MFE 409 Lecture 4
Measuring Value-at-Risk: Model-Building Approach

Valentin Haddad

Spring 2020

UCLA Anderson

Lecture Objectives

Measuring Value-at-Risk:

- How to judge validity of a VaR estimate?
- Historical approach
- Model-building approach
- How to get a measure for a given approach but also how to choose an appropriate approach

Outline

Model-Building Approach

- The main alternative to historical simulation is to make assumptions about the probability distributions of the returns on the market variables
- Sometimes called the variance-covariance approach

Normal Model

- Simplest and often-used assumption: normal distribution

Normal Model

- Simplest and often-used assumption: normal distribution

- VaR has a simple expression:

$$\text{VaR} = -\mu - \sigma \times z(c)$$

- Portfolios of normal returns are also normally distributed

- Estimation of normal distributions very developed

Portfolios

- With multivariate normal returns, portfolio returns are normally distributed
- Assume:
  - Each asset return R_i is normally distributed with mean 0 and variance σ_i^2
  - Pairwise correlations: ρ_{ij}
  - investment in each asset α_i

$$\underbrace{\Delta P}_{\text{portfolio gain}} = \sum_i \alpha_i R_i \sim \mathcal{N}(0, \sigma_P^2)$$

$$\sigma_P^2 = \sum_i \sum_j \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij}$$

Portfolios

- With multivariate normal returns, portfolio returns are normally distributed
- Assume:
  - Asset return $\mathbf{R}$ normally distributed with mean 0 and variance Σ
  - Investment vector α

$$\underbrace{\Delta P}_{\text{portfolio gain}} = \alpha'\mathbf{R} \sim \mathcal{N}(0, \sigma_P^2)$$

$$\sigma_P^2 = \alpha'\Sigma\alpha$$

Portfolios

- With multivariate normal returns, portfolio returns are normally distributed
- Assume:
  - Asset return $\mathbf{R}$ normally distributed with mean 0 and variance Σ
  - Investment vector α

$$\underbrace{\Delta P}_{\text{portfolio gain}} = \alpha'\mathbf{R} \sim \mathcal{N}(0, \sigma_P^2)$$

$$\sigma_P^2 = \alpha'\Sigma\alpha$$

- Multiply by $-z(c)$ to obtain VaR

Example: Imperfect Hedge

- Previous example:
  - Long position EUR 10m, $M_t =$ USD/EUR $= \$1.436$, $\sigma_M = 0.65\%$
  - Dollar position \$14.36m
  - VaR$= \$217,204$
- Suppose you want to hedge with Japanese Yens: $\sigma_J = 0.69\%$, $\rho_{MJ} = 0.2775$
  - What Yen position do you choose to hedge as well as possible?
  - What is your hedged VaR?

Buy x_y \$ in yens $\qquad x_E = 14.36$ m euros

$$\sigma_p^2 = x_E^2 \sigma_E^2 + x_y^2 \sigma_y^2 + 2\rho x_E x_y \sigma_y \sigma_E$$

Minimize with respect to x_y:

$$0 = 2 x_y \sigma_y^2 + 2\rho x_E \sigma_y \sigma_E$$

$$x_y = -\rho x_E \frac{\sigma_E}{\sigma_y}$$

$$VaR = 2.32\, x_E \sigma_E \sqrt{1-\rho^2}$$

What is my VaR?

$$\sigma_p^2 = x_E^2 \sigma_E^2 + \rho^2 x_E^2 \frac{\sigma_E^2}{\cancel{\sigma_y^2}} \cancel{\sigma_y^2} - 2\rho x_E \rho x_E \frac{\sigma_E}{\cancel{\sigma_y}} \cancel{\sigma_y} \sigma_E$$

$$= x_E^2 \sigma_E^2 + \rho^2 x_E^2 \sigma_E^2 - 2\rho^2 x_E^2 \sigma_E^2$$

$$= x_E^2 \sigma_E^2 (1-\rho^2)$$

Imperfect Hedge

- Want to hedge a position R_p using a hedging instrument R_h
- Optimal hedging position:

$$\alpha_{\text{hedge}} = -\rho \frac{\sigma_p}{\sigma_h}$$

- Variance of the hedged portfolio:

$$\text{Minimum variance} = \sigma_p^2 (1 - \rho^2)$$

Imperfect Hedge

- Want to hedge a position R_p using a hedging instrument R_h
- Optimal hedging position:

$$\alpha_{\text{hedge}} = -\rho \frac{\sigma_p}{\sigma_h}$$

- Variance of the hedged portfolio:

$$\text{Minimum variance} = \sigma_p^2(1 - \rho^2)$$

- VaR of the hedged portfolio:

$$\text{Minimum VaR} = \text{VaR}_p \sqrt{1 - \rho^2}$$

  - Only depends of correlation ρ

Volatility

- Often, volatility is defined as standard deviation of *log return*

$$\log\left(\frac{P_{t+1}}{P_t}\right)$$

- In risk management, typically the standard deviation of *simple return*

$$\frac{P_{t+1}}{P_t} - 1$$

Volatility

- Often, volatility is defined as standard deviation of *log return*

$$\underbrace{\log\left(\frac{P_{t+1}}{P_t}\right)}_{\text{log return}}$$

- In risk management, typically the standard deviation of *simple return*

$$\underbrace{\frac{P_{t+1}}{P_t} - 1}_{\text{simple return}}$$

- Some conventions:
  - Only count trading days: $\sigma_{\mathsf{yr}} = \sigma_{\mathsf{day}} \times \sqrt{252}$
  - Variance rate: σ^2

Estimating Volatility

- Assume today is date t and we have data for n past dates
- Unbiased estimates
  - Mean

$$\bar{R} = \frac{1}{n} \sum_{i=1}^{n} R_{t-i}$$

  - Volatility

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(R_{t-i} - \bar{R}\right)^2$$

Estimating Volatility

- Assume today is date t and we have data for n past dates
- Unbiased estimates
  - Mean

$$\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_{t-i}$$

  - Volatility

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n} \left(R_{t-i} - \bar{R}\right)^2$$

- Risk management practice:
  - Assume $\bar{R} = 0$: mean small relative to standard deviation for one day
  - Replace $n-1$ by n $\rightsquigarrow$ $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} R_{t-i}^2$

Estimating Volatility: Maximum Likelihood

Estimating Volatility: Maximum Likelihood

- Likelihood for one observation

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-R_i^2}{2\sigma^2}\right)$$

- Log likelihood

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \left[-\log(\sigma^2) - \frac{R_{t-i}^2}{\sigma^2}\right]$$

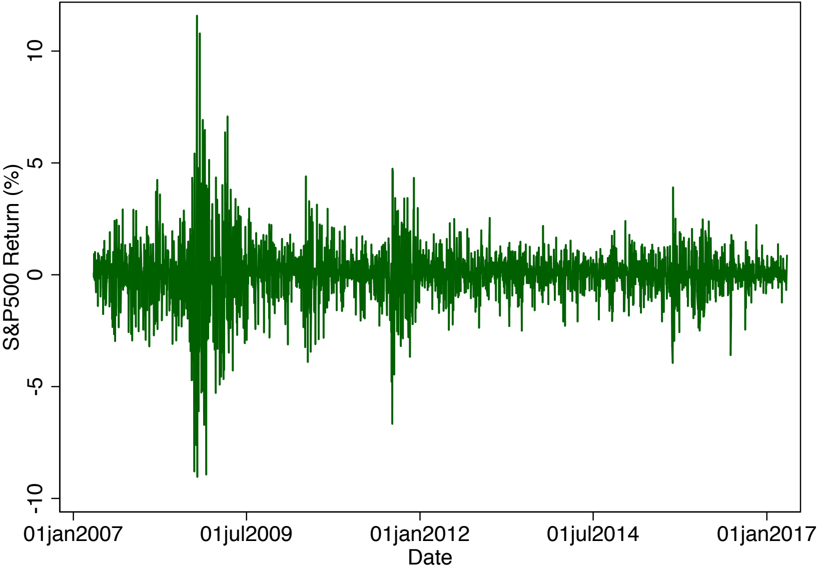
- First-order condition w.r.t. σ^2

$$0 = -\frac{n}{\sigma^2} + \frac{1}{\sigma^4} \sum_{i=1}^{n} R_{t-i}^2$$

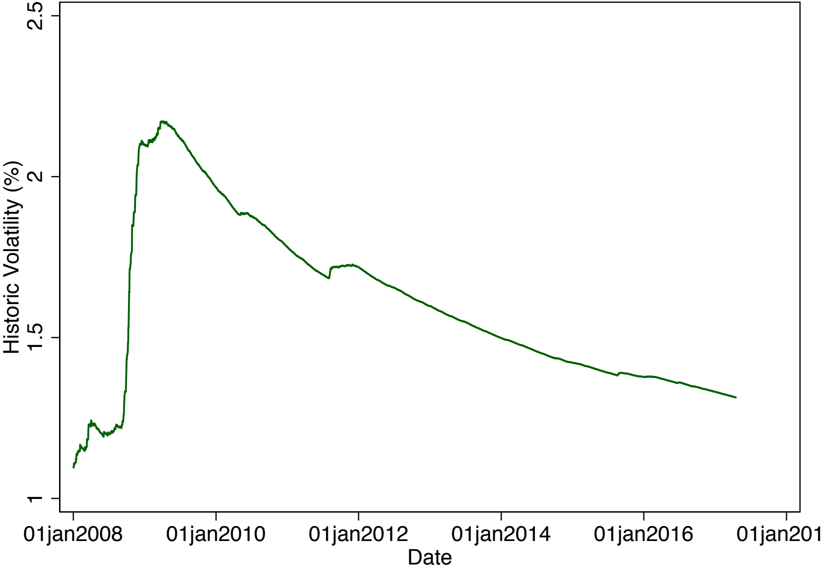
- Estimator

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} R_{t-i}^2$$

S&P500: Returns



S&P500: Historic Volatility



Weighting Schemes

- Volatility moves over time: recent information more relevant than older one

Weighting Schemes

- Volatility moves over time: recent information more relevant than older one
- Weighting scheme:

$$\sigma_t^2 = \sum_{i=1}^{n} \alpha_i R_{t-i}^2$$

$$\text{with } 1 = \sum_{i=1}^{n} \alpha_i$$

Weighting Schemes

- Volatility moves over time: recent information more relevant than older one
- Weighting scheme:

$$\sigma_t^2 = \sum_{i=1}^{n} \alpha_i R_{t-i}^2$$

$$\text{with } 1 = \sum_{i=1}^{n} \alpha_i$$

- Weighting scheme + long-run variance

$$\sigma_t^2 = \gamma V_L + \sum_{i=1}^{n} \alpha_i R_{t-i}^2$$

$$\text{with } 1 = \gamma + \sum_{i=1}^{n} \alpha_i$$

ARCH

- ARCH(m), autoregressive conditional heteroskedasticity:
  $R_t \sim \mathcal{N}(0, \sigma_t^2)$ with:
$$\sigma_t^2 = \omega + \sum_{i=1}^{m} \alpha_i R_{t-i}^2$$

ARCH

- ARCH(m), autoregressive conditional heteroskedasticity:
  $R_t \sim \mathcal{N}(0, \sigma_t^2)$ with:
  $$\sigma_t^2 = \omega + \sum_{i=1}^{m} \alpha_i R_{t-i}^2$$
  - If $\alpha_i = 1/m$ and $\omega = 0$, rolling window estimate

EWMA

$$\sigma_t^2 = \sum_{i=1}^{\infty} \alpha_i R_{t-i}^2$$

- EWMA, exponentially weighted moving average

$$\alpha_i = \frac{1-\lambda}{\lambda} \lambda^i$$

  - Simple volatility updating:

EWMA

$\sigma_2^2 = R_1^2$

$= (1-\lambda) R_1^2 + \lambda \sigma_1^2$

???

- EWMA, exponentially weighted moving average

$$\alpha_i = \frac{1-\lambda}{\underbrace{\lambda}_{*}} \lambda^i$$

  - Simple volatility updating:

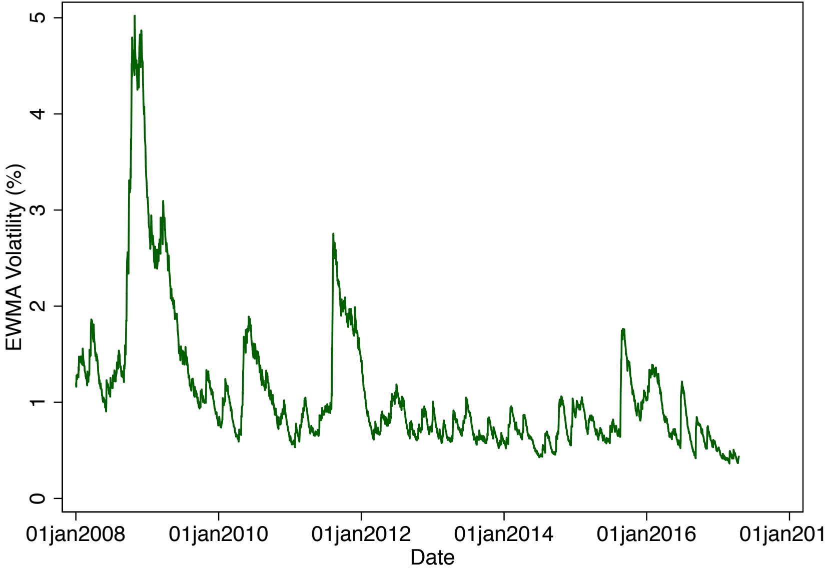
$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda) R_{t-1}^2$$

$\sigma_{t-1}^2 = * \lambda R_{t-2}^2 + * \lambda^2 R_{t-3}^2 + * \lambda^3 R_{t-4}^2 + \ldots$

$\sigma_t^2 = \lambda * R_{t-1}^2 + \lambda^2 * R_{t-2}^2 + \lambda^3 * R_{t-3}^3 \ldots$

  - RiskMetrics reported with $\lambda = 0.94$ until 2006

S&P500: EWMA Volatility



GARCH(1,1)

- GARCH(1,1), generalized autoregressive conditional heteroskedasticity

$$\sigma_t^2 = \underbrace{\gamma V_L}_{\omega} + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2$$

  - EWMA + long-run average
  - If $\gamma = 0$, EWMA
  - For stability, $\alpha + \beta < 1$

MLE Estimation of GARCH(1,1)

- Parameters: ω, α, β
- Log-likelihood

$$\sum_{i=1}^{n} \left[-\log(\sigma_{t-i}^2) - \frac{R_{t-i}^2}{\sigma_{t-i}^2} \right]$$

- Compute σ_{t-i}^2:
  - Initialize at $\sigma_0 = \sqrt{V_L} = \sqrt{\omega/(1-\alpha-\beta)}$
  - Use formula to iterate

$$\sigma_t^2 = \omega + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2$$

$$\mathcal{L}(\underbrace{R_1, \ldots\ldots, R_n}_{\text{data}} \mid \underbrace{\omega, \alpha, \beta, \sigma_1}_{\text{parameters of the model}})$$

$$= \mathcal{L}(R_1 \mid \omega, \alpha, \beta, \sigma_1)$$

$$\times \underbrace{\mathcal{L}(R_2 \mid \omega, \alpha, \beta, \sigma_1, R_1)} \to \mathcal{L}(R_2 \mid \sigma_2) \qquad R_2 \sim \mathcal{N}(0, \sigma_2^2) \quad R_1^2 \to$$

$$\times \underbrace{\mathcal{L}(R_3 \mid \omega, \alpha, \beta, \sigma_1, R_1, R_2)} \to \mathcal{L}(R_3 \mid \sigma_3)$$

$$\vdots$$

$$\times \underbrace{\mathcal{L}(R_n \mid \omega, \alpha, \beta, \sigma_1, R_1, R_2, \ldots R_{n-1})} \to \mathcal{L}(R_n \mid \sigma_n)$$

$$\mathcal{L}(R_n \mid \sigma_n) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{1}{2}\frac{R_n^2}{\sigma_n^2}\right) = \varphi\left(\frac{R_n}{\sigma_n}\right)$$

Estimation of GARCH(1,1)

- *Variance targetting*: set V_L to the sample variance or a reasonable number
- Goodness of fit ?

Estimation of GARCH(1,1)

- *Variance targetting*: set V_L to the sample variance or a reasonable number
- Goodness of fit :
  1. Remove time-varying volatility
  2. Remove autocorrelation of volatility
     - Compare autocorrelations $c_k = \mathrm{cor}(R_t^2/\sigma_t^2, R_{t-k}^2/\sigma_{t-k}^2)$

Estimation of GARCH(1,1)

- *Variance targetting*: set V_L to the sample variance or a reasonable number
- Goodness of fit :
  1. Remove time-varying volatility
  2. Remove autocorrelation of volatility
     - ⋆ Compare autocorrelations $c_k = \text{cor}(R_t^2/\sigma_t^2, R_{t-k}^2/\sigma_{t-k}^2)$
- Ljung-Box Statistic

$$n \sum_{k=1}^{K} w_k c_k^2$$

$$w_k = \frac{n+2}{n-k}$$

- For $K = 15$, 95% threshold is 25

Volatility Forecasts

- If we want to forecast k days in the future:

$$\mathbb{E}_t\left[\sigma_{t+k}^2\right] = V_L + (\alpha+\beta)^k(\sigma_t^2 - V_L)$$

  - Exponential mean reversion

Volatility Forecasts

- If we want to forecast k days in the future:

$$\mathbb{E}_t\left[\sigma^2_{t+k}\right] = V_L + (\alpha+\beta)^k(\sigma^2_t - V_L)$$

  - Exponential mean reversion

- *Remark:* If we want to hedge volatility risk, need to consider how shocks today will affect volatility during the lifetime of the option

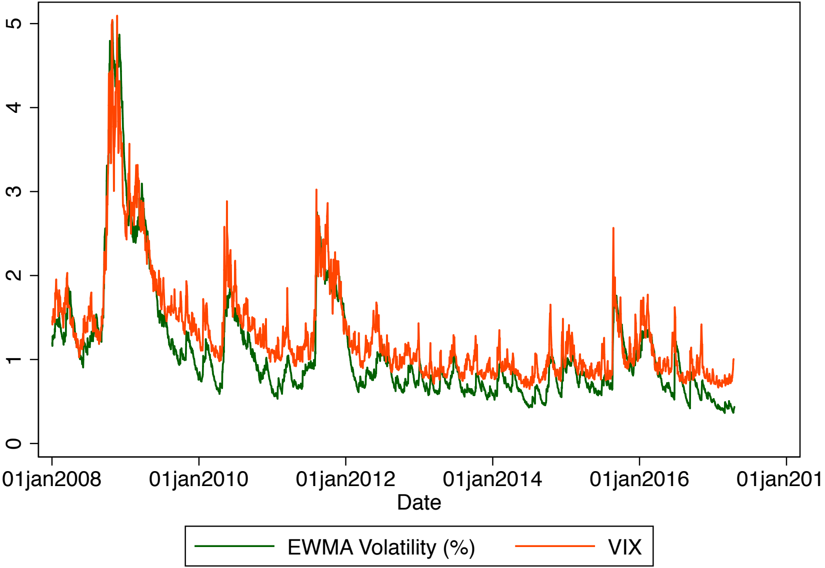
Implied volatiilty

- So far, we focused on backward-looking measures of volatility
- Can use market prices to obtain expectation of future volatility

Implied volatiilty

- So far, we focused on backward-looking measures of volatility
- Can use market prices to obtain expectation of future volatility
- Derivative contracts on volatility: VAR swaps, ...
- *Implied volatility* from calls and puts
  - Volatility so that Black-Scholes formula matches price

VIX


5
4
3
2
1
0
01jan2008
01jan2010
01jan2012
01jan2014
01jan2016
01jan201
Date
EWMA Volatility (%)
VIX

VIX

- The VIX index is published by CBOE
- It is no longer the Black and Scholes implied volatility
- But it is computed from a portfolio of options on the S&P500 index
  - It is deemed to better capture market "expected" volatility over the next 30 days without relying on any model

VIX

- The VIX index is published by CBOE
- It is no longer the Black and Scholes implied volatility
- But it is computed from a portfolio of options on the S&P500 index
  - It is deemed to better capture market "expected" volatility over the next 30 days without relying on any model
- Why is VIX systematically higher than realized volatility?

VIX

- The VIX index is published by CBOE
- It is no longer the Black and Scholes implied volatility
- But it is computed from a portfolio of options on the S&P500 index
  - It is deemed to better capture market “expected” volatility over the next 30 days without relying on any model
- Why is VIX systematically higher than realized volatility?
  - Risk adjustment implicit in options, that make VIX higher than future realized volatility
  - It does not mean that market expectations are systematically too high

Non-Normal Assumptions

- Market information can be useful beyond the normal distribution
- Directly obtain measures of downside risks from options

Using Options to Infer Downside Risk

- What can options tell us about downside risk?

Using Options to Infer Downside Risk

- What can options tell us about downside risk?
  - More generally, how can we measure the relative price of options?
- Only taking the ratio of prices may be imprecise, as different strike prices also imply different intrinsic values (how much we are in the money already), which mix up the computation.

Using Options to Infer Downside Risk

- What can options tell us about downside risk?
  - More generally, how can we measure the relative price of options?
- Only taking the ratio of prices may be imprecise, as different strike prices also imply different intrinsic values (how much we are in the money already), which mix up the computation.
- A standard methodology is to compare the implied volatility of options at different strikes
- Because higher volatility implies a higher price, if OTM options have higher implied volatility than ATM option $\rightarrow$ market expects negative skewness

Using Options to Infer Downside Risk

- What can options tell us about downside risk?
  - More generally, how can we measure the relative price of options?
- Only taking the ratio of prices may be imprecise, as different strike prices also imply different intrinsic values (how much we are in the money already), which mix up the computation.
- A standard methodology is to compare the implied volatility of options at different strikes
- Because higher volatility implies a higher price, if OTM options have higher implied volatility than ATM option $\rightarrow$ market expects negative skewness
  - Since the crash of October 1987, OTM put options have a higher implied volatility than ATM put options.
  - Moreover, the difference in implied volatilities is time varying.

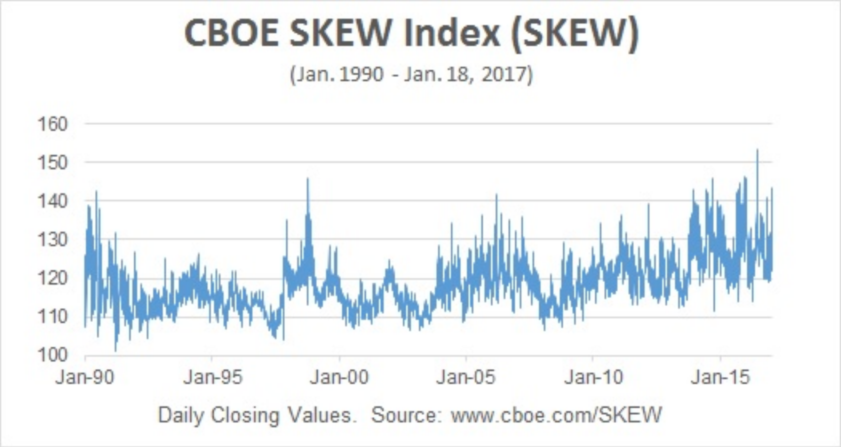
Skewness Index

- CBOE publishes a Skew Index
- Implied expected skewness is computed from option prices with a more elaborate methodology than the Black and Scholes implied volatilities, but the logic is similar.
- Recall that high *negative* skewness imply high downside risk.
  - CBOE define the Skew Index as

$$\text{Skew Index} = 100 - 10 \times \text{Implied Expected Skewness}$$

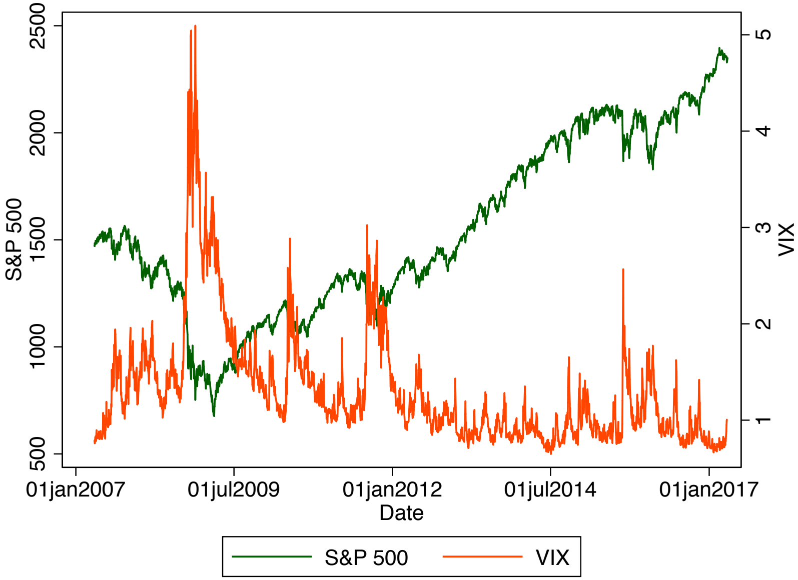
  - Higher positive index $\rightarrow$ higher downside risk

Skew Index



CBOE SKEW Index (SKEW)
(Jan. 1990 - Jan. 18, 2017)
160
150
140
130
120
110
100
Jan-90
Jan-95
Jan-00
Jan-05
Jan-10
Jan-15
Daily Closing Values. Source: www.cboe.com/SKEW

S&P500 and VIX



Model-Building

- To accurately model risk, necessary to understand interactions between different risks
- Lots of models, for each asset class
- Key question: what can go wrong?
- If model is too complex to compute VaR explicitly: Monte-Carlo simulations

Model-Building vs. Historical Simulation

Model-Building vs. Historical Simulation

- Model-building useful for
  - Large portfolios
  - Limited data
  - Taking account of nonlinearities
- Historical simulations useful for:
  - Non-normal situations
  - Unknown structure of investment performance

Model-Building vs. Historical Simulation

- Model-building useful for
  - Large portfolios
  - Limited data
  - Taking account of nonlinearities
- Historical simulations useful for:
  - Non-normal situations
  - Unknown structure of investment performance
- Key trade-off: making more assumptions vs. using a small part of the data
  - Always the same in statistics!