Risk Parity and Minimal Variance Portfolio based on a Regularized Estimate of Variance-Covariance Matrix

MFE 431-1 Quantitative Asset Management Final Project

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06/06/2020

Goals and Motivations

Literature has shown that investors exihbit leverage aversion. That is, investors would prefer to not lever up their portfolios. In order for investors to maximize their returns, it would follow that investors would over value high beta stocks. It would follow that the risk reward trade off for high beta stocks is lower than low beta stocks. Taking advantage of this empirical fact, Asness 2012 has shown[1] that a risk parity portfolio would out perform a 60/40[2] and value weighted CRSP stocks index, since risk parity gives every asset an equal weight according to their risk profile. Moreover, it would seem to follow that a minimum variance portfolio should also exihibit such a property. Hence, we propose two strategies, a risk parity and a minimum variance portfolio.

In Asness 2012, the constructed portfolio is a simple two asset portfolio formed between the value weighted CRSP stocks index and value weighted bonds index. Given the simplicity of this construction, and the low correlation between these two assets, the covariance matrix is very simple to estimate. However, if one wants to extend this to a multi-asset strategy, then we would run into an estimation problem. Since both proposed strategies heavily depends on the covariance matrix, it goes without saying that estimations errors pose a very real threat.

Ledoit 2004 shows that one should never use the sample covariance matrix for portfolio construction, as it contains a lot of estimation errors. Unless one can obtain such divine data on the universe of traded assets, it follows one would need a better estimator. Here, we use a nonlinear esimator proposed by de Lopez 2016. This estimator significantly boosts the performance of our strategies when data is not as readily available, but has a marginal performance increase when data is much more accessible.

[1] The paper actually uses the out performance of the risk parity portfolio to justify the existance of leverage aversion, but we digress.

[2] 60% value weighted CRSP stocks index, 40% value weighted CRSP bonds.

Proposed Strategy

Our strategy invovles construction a risk parity portfolio and a minimum variance portfolio on 49 assets. These assets are the traded industry factors constructed by Ken French[3]. We chose these assets because data is more readily available and we can test our hypothesis without needing intense computational power.

Both portfolios will be tested on daily and month returns, whose covariance matrix estimates are based on the respective time scales. We treat monthly data as data that's less "readily avaiable". We rebalance each portfolio monthly, quarterly, semi-annually, and annually. At each rebalance date, we re-estimate the covariance matrix and construct portfolio weights. We see that both portfolios exihibits similar properties, which is higher performance at quarterly rebalance, while applying the nonlinear covariance estimators gives much higher performance for monthly data, indicating superior estimation.

Backtest Assets and Sample Period

The data for our backtest is the 49 industry portfolio returns collected from the Kenneth R. French data Library from 1980 to 2020 at daily and monthly frequency[4]

We apply a rolling window of ten-year on monthly and daily returns to estimate the covariance matrix. We compare regressed the returns from our strategies on the Fama-French 3 factors and find significant α at the quarterly rebalance portfolio, for daily and monthly.

[3] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[4] The data starts on 1926, but some industries didn't exist until 1980.

Results and Deliverables

Risk Parity, Daily

	Return	Volatility	Sharpe Ratio
CRSP Value Weighted	0.0817	0.175	0.467
Equal Weighted Industry	0.0808	0.1658	0.490
RP*, Levered, Annual	0.0711	0.1334	0.535
RP*, Levered, Semi-Annual	0.0722	0.1310	0.554
RP*, Levered, Quarterly	0.0786	0.1328	0.590
RP*, Levered, Monthly	0.0803	0.1358	0.521
RP, Levered, Annual	0.0686	0.1323	0.519
RP, Levered, Semi-Annual	0.0688	0.1321	0.516
RP, Levered, Quarterly	0.0684	0.1323	0.517
RP, Levered, Monthly	0.0707	0.1358	0.521

Table 1: Annualized return and volatility of different risk parity portfolios from 1980 to 2020. * represents covariance matrix estimated from the shrinkage method mentioned above.

Minimum Variance, Monthly

	Return	Volatility	Sharpe Ratio
CRSP Value Weighted	0.0817	0.175	0.467
Equal Weighted Industry	0.0808	0.1658	0.490
MV*, Levered, Annual	0.0968	0.1574	0.616
MV*, Levered, Semi-Annual	0.0714	0.127	0.558
MV*, Levered, Quarterly	0.0808	0.127	0.637
RP*, Levered, Monthly	0.0745	0.126	0.581
MV, Levered, Annual	0.0771	0.432	0.179
MV, Levered, Semi-Annual	0.0789	0.37	0.294
MV, Levered, Quarterly	0.098	0.411	0.238
MV, Levered, Monthly	0.0996	0.3758	0.267

Table 2: Annualized return and volatility of different minimum variance portfolios from 1980 to 2020. * represents covariance matrix estimated from the shrinkage method mentioned above.

From the tables, we see that the shrinkage method gives significantly more boost towards data at the monthly level, but a small boost for data on the daily level. This shows that the shrinkage method is very useful for cases where there is insufficient data on estimating the covariance matrix.

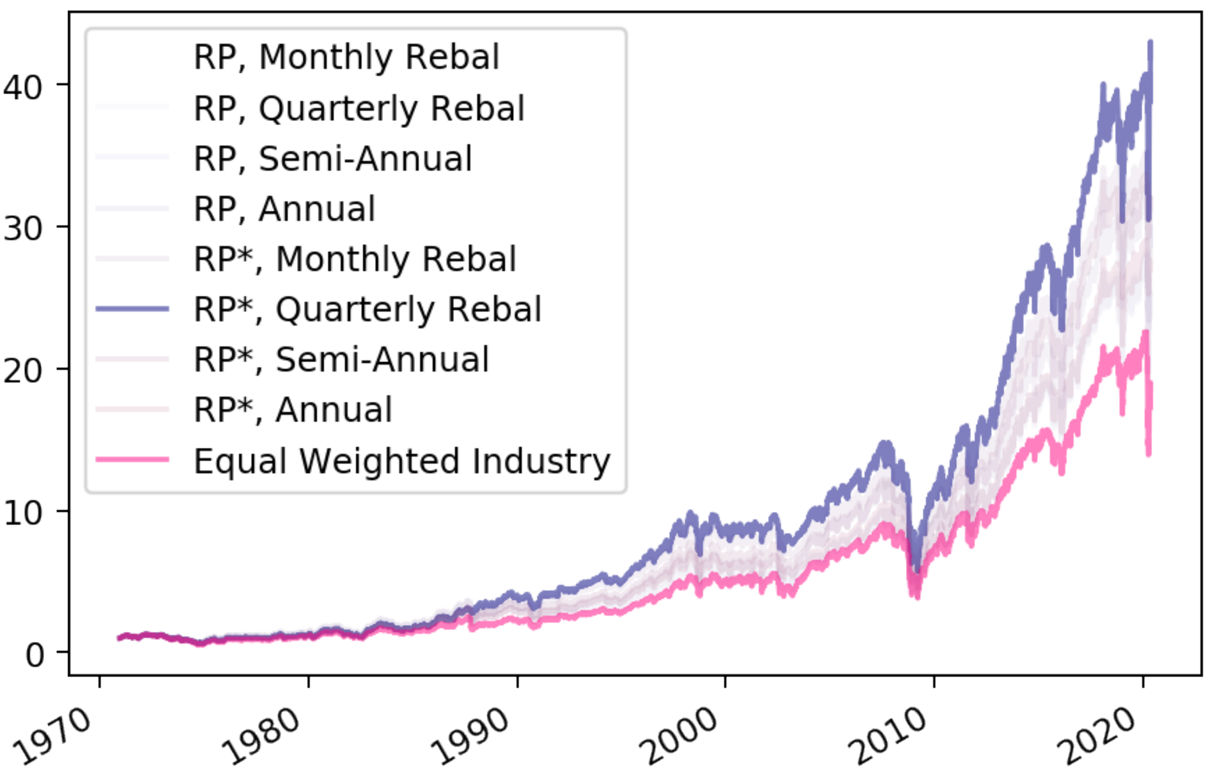

Figure 1: RP portfolio cumulative returns. Note that all RP portfolios are levered to match the risk of equal weighted industry portfolio.

	Risk Parity Portfolio							
	RP (M)	RP (Q)	RP (S)	RP (A)	RP* (M)	RP* (Q)	RP* (S)	RP* (A)
Market	0.766***	0.772***	0.772***	0.772***	0.794***	0.763***	0.750***	0.764***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
SMB	0.137***	0.134***	0.131***	0.135***	0.142***	0.151***	0.124***	0.122***
	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)	(0.005)
HML	0.139***	0.132***	0.125***	0.135***	0.157***	0.145***	0.106***	0.100***
	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)	(0.005)	(0.005)	(0.005)
α	0.00003	0.00002	0.00002	0.00003	0.00003	0.0001**	0.00004*	0.00003
	(0.00002)	(0.00002)	(0.00002)	(0.00002)	(0.00002)	(0.00002)	(0.00002)	(0.00002)
Observations	12,444	12,444	12,444	12,444	12,444	12,444	12,444	12,444
R^2	0.919	0.924	0.928	0.923	0.926	0.894	0.891	0.893
Adjusted R^2	0.919	0.924	0.928	0.923	0.926	0.893	0.891	0.893
Residual Std. Error (df = 12440)	0.002	0.002	0.002	0.002	0.002	0.003	0.003	0.003
F Statistic (df = 3; 12440)	47,111.970***	50,596.360***	53,319.350***	49,842.430***	51,627.450***	34,790.860***	34,032.180***	34,705.210***

Note: *p<0.1; **p<0.05; ***p<0.01

Table 3: Regression of different risk parity portfolios agains the Fama-French 3 factors. M, Q, S, A represents monthly, quarterly, semi-annually, and annually respectively.

Strategy Utilization

As we can see from the previous section, this strategy has a significant alpha at the quarterly rebalance level, so it's definitely worth while to invest in this strategy. Moreover, it has low transaction costs due to the infrequent rebalancing. Moreover, we can use sector ETFs as proxies for the stocks that form industry portfolios to redunce transaction costs even more.

This strategy is also easily implementable, and generlizable to more assets and strategies involving covariances. The significant improvement of the performance at the monthly level shows that covariance estimation errors can managed, so we can not only use this for risk parity and minimum variance portfolios, we can also use it for other type of strategies.

Costs and Associated Risks

In terms of costs, we might have high trading cost if we were to trade all stocks, including the less liquid and smaller cap stocks. However, these stratgies has high betas, so it's vulnerable to market downturns. Further, our main computation relies on the variance-covariance matrix, meaning that if we were to extend this to more assets, the estimation for the variance-covariance matrix is more prone to errors. Though the strategy is delivering alpha, the alpha seems to depend on the rebalancing frequency, but not in a monotonic way, so the alpha could be a fluke. Moreover, the source of alpha can be easily arbitraged away by institutions as its a simple strategy to implement.

Appendix: Shrinkage estimators

Ledoit-Wolf Shrinkage

Traditional covariance matrix shrinkage methods involves taking convex combination of the sample covariance matrix S with I and using cross validation to find the optimal parameter. Ledoit-Wolf shrinkage changes I with another matrix F, that has much more structure. See Ledoit and Wolf 2004 for the construction of F.

Diagonalization Shrinkage

The second shrinkage is to perform a Principal Component Analysis (PCA)-type compression on the sample correlation matrix. After having the correlation matrix estimated after LedoitWolf shrinkage,we perform a diagonalization, which is given by:

$$\hat{C} = Q\Lambda Q^T$$

where $\hat{C}$ is the sample correlatio matrix after LedoitWolf shrinkage, Λ is a diagnoal matrix with $\hat{C}$'s eigenvalues, and Q is a orthonormal matrix stacked by $\hat{C}$'s corresponded normalized eigenvectors.

The key step is to replace Λ with Λ', which is given by:

$$\Lambda' = \mathrm{diag}(\{\lambda_i'\})$$

where $\lambda_i' = \lambda_i$ if $\lambda_i >= \lambda_{max}$, $\lambda_i' = \bar{\lambda}$ if $\lambda_i < \lambda_{max}$, and $\bar{\lambda}$ is the arithmetic mean of all λ_i, which is samller than λ_{max}.

Then the shrinked correlation matrix estimate is given by:

$$\hat{C'} = Q\Lambda' Q^T$$

which is used as input for convex optimization.

The key idea here is to separate signals from noise in the sample correlation matrix, where large eigenvalue-eigenvector pairs are considered as signals, while small pairs are considered as noise and averaged.

One key issue here is to find the cutoff eigenvalue λ_{max}, which is given by fitting the empirical distribution of correlation matrix eigenvalues to the theoretical Marcenko-Pastur probability density function using a Gaussion Kernal Densitity Estimate algorithm. The Marcenko-Pastur probability density function of eigenvalues is given by:

$$f(\lambda) = \begin{cases} \frac{T}{N}\frac{\sqrt{(\lambda_{\max}-\lambda)(\lambda-\lambda_{\min})}}{2\pi\lambda\sigma^2}, & \text{if} \quad \lambda \in [\lambda_{\min}, \lambda_{\max}] \\ 0, & \text{Otherwise} \end{cases}$$

where T is the number of row and N is the number of colunms of the matrix, $\sigma^2 = 1$ when we are using correlation matrix. $\lambda_{\max} = \sigma^2(1 + \frac{N}{T})$ and $\lambda_{\min} = \sigma^2(1 - \frac{N}{T})$.

The idea is that for any eignvalues in $[\lambda_{\min}, \lambda_{\max}]$, we consider they are generated by i.i.d white noise with mean of 0 and variance of one, and therefore average them to denoise our estimate of correlation matrix.

Finding Portfolio Weights with Convex Optimization

The last step is to use Convex Optimization Algorithm to find the minimal variance portfolio and risk-parity portfolio.

For minimal variance portfolio, it is a typical Markwiz optimization problem:

$$\min_w \frac{1}{2} w^T \hat{V}' w$$

$$\text{s.t.} w^T 1_N = 1$$

where w is the portfolio weights and $\hat{V}'$ is our shrinked estimate of variance-covariance matrix.

The analytical solution is given by:

$$w^* = \frac{\hat{V}'^{-1} 1_N}{1_N \hat{V}'^{-1} 1_N}.$$

Reference

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