_2^2]+x_0^2c_2(y_1,y_2)=0.$$

We see at once that S is singular along a horizontal line C: $\mathbb{P}^1\times(1,0,0)$ with $\pi'(C)$ is a point o: $(1,0,0)\in\mathbb{P}^2$. And the fiber over $(1,0)\in\mathbb{P}^1$ is given by $\{c_2(y_1,y_2)=0\}$, which is the union of three lines intersecting at o. In addition, $\overline{\mathbf{C}}_o\mathbf{B}(S)$ is a quadruple line $\{y_2^4=0\}$ if μ is nonzero. Otherwise, the degree of $\overline{\mathbf{C}}_o\mathbf{B}(S)$ is at least 5.

Conversely, up to a coordinate change, one can assume that S is singular along a horizontal line C: $\mathbb{P}^1\times(1,0,0)$ with $\pi'(C)$ is a point o: $(1,0,0)\in\mathbb{P}^2$. In addition, the fiber over $(1,0)\in\mathbb{P}^1$ is the union of three lines intersecting at o. So the equation of S can be written as

$$x_1^2[c_0(y_1,y_2)+y_0q_1(y_1,y_2)]+x_0x_1[c_1(y_1,y_2)+y_0q_2(y_1,y_2)]+x_0^2c_2(y_1,y_2)=0.$$

If $q_2=0$, then obviously S is of type (U5). If $q_2\neq 0$, then $\overline{\mathbf{C}}_o\mathbf{B}(S)$ is given by $q_2(y_1,y_2)^2$. Thus one can assume $q_2(y_1,y_2)=\mu y_2^2$ up to a coordinate change of y_1 and y_2 since $\overline{\mathbf{C}}_o\mathbf{B}(S)$ is a quadruple line.

(3) If S is of type (U7), then the equation of S is given by

$$\begin{aligned}x_1^2[c(y_1,y_2)+y_0y_2l(y_1,y_2)+\mu y_0^2y_2]+x_0x_1(y_2^2l_0(y_0,y_1,y_2)+\nu y_1^2y_2)&\\+x_0^2y_2^2l_1(y_0,y_1,y_2)=0&\end{aligned}$$

with $\mathbf{B}(S)$ given by

$$y_2^2(y_2l_0(y_0,y_1,y_2)+\nu y_1^2)^2-y_2^2l_1(y_0,y_1,y_2)(c(y_1,y_2)+y_0y_2l(y_1,y_2)+\mu y_0^2y_2)=0.$$

One observes that S is singular along a vertical line $\{x_1=y_2=0\}$ with $L\colon\{y_2=0\}$ the projection to $\mathbb{P}^2$. Also one can notice that $\mathbf{B}(S)=2L\cup B'$. If the coefficient of $x_1^2y_1^3$ is zero, then the projection of every fiber under π' contains L. One can see that $L\cap B'$ is a quadruple point $(1,0,0)$. If the coefficient of $x_1^2y_1^3$ is nonzero, it is clear that $L\cap\pi'(F)$ is a triple point $o\colon(1,0,0)$ for any fiber F. Finally, $L\cap B'$ is given by $\{\nu^2y_1^4-y_1^3l_1(y_0,y_1)=y_2=0\}$, which contains a triple point $(1,0,0)$.

Conversely, up to a coordinate change, one may assume that S is singular along a type horizontal line $\{x_1=y_2=0\}$ with $L\colon\{y_2=0\}$ the projection to $\mathbb{P}^2$. One can then write the equation of S as

$$\begin{aligned}x_1^2[c(y_0,y_1)+y_2q_0(y_0,y_1)+y_2^2l_0(y_0,y_1)+y_2^3]+x_0x_1(y_2^2l_1(y_0,y_1,y_2)+y_2q(y_0,y_1))\\+x_0^2y_2^2l_2(y_0,y_1,y_2)=0.\end{aligned}$$

If the projection of every fiber under π' contains L, then there are no $x_1^2c(y_0,y_1)$ terms. One can compute $\mathbf{B}(S)=2L\cup B'$ with $B'\cap L=\{q(y_0,y_1)^2=0\}\subset\mathbb{P}^1$. By our assumption, we see $q(y_0,y_1)=l(y_0,y_1)^2$ and we may set $l=y_1$ by passing to coordinate change. This completes the proof.

From here we assume that there is a fiber F whose projection under π' does not contain L. One can assume that F is over $(0,1)$. Then $L\cap\pi'(F)$ is given by $\{c(y_0,y_1)=0\}$. By our assumption, we see $c(y_0,y_1)=y_1^3$ after applying coordinate change and the triple point o is $(1,0,0)$. Still $\mathbf{B}(S)=2L\cup B'$ and $L\cap B'$ is given by $\{q(y_0,y_1)^2-y_1^3l_2(y_0,y_1,0)=0\}$. It follows that $q=y_1^2$ since $L\cap B'$ contains the triple point o.

♣

4.4. **Proof of Theorem 1.1 and Corollary 1.3.** By Theorem 4.2, the $(2,3)$-hypersurface of $\mathbb{P}^1\times\mathbb{P}^2$ with simple singularity is GIT-stable. Hence it's it makes sense to talk about its moduli space $\mathcal{M}_2$. Let $\mathcal{U}_2$ be the open part of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}|^s$ parameterizing such hypersurfaces in $\mathbb{P}^1\times\mathbb{P}^2$. Then we have $\mathcal{M}_2=\mathcal{U}_2/\operatorname{SL}_2\times\operatorname{SL}_3(\mathbb{C})$. By [Zho21, Theorem 1.1], there exists some rational number $0<c<1$ such that one can identify the K-stability of the log Fano pair $(\mathbb{P}^1\times\mathbb{P}^2,\epsilon S)$ for $\epsilon\in(0,c)$ with the GIT-stability of S under the action of $\operatorname{Aut}(\mathbb{P}^1\times\mathbb{P}^2)$ for S any $(2,3)$-hypersurface. Moreover, there is a isomorphism

$$\overline{\mathcal{M}}_2^K:=M^K_{\mathbb{P}^1\times\mathbb{P}^2,\epsilon}\xrightarrow{\cong}|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}|/\!\!/\operatorname{SL}_2\times\operatorname{SL}_3:=\overline{\mathcal{M}}_2^{GIT}$$

Note that the $(2,3)$ surfaces with simple singularities correspond to T_1 polarized K3 surfaces containing (-2) curves. So there is an open immersion $p:\mathcal{M}_2\to\mathcal{F}_{T_1}$ by Torelli theorem of lattice polarized K3 surfaces (see [Dol96, Section 3]). By Proposition 2.5, the boundary divisors of the image $\mathcal{P}_2\left(\mathcal{F}_{T_1}\right)$ are the union of $\mathbf{H}_u$ and $\mathbf{H}_h$. The last statement follows from Corollary 7.7.

5. Minimal orbits

In this section, we study the strictly semistable locus of the GIT compactification . According to Subsection 3.2, it suffices to discuss the points of type $(\alpha)-(\delta)$. The results are obtained by using the following criterion of Luna.

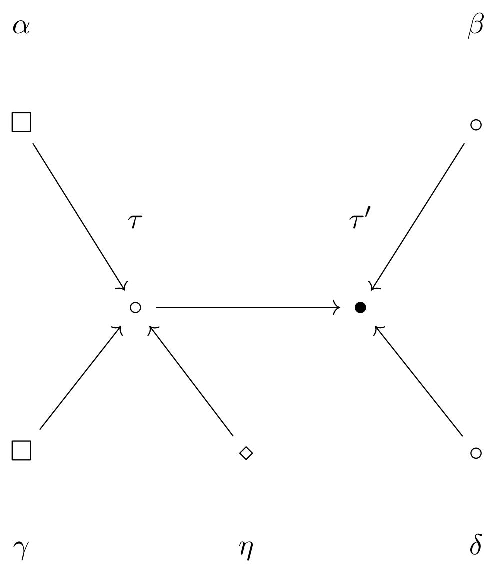

FIGURE 1. Incidence of the boundary components

Lemma 5.1 (Luna's criterion [Lun75])**.** *Let G be a reductive group acting on an affine variety X. If H is a reductive subgroup of G and $x \in X$ is stabilized by H, then the orbit $G \cdot x$ is closed if and only if $C_G(H) \cdot x$ is closed.*

As an immediate application, one can calculate the dimensions of the boundary strata (see subsection 6.3).

Lemma 5.2. *A generic cubic fourfold of type $(\alpha) - (\delta)$ gives a closed orbit. Therefore, each of the types $(\alpha) - (\delta)$ gives an irreducible boundary component for the GIT compactification. The dimensions of these boundary strata are 4 for (α) and (γ), 3 for (η), and 1 for (β) and (δ).*

Before starting the detailed analysis of the strata $(\alpha) - (\delta)$, we describe the relations of their common degeneration. One observes that the types (α), (γ) and (η) have a common specialization

$$(\tau)\colon a_1x_1^2y_0^2y_1 + a_2x_0x_1y_0y_1y_2 + a_3x_0^2y_1y_2^2 = 0.$$

The stabilizer of (τ) contains a 1-PS $(4, -4, 3, 2, -5)$. Its equation further degenerates (for $a_1a_3 = 0$) to

$$(\tau')\colon x_0x_1y_0y_1y_2 = 0.$$

In addition, (τ') is also a specialization of the cases (β), (γ) and (δ). The resulting incidence diagram is given in the figure 1.

Lemma 5.3. *A generic hypersurface of type (τ) and (τ') is semi-stable with closed orbit.*

Proof. This follows from Luna's criterion cited above. The stabilizer of (τ) and (τ') both contain a 1-PS H of distinct weights. Thus it suffices to check the semi-stability with respect to the standard maximal torus $T = C_G(H)$ in G. The proposition follows easily. For example, the fact that (τ) is semi-stable follows from the simple observation that $(-2a+2b+c)+(2a+c+2(-b-c)) = 0$ implies either $-2a + 2b + c \geq 0$ or $2a + c + 2(-b - c) \geq 0$, where $(a, -a, b, c, -(b + c))$ are the weights of a 1-PS of T. ♣

We now do the case-by-case analysis of the minimal orbits of type $(\alpha)-(\delta)$. The common feature of all these cases is that the analysis reduces to some lower dimensional GIT problem. Generally speaking, as the dimension of stratum increases, the analysis gets more involved. We first consider several cases with higher dimension.

Proposition 5.4. *Let S be a hypersurface of type (α). Then S is singular along a vertical line L in fiber F . And S has a corank 3 singularity p such that $p \notin F$ and $\pi'(p) \notin \pi'(L)$. Moreover, we have*

(1) S is unstable if one of the following conditions hold
- (i) *$S = \mathbb{P}^2 \cup S'$ for some $S' \in |\mathcal{O}(1,3)|$, where $\mathbb{P}^2 = \pi^{-1}(\pi(F))$. And $\mathbb{P}^2 \cap S' = l_1^2\pi'(L)$.*
- (ii) *S has an another triple point o such that $o \in L$. And S is singular along a horizontal line containing o. In addition, the branch locus $\mathbf{B}(S)$ contains a triple line.*
- (iii) *The fiber over $\pi(p)$ is a triple line $3L_1$ and the branch locus $\mathbf{B}(S)$ also contains $3L_1$.*

(2) The orbit of S is not closed if one of the following conditions hold
- (i) *S is singular along a vertical line in the fiber over $\pi(p)$. It degenerates to type (τ).*
- (ii) *$S = (\mathbb{P}^1 \times \mathbb{P}^1) \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,2)|$. It degenerates to type (τ).*
- (iii) *$S = \mathbb{P}^2 \cup S'$ for some $S' \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,3)|$. It degenerates to type (τ').*

Otherwise, S is semistable with closed orbit.

Proof. By inspecting the equation of (α), it is easy to see that S is singular along a vertical line $\{x_1 = y_2 = 0\}$, whose projection under π' is given by $L\colon \{y_2 = 0\}$. And S has a corank 3 singularity at $(0,1,0,0,1)$. The converse is trivial.

Since the stabilizer of type (α) contains a 1-PS:

$$H_1 = \{\operatorname{diag}(t^3, t^{-3}, t^2, t^2, t^{-4}) \mid t \in \mathbb{C}^*\}$$

and the center $C_G(H_1) \cong \mathbb{C}^* \times \mathbb{C}^* \times SL_2(\mathbb{C}) \times \mathbb{C}^*$, we can reduce our problem to a simpler GIT problem $V^{H_1} /\!\!/ C_G(H_1)$ by Luna's criterion where $V^{H_1} = \langle x_0^u x_1^{2-u} y_0^v y_1^{3-u-v} y_2^u \rangle$. Then following the Hilbert-Mumford criterion, by diagonalizing 1-PS λ into the form

$$\lambda(t) = \operatorname{diag}(t^a, t^{-a}, t^b, t^c, t^{-b-c}) \ \ (b \geq c)$$

we can deduce that S is unstable if and only if its defining equation is one of the following forms up to a coordinate change of y_0 and y_1

- $x_1^2 c(y_0, y_1) + \mu x_0 x_1 y_1^2 y_2 = 0$.
- $x_1^2 y_1^2 l(y_0, y_1) + \mu x_0 x_1 y_1^2 y_2 + \nu x_0^2 y_1 y_2^2 = 0$.
- $\mu x_1^2 y_1^3 + \nu x_0 x_1 y_1^2 y_2 + x_0^2 y_2^2 l(y_0, y_1) = 0$.

And the orbit of S is not closed if its equation has one of the following forms

- $x_1(x_1 c(y_0, y_1) + x_0 q(y_0, y_1) y_2) = 0$.
- $x_0(x_1 q(y_0, y_1) y_2 + x_0 l(y_0, y_1) y_2^2) = 0$.
- $y_1(x_1^2 q_1(y_0, y_1) + x_0 x_1 l_1(y_0, y_1) y_2 + \mu x_0^2 y_2^2) = 0$.
- $x_1^2 y_1^2 l_0(y_0, y_1) + x_0 x_1 y_1 y_2 l_1(y_0, y_1) + x_0^2 l_2(y_0, y_1) y_2^2 = 0$.

Then the first two cases degenerate to type (τ'), the last two cases degenerates to type (τ).

The proof of the geometric description is similar to Theorem 4.2 and Theorem 4.3. To give an example, we show the third unstable case. If the equation of S is given by $\mu x_1^2 y_1^3 + \nu x_0 x_1 y_1^2 y_2 + x_0^2 y_2^2 l(y_0, y_1) = 0$. Then it is clear that the fiber over $(0,1)$ is a triple line $\{y_1^3 = 0\}$ and the branch locus also contains $\{y_1^3 = 0\}$.

Conversely, if S is of type (α) whose fiber over $\pi(p)$ is a triple line $3L_1$. Then one can assume that $L_1\colon \{y_1=0\}$ up to a coordinate change. Its equation has the form

$$x_1^2y_1^3+x_0x_1q(y_0,y_1)y_2+x_0^2l(y_0,y_1)y_2^2=0$$

with $\mathbf{B}(S)$ given by $y_2^2(q(y_0,y_1)^2-y_1^3l(y_0,y_1))=0$. Thus one can deduce that $q=y_1^2$ by our assumption. In conclusion, the equation of S can be written as

$$x_1^2y_1^3+x_0x_1y_1^2y_2+x_0^2y_2^2l(y_0,y_1)=0.$$

♣

Proposition 5.5. *Let S be a hypersurface of type (γ), then $S=(\mathbb{P}^1\times\mathbb{P}^1)\cup S'$ for some $S'\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,2)|$ and S is singular along a horizontal line C such that the intersection of $\mathbb{P}^1\times\mathbb{P}^1$ and C is empty. Moreover, we have*

- (i) *S is unstable if the intersection $S'\cap(\mathbb{P}^1\times\mathbb{P}^1)$ is of type $\mathcal{O}(1,0)\cup\mathcal{O}(1,2)$ or $\mathcal{O}(0,1)\cup\mathcal{O}(2,1)$, where the two components meet at a double point.*
- (ii) *The orbit of S is not closed if the intersection $S'\cap(\mathbb{P}^1\times\mathbb{P}^1)$ is a singular curve. It degenerates to type (τ).*

Otherwise, S is semistable with closed orbit.

Proof. The stabilizer of type (γ) contains a 1-PS:

$$H_2=\{\mathrm{diag}(1,1,t^1,t^1,t^{-2})\mid t\in\mathbb{C}^*\}.$$

The center $C_G(H_2)\cong SL_2(\mathbb{C})\times SL_2(\mathbb{C})\times\mathbb{C}^*$. By Luna's criterion, we can reduce our problem to a simpler GIT analysis $V^{H_2}/\!\!/C_G(H_2)$ where $V^{H_2}=\langle x_0^ux_1^{2-u}y_0^vy_1^{2-v}y_2\rangle$. Any 1-PS λ can be diagonalized in the form

$$\lambda(t)=\mathrm{diag}(t^a,t^{-a},t^b,t^c,t^{-b-c}).$$

where $a\geq 0$ and $b\geq c$. Then following the Hilbert-Mumford criterion, we deduce that S is unstable if and only if its defining equation is one of the following forms after we make a coordinate change.

- $y_1y_2(x_1^2l(y_0,y_1)+\mu x_0x_1y_1+\nu x_0^2y_1)=0.$
- $x_1y_2(x_1q(y_0,y_1)+\mu x_0y_1^2)=0.$

And the orbit of S is not closed if its equation has the form

- $y_2(x_1^2q(y_0,y_1)+x_0x_1y_1l(y_0,y_1)+\mu x_0^2y_1^2)=0.$
- $y_1y_2(x_1^2l_0(y_0,y_1)+x_0x_1l_1(y_0,y_1)+x_0^2l_2(y_0,y_1))=0.$
- $x_1y_2(x_1q_1(y_0,y_1)+x_0q_0(y_0,y_1))=0.$

Then first case degenerates to type (τ), and the other two cases are unstable.

The remaining part follows by an analysis of the geometric description of the possible degenerations similar to before. ♣

The remaining cases are quite similar, we omit the details.

Proposition 5.6. *Let S be a hypersurface of type (η) given by the equation (3.4). Then S has two isolated $\widetilde{E}_7$-type singularities p and q such that $\pi(p)\neq\pi(q)$ and $\pi'(p)\neq\pi'(q)$. The fiber over $\pi(p)$ and $\pi(q)$ contain a line L_1 and L_2 respectively with $L_1\neq L_2$. And $\mathrm{mult}_{\pi'(p)}((\pi'(F)),L_1)\geq 2$, $\mathrm{mult}_{\pi'(q)}((\pi'(F)),L_2)\geq 2$ for any fiber F. Moreover, we have*

(1) S is unstable if one of the following conditions holds

- (i) *$S=\mathbb{P}^2\cup S'$ for some $S'\in|\mathcal{O}(1,3)|$, meeting along a triple line.*

(ii) *The fiber over p and q contain double line $2L_1$ and $2L_2$ respectively, with* $\mathbf{B}(S) = 3L_1 \cup 3L_2$.

(iii) *The fiber over p and q are l^2L_1 and l^2L_2 respectively. And the branch locus* $\mathbf{B}(S)$ *contains $4l$.*

(2) The orbit of S is not closed if the fiber over p or q contains a double line $2l$ and the branch locus also contains the double line $2l$. It degenerates to type (τ).

Otherwise, it is semistable with closed orbit.

Proposition 5.7. *Let S be a hypersurface of type (β), then S has two isolated $\widetilde{E}_8$-type singularities p and q such that $\pi(p) \neq \pi(q)$ and $\pi'(p) \neq \pi'(q)$. The fiber over p and q are $3L_1$ and $3L_2$ respectively with $L_1 \neq L_2$. Moreover, we have*

(i) *S is unstable if the branch locus contains a triple line.*

(ii) *The orbit of S is not closed if the branch locus contains a double line. It degenerates to type (τ').*

Otherwise, S is semistable with closed orbit.

Proposition 5.8. *Let S be a hypersurface of type (δ), then $S = \mathbb{P}^2 \cup \mathbb{P}^2 \cup (\mathbb{P}^1 \times C)$ for some cubic curves C in $\mathbb{P}^2$. Moreover, we have*

(i) *S is unstable if if C has cusps or triple points, or is reducible with two components tangent at a point.*

(ii) *The orbit of S is not closed if C is the union of a conic and a transversal line.*

Otherwise, it's semistable with closed orbit.

Proof. One can deduce this from the well-known GIT analysis of plane cubic curves, for example see [Mum77, Section 1.11].

♣

6. Boundary of $\overline{\mathcal{M}}_2^{\mathrm{GIT}}$

6.1. **Stable surface with isolated singularity.** The following is the direct consequence of Theorem 4.2.

Proposition 6.1. *Let $S \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,3)|$ with only isolated singularity. Then S is stable if and only if the non-ADE singularities are in one of the following types*

(ζ). *$p \in \mathrm{Sing}(S)$ is of $\widetilde{E}_7$ type and there exists a fiber F such that $\pi'(F)$ does not contain $\pi'(p)$.*

(η). *$p \in \mathrm{Sing}(S)$ is of $\widetilde{E}_8$ type and the fiber containing p is not a triple line.*

Proof. Suppose S has only at worst isolated singularities of type (ζ) or (η). By Theorem 4.2, we immediately know that S is stable. Conversely, suppose S is stable and it has a non-ADE corank 2 isolated singularity at p: $(1, 0, 1, 0, 0)$. Then p is an isolated $\widetilde{E}_7$ or $\widetilde{E}_8$ singularity. Let us analyse the local equation of p. Up to a change of coordinates preserving p, the defining equation has three possibilities :

- $\mu x_1^2 + 3$ jets $+4$ jets $+5$ jets,
- $\mu y_1^2 + 3$ jets $+4$ jets $+5$ jets,
- $\mu(x_1 + y_1)^2 + 3$ jets $+4$ jets $+5$ jets.

Direct computation shows that the only possible case is the third one.

If p is $\widetilde{E}_7$ type, take the change of coordinate: $x_1+y_1\to z$. Then there is no term $y_1^ky_2^{3-k}$ in the third jets. The equation of S is of the form

$$\begin{aligned}x_1^2(c_0(y_1,y_2)+y_0q_0(y_1,y_2)+y_0^2l_0(y_1,y_2))+x_0x_1(c_1(y_1,y_2)+y_0q_1(y_1,y_2))\\+x_0^2(y_1^2l_1(y_1,y_2)+y_1y_2^2)+\mu(x_1^2y_0^3+2x_0x_1y_0^2y_1+x_0^2y_0y_1^2)=0,\end{aligned}\tag{6.1}$$

satisfying $a_{210}-a_{120}+a_{030}=0$, $a_{201}-a_{111}+a_{021}=0$ and $a_{102}=a_{012}$. Where a_{ijk} represents the coefficient of $x_0^{2-i}x_1^iy_0^{3-j-k}y_1^jy_2^k$. Conversely, for the general equations of the form (6.1), take the change of coordinate: $x_1+y_1\to z$. Then the weights on variables z, y_1 and y_2 are $(\frac{1}{2},\frac{1}{4},\frac{1}{4})$. Thus p is $\widetilde{E}_7$ type. And one can see that $\pi'(F)$ does not contain $\pi'(p)=(1,0,0)$, where F is the fiber over $(0,1)$.

If p is $\widetilde{E}_8$ type, take the change of coordinate: $x_1+y_1\to z$. Suppose that the weight of y_1 is less than $\frac{1}{6}$, then there is no term y_1^3, $y_2^2y_1$, $y_2y_1^2$, zy_1^2 in the third jets. No term y_1^4, $y_1^3y_2$, and y_1^5 in the fourth and fifth jets. Thus the equation of S is of the form

$$\begin{aligned}x_1^2(y_2^3+y_2^2y_1+y_2y_1^2+y_0q_0(y_1,y_2)+y_0^2l_0(y_1,y_2))\\+x_0x_1(c_0(y_1,y_2)+y_0q_1(y_1,y_2))+x_0^2c_1(y_1,y_2)\\+\mu(x_1^2y_0^3+2x_0x_1y_0^2y_1+x_0^2y_0y_1^2)=0,\end{aligned}\tag{6.2}$$

satisfying $a_{201}-a_{111}+a_{021}=0$, $a_{220}=a_{130}$, $a_{211}=a_{121}$, $a_{120}=2a_{210}=2a_{030}$ and $a_{102}=a_{012}$. Conversely, for the general equations of the form (6.2), take the change of coordinate: $x_1+y_1\to z$. Then the weights on variables z, y_1 and y_2 are $(\frac{1}{2},\frac{1}{6},\frac{1}{3})$. And one can see that the fiber containing p is not a triple line.

Similarly, if the weight of y_2 is less than $\frac{1}{6}$, the equation of S is of the form

$$\begin{aligned}x_1^2(c_0(y_1,y_2)+y_0q_0(y_1,y_2)+y_0^2l_0(y_1,y_2))+x_0^2y_1^2l_1(y_1,y_2)\\+x_0x_1(y_1^3+y_1^2y_2+y_1y_2^2+y_0y_1l_2(y_1,y_2))\\+\mu(x_1^2y_0^3+2x_0x_1y_0^2y_1+x_0^2y_0y_1^2)=0.\end{aligned}$$

satisfy $a_{201}-a_{111}+a_{021}=0$. One can easily see that S is singular along $\{x_1=y_1=0\}$. Thus p is not an isolated singularity. ♣

6.2. Stable surface with non-isolated singularity.

Proposition 6.2. *Let S be a bidegree $(2,3)$ hypersurface of $\mathbb{P}^1\times\mathbb{P}^2$ with non-isolated singularities. Then one of the following situations holds:*

(i) $\mathrm{Sing}(S)$ *contains a curve of type* $(0,1)$ *and S is not properly stable.*
(ii) $\mathrm{Sing}(S)$ *contains a curve of type* $(1,0)$ *and S is not properly stable.*
(iii) $\mathrm{Sing}(S)$ *contains an elliptic curve and S is not properly stable unless S is of type* $(1,1)+(1,2)$ *or* $(0,2)+(2,1)$.
(iv) $\mathrm{Sing}(S)$ *contains a section of degree* 1 *and S is stable.*
(v) $\mathrm{Sing}(S)$ *contains a section of degree* 2 *and S is stable.*

Proof. Let $C\subseteq\mathrm{Sing}(S)$ be an irreducible curve. If $S=S_1\cup S_2$ is reducible, one can assume the bidegree of S_1 is $(1,0)$, $(0,1)$, $(0,2)$ or $(1,1)$. This gives the elliptic curve cases. For the first two cases, S is not properly stable by Theorem 4.2.

If S is irreducible, then C is either a section or contained in some fiber as $C\cap$ fiber is at most 1 by calculating the genus. Suppose that C is contained in some fiber. Then the only possible case is that S is singular along a vertical line by some simple computation. Next, we assume that C is a section. Let d be the degree of C as a section. Take a general hypersurface

H of type $(0,1)$, then $H\cap S$ is an irreducible curve and it is singular along $H\cap C$. Since the arithmetic genus of $H\cap S$ is at most 2, then $H\cap C$ has at most 2 points. It follows that d is at most 2.

The case $d=0$. This is just (N4) type, which is not properly stable.

The case $d=1$. We claim that up to change of coordinate, the equation of S is of the form

$$(\theta): y_2^2c_{2,1}(x_0,x_1|y_0,y_1,y_2)+y_2(x_0y_1-x_1y_0)q_{1,1}(x_0,x_1|y_0,y_1)+(x_0y_1-x_1y_0)^2l(y_0,y_1)=0,$$

where $c_{2,1}(x_0,x_1|y_0,y_1,y_2)$ represents the bidegree $(2,1)$ homogeneous polynomial of x_0, x_1 and y_0, y_1, y_2. And $q_{1,1}(x_0,x_1|y_0,y_1)$ is similar. After changing coordinates, we may assume that S is singular along a curve C with parametric expression $\{(x_0,x_1,x_0,x_1,0)|(x_0,x_1)\in\mathbb{P}^1\}$. Its eqution is given by

$$C\colon\{y_2=x_0y_1-x_1y_0=0\}.$$

Since S contains C, the equation of S is of the form

$$f=y_2g+(x_0y_1-x_1y_0)h=0.$$

where $g\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,2)|$ and $h\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$. Via computing $\frac{\partial f}{\partial y_2}$ and $\frac{\partial f}{\partial y_1}$ on C, one can deduce that both $\{g=0\}$ and $\{x_0h=0\}$ contain C. This gives the equation of type (θ).

Conversely, if the equation of S is of type (θ), the Jacobi of S is zero along C.

The case $d=2$. We claim that the general equation is of the form

$$\begin{aligned}(\phi): (x_0y_1-x_1y_0)^2l_0(y_0,y_1,y_2)+(x_0y_1-x_1y_0)(x_0y_2-x_1y_1)l_1(y_0,y_1,y_2)\\ +(x_0y_2-x_1y_1)^2l_2(y_0,y_1,y_2)=0.\end{aligned}$$

As above, one can assume that S is singular along the curve

$$C:\{x_0y_2-x_1y_1=x_0y_1-x_1y_0=0\}.$$

Since C is contained in S, the equation of S is of the form

$$f=(x_0y_1-x_1y_0)g+(x_0y_2-x_1y_1)h=0,$$

where g, $h\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,2)|$. One can deduce that both $\{x_1g=0\}$ and $\{x_0h=0\}$ contain C because $\frac{\partial f}{\partial y_2}$ and $\frac{\partial f}{\partial y_0}$ are zero on C. Furthermore, using the parametric equation of C, one can get that the only possibility is that both $\{g=0\}$ and $\{h=0\}$ contain C. Thus the equation of S is of type (ϕ).

Conversely, if the equation of S is of type (ϕ), the Jacobi of S is zero along C.

Finally, the assertion of stability follows directly from Theorem 4.2. ♣

6.3. **Proof of Theorem 1.2.** Recall that in the proof of Theorem 1.1, the K-moduli compactification $M^K_{\mathbb{P}^1\times\mathbb{P}^2,\epsilon}$ is isomorphic to the GIT compactification M_2^{GIT}. And the description of the boundary components simply follows from the combination of Proposition 5.4-5.8, 6.1 and 6.2. The dimension of the strictly semistable boundary $(\alpha)-(\delta)$ can be computed via Luna's criterion. For example,

$$\dim(\alpha)=\dim\mathbb{P}\left(V^{H_1}\right)/\!\!/C_G(H_1)=4,$$

where $V^{H_1}=\langle x_0^ux_1^{2-u}y_0^vy_1^{3-u-v}y_2^u\rangle$ and $C_G(H_1)\cong\mathbb{C}^*\times\mathbb{C}^*\times SL_2(\mathbb{C})\times\mathbb{C}^*$. For stable components, we can also compute the dimension as follows:

- For (ζ), it can be viewed as the quotient space $\mathbb{P}(V_1)/G_1$, where V_1 is the vector space spanned by monomials in the equation (ζ) and G_1 is the group fixing the singular point p: $(1,0,1,0,0)$ and the 2- jets $(x_1+y_1)^2$. As $\dim \mathbb{P}(V_1)=16$ and $\dim G=6$, we get $\dim(\zeta)=10$. Similarly, (ξ) is the quotient space $\mathbb{P}(V_2)/G_2$ with $\dim \mathbb{P}(V_2)=13$ and $\dim G_2=6$. It follows that $\dim(\xi)=7$.
- Similar as above, (θ) is the quotient space $\mathbb{P}(V_3)/G_3$, where V_3 is the vector space spanned by monomials in the equation (θ) and G_3 is the group fixing the section of degree 1 with the parametric expression $\{(x_0,x_1,x_0,x_1,0)|(x_0,x_1)\in\mathbb{P}^1\}$. One can calculate that $\dim\mathbb{P}(V_3)=14$ and $\dim G_3=6$. It follows that $\dim(\theta)=8$. Similarly, one can see that $\dim(\phi)=5$.
- (r_1) is a union of a $(1,1)$ hypersurface and a $(1,3)$ hypersurface. The elements of (r_1) are parameterized by the product of two projective spaces $\mathbb{P}(V)\times\mathbb{P}(V')$, where $V=H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,1))$ and $V'=H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,3))$. Thus $\dim(r_1)=\dim(V)+\dim(V')-\dim \mathrm{PGL}_2\times\mathrm{PGL}_3=13$. with the same method, one can get that $\dim(r_2)=2$.

♣

7. Arithmetic compactifications

7.1. **Baily-Borel compactifications of $\mathcal{F}_{T_n}$.** Recall that the type II and III boundary components of the Baily-Borel compactifications of $\mathcal{F}_{T_n}$ correspond one-to-one to the classes of rank 2 and 1 isotropic sublattices of Σ_n modulo Γ_n.

Let $I_2(\Sigma_n)=\bigcup_e I_{2,e}(\Sigma_n)$ be the set-theoretic partition of rank 2 isotropic sublattices of Σ_n where

$$I_{2,e}(\Sigma_n)=\left\{J\in I_2(\Sigma_n)\,\middle|\,|H_J|=e\right\}.$$

Here $H_J=(J^\perp)^\perp_{\Sigma_n^*}$ with both the orthogonals taken inside Σ_n^*, is a isotropic subgroup of A_{Σ_n} with the induced finite quadratic form. Therefore $e^2\,\big|\,(9=|A_{\Sigma_n}|)$ and $e=1,3$.

Lemma 7.1. *For $n=1,2,3$ and $e=1,3$ fixed, all $J^\perp/J$ have the signature $(0,16)$ and the same discriminant form $(A_{n,e},q_{n,e})$ for $J\in I_{2,e}(\Sigma_n)$, i.e., are in the same genus which we denote by $\mathcal{G}(n,e)$. Moreover, for $J\in I_{2,e}(\Sigma_n)$ and $e=1$ or $e=3$, there exists $\{v_1,\dots,v_{20}\}$ a basis of the lattice Σ_n such that $J=\langle v_1,v_2\rangle$, $J^\perp=\langle v_1,\dots,v_{18}\rangle$ and the quadratic form in this basis is of the form*

$$Q=\begin{pmatrix}0&0&A\\0&B&0\\{}^tA&0&D\end{pmatrix}$$

where

$$A=\begin{pmatrix}0&1\\e&0\end{pmatrix},D=\begin{pmatrix}2t&0\\0&0\end{pmatrix},$$

$0\le t<e$ is uniquely determined by n and B is any matrix representing the quadratic form on $J^\perp/J$.

Proof. It's well-known that we have the isomorphism of finite quadratic form

$$A_{J^\perp/J}\cong H_J^\perp/H_J$$

(c.f. [Cam18, Proposition 6.5] and [Sca87, Proof of Lemma 5.1.3]). For $e=1$, we have $A_{J^\perp/J}\cong H_J^\perp/H_J\cong A_{\Sigma_n}$. For $e=3$, one can find that $H_J^\perp=H_J$ for $n=1,2,3$ and we have $A_{J^\perp/J}=\{0\}$. So $\mathcal{G}(n,e)$ is well-defined.

For the second statement, the proof of [Sca87, lemma 5.2.1] works and we sketch the proof here for the convenience of readers. Since $J \subset J^{\perp}$ are primitive sublattices of Σ_n, we can choose a basis of Σ_n in which $J = \langle v_1, v_2 \rangle$, $J^{\perp} = \langle v_1, \ldots, v_{20} \rangle$ The matrix $Q(v_i, v_j)$ will then have the form

$$Q = \begin{pmatrix} 0 & 0 & A_0 \\ 0 & B & C_0 \\ A_0^t & C_0^t & D_0 \end{pmatrix}$$

where B represents the bilinear form of $J^{\perp}/J$. Recalling that $J \in I_{2,e}(\Sigma_n)$, $H_J \cong Z/eZ$, a direct application of elementary divisors produces matrices U,$Z \in \mathrm{GL}_2(\mathbb{Z})$ such that $U^t A_0 Z = \left(\begin{smallmatrix} 0 & 1 \\ e & 0 \end{smallmatrix}\right)$. Therefore, the change of basis described by the matrix $g = \mathrm{diag}(U, \mathrm{Id}_{16}, Z) \in \mathrm{GL}_{20}(\mathbb{Z})$ transform A_0 into A, C_0 into C_1 and preserves B.

Next Scattone showed that there exist integral matrices V and Y such that $BY + VA + C_1 = 0$ if $\gcd(e, \det B) = 1$, which is satisfied since $e^2 \cdot \det B = 9$ and we always have e or $\det B$ equals 1. By choosing V and Y as above, and applying change of basis $g = \begin{pmatrix} I & V^t & \\ & I & Y \\ & & I \end{pmatrix}$ we put Q into the form $\begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ A^t & 0 & D_2 \end{pmatrix}$. Finally, by applying $g = \begin{pmatrix} I & 0 & W \\ & I & 0 \\ & & I \end{pmatrix}$ and choose a appropriate W, we can put D_2 into the required form. For $e = 1$, t can only be 1. ♣

Let $N_H(J) := \mathrm{Im}\left(\mathrm{Stab}_H(J) \xrightarrow{r} \mathrm{GL}(J) \cong \mathrm{GL}_2(\mathbb{Z})\right)$ be image of the stabilizer of J under the action of the group $H < O(\Sigma_n)$ in $\mathrm{GL}_2(\mathbb{Z})$.

Lemma 7.2. *For $J \in I_{2,e}(\Sigma_n)$, we have:*

$$N_{O(\Sigma_n)}(J) \cong \left\{ \begin{pmatrix} a & be \\ c & d \end{pmatrix} \in \mathrm{GL}_2(Z) \,\middle|\, a^2 \equiv 1 \pmod{e} \right\}$$

and

$$N_{O^+(\Sigma_n)}(J) \cong \left\{ \begin{pmatrix} a & be \\ c & d \end{pmatrix} \in \mathrm{SL}_2(Z) \,\middle|\, a^2 \equiv 1 \pmod{e} \right\}.$$

Proof. For the case of polarized K3 surface, this is [Sca87, Lemma 5.6.3, 5.6.6]). The proof for our case is basically the same. Let g be a general element of $\mathrm{Stab}_{O(\Sigma_n)}(J)$ of the form

$$g = \begin{pmatrix} U & V & UW \\ & X & Y \\ & & Z \end{pmatrix}.$$

Take Q as in the previous lemma. The condition $g^t Q g = Q$ gives

$$A = U^t AZ, B = X^t BX, C = X^t BY + V^t AZ, D = Z^t DZ + Y^t BY + W^t U^t AZ + Z^t A^t UW$$

which implies $U \in N_{O(\Sigma_n)}(J)$ is of the form $\begin{pmatrix} a & be \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ and $D \equiv \begin{pmatrix} 2a^2 t & * \\ * & * \end{pmatrix} \pmod{2e}$. Note that in our case we have $\gcd(t, e) = 1$, which implies $a^2 \equiv 1 \pmod{e}$ is a necessary condition.

Conversely, if $U \in \mathrm{GL}_2(\mathbb{Z})$ is an arbitrary matrix of the form $U = \begin{pmatrix} a & be \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ with $a^2 \equiv 1 \pmod{e}$, let $Z = A^{-1}\left(U^{-1}\right)^t A$ and then

$$g = \begin{pmatrix} U & 0 & UW \\ & I & 0 \\ & & Z \end{pmatrix}$$

gives a lift of U in $\mathrm{Stab}_{O(\Sigma_n)}(J)$ with an appropriately chosen W.

The second statement holds naturally by considering the sign of the determinant.

♣

Proposition 7.3. *There is a bijection between*

$$I_{2,e}(\Sigma_n)/\Gamma_n \xleftrightarrow{1:1} \mathcal{G}(n,e).$$

Proof. We divide the proof into three steps.

Firstly we show that $I_{2,e}(\Sigma_n)/O(\Sigma_n) \xleftrightarrow{1:1} \mathcal{G}(n,e)$. Let $L(e,t)$ be the lattice given by the Gram matrix $\left(\begin{smallmatrix} 0 & e \\ e & 2t \end{smallmatrix}\right)$. By Lemma 7.1 above, given $J \in I_{2,e}(\Sigma_n)$, one get $\Sigma_n \cong U \oplus L(e,t) \oplus \left(J^\perp/J\right)$ where $J^\perp/J$ belongs to $\mathcal{G}(n,e)$. Conversely, for any $M \in \mathcal{G}(n,e)$ since $U \oplus L(e,t) \oplus M$ and Σ_n belong to the same genus, we have $\Sigma_n \cong U \oplus L(e,t) \oplus M$ so by the uniqueness of indefinite even lattice (see [Nik79, Corollary 1.13.3]). With this isomorphism one can find $J \in I_{2,e}(\Sigma_n)$ such that $J^\perp/J \cong M$. Therefore,

$$\mathcal{G}(n,e) \to I_{2,e}(\Sigma_n)/O(\Sigma_n)$$

is a bijection.

Next we show that

$$I_{2,e}(\Sigma_n)/O^+(\Sigma_n) \to I_{2,e}(\Sigma_n)/O(\Sigma_n)$$

is a bijection. Let $H_1 < H_2$ be a finite index subgroup acting on set $\mathscr{S}$. Note that the fiber of $\mathscr{S}/H_1 \to \mathscr{S}/H_2$ over $[a]/H_2$ has cardinality $\frac{[H_2:H_1]}{[\mathrm{Stab}_{H_2}(a):\mathrm{Stab}_{H_1}(a)]}$ for $a \in \mathscr{S}$. In our case, $[O(\Sigma_n) : O^+(\Sigma_n)] = 2$. By lemma 7.2 we have $N_{O^+(\Sigma_n)}(J) \neq N_{O(\Sigma_n)}(J)$, hence $\mathrm{Stab}_{O^+(\Sigma_n)}(J) \neq \mathrm{Stab}_{O(\Sigma_n)}(J)$. Then one can conclude that $I_2(\Sigma_n)/O(\Sigma_n) \cong I_2(\Sigma_n)/O^+(\Sigma_n)$.

Finally we consider the map

$$I_{2,e}(\Sigma_n)/\Gamma_n \to I_{2,e}(\Sigma_n)/O^+(\Sigma_n).$$

For $n = 1$ and 2, one have $O^+(\Sigma_n)/\Gamma_n \cong O(A_{\Sigma_n}) = \{\mathrm{id}, -\mathrm{id}\}$. Since $-\mathrm{id}$ doesn't change the Γ_n-class of J, the conclusion follows.

For $n = 3$, we have $O^+(\Sigma_3)/\Gamma_n \cong O(\Sigma_3)/\widetilde{O}(\Sigma_3) \cong O(A_{\Sigma_3}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by [Nik79, Theorem 1.14.2]. To determine $\mathrm{Stab}_{O^+}(J)/\mathrm{Stab}_{\Gamma_n}(J)$, we consider the exact sequence

$$1 \to \mathrm{Fix}_{O^+(\Sigma_n)}(J)/\mathrm{Fix}_{\Gamma_n}(J) \xrightarrow{i} \mathrm{Stab}_{O^+}(J)/\mathrm{Stab}_{\Gamma_n}(J) \xrightarrow{r} N_{O^+}(J)/N_{\Gamma_n}(J) \to 1.$$

Note that

$$\begin{tikzcd} 1 \arrow[r] & \ker\tilde{\beta} \arrow[r] \arrow[d, hook] & \mathrm{Fix}_{\Gamma_n}(J) \arrow[r, "\tilde{\beta}"] \arrow[d, hook] & \widetilde{O}^+(W_e) \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \ker\beta \arrow[r] & \mathrm{Fix}_{O^+(\Sigma_n)}(J) \arrow[r, "\beta"] & O^+(W_e) \arrow[r] & 1 \end{tikzcd}.$$

We have

$$1 \to \ker\beta/\ker\tilde{\beta} \to \mathrm{Fix}_{O^+(\Sigma_n)}(J)/\mathrm{Fix}_{\Gamma_n}(J) \to O^+(W_e)/\widetilde{O}^+(W_e) \to 1$$

for $W_e = J^{\perp}/J$ where $J \in I_{2,e}(\Sigma_n)$ and $\beta, \tilde{\beta}$ are the corresponding restriction maps.

For the case $e = 1$, note that $\Sigma_3 \cong U^2 \oplus W_1$, we have $\ker \beta = \ker \tilde{\beta}$. Hence

$$\mathrm{Fix}_{O^+(\Sigma_n)}(J)/\,\mathrm{Fix}_{\Gamma_n}(J) \cong O^+(W_e)/\widetilde{O}^+(W_e).$$

Since there are only two types of even (negative) definite unimodular lattice of rank 16, $D_{16} \subset D_{16}^+$ and $E_8 \oplus E_8$, Since the only two isotropic subgroups of A_{Σ_3} are differed by $\mathrm{id} \times (-\mathrm{id}) \in O(A_{W_1})$, one see that $\mathrm{id} \times (-\mathrm{id})$ is in the image of $O(W_1) \to O(A_{W_1})$ by Lemma 7.4 below, which generates $O(A_{W_1})$. Hence $O(W_1) \to O(A_{W_1})$ is surjective and $O^+(W_1)/\widetilde{O}^+(W_1) \cong O(A_{\Sigma_3})$. Form the exact sequence, the cardinality of the fiber over $[J]_{O^+(\Sigma_n)}$ is less or equal than 1 which gives the conclusion.

For $e = 3$, the lattice W_3 is unimodular and we have $\Sigma_3 \cong U \oplus U(3) \oplus W_3$. Using normalized basis, one can easily show that

$$N_{\Gamma_3}(J) \cong \left\{ \begin{pmatrix} a & 3b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(Z) \;\middle|\; a \equiv 1 \pmod 3 \right\}.$$

and $\left|N_{O^+}(J)/N_{\Gamma_3}(J)\right| = 2$ for $J \in I_{2,3}(\Sigma_3)$. One the other hand we have

$$\mathrm{Fix}_{O^+(\Sigma_n)}(J)/\,\mathrm{Fix}_{\Gamma_n}(J) \cong \left(\ker \beta / \ker \tilde{\beta}\right) \neq 1$$

So $\left[\mathrm{Stab}_{O^+}(J) : \mathrm{Stab}_{\Gamma_3}(J)\right] \geq 4$. The conclusion follows from the cardinality formula of the fiber.

♣

Lemma 7.4. [Ebe13, Proposition 3.6] *Let $\Lambda \subset \mathbb{R}^n$ be a lattice. There is a natural one-to-one correspondence between isomorphism classes of even overlattices $\Lambda \hookrightarrow \Gamma$ and orbits of isotropic subgroups $H \subset A_\Lambda$ under the image of the natural homomorphism $O(\Lambda) \to O(q_\Lambda)$. Unimodular lattices correspond to subgroups H with $|H|^2 = |A_\Lambda|$.*

In other words, there is a bijection between isotropic subgroups of discriminant group G_Λ and overlattices of Λ.

With Proposition 7.3, we have

Theorem 7.5. *Let $\mathcal{F}_{T_n}^*$ be the Baily-Borel compactification of $\mathcal{F}_{T_n}$. Then the boundary $\partial\mathcal{F}_{T_n} := \mathcal{F}_{T_n}^* - \mathcal{F}_{T_n}$ is given as follows:*

(i) *$n = 1$, $\partial\mathcal{F}_{T_1}$ consists of 14 modular curves and 2 points. All curves meet at one point. There are 2 curves meet at another point.*
(ii) *$n = 2$, $\partial\mathcal{F}_{T_2}$ consists of 3 modular curves and 2 points. All curves meet at one point. There are 2 curves meet at another point.*
(iii) *$n = 3$, $\partial\mathcal{F}_{T_3}$ consists of 10 modular curves and 2 points. All curves meet at one point. There are 2 curves meet at another point.*

Proof. By Proposition 7.3, there is a one-to-one correspondence between isomorphic classes of lattices in the genus $\mathcal{G}(n,e)$ and modular curves. To determine the classes L in $\mathcal{G}(n,e)$, by Nikulin's result [Nik79, Proposition 1.6.1], it's sufficient to classify all the the primitive embeddings of F of signature $(0,8)$ with $A_L \cong A_F$ and $q_L \cong -q_F$, into some $H \in \mathrm{II}_{0,24}$ unimodular signature $(0,24)$ lattices up to $O(H)$-equivalence.

For $n = 1$, we have $(A_{\Sigma_1}, q_{\Sigma_1}) \cong (A_{A_8}, -q_{A_8})$. So we consider the all possible primitive embeddings of A_8 for $e = 1$ and the primitive embedding of E_8 for $e = 3$, into some $H \in \mathrm{II}_{0,24}$. By simple facts about root lattice embeddings (see, for example [Nis96, Section 4.1]), it turns out that the modular curves are given by the following 14 isomorphic classes of lattices $L = J^{\perp}/J \in \mathcal{G}(1,e)$ labeled by its root system:

- $e = 3$: E_8^2, D_{16},
- $e = 1$: $E_8 \oplus D_7$, $E_7^2 \oplus A_1$, $E_7 \oplus A_8$, D_{15}, $D_{12} \oplus A_3$, $D_9 \oplus A_6\,(A_8 \hookrightarrow A_{15} \subset D_9 \oplus A_{15})$, $A_{15}\,(A_8 \hookrightarrow D_9 \subset D_9 \oplus A_{15})$, $A_9 \oplus D_6$, $E_6 \oplus D_7 \oplus A_2$, A_8^2, $A_{12} \oplus A_3$, $A_{15}\,(A_8 \hookrightarrow A_{24})$.

Note that there are two isomorphic classes with the same root system A_{15}. Since there is only one orbit of isotropic subgroup of $A_{A_8} \oplus A_{1,1}$ giving primitive embedding of A_8 into $H \in \mathrm{II}_{0,24}$ and their direct sum with A_8 are contained in the different overlattice, by Lemma 7.4, the two classes are different.

For $n = 3$, we have $(A_{\Sigma_3}, q_{\Sigma_3}) \cong (A_{E_6 \oplus A_2}, -q_{E_6 \oplus A_2})$. So we consider the primitive embedding of $E_6 \oplus A_2$ and E_8 respectively. There are actually 10 isomorphic classes of lattices $L = J^{\perp}/J \in \mathcal{G}(3,e)$ labeled by its root system:

- $e = 3$: E_8^2, D_{16}.
- $e = 1$: $E_8 \oplus E_6 \oplus A_2$, $D_{13} \oplus A_2$, $D_{10} \oplus A_5$, $E_7 \oplus D_7$, A_{14}, $E_6^2 \oplus A_2^2$, $D_4 \oplus A_{11}$, $D_7 \oplus A_8$.

For $n = 2$, we have $(A_{\Sigma_2}, q_{\Sigma_2}) \cong (A_{L_8}, -q_{L_8}) \cong (A_{A_8}, q_{A_8})$ where L$_8$ is the even lattice of rank 8 with the following Gram matrix under the basis $\{e_2, \cdots e_7, w\}$:

$$\begin{pmatrix} -2 & 1 & & & & & & 1 \\ 1 & -2 & 1 & & & & & 0 \\ & 1 & -2 & 1 & & & & 0 \\ & & 1 & -2 & 1 & & & 0 \\ & & & 1 & -2 & 1 & 1 & -1 \\ & & & & 1 & -2 & & 1 \\ & & & & 1 & & -2 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & -4 \end{pmatrix}.$$

where $\{e_1, \cdots, e_8\}$ are the bases of D_8. Consider the primitive embedding of L_8 and E_8 and we claim that the modular curves are given by the following isomorphic classes of lattices labeled by its root system:

- $e = 3$: E_8^2, D_{16}.
- $e = 1$: $E_8 \oplus A_8$ (given by $L_8 \hookrightarrow D_{16} \oplus E_8$).

The $e = 3$ case is as the same as above. For $e = 1$, note that we have $D_7 \subset L_8$ as a sublattice (not necessarily primitive), it's sufficient to consider the embedding of L_8 into lattices in $\mathrm{II}_{0,24}$ with root system E_8^3, $E_8 \oplus D_{16}$, $E_7^2 \oplus D_{10}$, D_{24}, D_{12}^2, D_8^3, $D_9 \oplus A_{15}$ and $E_6 \oplus D_7 \oplus A_{11}$. We denote the corresponding unique unimodular lattice by $M(R)$ for R the root lattice in the above list. The strategy is to cast out all the possibilities except for $L_8 \to M(E_8 \oplus D_{16})$ through explicit computation.

For example, suppose we have primitive embedding $\iota\colon L_8 \hookrightarrow L := M(D_8^3)$. Denote δ_1, δ_8 the two generators of A_{D_8} with $\delta_1^2 = -2$, $\delta_8^2 = -1$ and $\delta_1 \cdot \delta_8 = -\frac{1}{2}$. Then

$$L \cong \mathrm{Span}\{v_1 = (\delta_1, 0, 0),\ v_2 = (0, \delta_1, 0), v_3 = (0, 0, \delta_1)\} + D_8^3 \subset (D_8^*)^3.$$

One can write

$$\iota(w) = \sum_{i=1}^{3} m_i v_i + \sum_{i,j} c_j^{(i)} e_j^{(i)}$$

where $m_i, c_j^{(i)} \in \mathbb{Z}$, $i \in \{1,2,3\}, 1 \le j \le 8$ and $\{e_j^{(i)}\}_j$ is the basis of i-th copy of D_8. By the inner product on L_8, we get a group of integral coefficient equations of $m_i, c_j^{(i)}$. By simple linear algebra we deduce the equations have no integral solutions and get the contradiction.

The same method indicates that L_8 can not be primitively embedded into $E_8 \oplus E_8$ or $E_8 \oplus E_8 \oplus E_8$. However, since L_8 and A_8 have opposite discriminants, by Nikulin's result, L_8 can be primitively embedded into a unimodular lattice Y of rank 16. As there are only two choices for Y, namely D_{16}^+ and $E_8 \oplus E_8$, we have $Y = D_{16}^+$. Thus L_8 can be primitively embedded into D_{16}^+, hence also $M(D_{16} \oplus E_8)$.

For type III boundary components, by [Sca87, Lemma 4.1.2, Proposition 4.1.3], they are bijective to

$$\{\text{isotropic elements of } A_{\Sigma_n}\}/\{\pm 1\}.$$

For every n, there are only 2 such elements class, one is 0. Any isotropic rank 2 sublattice contain the vector corresponding to 0, and only those with $J^\perp/J \in \mathcal{G}(n,3)$ contain the vector corresponding to the nontrivial discriminant class. This completes the proof.

♣

7.2. **Looijenga's compactifications.** Let D be the type IV domain associated to an integral lattice (L,q) of signature $(2,n)$. Let Γ be a congruence arithmetic subgroup of $\widetilde{O}(L)$ the stable orthogonal group. Following [Loo03], let $\mathcal{H}$ be a Γ-invariant hyperplane arrangement, we set

$$\mathrm{D}^\circ = \mathrm{D} - \bigcup_{H\in\mathcal{H}} H$$

and define

- $\mathrm{PO}(\mathcal{H})$: the collection of subspaces $M \subseteq L$ which are intersection of members in $\mathcal{H}$ meeting D. Denote by

  $$\pi_M : \mathbb{P}(L_{\mathbb{C}}) - \mathbb{P}(M) \longrightarrow \mathbb{P}(L_{\mathbb{C}}/M)$$

  the natural projection. The projection also defines a natural subdomain $\pi_M \mathrm{D}^\circ \subseteq \mathrm{D}^\circ$ (*cf.* [Loo03, §7]).
- $\Sigma(\mathcal{H})$: the collection of the common intersection of $I^\perp$ and members in $\mathcal{H}$ containing I, where I is a $\mathbb{Q}$-isotropic line or planes of L.

Then Looijenga's compactification of $\Gamma\backslash\mathrm{D}^\circ$ can be interpreted as below: let

$$\widehat{\mathrm{D}} = \mathrm{D}^\circ \cup \coprod_{M\in\mathrm{PO}(\mathcal{H})} \pi_M\mathrm{D}^\circ \cup \coprod_{V\in\Sigma(\mathcal{H})} \pi_V\mathrm{D}^\circ, \tag{7.1}$$

we define $\overline{\Gamma\backslash\mathrm{D}}^{\mathcal{H}}$ to be the quotient $\Gamma\backslash\widehat{\mathrm{D}}$, which compactifies $\Gamma\backslash\mathrm{D}^\circ$ and boundary decomposes into finitely many stratas. The birational map $\overline{\Gamma\backslash\mathrm{D}}^{\mathcal{H}} \dashrightarrow (\Gamma\backslash\mathrm{D})^*$ can be resolved by the following diagram

$$\begin{tikzcd}
\widetilde{\Gamma\backslash\mathrm{D}}^{\mathcal{H}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\widetilde{\pi}"] & (\Gamma\backslash\mathrm{D})^{\Sigma(\mathcal{H})} \arrow[d, "\pi_{\mathcal{H}}"] \\
\overline{\Gamma\backslash\mathrm{D}}^{\mathcal{H}} \arrow[r, dotted] & \Gamma\backslash\mathrm{D}^*
\end{tikzcd} \tag{7.2}$$

where $\pi_{\mathcal{H}} : (\Gamma\backslash\mathrm{D})^{\Sigma(\mathcal{H})} \to (\Gamma\backslash\mathrm{D})^*$ is the $\mathbb{Q}$-Cartierization of the hyperplane arrangement in $\mathcal{H}$ and π_i are the blow up and blow down respectively. Note that when $\mathcal{H}^{(r)} \neq \emptyset$, i.e. there exists non-empty common intersection of D with r linearly independent hyperplanes in $\mathcal{H}$, the dimension of the boundary

$$\dim(\overline{\Gamma\backslash D}^{\mathcal{H}} - \Gamma\backslash D^\circ) \geq r - 1 \tag{7.3}$$

.

For $L = \Sigma_1$, $\Gamma = \widetilde{O}(\Sigma_1)$, let $\mathcal{H}_1$ be the collection of hyperplanes such that $\Gamma\backslash D^\circ$ is the complement of the Heegner divisors $\mathbf{H}_u$ and $\mathbf{H}_h$. To be specific, we have $\mathcal{H}_1 = \mathcal{H}_u \cup \mathcal{H}_h$ where

$$\mathcal{H}_u = \bigcup_{\substack{v\in\Sigma_1^*,\, v^2=-\frac{4}{9}\\ v\in 4\xi_1+\Sigma_n}} v^\perp, \quad \mathcal{H}_h = \bigcup_{\substack{v\in\Sigma_1^*,\, v^2=-\frac{10}{9}\\ v\in 2\xi_1+\Sigma_n}} v^\perp$$

Then $\overline{\mathcal{F}}_{T_1}^{\mathcal{H}_1}$ is the Looijenga's compactification of $\mathcal{F}_{T_1} - \mathbf{H}_u \cup \mathbf{H}_h$.

Lemma 7.6. *When $n = 1$, the boundary $\overline{\mathcal{F}}_{T_1}^{\mathcal{H}_1} - \Gamma\backslash D^\circ$ has codimension* 1.

Proof. Consider the signature $(1, 19)$ lattice N given by the span of $C, E, e_1, \cdots, e_{18}$ in where $e_i \cdot C = 2$, $e_i \cdot E = 1$ and $e_i \cdot e_j = \delta_{ij}$. Since $A_N \cong \mathbb{Z}/27\mathbb{Z}$, it's easy to show that N can be primitively embedded into the K3 lattice Λ by Nikulin 's results (see [Nik79, Corollary 1.12.3]). Then N can represent the common intersection of 18 hyperplanes in $\mathcal{H}_h$. This proves the assertion by (7.3). ♣

Corollary 7.7. *For $n = 1$, the GIT quotient $\overline{\mathcal{M}_2}^{GIT}$ is not isomorphic to the Looijenga's compactification $\overline{\mathcal{F}}_{T_1}^{\mathcal{H}_1}$.*

Proof. Since $\mathcal{M}_2$ is isomorphic to $\Gamma\backslash D^\circ \cong \mathcal{F}_{T_1} - \mathbf{H}_u \cup \mathbf{H}_h$, this is obtained by comparing the dimension of the boundary of $\overline{\mathcal{M}_2}^{GIT} - \mathcal{M}_2$ and $\overline{\mathcal{F}}_{T_1}^{\mathcal{H}_1} - \Gamma\backslash D^\circ$. ♣

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