Algorithms

FOURTH EDITION

Robert Sedgewick | Kevin Wayne

3.2 Binary Search Trees

▸ BSTs

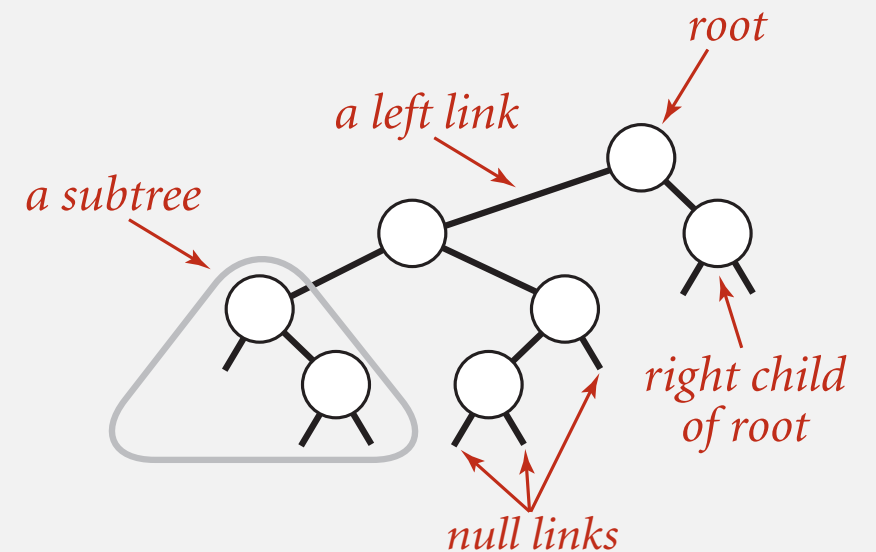▸ *ordered operations*

▸ *deletion*

3.2 BINARY SEARCH TREES

- ▶ *BSTs*
- ▶ *ordered operations*
- ▶ *deletion*

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

Binary search trees

Definition. A BST is a binary tree in symmetric order.

A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).

root

a left link

a subtree

right child of root

null links

Symmetric order. Each node has a key,
and every node's key is:

- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.

parent of A and R

left link of E

key

value associated with R

keys smaller than E

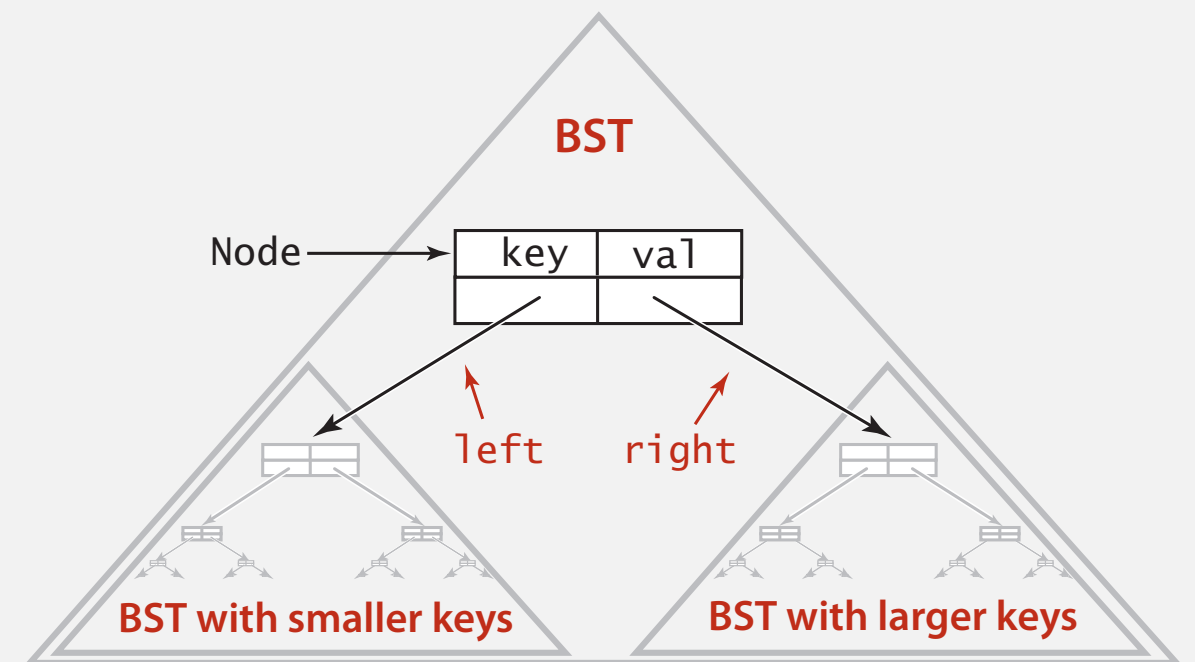*keys larger than E*
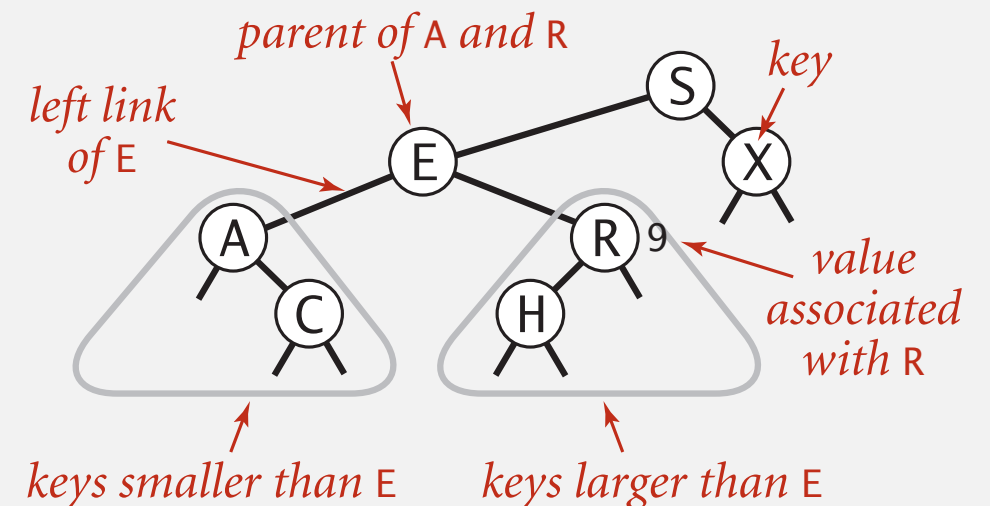
BST representation in Java

Java definition. A BST is a reference to a root Node.

A Node is composed of four fields:

- A Key and a Value.
- A reference to the left and right subtree.

smaller keys   larger keys

```
private class Node
{
    private Key key;
    private Value val;
    private Node left, right;
    public Node(Key key, Value val)
    {
        this.key = key;
        this.val = val;
    }
}
```

Key and Value are generic types; Key is Comparable

Binary search tree

BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
{
    private Node root;                                      ← root of BST

    private class Node
    {  /* see previous slide */  }

    public void put(Key key, Value val)
    {  /* see next slides */  }

    public Value get(Key key)
    {  /* see next slides */  }

    public void delete(Key key)
    {  /* see next slides */  }

    public Iterable<Key> iterator()
    {  /* see next slides */  }

}
```
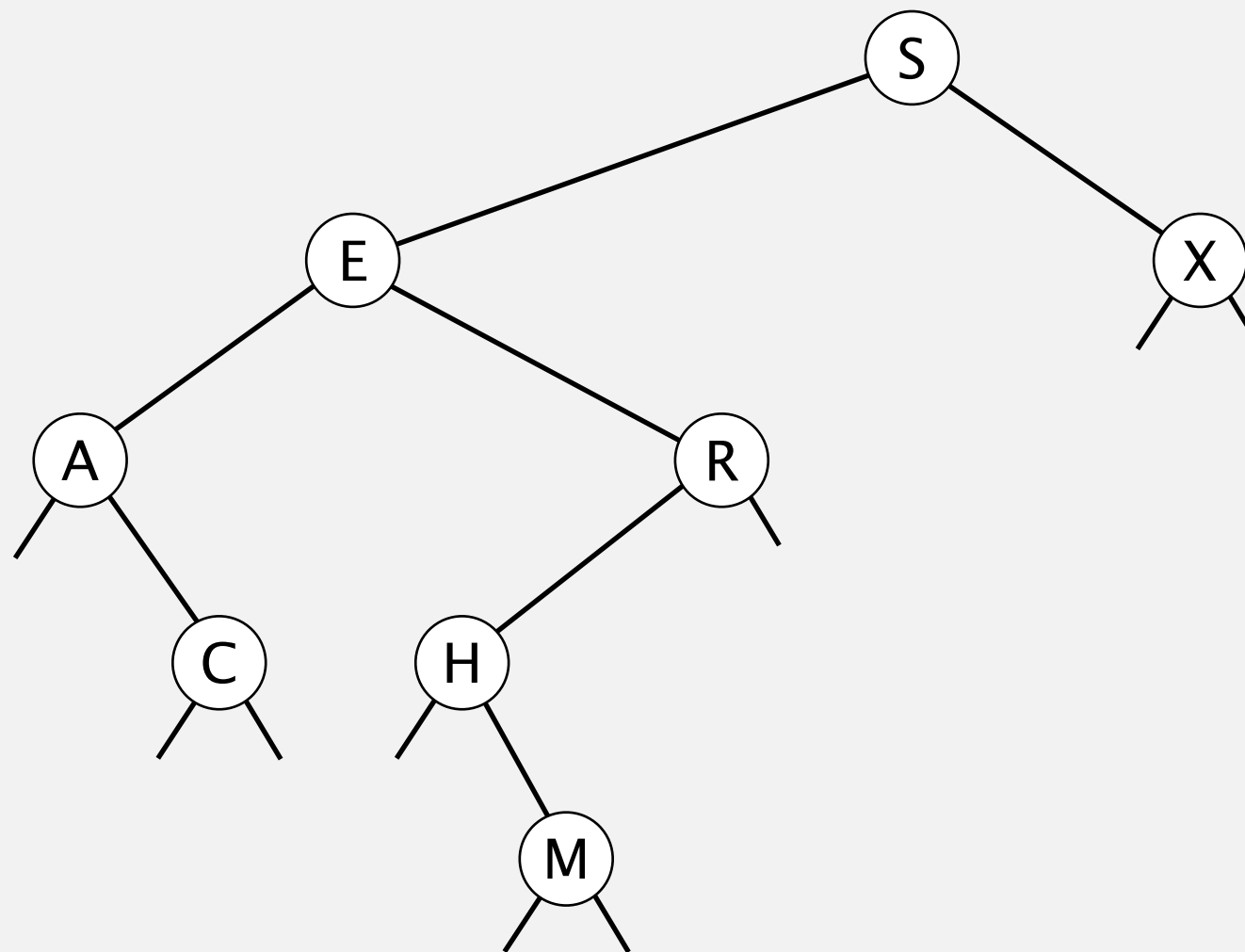
Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.
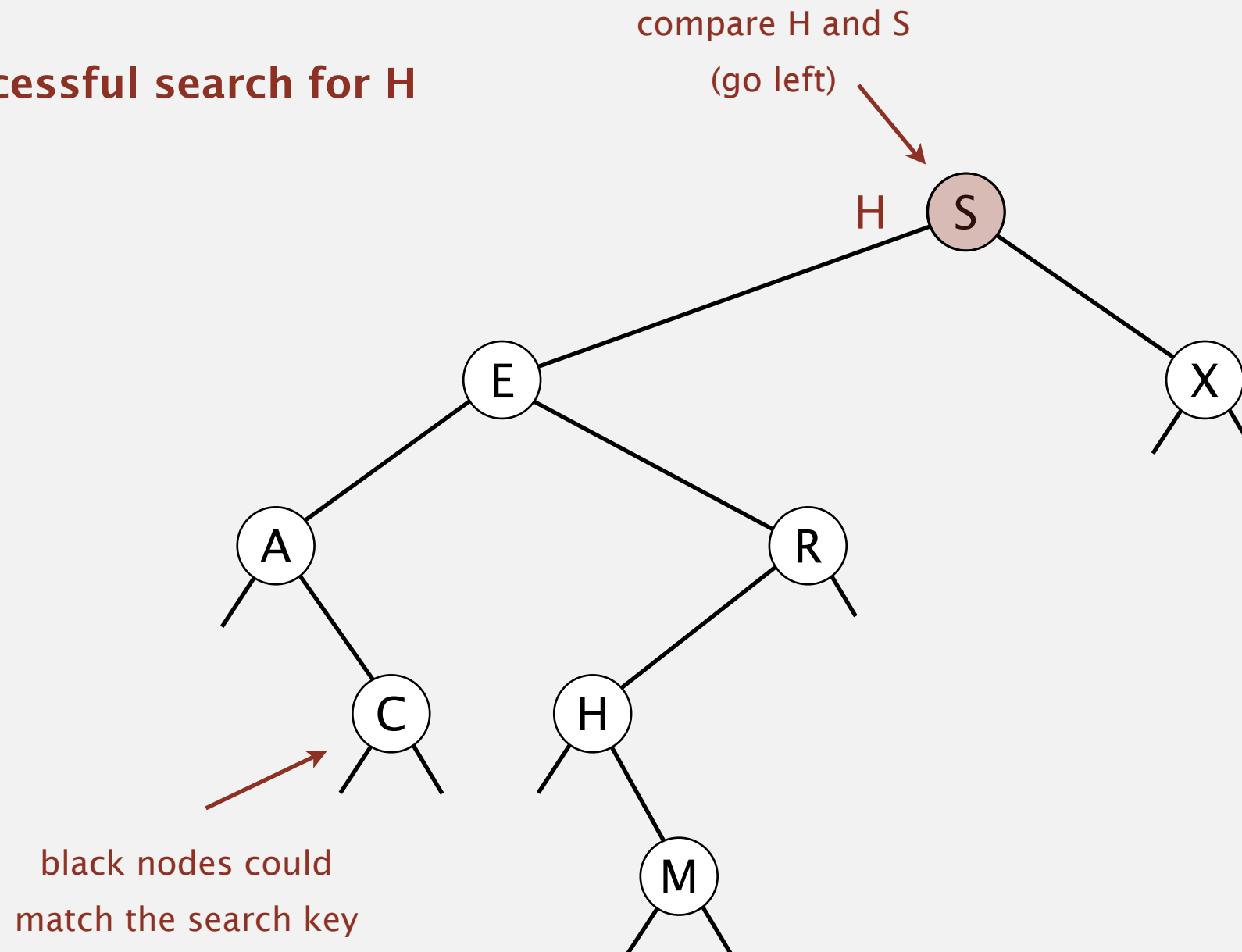
successful search for H

Binary search tree demo

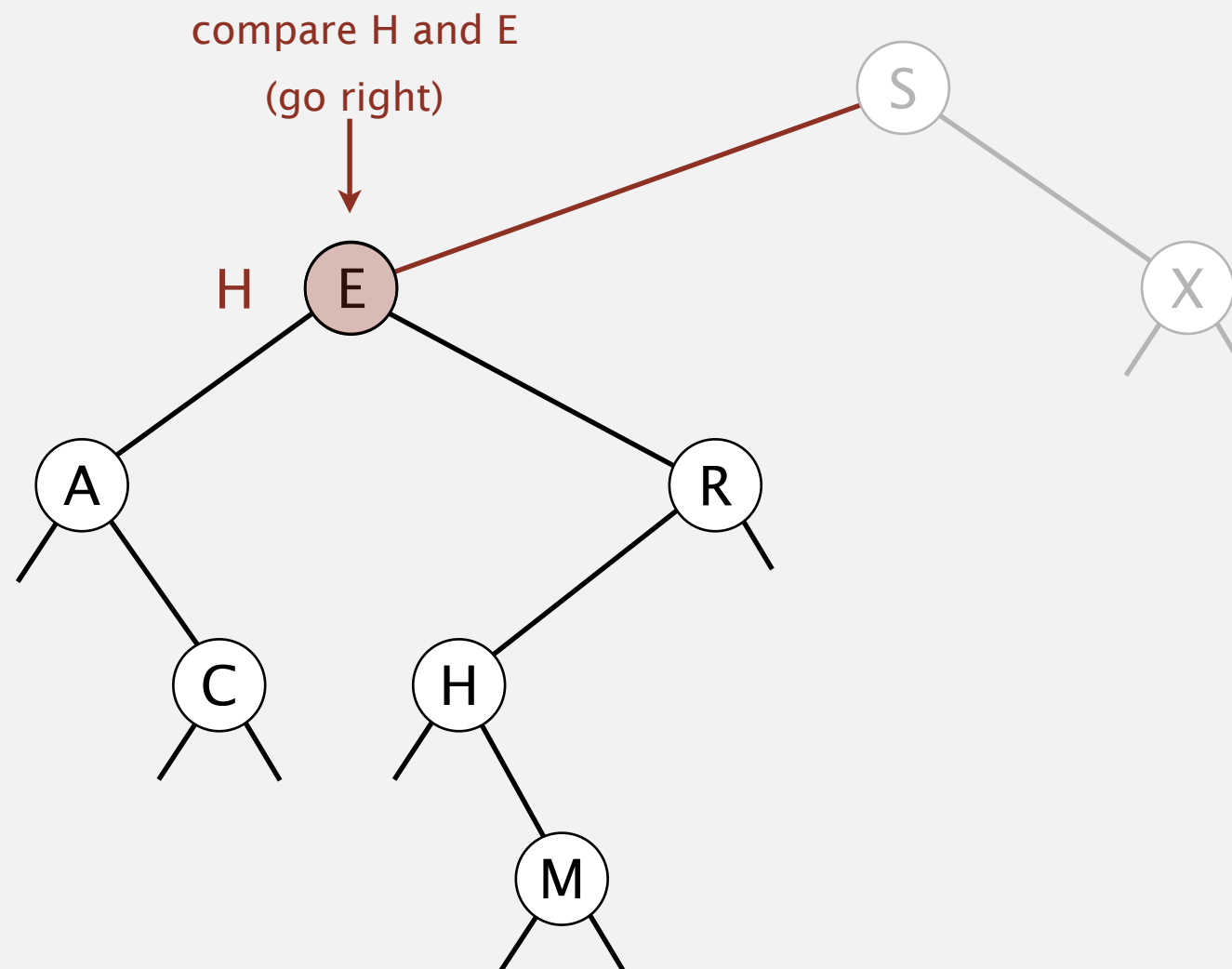Search. If less, go left; if greater, go right; if equal, search hit.

successful search for H

compare H and S
(go left)

black nodes could
match the search key

Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

successful search for H

Binary search tree demo

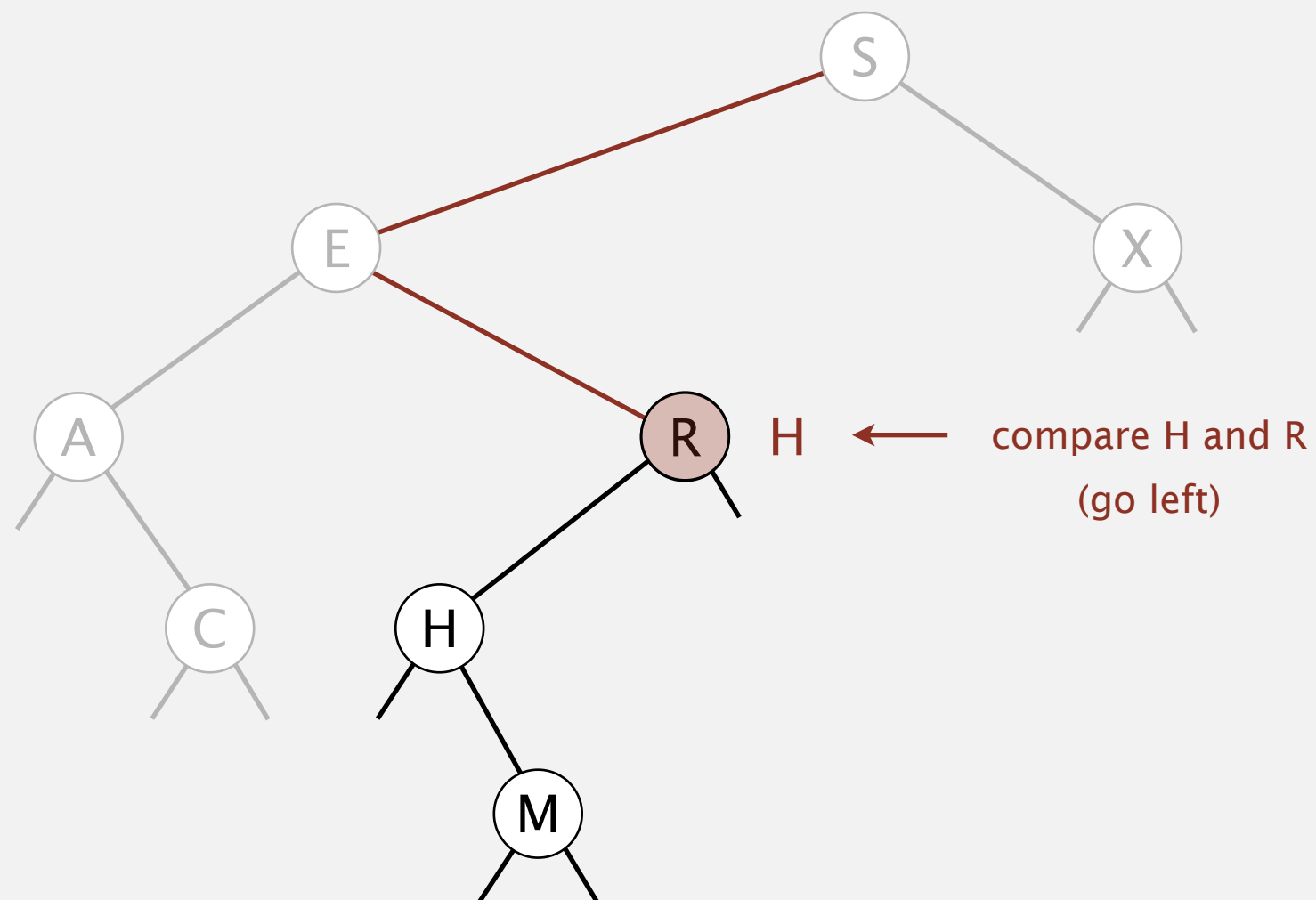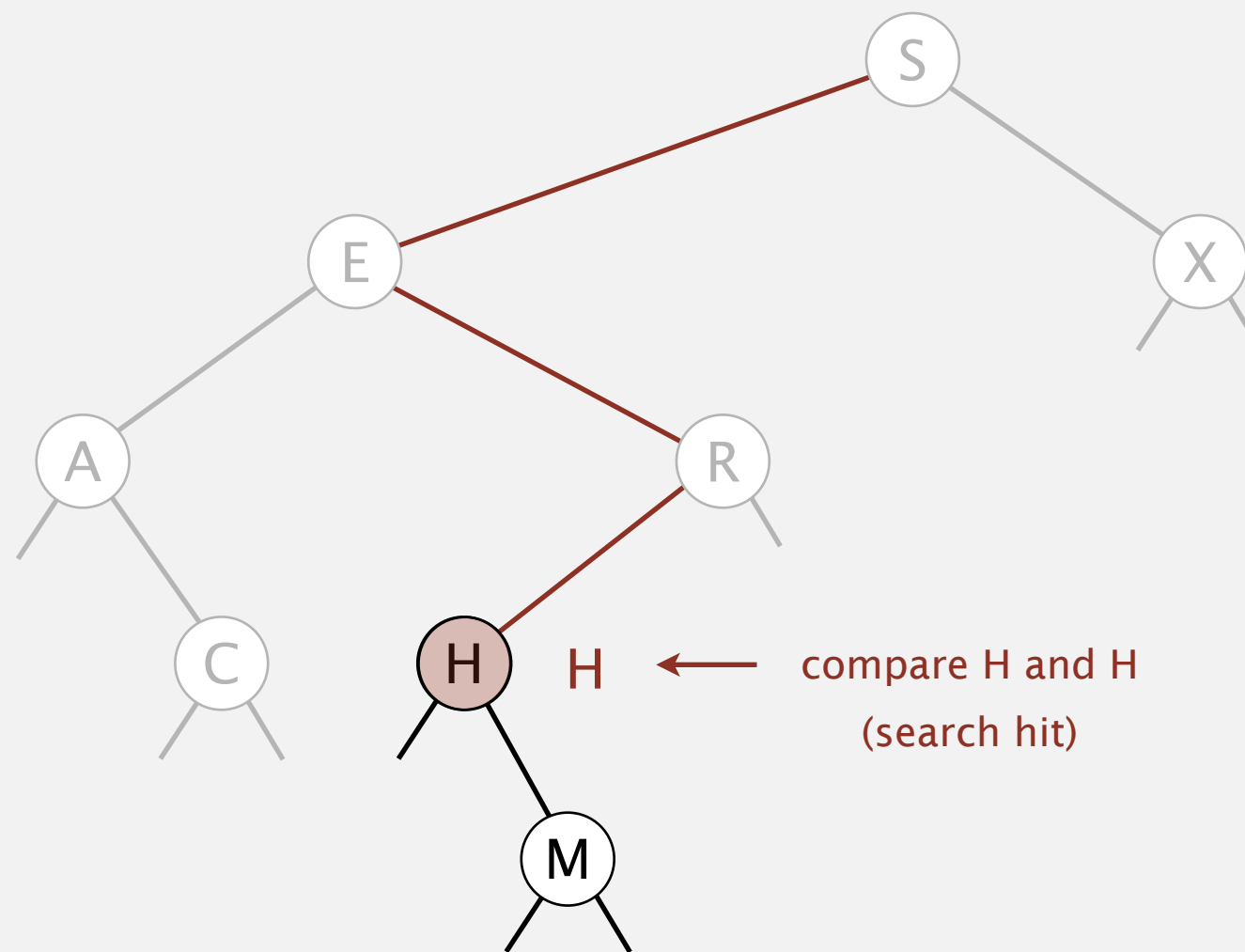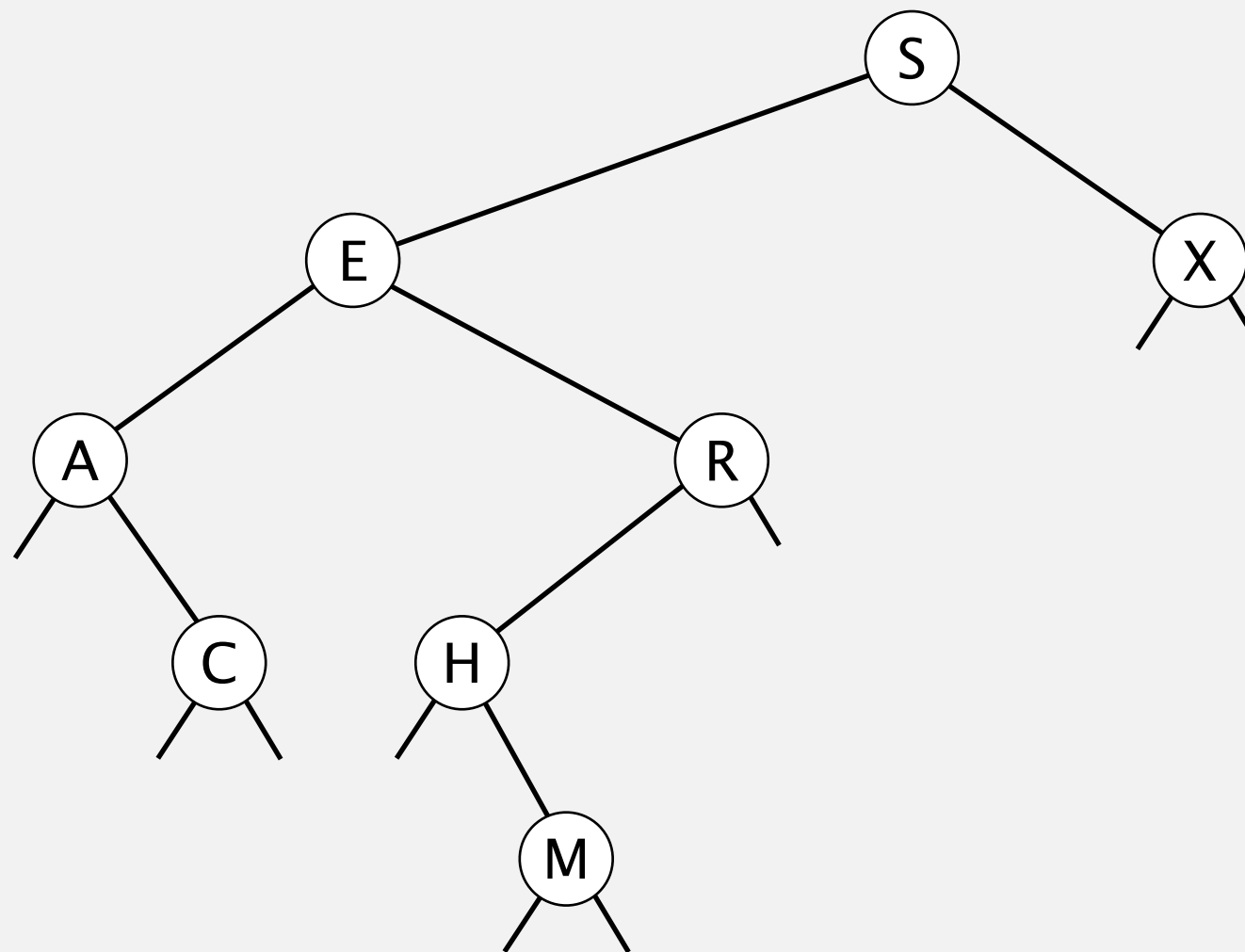Search. If less, go left; if greater, go right; if equal, search hit.

successful search for H

compare H and R
(go left)

Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

successful search for H

compare H and H
(search hit)

Search. If less, go left; if greater, go right; if equal, search hit.

unsuccessful search for G

Binary search tree demo

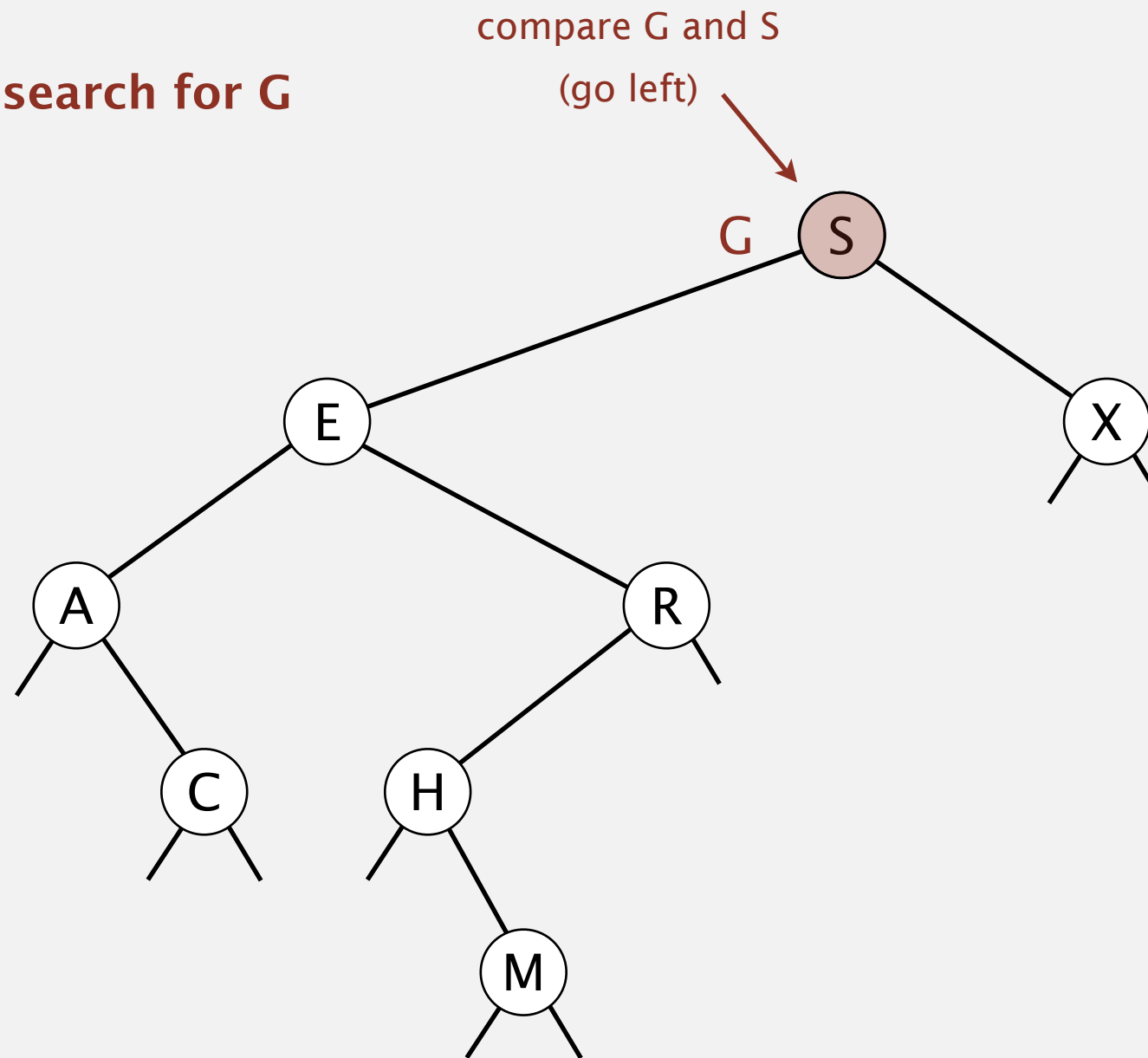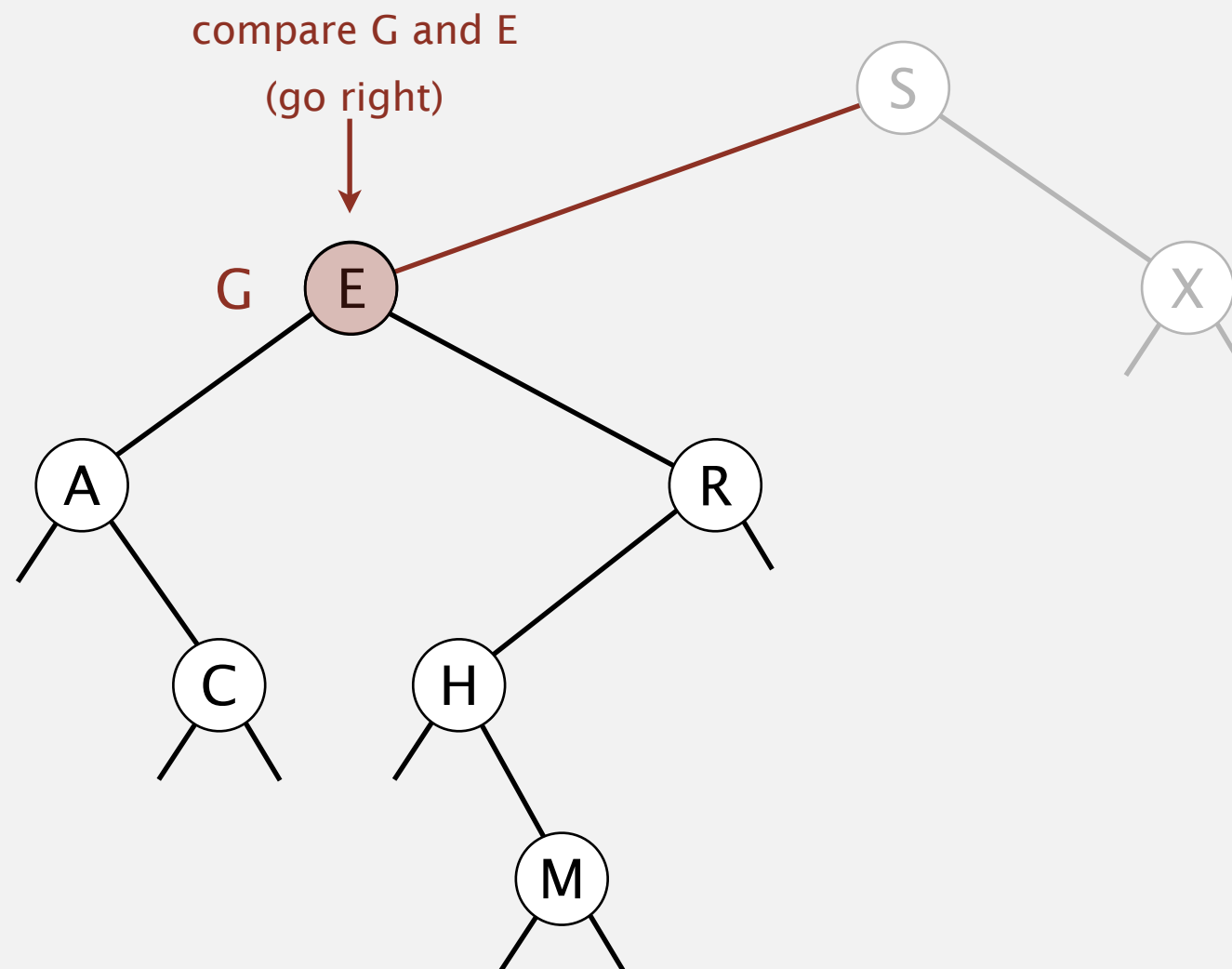Search. If less, go left; if greater, go right; if equal, search hit.

compare G and S
(go left)

unsuccessful search for G

G S

E

X

A

R

C

H

M

Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

unsuccessful search for G

compare G and E
(go right)

Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

unsuccessful search for G

compare G and R
(go left)

Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.
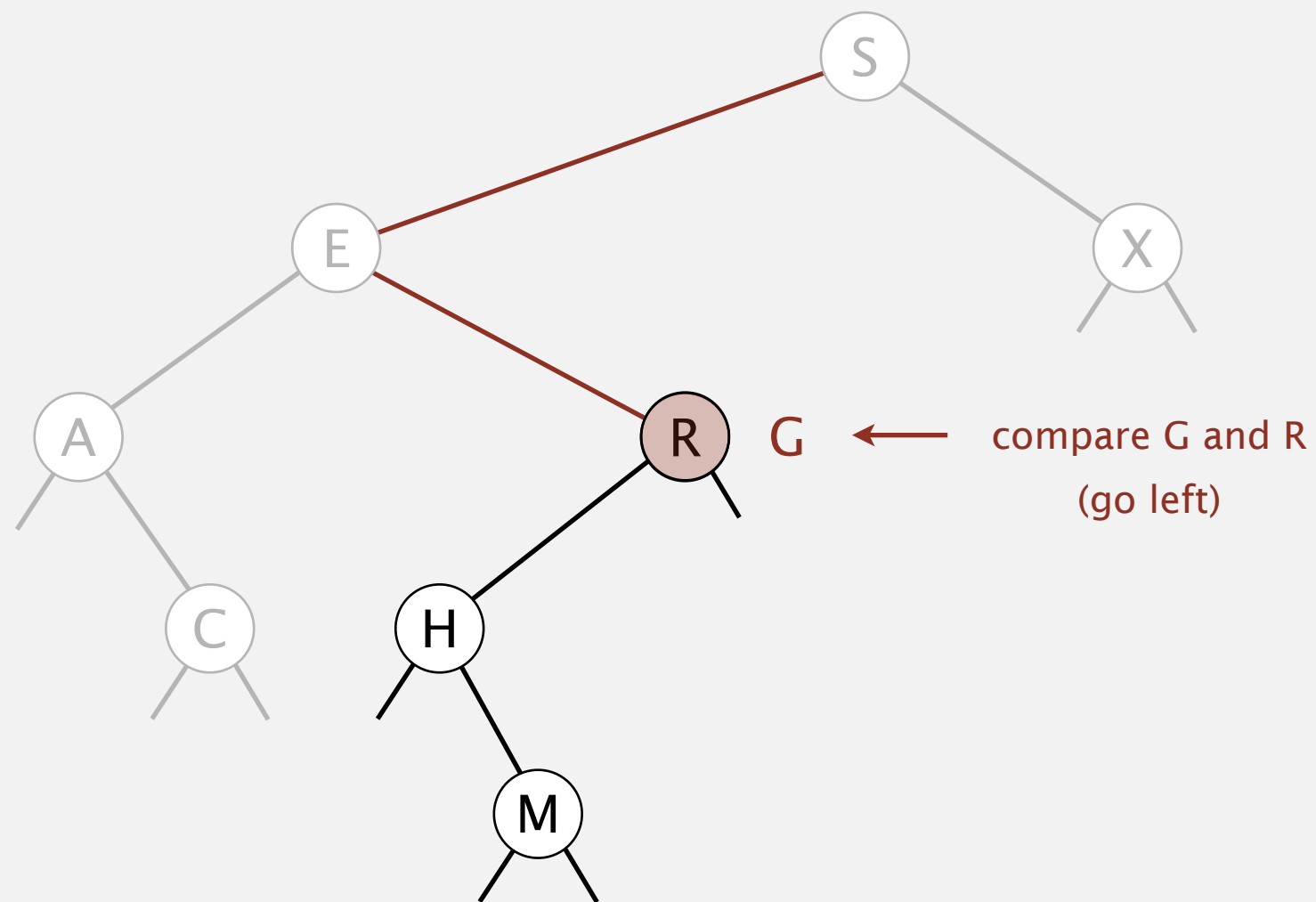
unsuccessful search for G

compare G and H
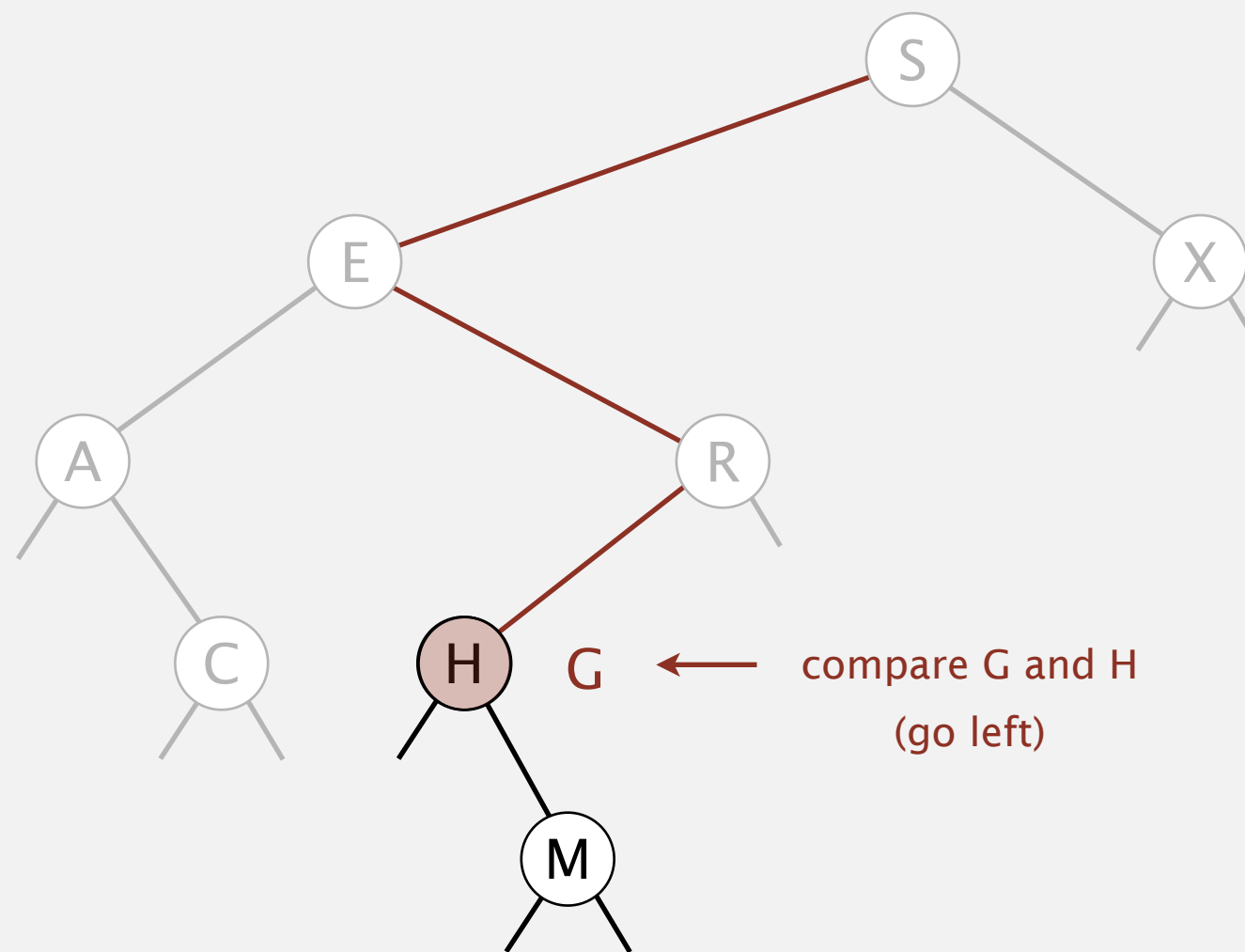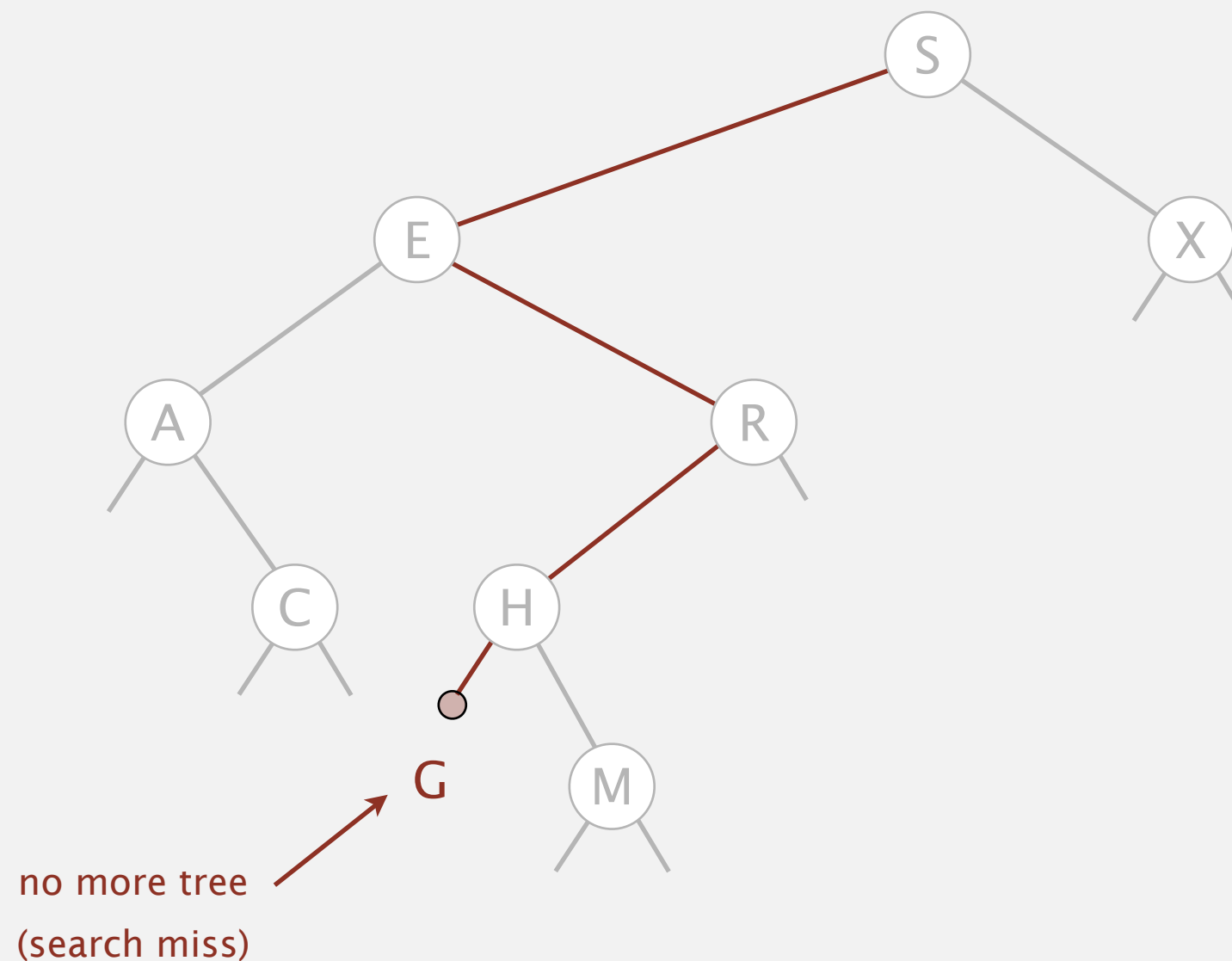(go left)

Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.
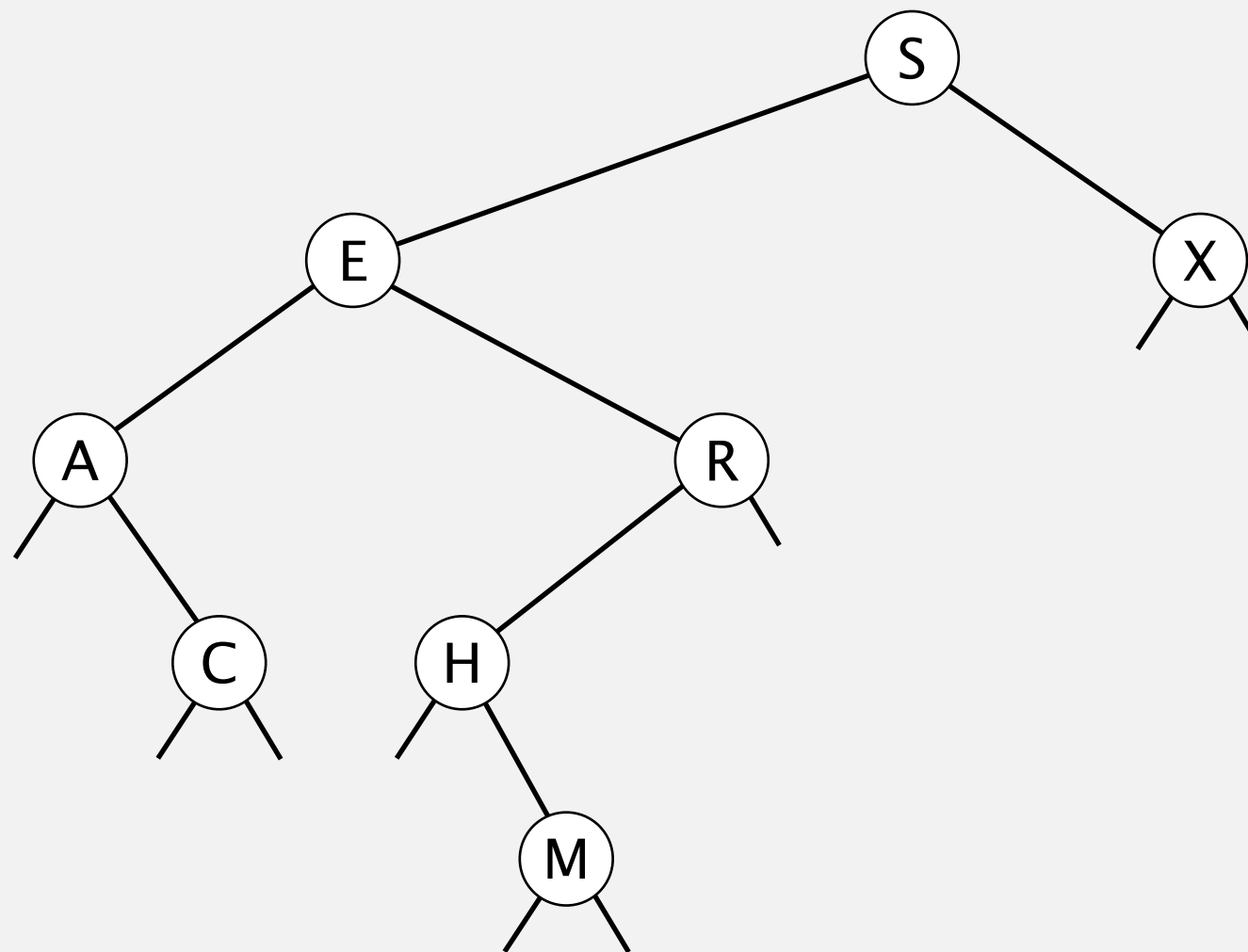
unsuccessful search for G

no more tree
(search miss)

Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G

Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G

compare G and S
(go left)

Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G

compare G and E
(go right)

Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G

compare G and R
(go left)

Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G

compare G and H
(go left)

Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G

no more tree

(insert here)

Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G

S

E   X

A   R

C   H

G   M

no more tree
(insert here)

Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G

BST search Quick Summary

Get. Return value corresponding to given key, or `null` if no such key.

successful search for R

R *is less than* S
so look to the left

*black nodes could
match the search key*

R *is greater than* E
so look to the right

*gray nodes cannot
match the search key*

found R
*(search hit)
so return value*

unsuccessful search for T

T *is greater than* S
so look to the right

T *is less than* X
so look to the left

*link is null
so* T *is not in tree
(search miss)*
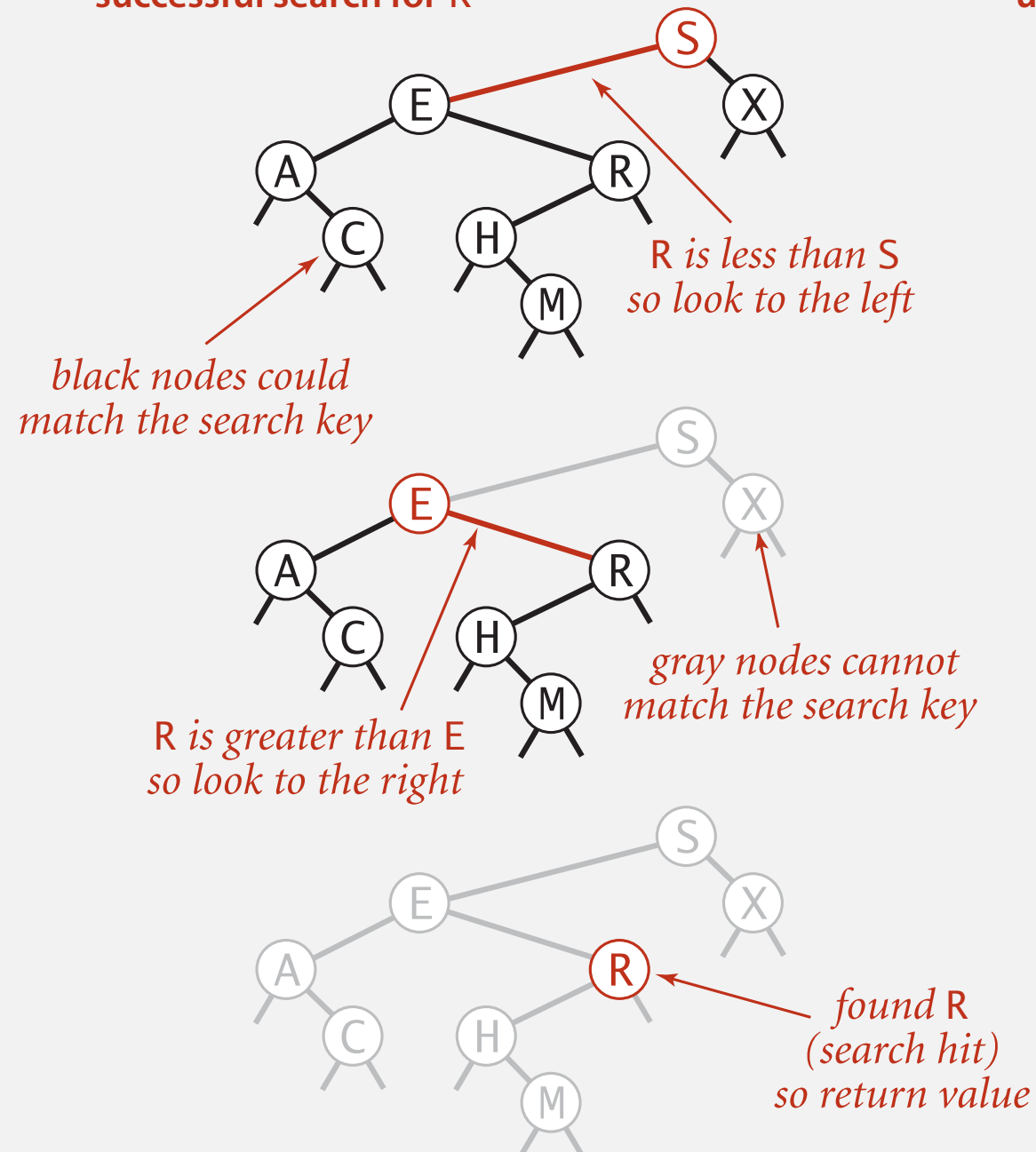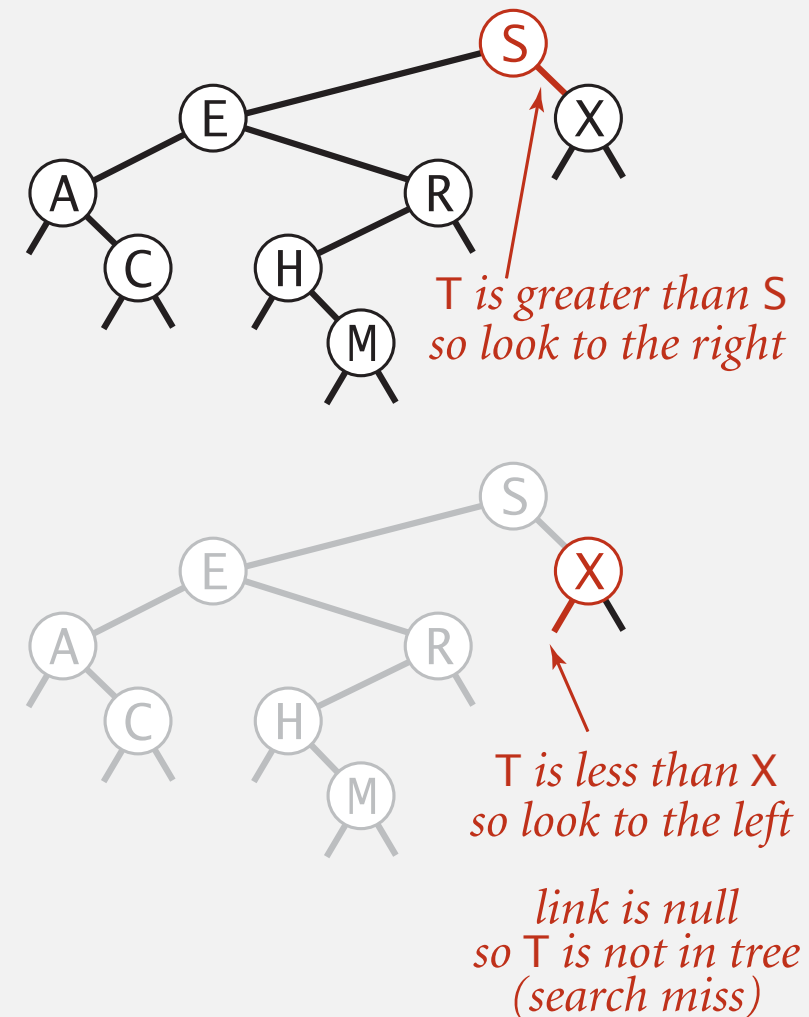
BST search: Java implementation

Get. Return value corresponding to given key, or `null` if no such key.

```java
public Value get(Key key)
{
    Node x = root;
    while (x != null)
    {
        int cmp = key.compareTo(x.key);
        if      (cmp  < 0) x = x.left;
        else if (cmp  > 0) x = x.right;
        else if (cmp == 0) return x.val;
    }
    return null;
}
```

Cost. Number of compares is equal to 1 + depth of node.

Put. Associate value with key.

Search for key, then two cases:
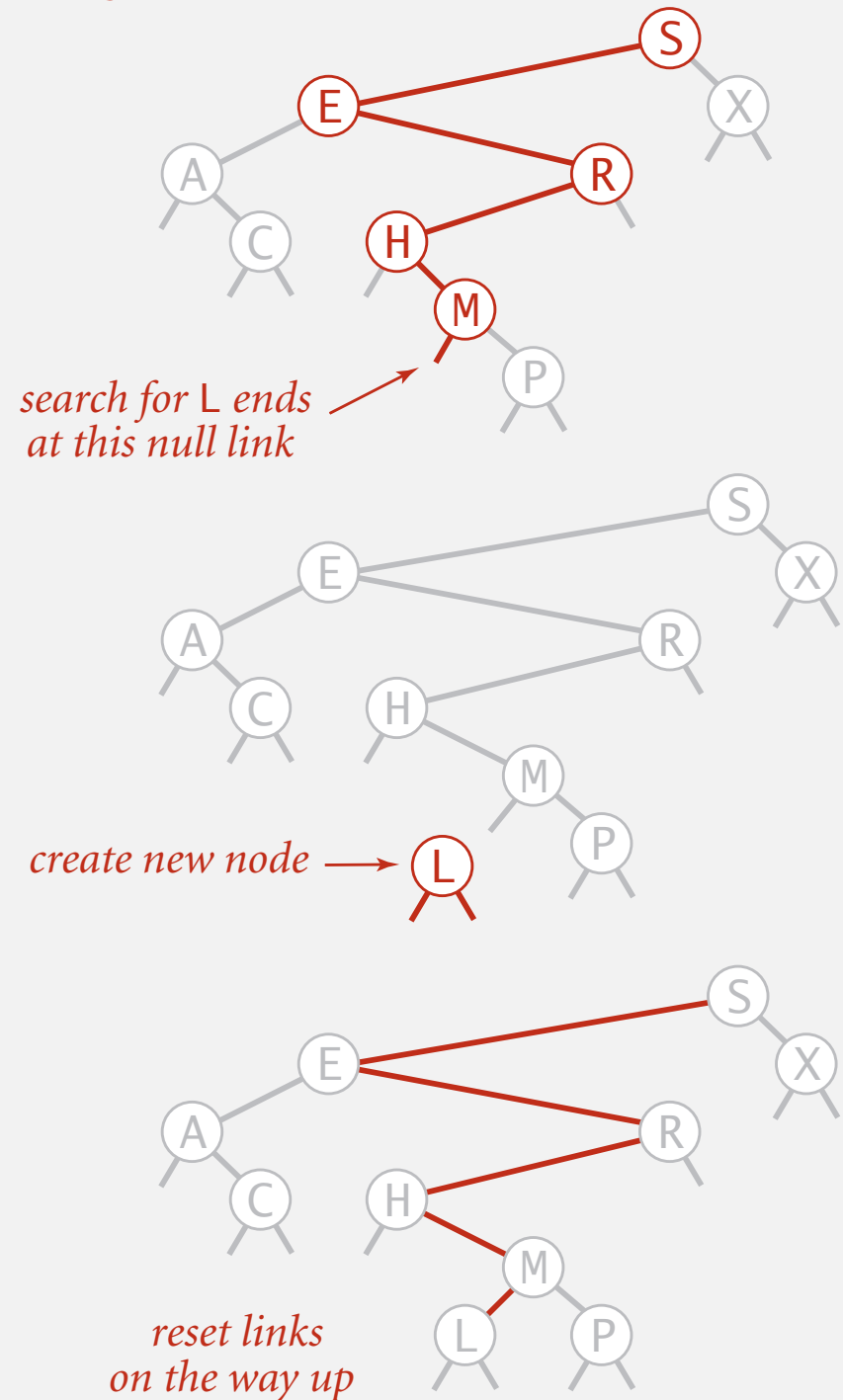- Key in tree ⇒ reset value.
- Key not in tree ⇒ add new node.

inserting L

search for L ends at this null link

create new node ⟶ L

reset links on the way up

Insertion into a BST

BST insert: Java implementation

Put. Associate value with key.

concise, but tricky, recursive code; read carefully!

```java
public void put(Key key, Value val)
{   root = put(root, key, val);   }

private Node put(Node x, Key key, Value val)
{
    if (x == null) return new Node(key, val);
    int cmp = key.compareTo(x.key);
    if        (cmp  < 0)
        x.left  = put(x.left,  key, val);
    else if (cmp  > 0)
        x.right = put(x.right, key, val);
    else if (cmp == 0)
        x.val = val;
    return x;
}
```

Cost. Number of compares is equal to 1 + depth of node.

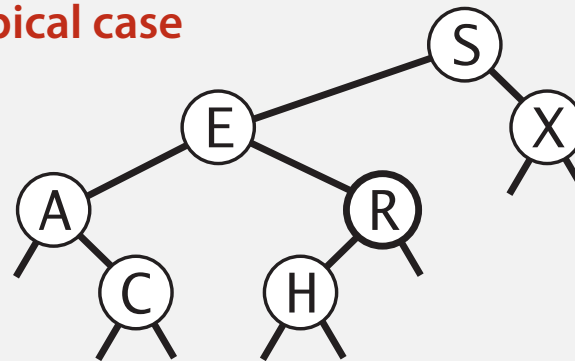# Tree shape

- Many BSTs correspond to same set of keys.

- Number of compares for search/insert is equal to 1 + depth of node.
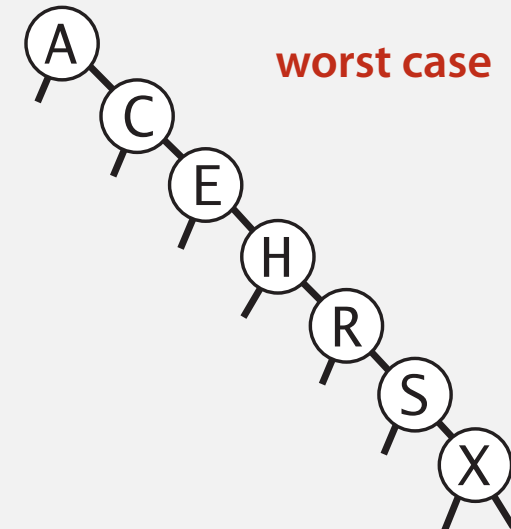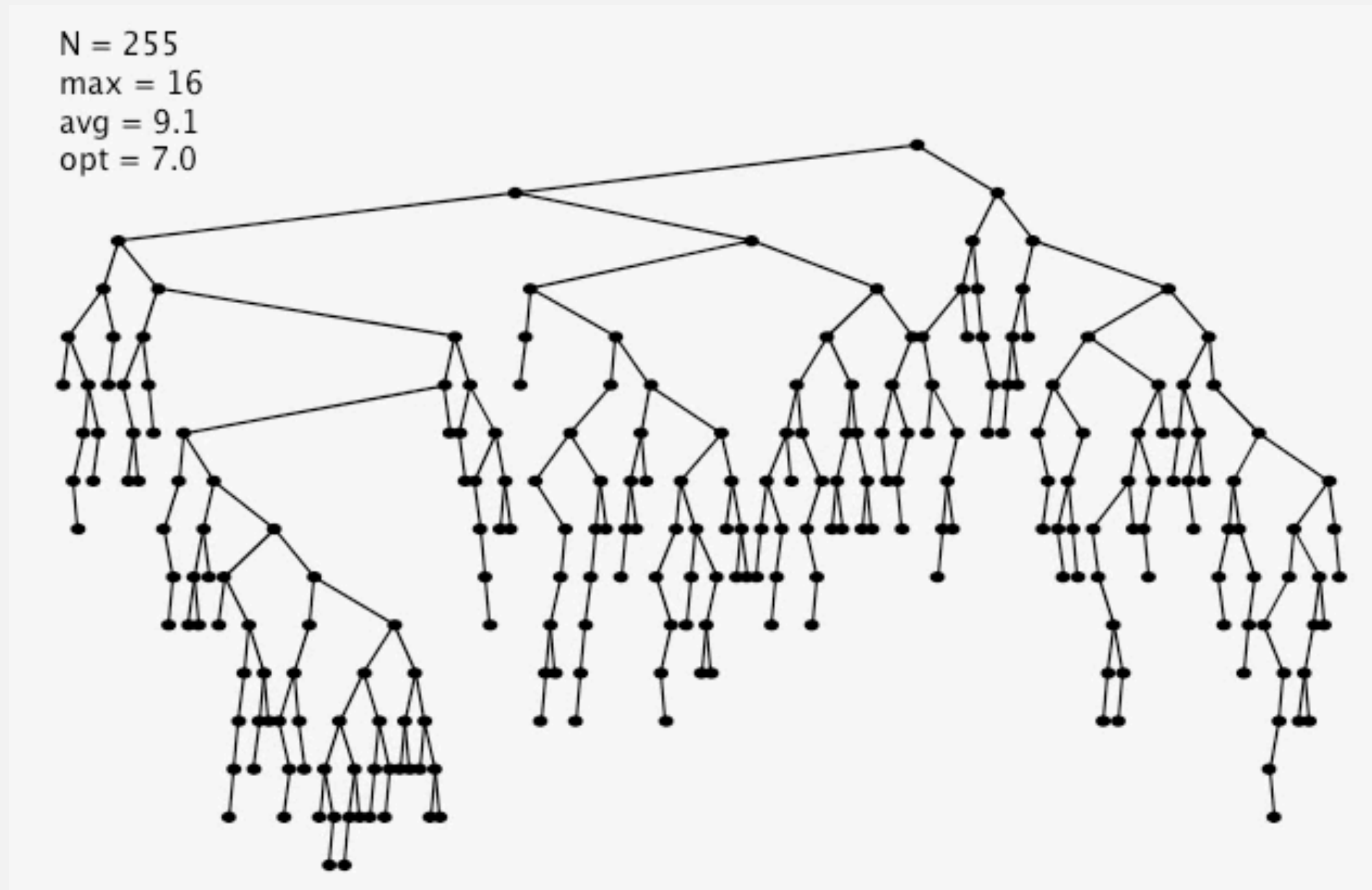

best case
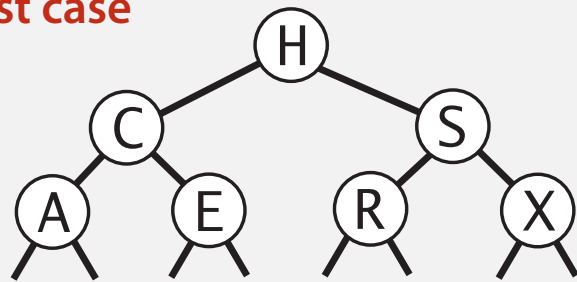
typical case

worst case

Bottom line. Tree shape depends on order of insertion.

BST insertion: random order visualization

Ex. Insert keys in random order.

N = 255
max = 16
avg = 9.1
opt = 7.0

Correspondence between BSTs and quicksort partitioning

0	1	2	3	4	5	6	7	8	9	10	11	12	13
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
H	L	E	A	D	O	M	C	I	P	T	Y	U	S
D	C	E	A	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
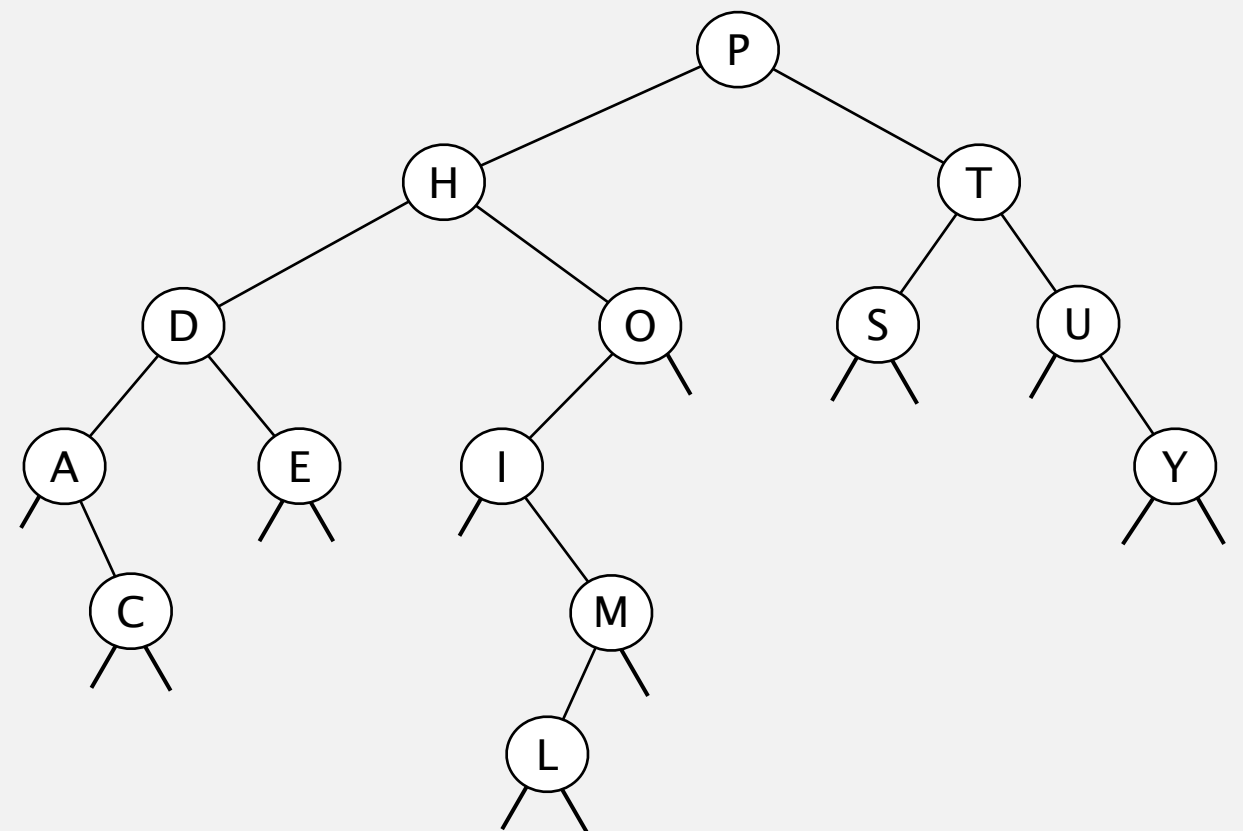
Remark. Correspondence is 1–1 if array has no duplicate keys.

BSTs: mathematical analysis

Proposition. If N distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is ~ $2 \ln N$.

Pf. 1–1 correspondence with quicksort partitioning.

Proposition. [Reed, 2003] If N distinct keys are inserted in random order, expected height of tree is ~ $4.311 \ln N$.

How Tall is a Tree?

Bruce Reed
CNRS, Paris, France
reed@moka.ccr.jussieu.fr

ABSTRACT

Let H_n be the height of a random binary search tree on n nodes. We show that there exists constants $\alpha = 4.31107\ldots$ and $\beta = 1.95\ldots$ such that $\mathbf{E}(H_n) = \alpha \log n - \beta \log \log n + O(1)$, We also show that $\mathrm{Var}(H_n) = O(1)$.

But... Worst-case height is N.

[exponentially small chance when keys are inserted in random order]

ST implementations: summary

implementation	guarantee		average case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	N	N	½ N	N	`equals()`
binary search (ordered array)	lg N	N	lg N	½ N	`compareTo()`
BST	N	N	1.39 lg N	1.39 lg N	`compareTo()`

Why not shuffle to ensure a (probabilistic) guarantee of 4.311 ln N?

3.2 BINARY SEARCH TREES

- ▸ *BSTs*
- ▸ *ordered operations*
- ▸ *deletion*

Algorithms

Robert Sedgewick | Kevin Wayne

Minimum and maximum

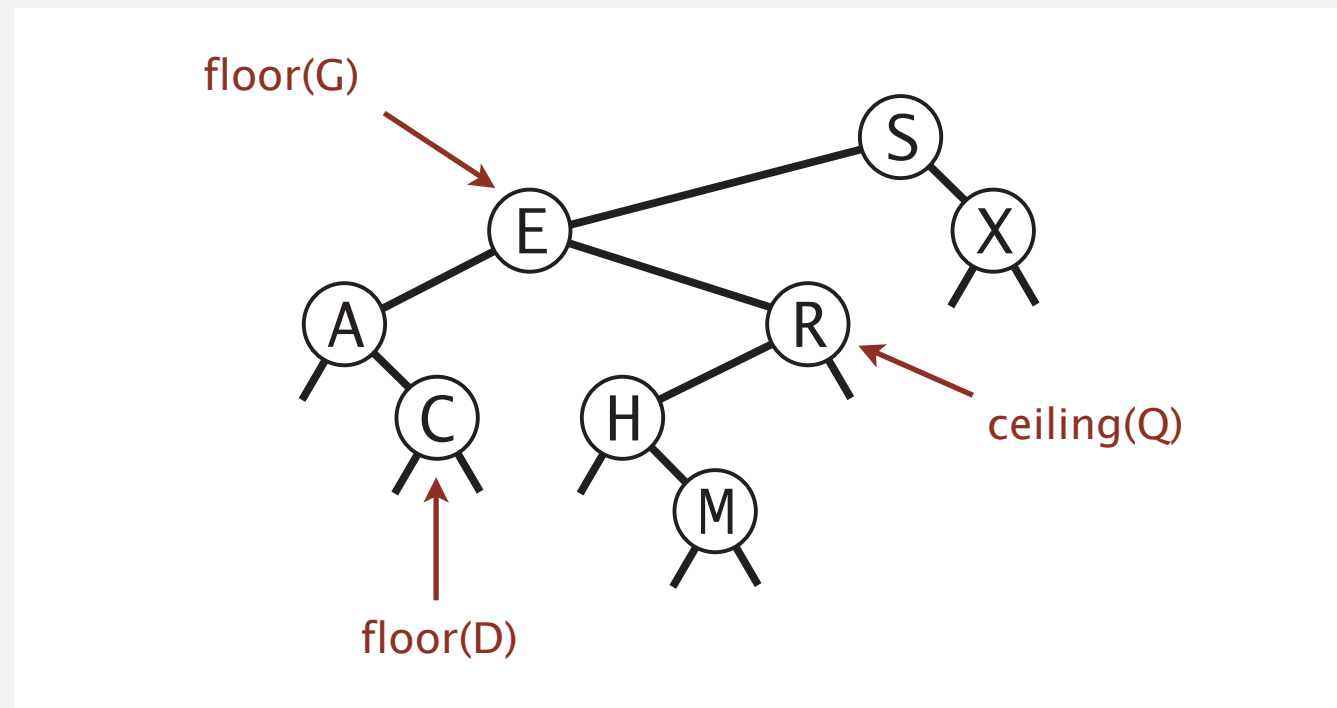Minimum. Smallest key in table.

Maximum. Largest key in table.
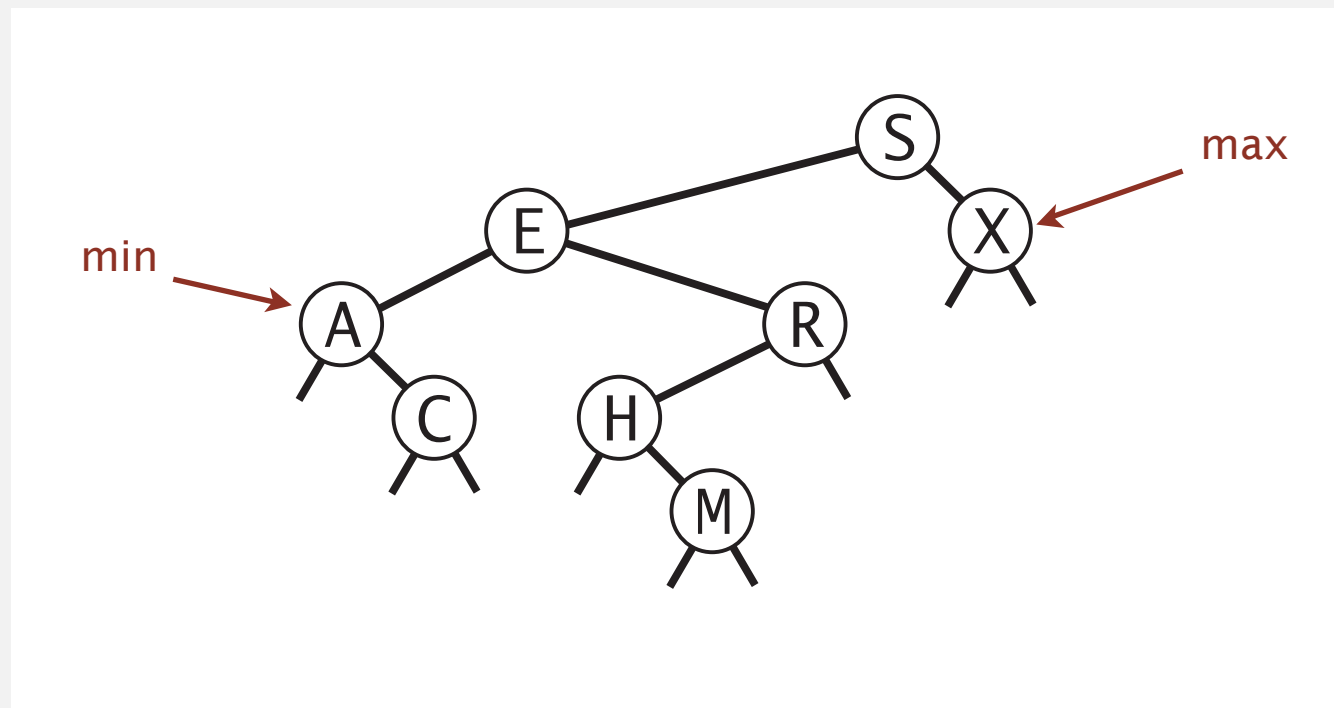
Q. How to find the min / max?

Floor and ceiling

Floor. Largest key ≤ a given key.

Ceiling. Smallest key ≥ a given key.

Q. How to find the floor / ceiling?

Computing the floor

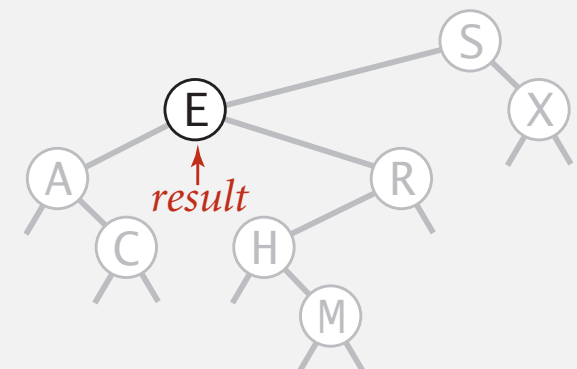**Case 1.** [*k* equals the key in the node]

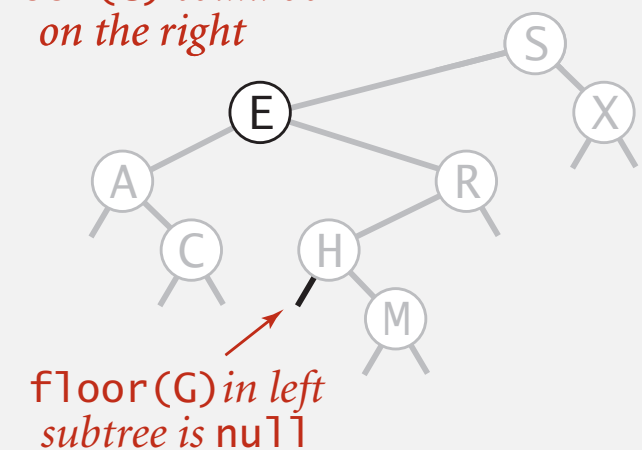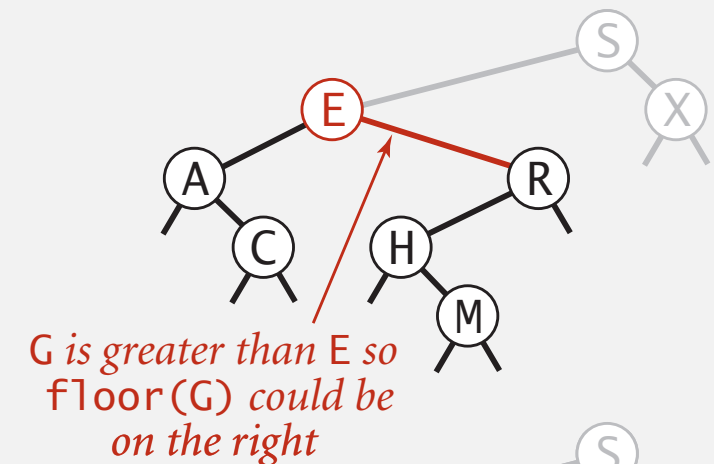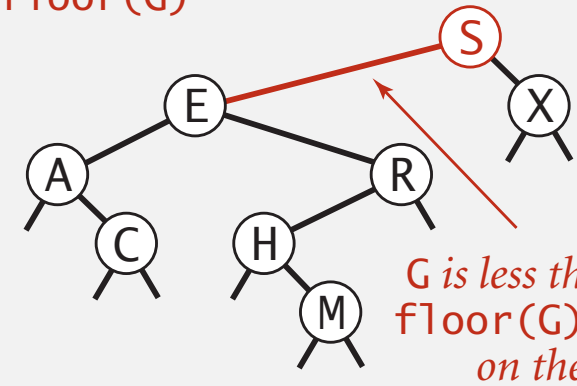The floor of *k* is *k*.

Case 2. [*k* is less than the key in the node]

The floor of *k* is in the left subtree.

Case 3. [*k* is greater than the key in the node]

The floor of *k* is in the right subtree
(if there is any key ≤ *k* in right subtree);

otherwise it is the key in the node.

finding `floor(G)`

G is less than S so `floor(G)` *must be on the left*

G is greater than E so `floor(G)` *could be on the right*

`floor(G)` *in left subtree is* `null`

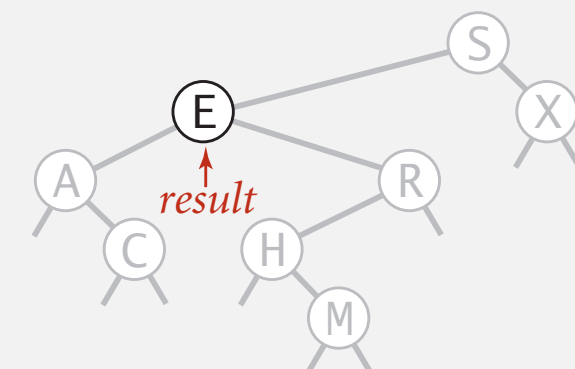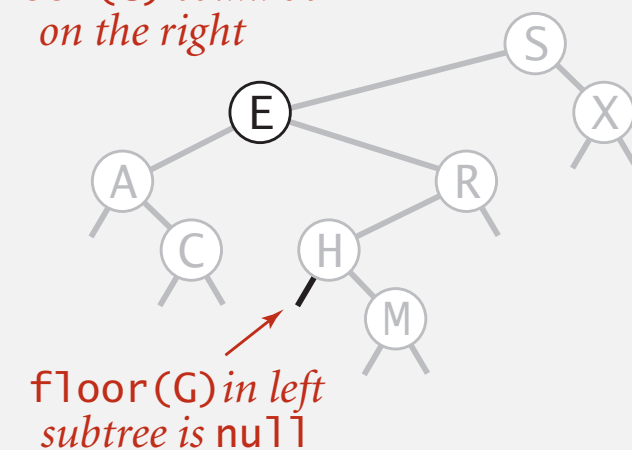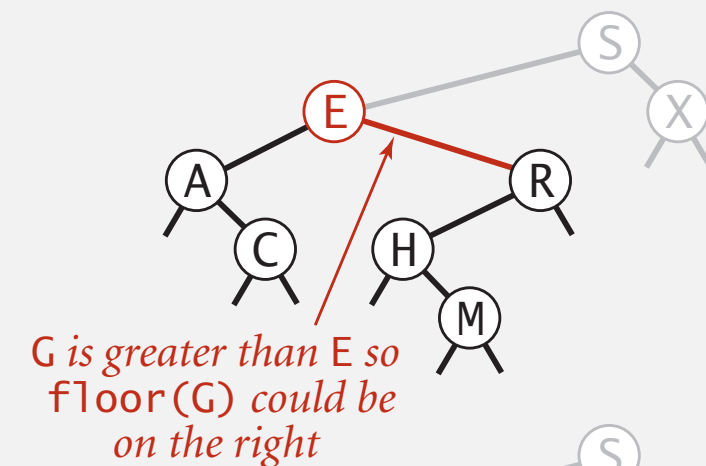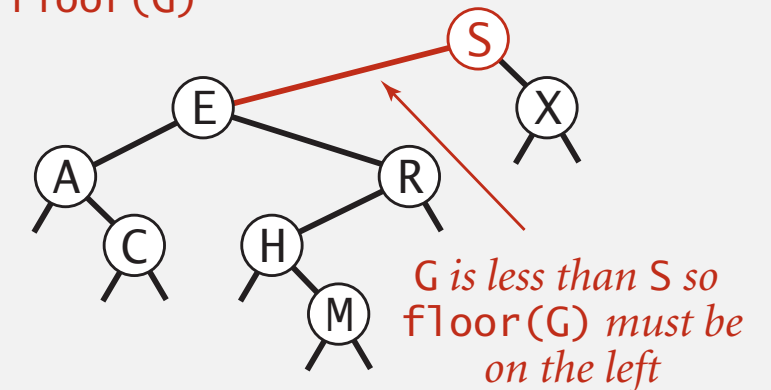*result*

Computing the floor

```
public Key floor(Key key)
{
    Node x = floor(root, key);
    if (x == null) return null;
    return x.key;
}
private Node floor(Node x, Key key)
{

    if (x == null) return null;
    int cmp = key.compareTo(x.key);


    if (cmp == 0) return x;


    if (cmp < 0)  return floor(x.left, key);


    Node t = floor(x.right, key);
    if (t != null) return t;
    else           return x;


}
```

finding `floor(G)`

G *is less than S so*
`floor(G)` *must be
on the left*

G *is greater than E so*
`floor(G)` *could be
on the right*

`floor(G)` *in left
subtree is* `null`

result

BST implementation: subtree counts

```
private class Node
{
    private Key key;
    private Value val;
    private Node left;
    private Node right;
    private int count;
}
```

number of nodes in subtree

```
public int size()
{   return size(root);   }
```

```
private int size(Node x)
{
    if (x == null) return 0;
    return x.count;
}
```

ok to call
when x is null

```
private Node put(Node x, Key key, Value val)
{
    if (x == null) return new Node(key, val, 1);
    int cmp = key.compareTo(x.key);
    if      (cmp  < 0) x.left  = put(x.left,  key, val);
    else if (cmp  > 0) x.right = put(x.right, key, val);
    else if (cmp == 0) x.val = val;
    x.count = 1 + size(x.left) + size(x.right);
    return x;
}
```
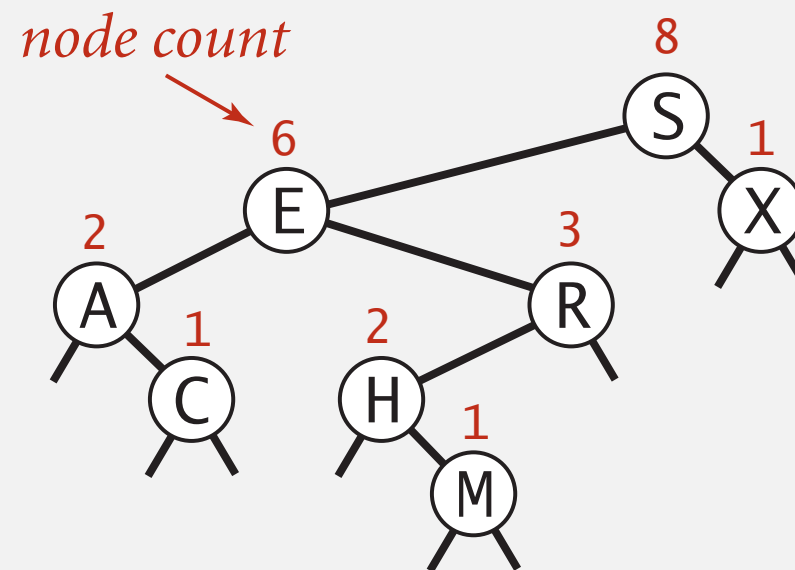
initialize subtree
count to 1

Rank and select

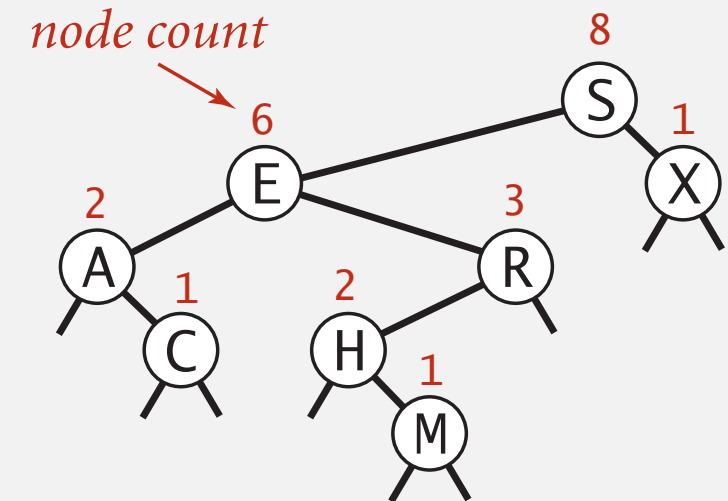Q. How to implement `rank()` and `select()` efficiently?

A. In each node, we store the number of nodes in the subtree rooted at that node; to implement `size()`, return the count at the root.

node count

Rank

Rank. How many keys $< k$?

Easy recursive algorithm (3 cases!)

node count

```
public int rank(Key key)
{   return rank(key, root);  }

private int rank(Key key, Node x)
{
    if (x == null) return 0;
    int cmp = key.compareTo(x.key);
    if       (cmp  < 0) return rank(key, x.left);
    else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
    else if (cmp == 0) return size(x.left);
}
```
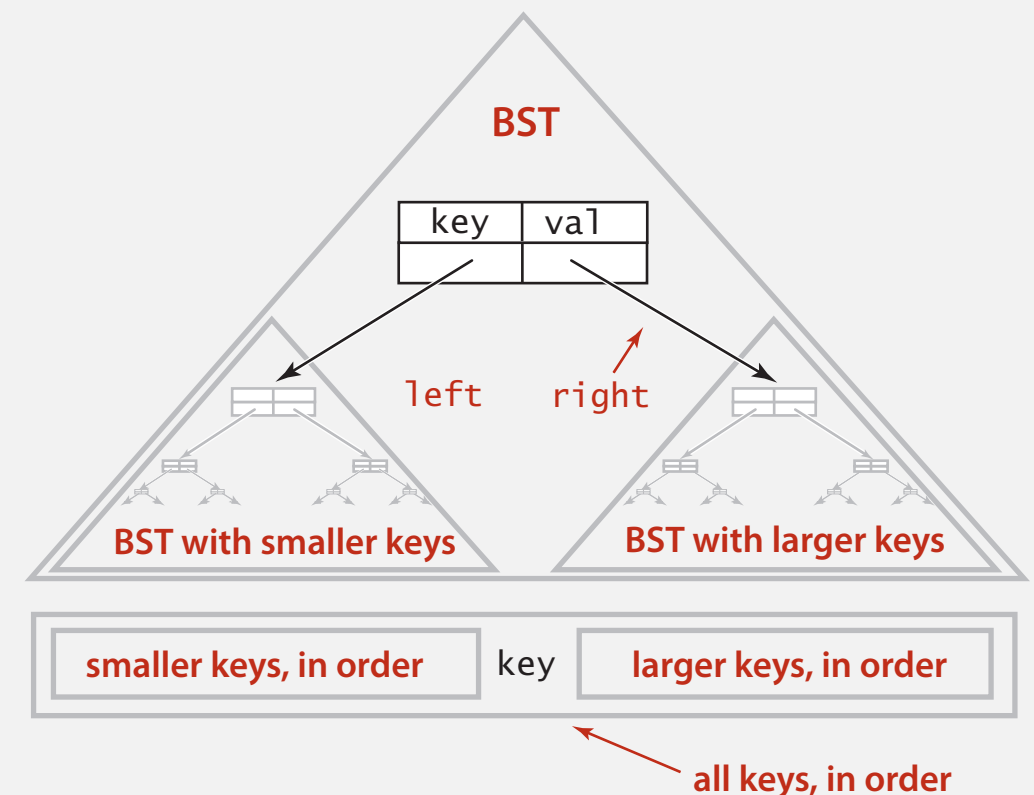
Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
    Queue<Key> q = new Queue<Key>();
    inorder(root, q);
    return q;
}

private void inorder(Node x, Queue<Key> q)
{
    if (x == null) return;
    inorder(x.left, q);
    q.enqueue(x.key);
    inorder(x.right, q);
}
```


BST

key | val

left   right

BST with smaller keys   BST with larger keys

smaller keys, in order | key | larger keys, in order

all keys, in order

Property. Inorder traversal of a BST yields keys in ascending order.

BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	N	$\lg N$	h
insert	N	N	h
min / max	N	1	h
floor / ceiling	N	$\lg N$	h
rank	N	$\lg N$	h
select	N	1	h
ordered iteration	$N \log N$	N	N

h = height of BST
(proportional to log N
if keys inserted in random order)

order of growth of running time of ordered symbol table operations

3.2 BINARY SEARCH TREES

- ▸ *BSTs*
- ▸ *ordered operations*
- ▸ **deletion**

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

ST implementations: summary

implementation	guarantee			average case			ordered ops?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	½ N	N	½ N		`equals()`
binary search (ordered array)	lg N	N	N	lg N	½ N	½ N	✔	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	???	✔	`compareTo()`

Next. Deletion in BSTs.

BST deletion: lazy approach

To remove a node with a given key:

- Set its value to `null`.

- Leave key in tree to guide search (but don't consider it equal in search).
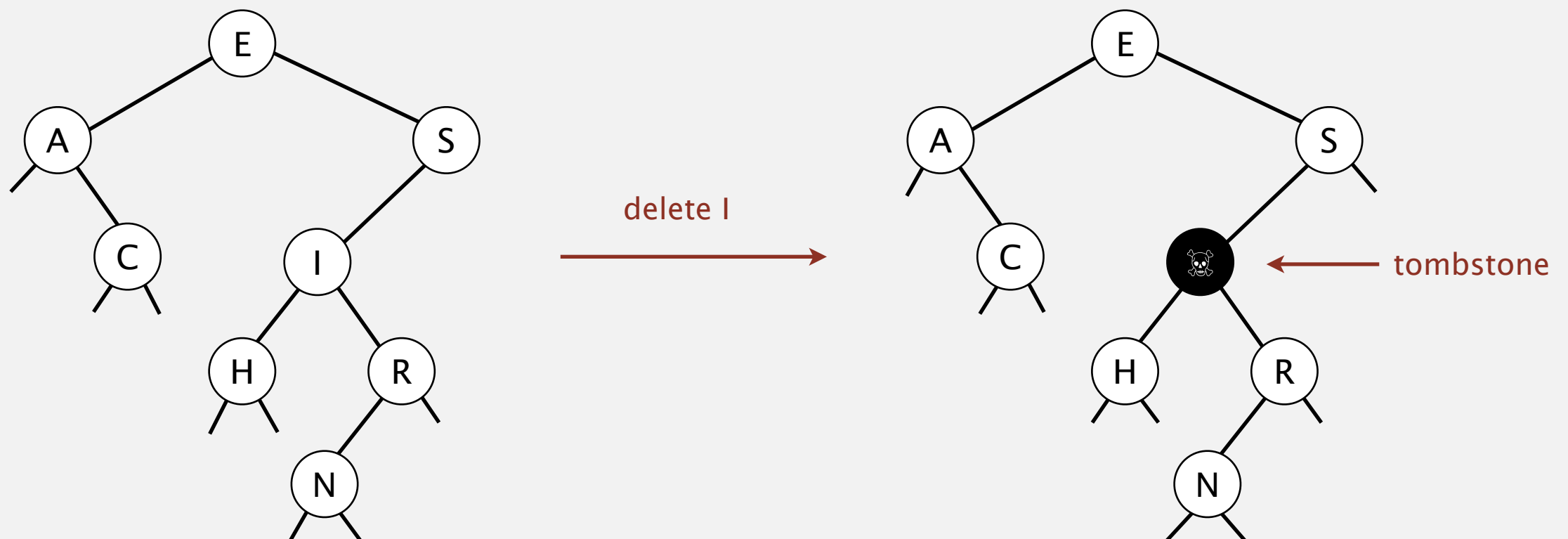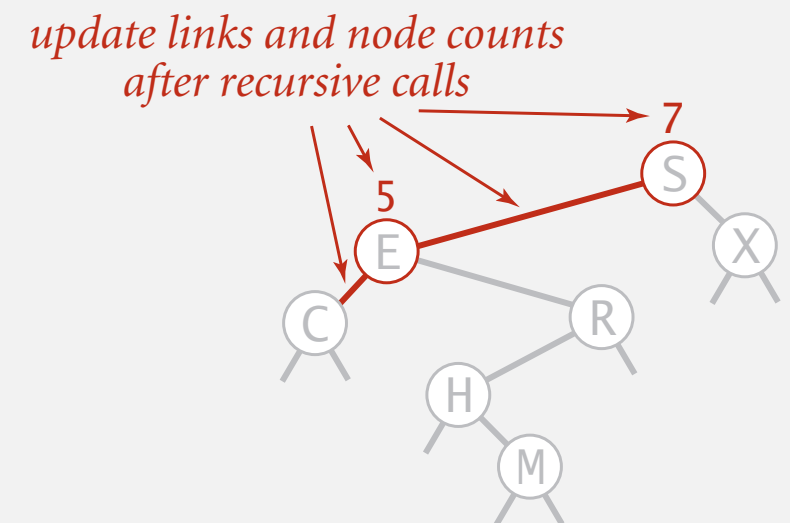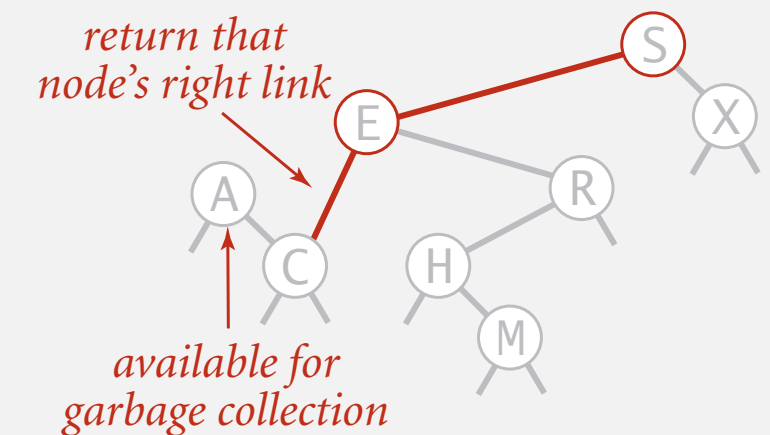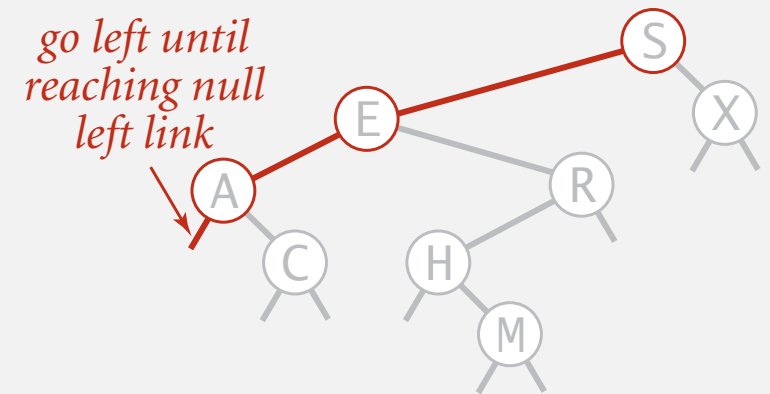

delete I

tombstone

Cost. $\sim 2 \ln N'$ per insert, search, and delete (if keys in random order), where N' is the number of key-value pairs ever inserted in the BST.

Unsatisfactory solution. Tombstone (memory) overload.

Deleting the minimum

To delete the minimum key:

- Go left until finding a node with a null left link.
- Replace that node by its right link.
- Update subtree counts.

go left until reaching null left link

return that node's right link

available for garbage collection

update links and node counts after recursive calls

```java
public void deleteMin()
{   root = deleteMin(root);   }

private Node deleteMin(Node x)
{
    if (x.left == null) return x.right;
    x.left = deleteMin(x.left);
    x.count = 1 + size(x.left) + size(x.right);
    return x;
}
```
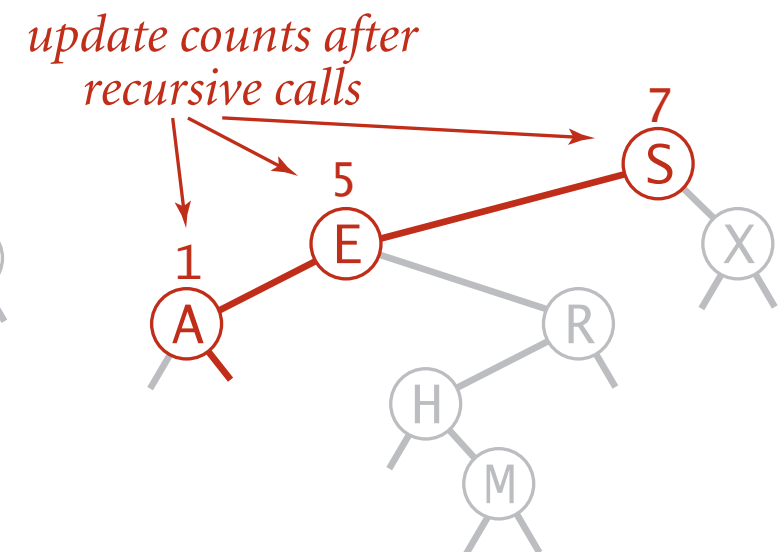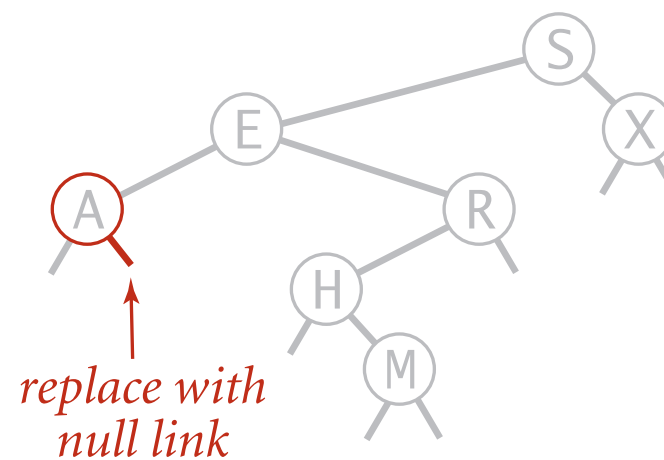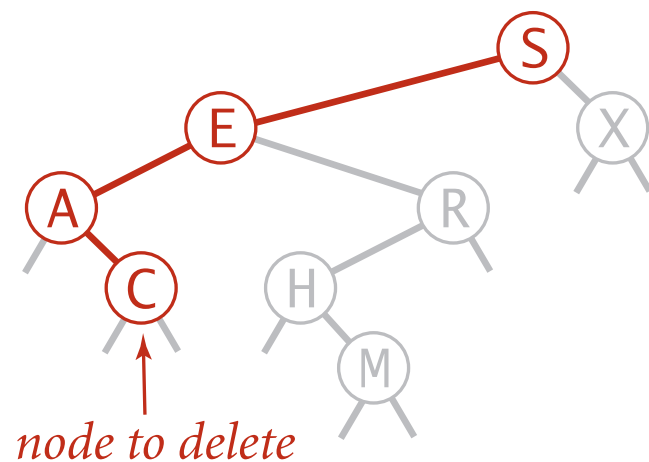
Hibbard deletion

To delete a node with key k: search for node t containing key k.

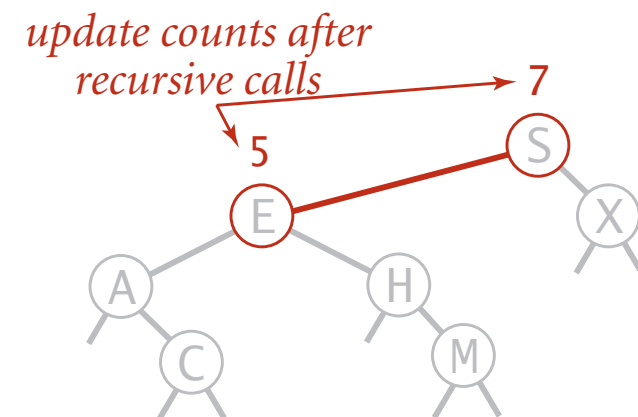Case 0. [0 children] Delete t by setting parent link to null.

deleting C

node to delete

replace with null link

available for garbage collection

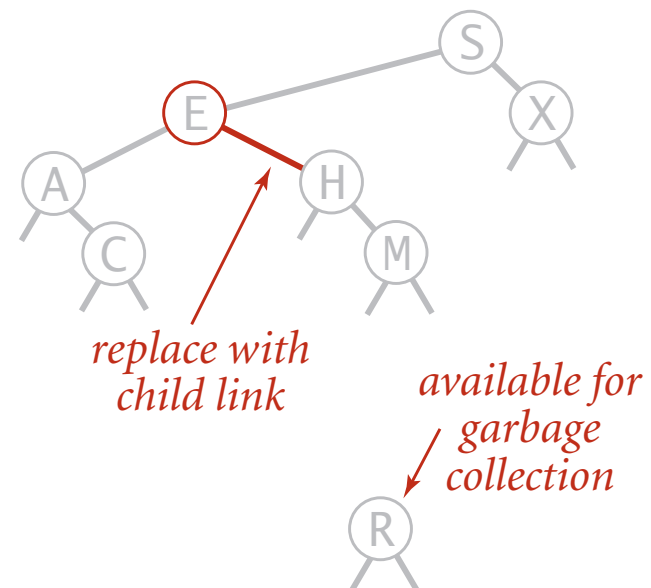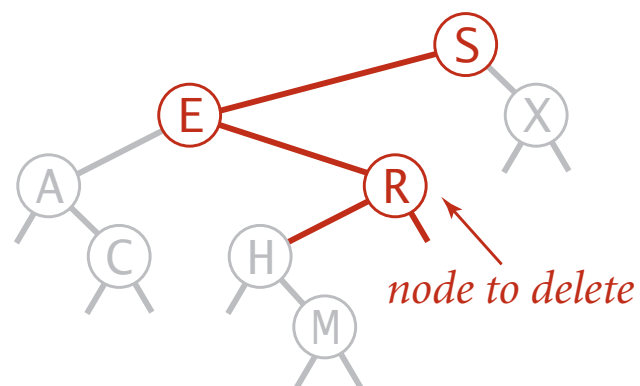update counts after recursive calls

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1

Hibbard deletion

To delete a node with key `k`: search for node `t` containing key `k`.

Case 1. [1 child] Delete `t` by replacing parent link.

deleting R

node to delete

replace with
child link

available for
garbage
collection

update counts after
recursive calls

Hibbard deletion

To delete a node with key `k`: search for node `t` containing key `k`.

Case 2. [2 children]

- Find successor `x` of `t`.   ⟵ x has no left child
- Delete the minimum in `t`'s right subtree.   ⟵ but don't garbage collect x
- Put `x` in `t`'s spot.   ⟵ still a BST
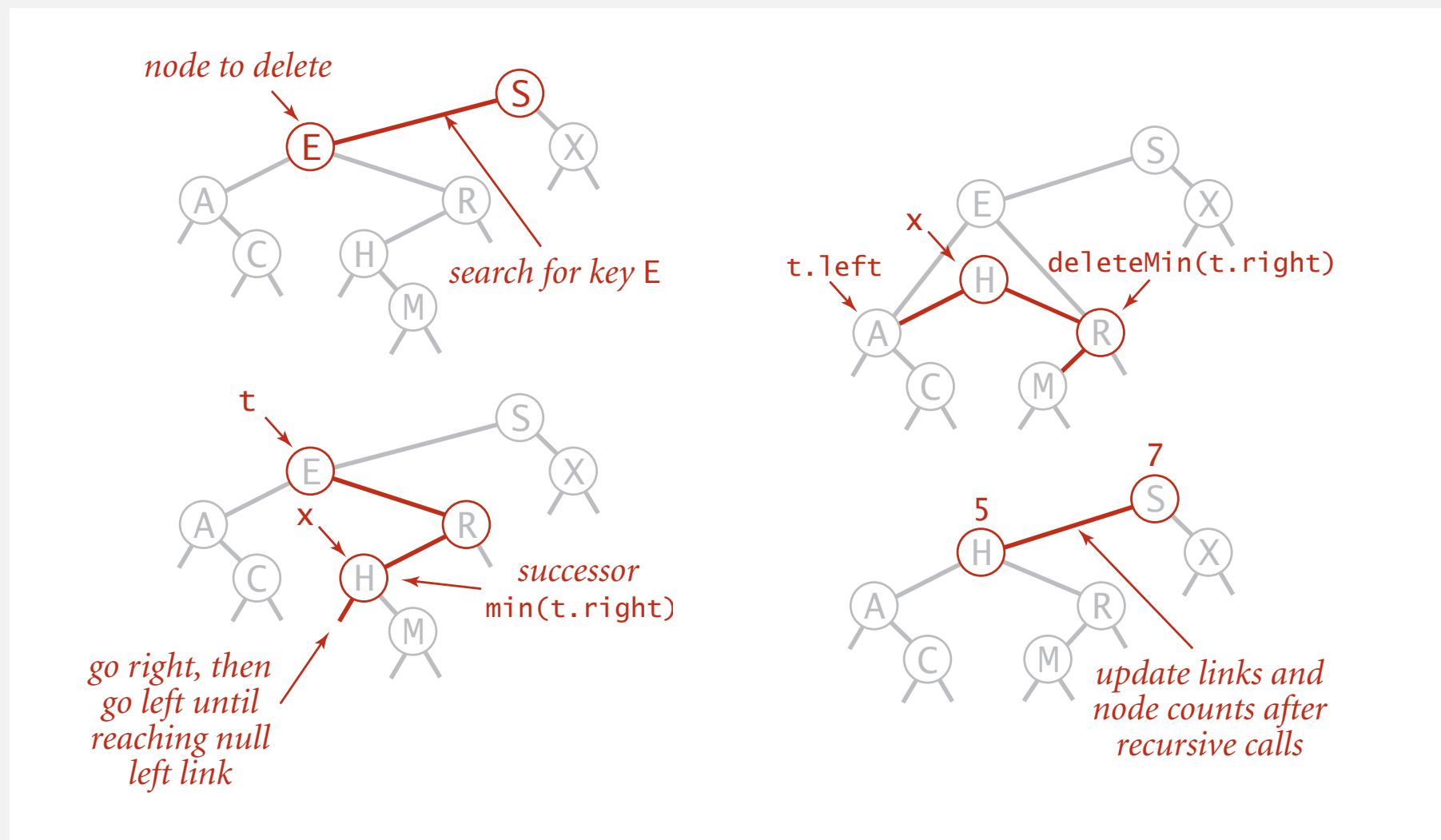
Hibbard deletion: Java implementation

```java
public void delete(Key key)
{   root = delete(root, key);   }

private Node delete(Node x, Key key) {
    if (x == null) return null;
    int cmp = key.compareTo(x.key);
    if      (cmp < 0) x.left  = delete(x.left,  key);    ← search for key
    else if (cmp > 0) x.right = delete(x.right, key);
    else {
        if (x.right == null) return x.left;              ← no right child
        if (x.left  == null) return x.right;             ← no left child

        Node t = x;
        x = min(t.right);                                ← replace with successor
        x.right = deleteMin(t.right);
        x.left = t.left;
    }
    x.count = size(x.left) + size(x.right) + 1;          ← update subtree counts
    return x;
}
```

Unsatisfactory solution. Not symmetric.

N = 150
max = 16
avg = 9.3
opt = 6.4
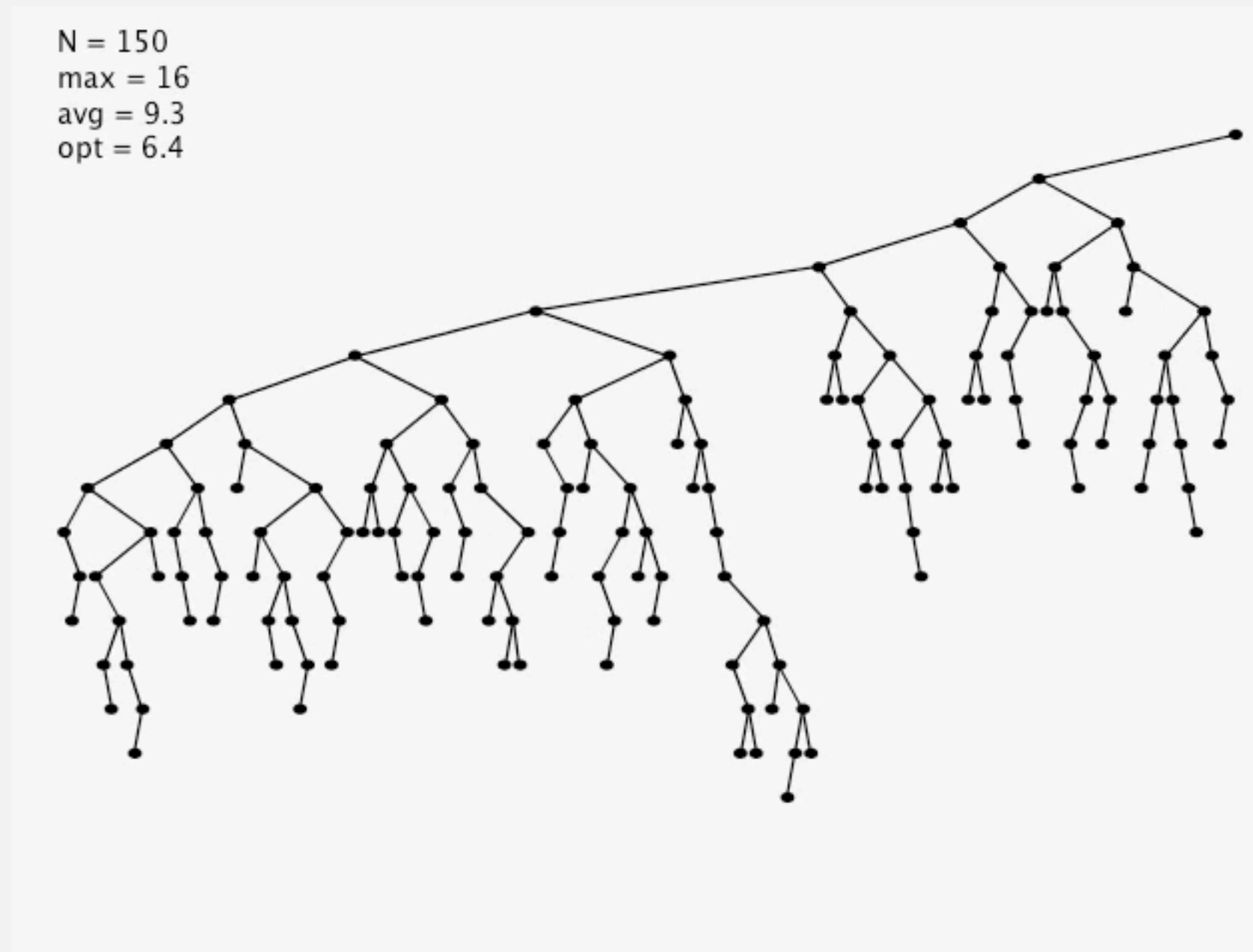
Surprising consequence. Trees not random (!) $\Rightarrow \sqrt{N}$ per op.

Longstanding open problem. Simple and efficient delete for BSTs.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	½ N	N	½ N		`equals()`
binary search (ordered array)	lg N	N	N	lg N	½ N	½ N	✔	`compareTo()`
BST	N	N	N	1.39 lg N	1.39 lg N	$\sqrt{N}$	✔	`compareTo()`

other operations also become √N
if deletions allowed

Next lecture. Guarantee logarithmic performance for all operations.