Design and Analysis of Algorithms
Flipped Class Offering

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Course **_Comp 582 (and Elec 420)_**

Time **_1050AM to 1205PM_**

Location **_DCH 1055_**

Part II: Algorithmic Foundations of Data Science

Pre-requisite Linear Algebra

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Source:
These slides are based on source material (including notation, concepts, and presentation approach) from *Basic Vector Space Methods in Signal and Systems Theory* By *C. Sidney Burrus* (Rice University, Houston, Texas, USA)

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Topics in this Lecture

Set Theory

Set Theory

- A *set* is a collection of *distinct objects.*
  - Example: Set $A = \{1,2,3,4\}$.
- A *finite set* has a finite number of objects.
  - Set A in the above example is a finite set.
- An *infinite set* is a set that has an infinite number of objects.
  - Example: $\mathbb{Z} = \{\cdots -2, -1, 0, 1, 2 \cdots\}$ is the set of all integers.
  - Similarly, $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers, respectively.

Set Theory (contd.)

- A set A is a *subset* of a set B, if all elements of A are also elements of B.
  - It is denoted by $A \subseteq B$.
  - An empty set is always a subset of every set.
- The *power set* of a set S is the set of all subsets of S.
  - It is denoted by 2^S.
  - Example: If $S = \{1,2,3\}$, then
    $$2^S = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Vectors and Vector Spaces

Vectors

- A vector is a mathematical structure that has the ability to represent both *magnitude* and *direction* simultaneously.
- A point (x, y) on a Cartesian coordinate plane is an example of a vector.
- A *vector of dimension* n is an *ordered* collection of n elements called *components* or *entries*.
  - Components can be real or complex numbers.
- A *scalar* is a dimensionless quantity.
  - For example, one that can be described by a single real number.

A Signal as a Vector

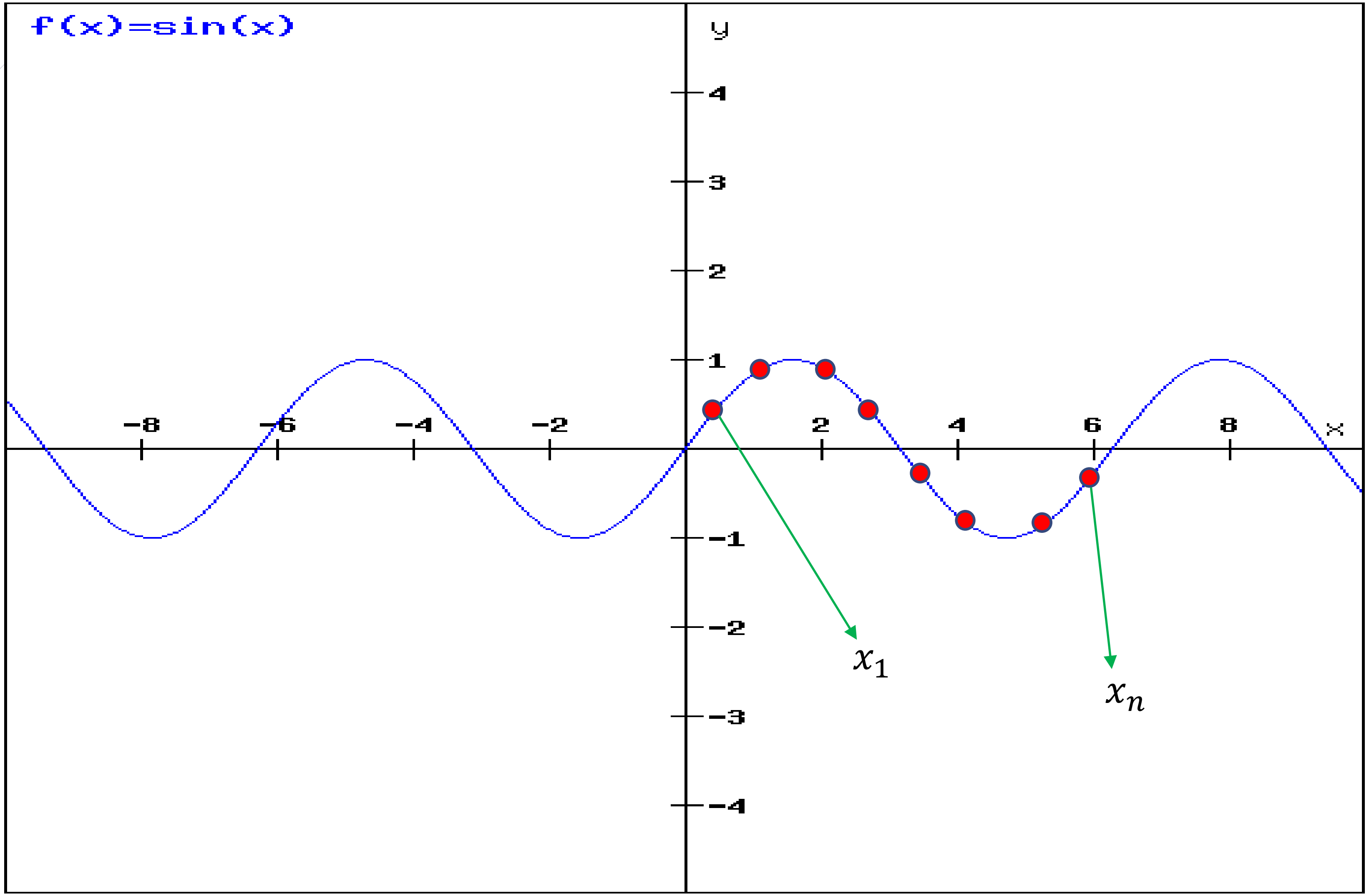
- Information about many real world phenomena (audio, video, radar, etc.) are conveyed as signals
  - Typically patterns that repeat overtime represented as periodic functions.
- One period of the signal can be represented as a vector

$$\boldsymbol{x} = (x_1, x_2, \dots, x_t, \dots, x_n)$$

Where x_1 is the sample representing the beginning of the period, x_n is the sample representing the end. All other x_i, $2 \leq i < n$, represent intermediate samples.

- This perspective will serve as a running example.

A Signal as a Vector



Vector Operations Leading to Vectors

- *Vector Addition*: For two vectors $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \cdots, b_n)$, the vector sum is given by
$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n).$$
- *Scalar Multiplication*: Let ℓ be a scalar. The scalar multiplication is given by $\ell\boldsymbol{a} = (\ell a_1, \ell a_2 \cdots \ell a_n)$.
- *Linear Combination*: When ℓ_1 and ℓ_2 are two scalars, $\ell_1\boldsymbol{a} + \ell_2\boldsymbol{b}$ is called a *linear combination* of $\boldsymbol{a}$ and $\boldsymbol{b}$.
- *Span*: Let $\boldsymbol{X}$ be a set of vectors. The set of all linear combinations of vectors of $\boldsymbol{X}$ is called the span of $\boldsymbol{X}$.
  - *Notice: the span leads to a whole "space" of vectors!*

Vector Operations Resulting in Scalars

- An *Inner Product* of two vectors denoted by $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is any notion of multiplication of $\boldsymbol{a}$ and $\boldsymbol{b}$ resulting in a scalar and obeys the following rules.
- Given the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$
  1. Commutative, i.e., $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \boldsymbol{a} \rangle$,
  2. Distributive, i.e., $\langle \boldsymbol{a}, \boldsymbol{b} + \boldsymbol{c} \rangle = \langle \boldsymbol{a}, \boldsymbol{b} \rangle + \langle \boldsymbol{a}, \boldsymbol{c} \rangle$.
  3. Positive-Definite, i.e., $\langle \boldsymbol{a}, \boldsymbol{a} \rangle \geq 0$ with equality iff $\boldsymbol{a} = \boldsymbol{0}$, and
  4. For any scalar ℓ, $\langle \ell \boldsymbol{a}, \boldsymbol{b} \rangle = \ell \langle \boldsymbol{a}, \boldsymbol{b} \rangle$.
- The *Dot Product* is the most common inner product. Consider $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \cdots, b_n)$ then the dot product is given by

$$\boldsymbol{a}.\boldsymbol{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Vector Space

- A *vector space* $\mathbf{V}$ is a set of vectors that is closed under finite vector addition and *scalar* multiplication, that satisfy the following eight conditions.
  - Example: n-dimensional Euclidean space $\mathbb{R}^n$.
  - In $\mathbb{R}^n$, every element is represented by a set of n real numbers which is a vector in $\mathbb{R}^n$.
  - Closure: Given any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}+\boldsymbol{b}$ is also in $\mathbb{R}^n$.
- In order for $\mathbf{V}$ to be a vector space, the following properties must hold for all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbf{V}$ and any scalars α, β.
  1. Commutativity:
  $$\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}.$$
  2. Associativity of vector addition:
  $$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w}).$$

Vector Space (contd.)

3. Additive identity: $\exists\ 0 \in \boldsymbol{V}$ s.t. $\forall\ \boldsymbol{u} \in \boldsymbol{V}$
$$\mathbf{0} + \boldsymbol{u} = \boldsymbol{u} + \mathbf{0} = \boldsymbol{u}.$$
4. Additive inverse: $\forall\ \boldsymbol{u} \in \boldsymbol{V},\ \exists\ -\boldsymbol{u} \in \boldsymbol{V}$ s.t.
$$\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}.$$
5. Associativity of scalar multiplication:
$$\alpha(\beta\boldsymbol{u}) = (\alpha\beta)\boldsymbol{u}.$$
6. Distributivity of scalar sums:
$$(\alpha + \beta)\boldsymbol{u} = \alpha\boldsymbol{u} + \beta\boldsymbol{u}.$$
7. Distributivity of vector sums:
$$\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v}.$$
8. Scalar multiplication identity:
$$1\boldsymbol{u} = \boldsymbol{u}.$$

"$\mathbf{0}$" is a zero vector or null vector having magnitude equal to zero, and thus has all components equal to zero.

This additive inverse of $\boldsymbol{u}$ denoted by "$-\boldsymbol{u}$" has the same components as $\boldsymbol{u}$, but negated.

Linear Dependence and Independence of Vectors

- A set of vectors is said to *linearly dependent* iff one of the vectors in the set can be written as a linear combination of the other vectors.
- Whenever no vector in the set can be written as a linear combination of rest of the vectors in the set, then the vectors are said to be *linearly independent*.
- Exercise: Prove that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent and $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent.

$$\boldsymbol{u} = (1, 1), \qquad \boldsymbol{v} = (-3, 2), \qquad \boldsymbol{w} = (2, 4).$$

Subspace of a Vector Space

- Let $\boldsymbol{V}$ be a vector space. Then $\boldsymbol{W} \subseteq \boldsymbol{V}$ is a *subspace* of $\boldsymbol{V}$ if $\boldsymbol{W}$ is itself a vector space.
- Example: Consider a simple vector space $\mathbb{R}^3$, which is the 3-dimensional Euclidean space. Its subspaces are
  - The origin,
  - Any line passing through the origin,
  - Any plane passing through the origin, and finally
  - $\mathbb{R}^3$.
- Exercise: Would the unit cube centered at the origin be a subspace? In particular, would it be closed?

Basis and Dimension of a Vector Space

- A *basis* of a vector space $\boldsymbol{V}$ is defined as a subset of vectors in $\boldsymbol{V}$ that are linearly independent and span $\boldsymbol{V}$.
- These are called *basis vectors.*
- A vector space $\boldsymbol{V}$ can have many different bases, but there are always the same number of basis vectors in each of them.
- The number of basis vectors in a basis of $\boldsymbol{V}$ is the *dimension* of $\boldsymbol{V}$.
- Example: In $\mathbb{R}^2$, $\boldsymbol{u} = (1,0)$ and $\boldsymbol{v} = (0,1)$ forms a basis. Any vector $\boldsymbol{w} = (a, b)$ can be uniquely written as the linear combination $\boldsymbol{w} = a\boldsymbol{u} + b\boldsymbol{v}.$ Its dimension is 2.

Standard Basis

- In a *standard basis,* each basis vector has only one nonzero entry whose magnitude is 1.
- Example: In the real vector space $\mathbb{R}^3$, the standard basis is the set of vectors

  $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}.$
- On the other hand, vectors $(3,2)$ and $(2,1)$ are also a basis for $\mathbb{R}^2$ but not a standard basis.
  - This means any vector in $\mathbb{R}^2$ can be generated as a linear combination of vectors $(3,2)$ and $(2,1)$ but it is not a standard basis for obvious reasons.

Change of Basis

- We know that there are many different bases other than standard basis.
  - It is often convenient to work with more than one basis for a vector space.
- Let $\boldsymbol{V}$ be a vector space and let $S = \{\boldsymbol{v_1} \cdots \boldsymbol{v_n}\}$ be a set of vectors which forms a basis for $\boldsymbol{V}$.
  - If S is a basis for $\boldsymbol{V}$, then every vector $\boldsymbol{v} \in \boldsymbol{V}$ can be expressed as:

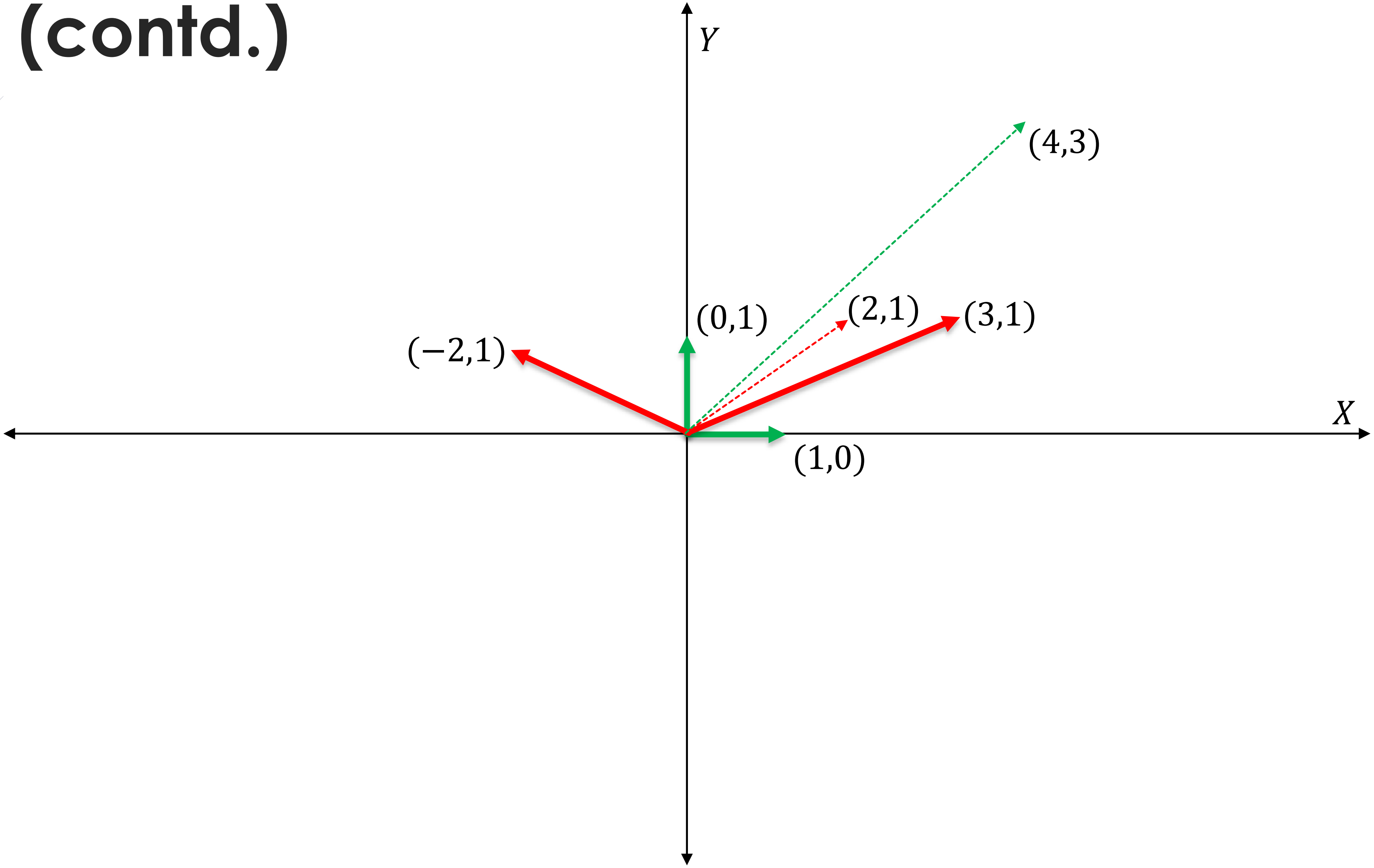
$$\boldsymbol{v} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n}.$$

  - Think of $(c_1, c_2, \cdots, c_n)$ as the coordinates of $\boldsymbol{v}$ *relative* to the basis S.
    - These coordinates are easily understood if we choose the standard basis of course.
  - Let $(c'_1, c'_2, \cdots, c'_n)$ be the coordinates of $\boldsymbol{v}$ relative to some other basis S'.
    - This process of transformation of coordinates from one basis to another is called *change of basis.*

Change of Basis: A Simple Example

- $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$
  - Two different bases in $\mathbb{R}^2$.
- The Change of basis from B' to B can be represented as a matrix
$$A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}.$$
- Suppose the coordinates of vector $\boldsymbol{v} \in \mathbb{R}^2$ relative to the basis B' are $[\boldsymbol{v}]_{B'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
- The coordinates of vector $\boldsymbol{v} \in \mathbb{R}^2$ relative to basis B are
$$[\boldsymbol{v}]_B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$
- Notice that change of basis is a transformation which requires multiplication of a matrix and a vector.

Change of Basis: A Simple Example (contd.)



A Matrix Times a Vector

Introducing the Role of Matrices to Change the Basis

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \cdots (1)$$

is a simple matrix equation, where $\boldsymbol{x}$ and $\boldsymbol{b}$ are vectors from the same or perhaps different vector spaces and A is a matrix.

- This equation has variety of special cases.
  - The matrix A may be square or may be rectangular
  - be symmetric
  - orthogonal
  - or have some of many other characteristics which would be interesting.
  - The entries in A could be complex for some important applications.
- Obvious observation: Given any two elements of $eq. (1)$ we can find the third element if a solution exists.
- We will focus on finite dimensions.

Dual Basis contd.

- Inverse of a *square* matrix A is denoted by A^{-1} and it is defined as
  - Remember, $AA^{-1} = I$ where I is a unit matrix.
- The rows of matrix A^{-1} is defined to be the *dual basis* (of our original basis) for **V**.
  - Exists only if A is a square matrix.
- Example : Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ then the inverse of A is

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ -1/5 & 3/5 \end{bmatrix}$$

(Use determinant x adjugate)

Introduction to $Ax = b$ (contd.)

- In $eq. (1)$, if we take the vectors (i.e. $\boldsymbol{x}$ and $\boldsymbol{b}$) and the matrix A.
  - Following our earlier example the product $A\boldsymbol{x}$ can be interpreted as a change of basis.
- The vector stays the same but its basis changes.

Introduction to $\boldsymbol{Ax} = \boldsymbol{b}$ (contd.)

- Continuing with our view of matrix multiplication as a change of basis:
  - The operation $A\boldsymbol{x} = \boldsymbol{b}$ can now be viewed as $\boldsymbol{x}$ being a set of weights so that $\boldsymbol{b}$ is a weighted sum of the columns of A.

This view is shown below. If the vector a_i is the i^{th} column of A, the product takes the form

$$A\boldsymbol{x} = x_1 \begin{bmatrix} \vdots \\ a_1 \\ \vdots \end{bmatrix} + x_2 \begin{bmatrix} \vdots \\ a_2 \\ \vdots \end{bmatrix} + \cdots + x_n \begin{bmatrix} \vdots \\ a_n \\ \vdots \end{bmatrix} = \boldsymbol{b}$$

In other words, $\boldsymbol{b}$ will lie in the space spanned by the columns of A at a location determined by $\boldsymbol{x}$.

A Basis and *Dual* Basis

- Recall that, a set of linearly independent vectors x_n forms a basis for a vector space ***V*** if every vector $\boldsymbol{x}$ in the space can be uniquely written as

$$\boldsymbol{x} = \sum_{n} a_n \, x_n \qquad \cdots (2)$$

- We also know that a basis can be represented as a matrix where the i^{th} column of that matrix is the an individual basis from the set.
  - As before, let A be that matrix of basis vectors.

Dual Basis contd.

- Recall that, column vectors of matrix A in previous slide forms a basis in $\mathbb{R}^2$.

- Let us give the same name to this basis as before namely $\boldsymbol{B'}$.
  - $\boldsymbol{B'} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

- Let $\boldsymbol{B''}$ be the dual basis of $\boldsymbol{B'}$ and from the determined by taking the inverse of $\boldsymbol{B'}$
  - Exercise: Verify that $\boldsymbol{B''} = \left\{ \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}, \begin{bmatrix} -1/5 \\ 3/5 \end{bmatrix} \right\}$.

Orthogonal and Orthonormal Basis

- *Orthogonal:* Two vectors are said to be orthogonal if their dot product is zero.
  - Example: $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ are orthogonal vectors.
  - $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} . \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = (1 \times 1) + \left(\sqrt{2} \times -\sqrt{2}\right) + (1 \times 1) = 1 - 2 + 1 = 0$
- *Orthonormal:* If the two orthogonal vectors are unit vectors then they are called orthonormal vectors.
  - Example: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are unit vectors and their dot product is zero and hence they are orthonormal vectors.

Orthogonal Basis Observation

- *A given basis is called an orthogonal basis if the basis vectors are mutually (pair-wise) orthogonal.*

  - Given any two vectors from an orthogonal basis, their dot product is zero, i.e. they are orthogonal vectors.

Orthonormal Basis Observations

- When the vectors of an orthogonal basis are unit vectors the resulting basis is an orthonormal basis.
  - Example: The set of vectors $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ forms an orthonormal basis of $\mathbb{R}^3$.

Orthonormal Basis Observations (contd.)

- Example: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ where the column vectors of A forms an orthonormal basis in $\mathbb{R}^2$.
- Let this basis is denoted by $\boldsymbol{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
- Here, $A^{-1} = A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and
- The row vectors of A^{-1} forms a dual basis and let this dual basis is denoted by $\boldsymbol{B'} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.
- From above example, we note that that orthonormal basis $\boldsymbol{B}$ is it's own dual.