3.3 Balanced Search Trees

- *2-3 search trees*
- *red-black BSTs*
- *B-trees*

Symbol table review

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	½ N	N	½ N		`equals()`
binary search (ordered array)	$\lg N$	N	N	$\lg N$	½ N	½ N	✔	`compareTo()`
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✔	`compareTo()`

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sequential search (unordered list)	N	N	N	$\frac{1}{2} N$	N	$\frac{1}{2} N$		`equals()`
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BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✔	`compareTo()`
goal	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`

Challenge. Guarantee performance.

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goal	log N	log N	log N	log N	log N	log N	✔	`compareTo()`

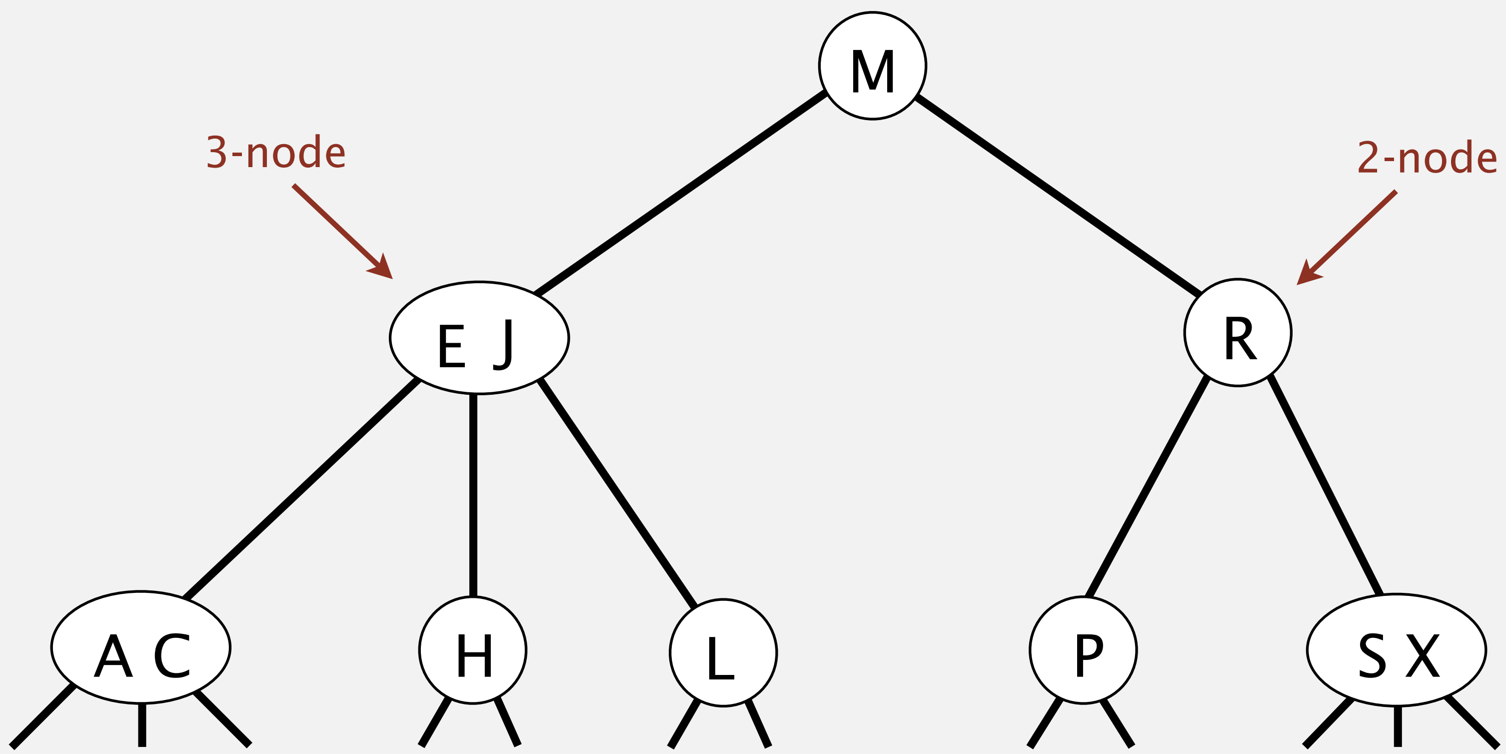
Challenge. Guarantee performance.

This lecture. 2-3 trees, left-leaning red-black BSTs, B-trees.

2-3 tree

Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.



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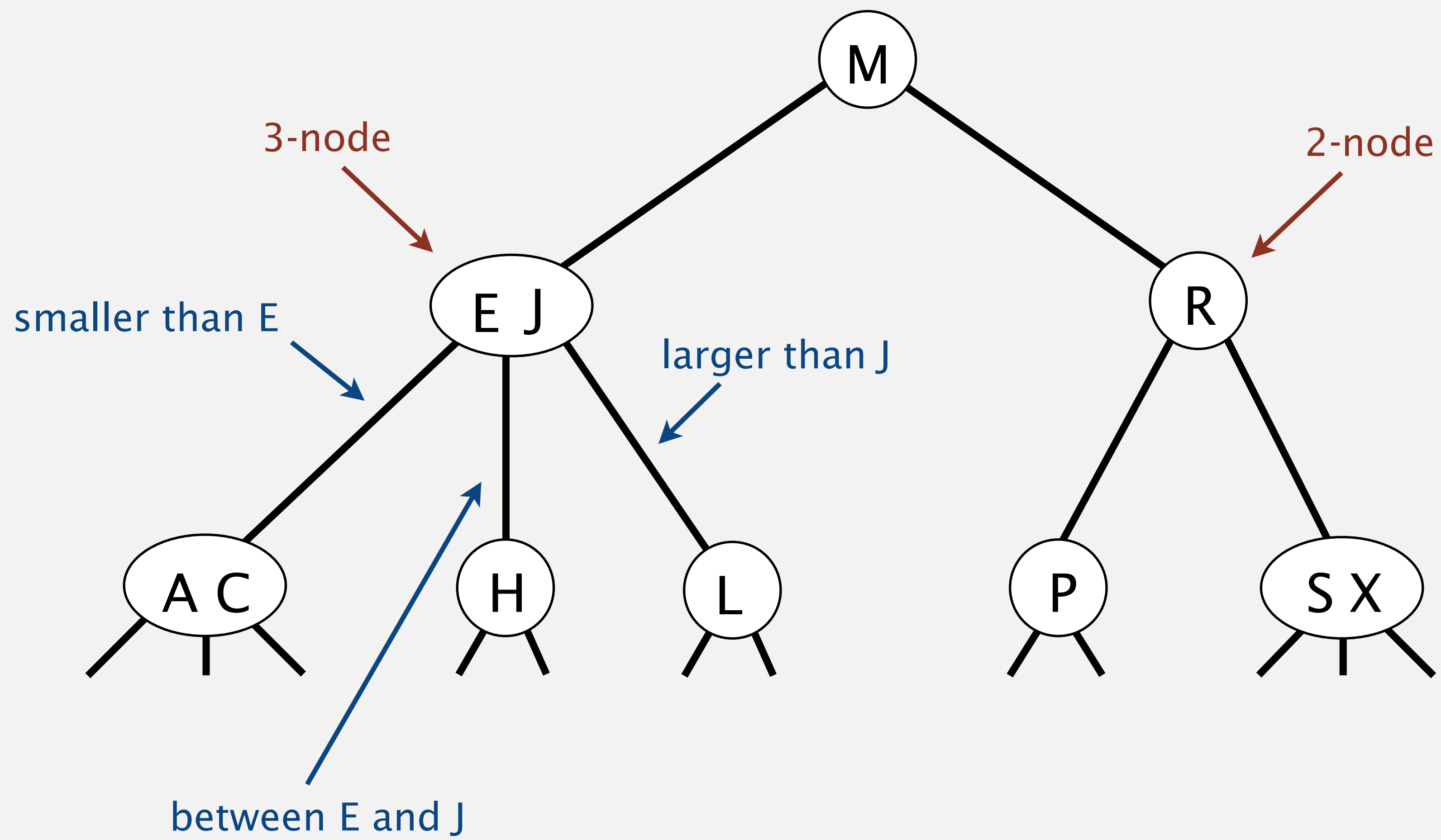
Symmetric order. Inorder traversal yields keys in ascending order.



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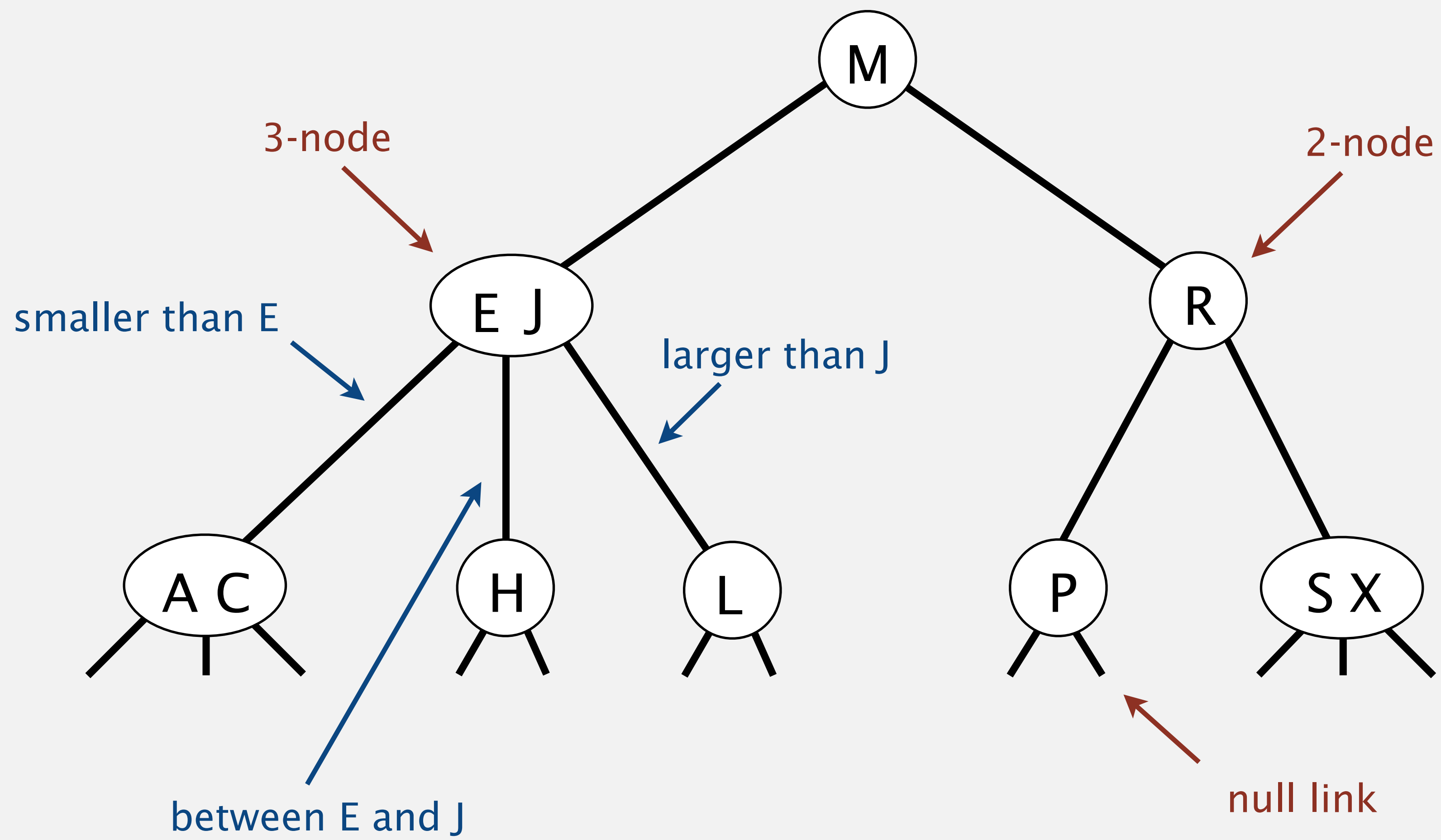
Perfect balance. Every path from root to null link has same length.



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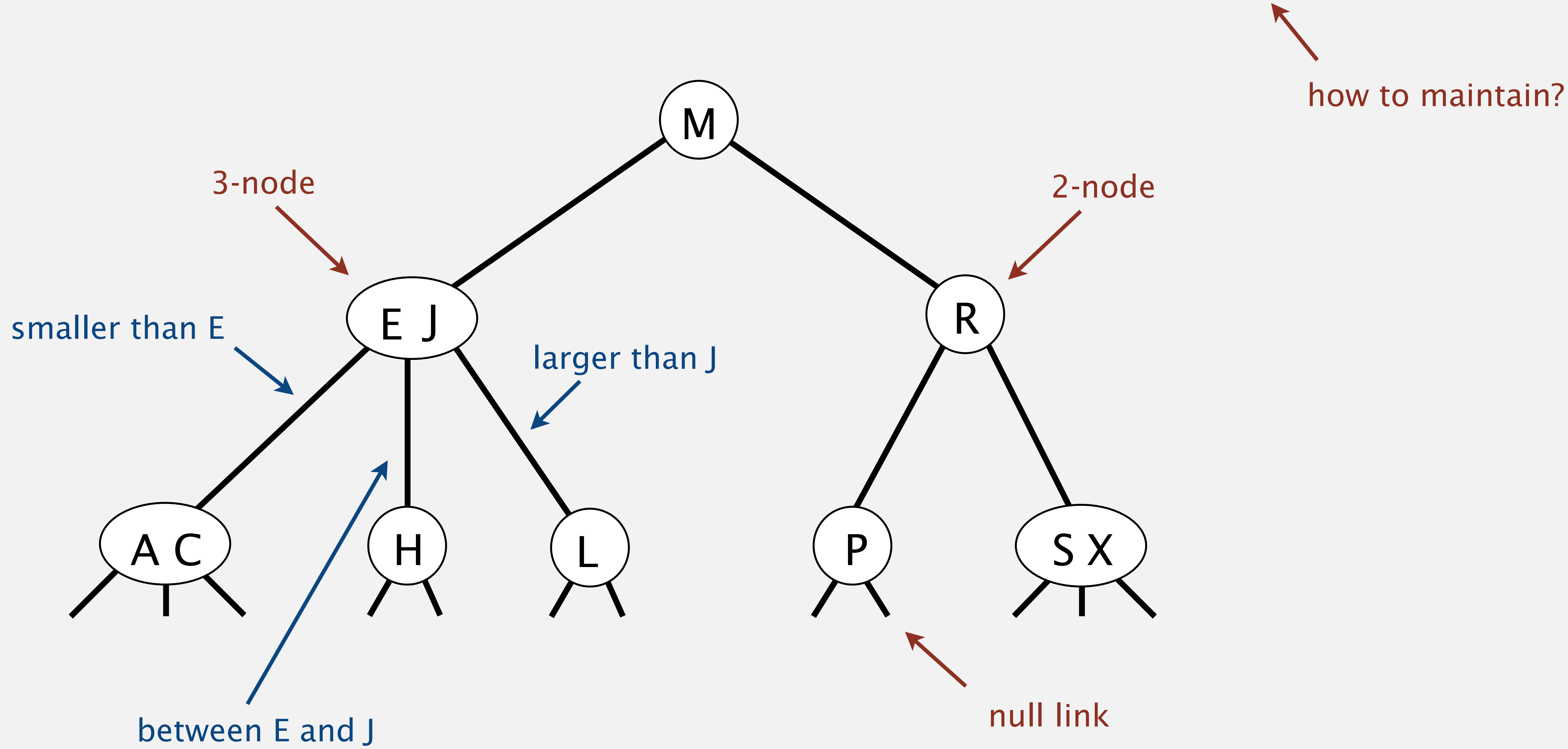
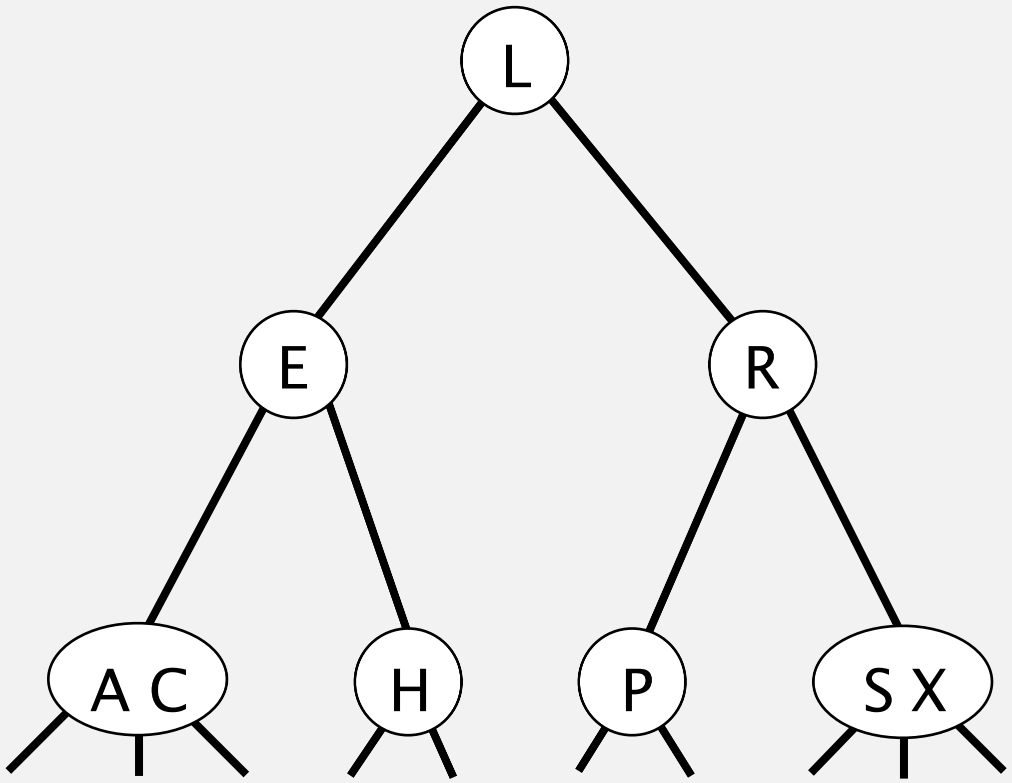


Insertion into a 2-3 tree

Insertion into a 2-node at bottom.

- Add new key to 2-node to create a 3-node.

insert G



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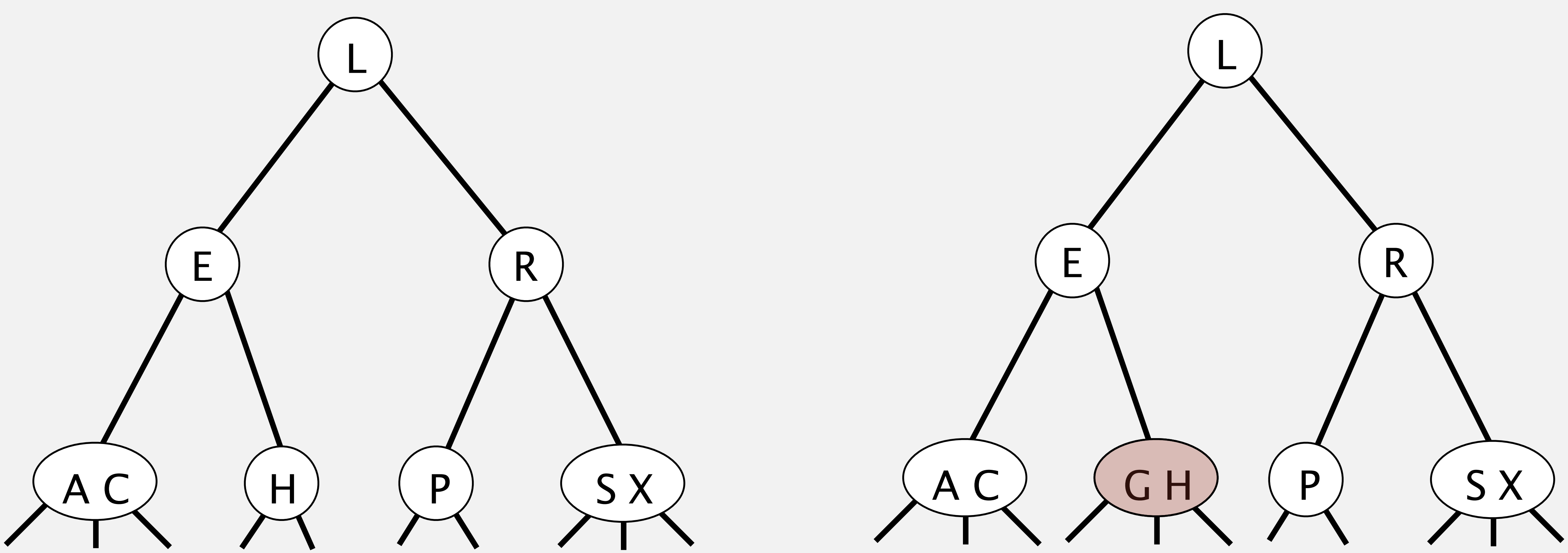
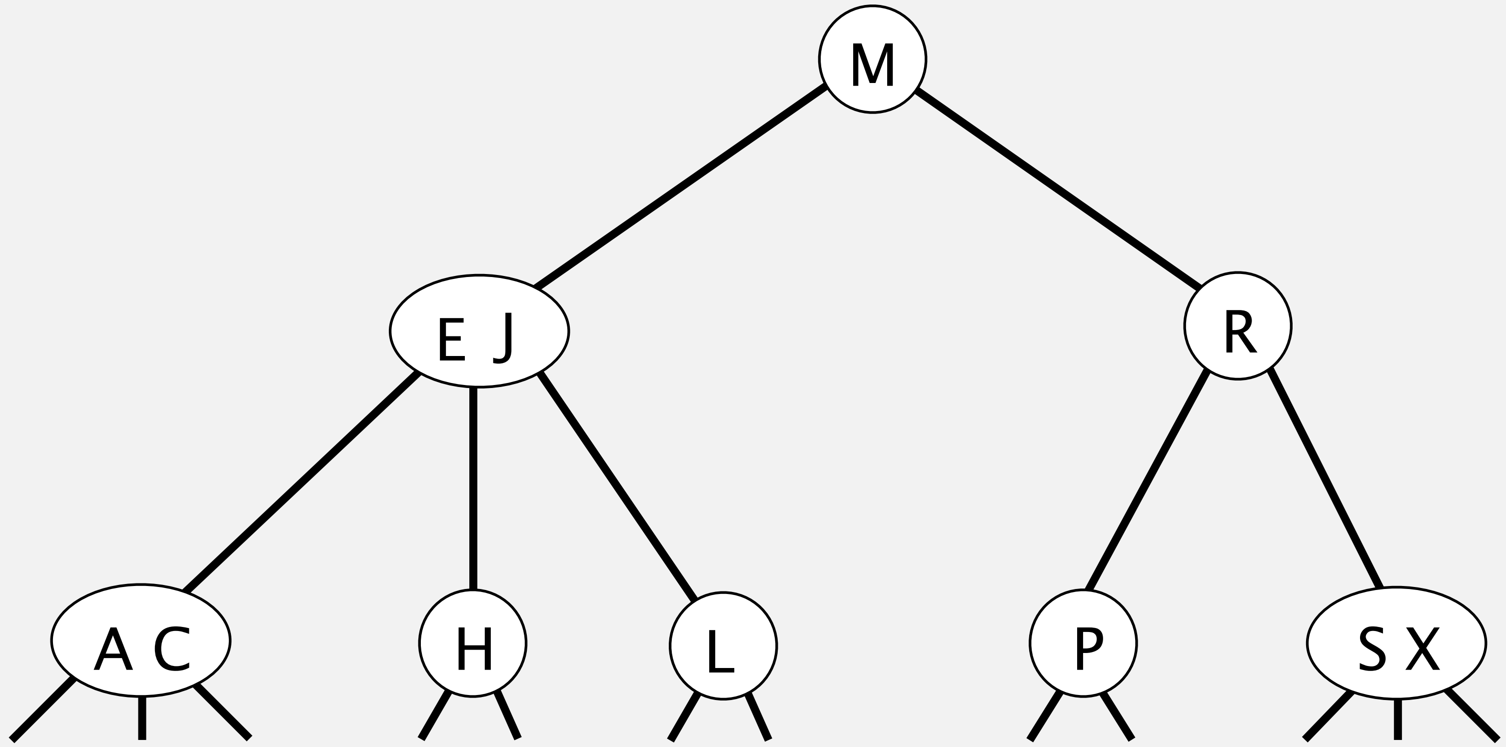


2-3 tree demo: search

Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for H



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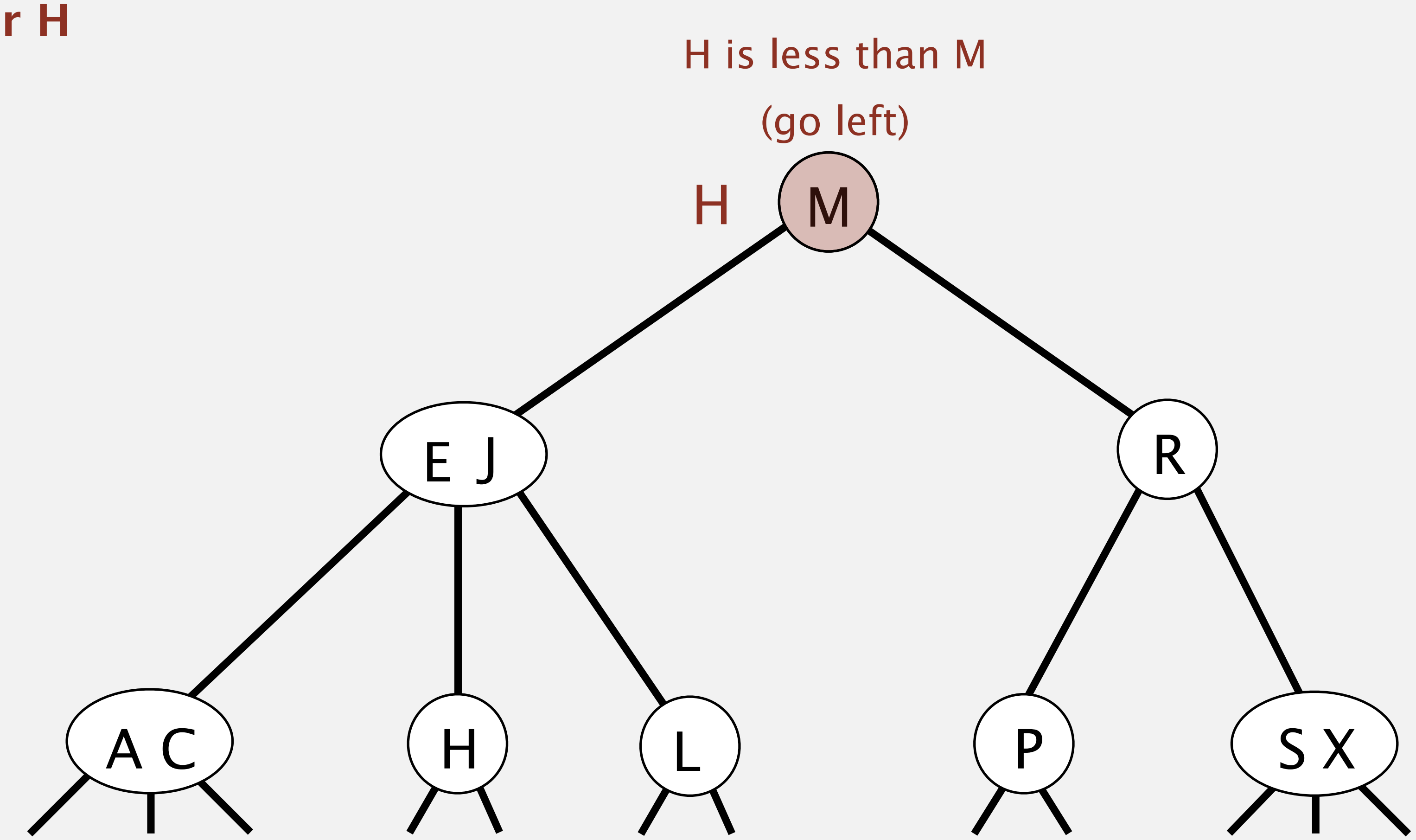

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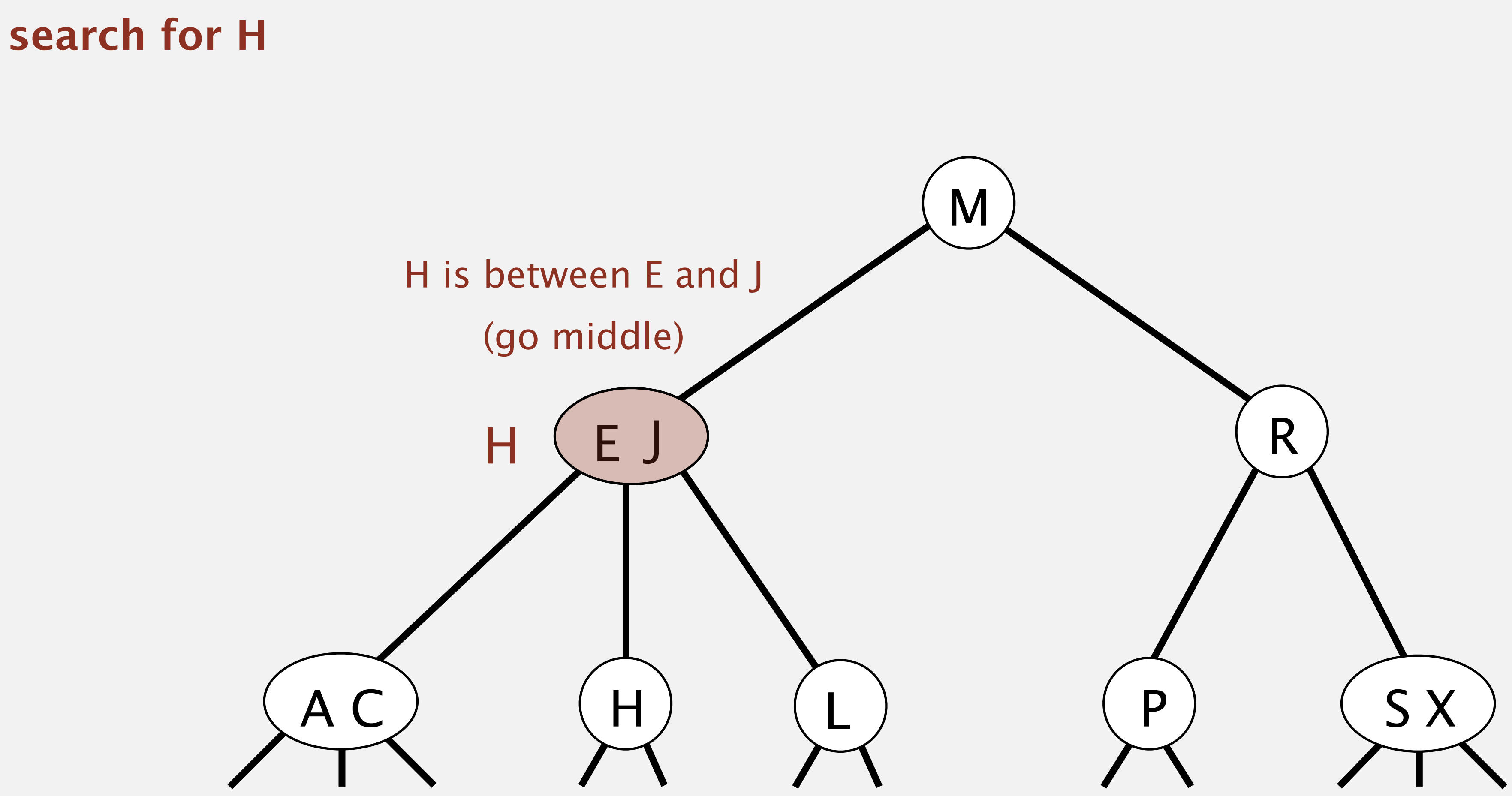

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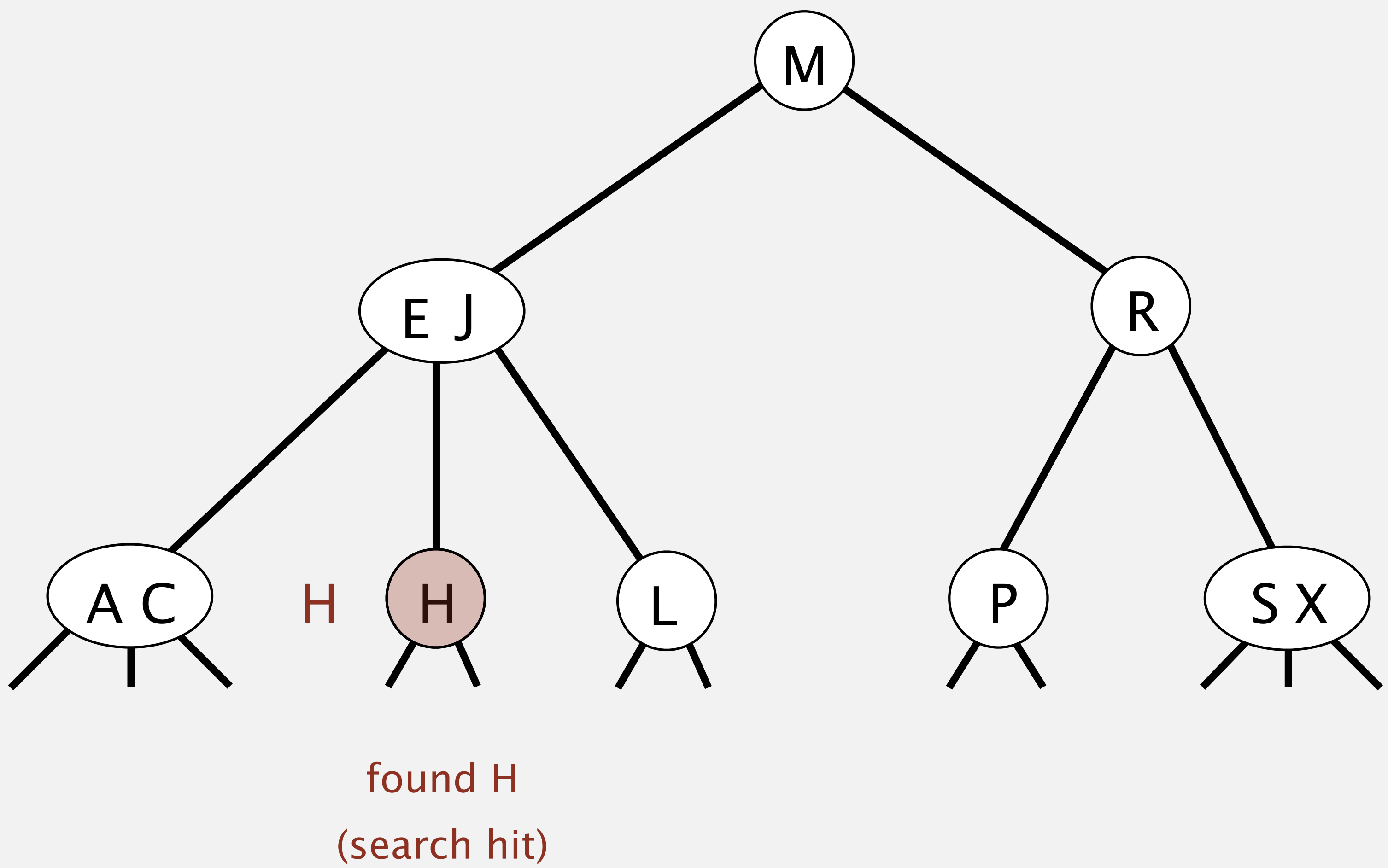

2-3 tree demo: search

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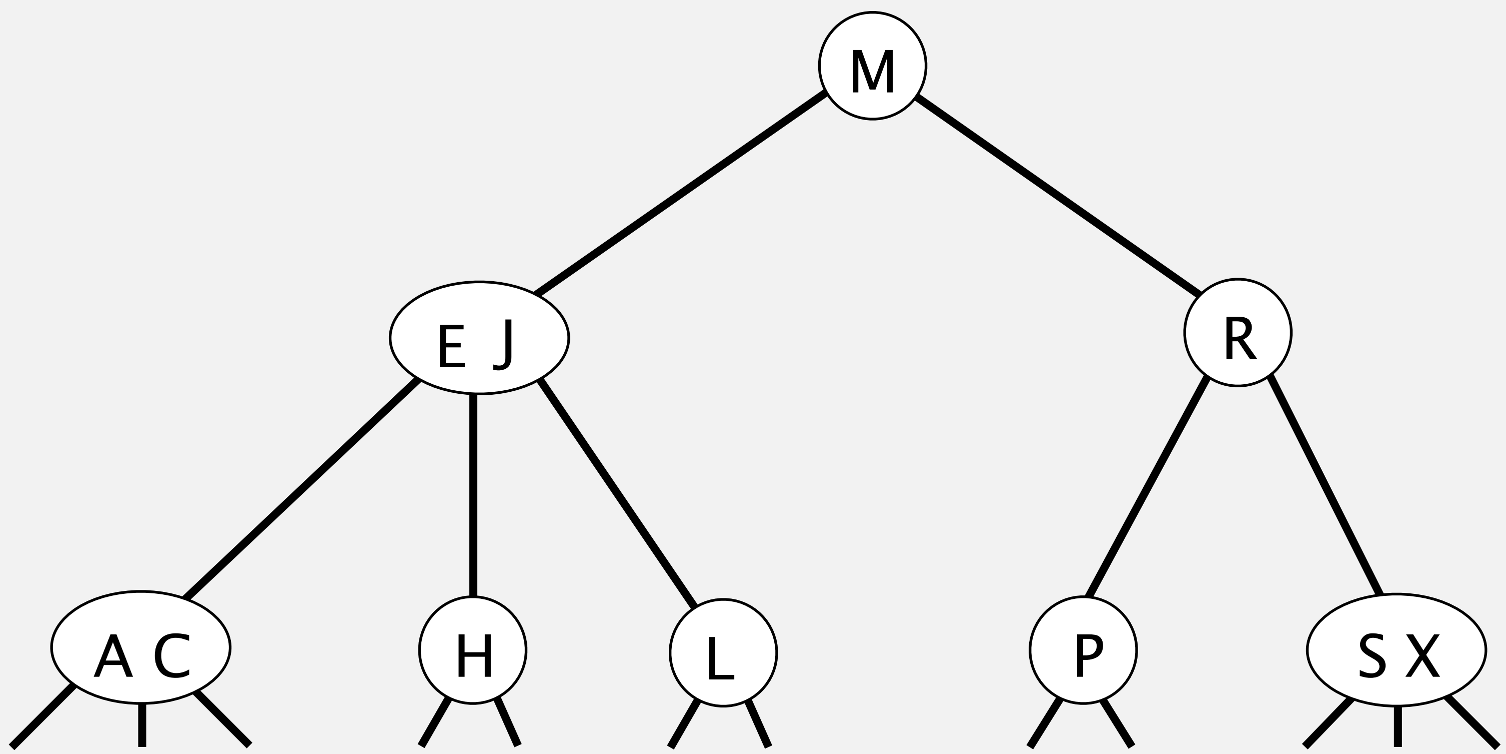
search for B



2-3 tree demo: search

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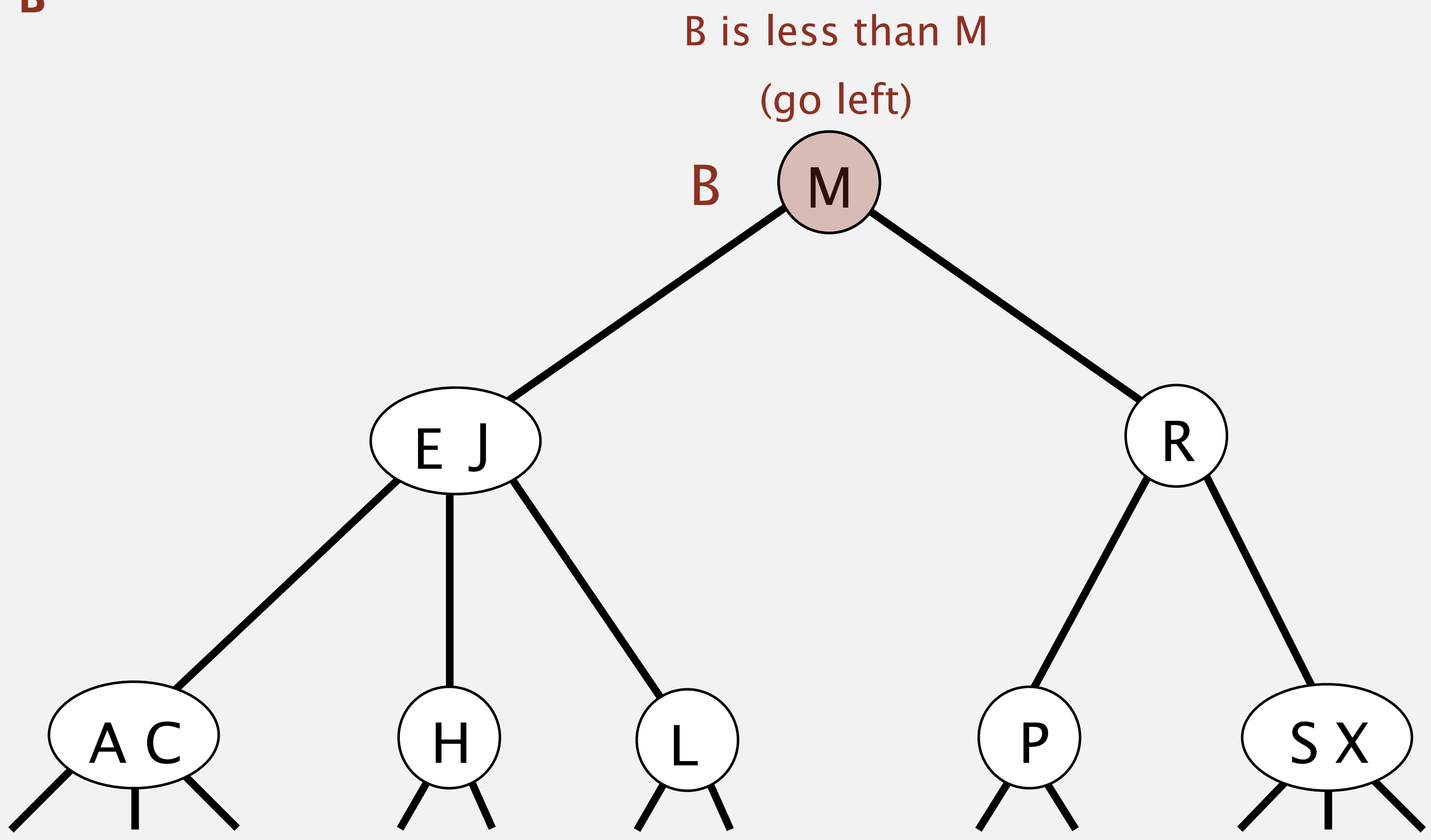
- Follow associated link (recursively).

search for B



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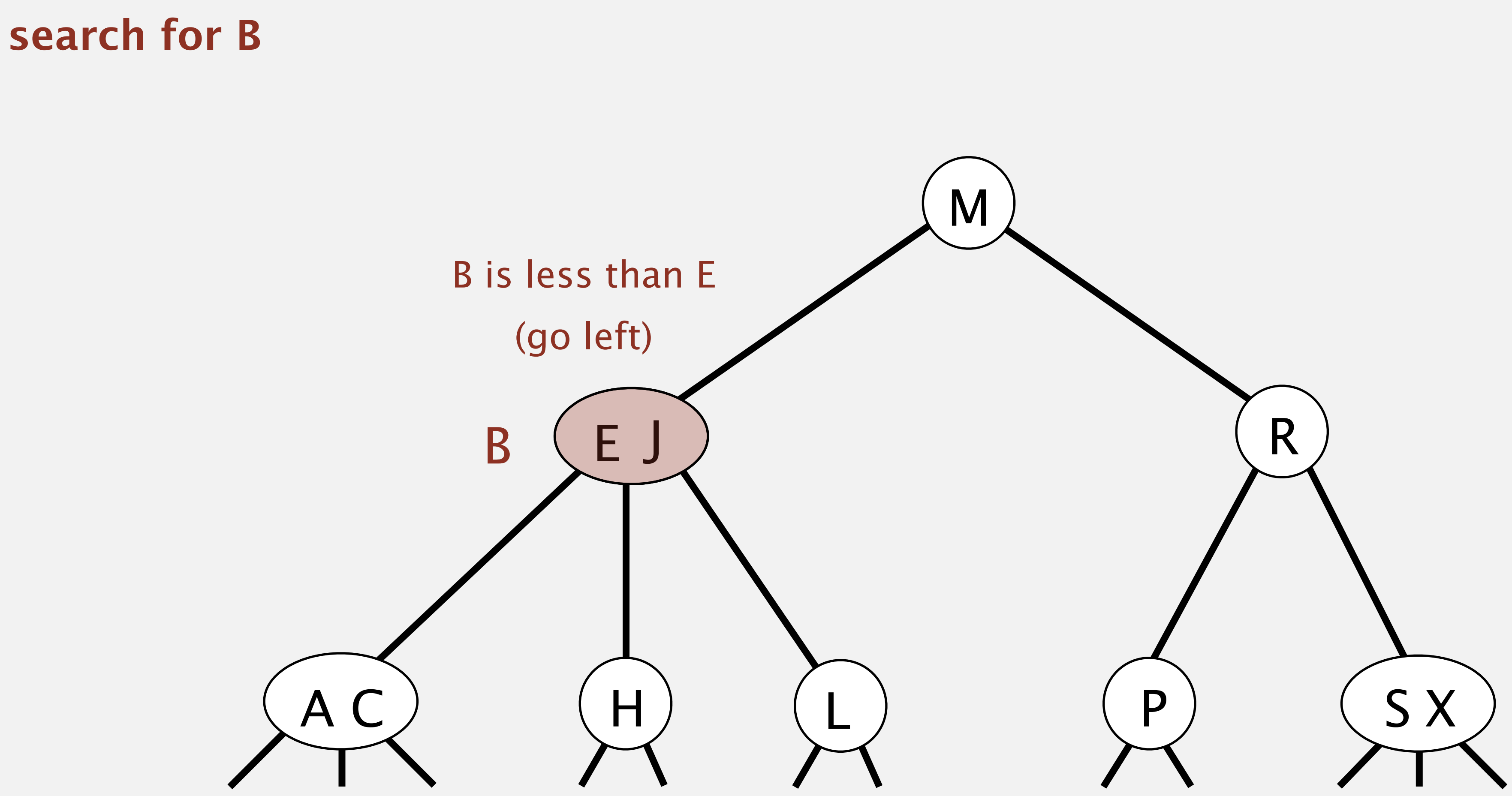

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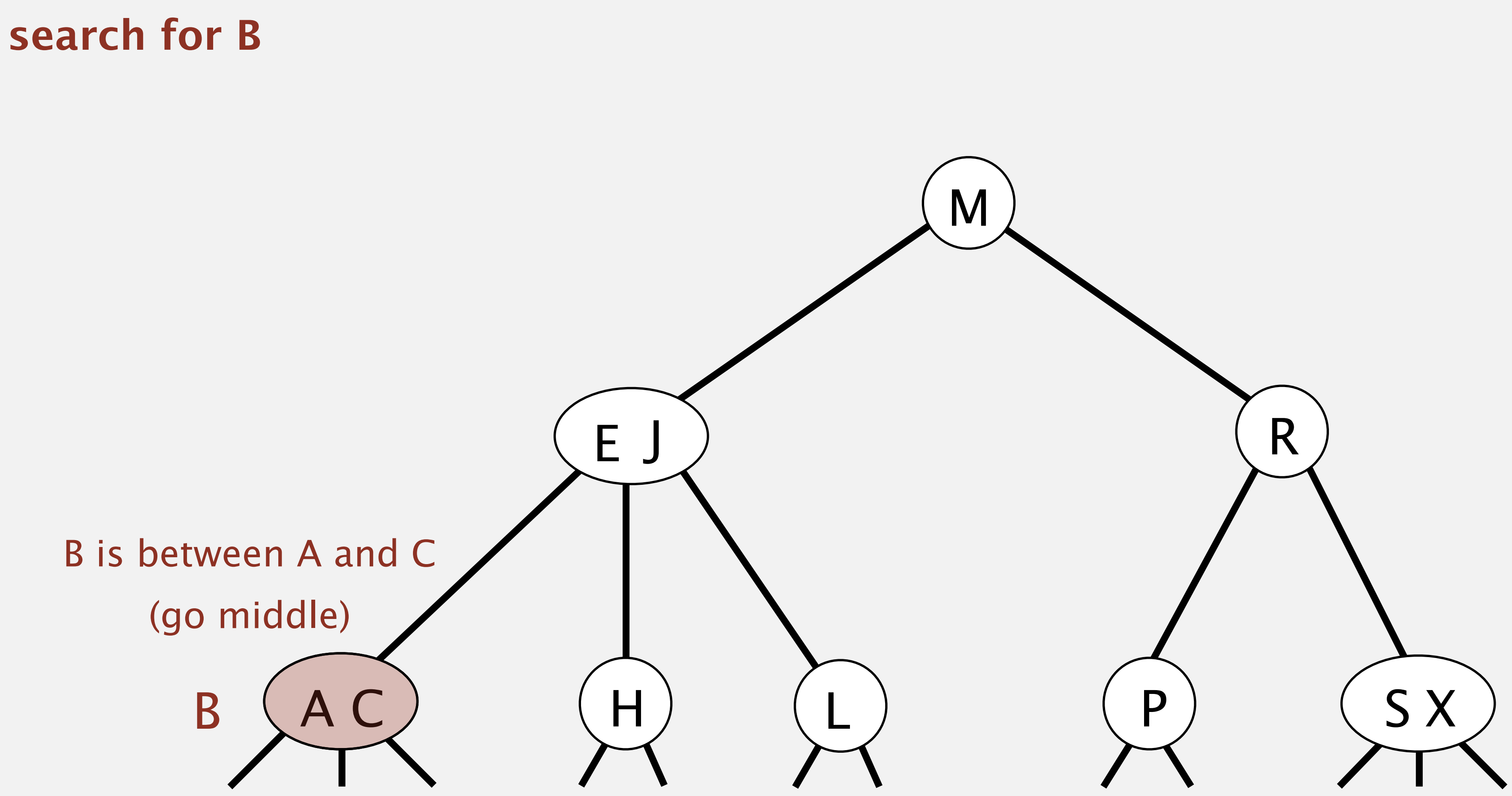

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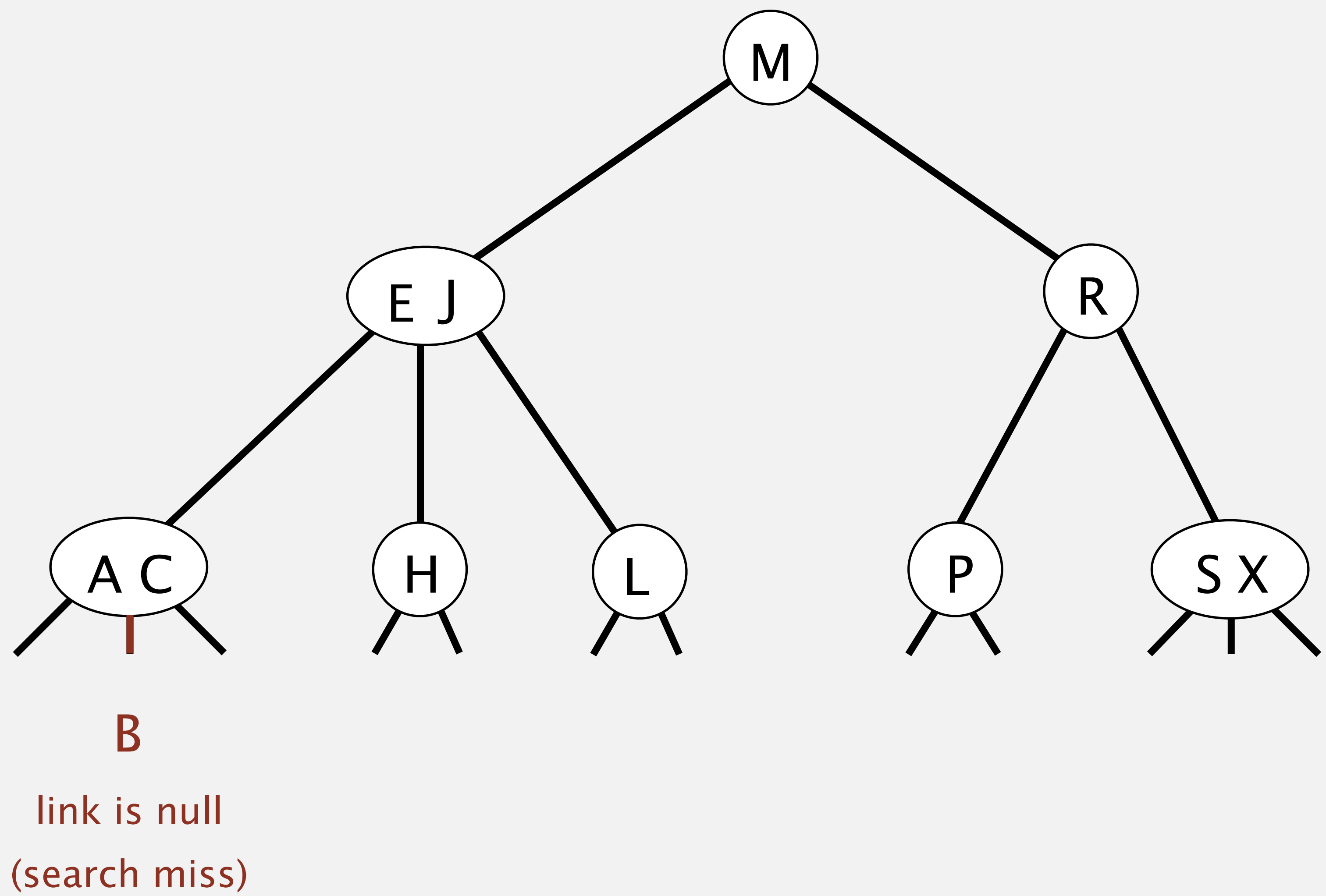
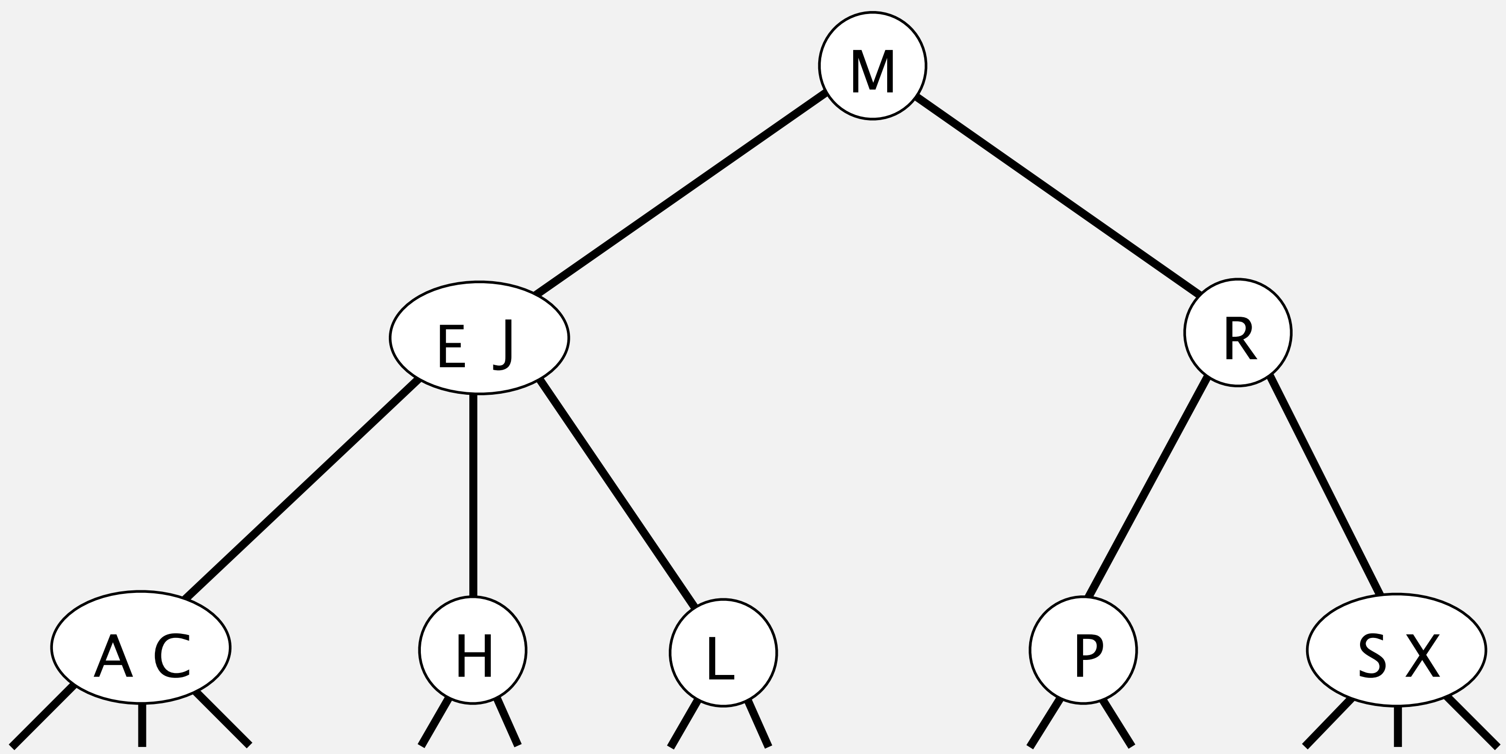


2-3 tree demo: insertion

Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.

insert K



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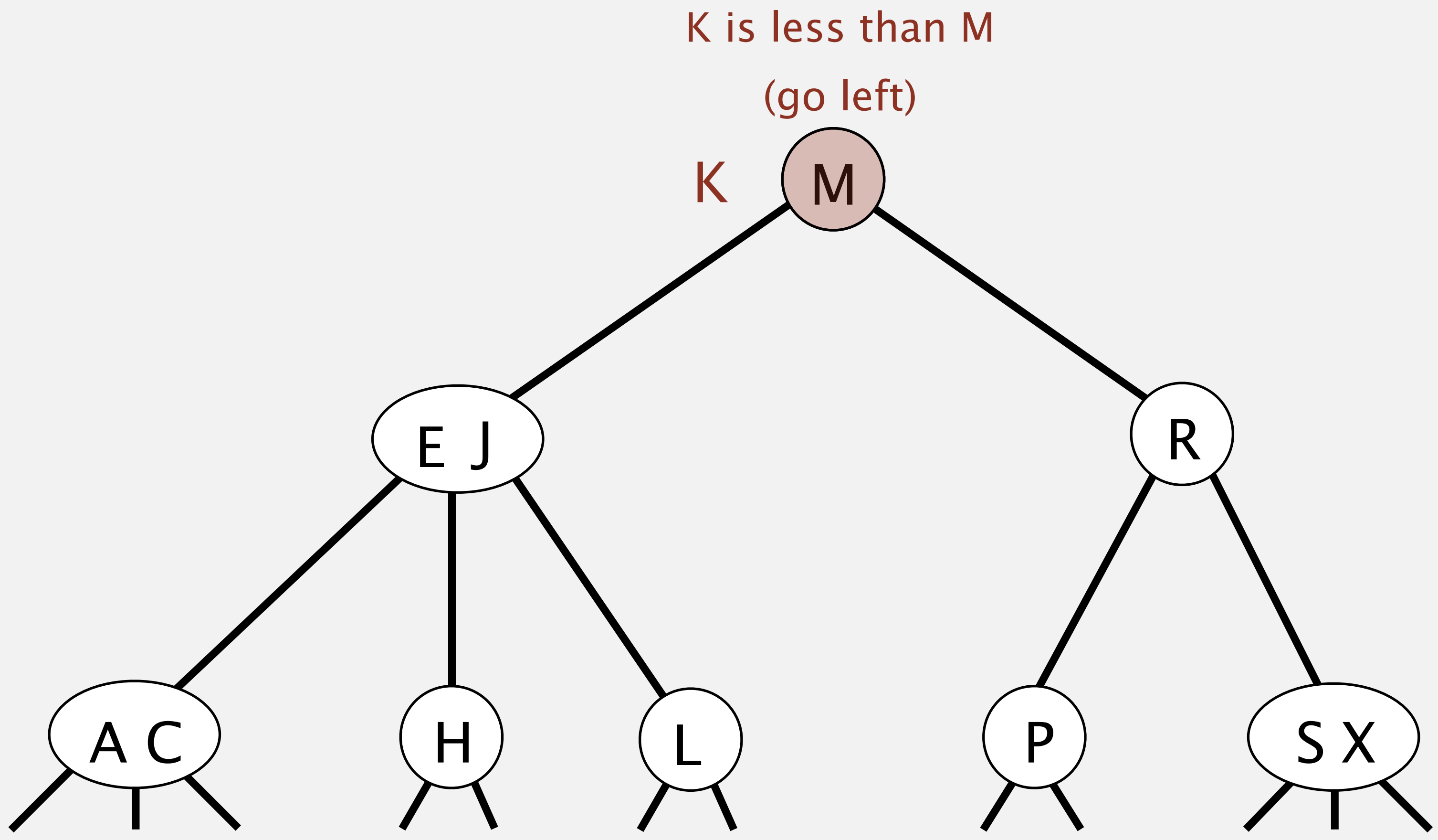

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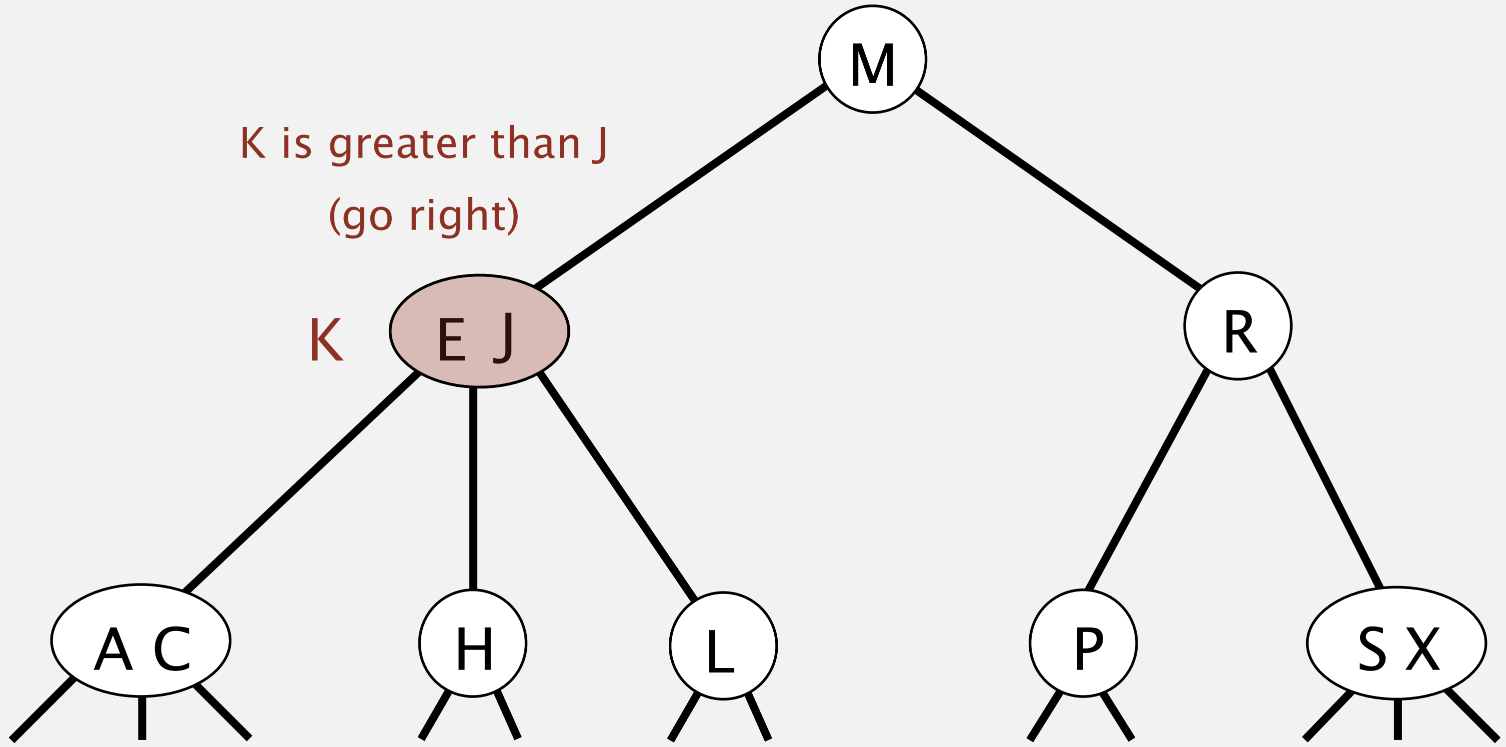

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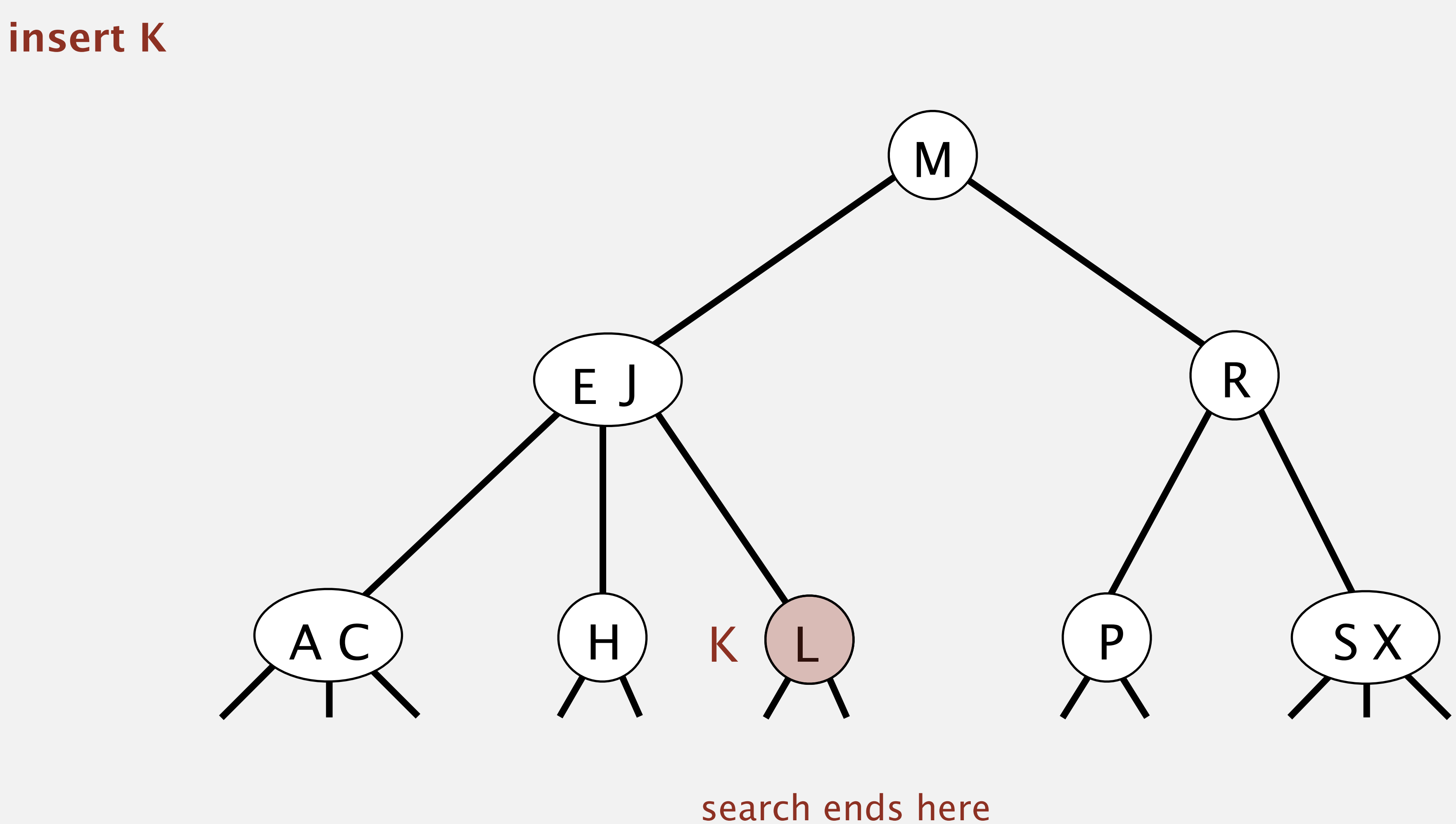

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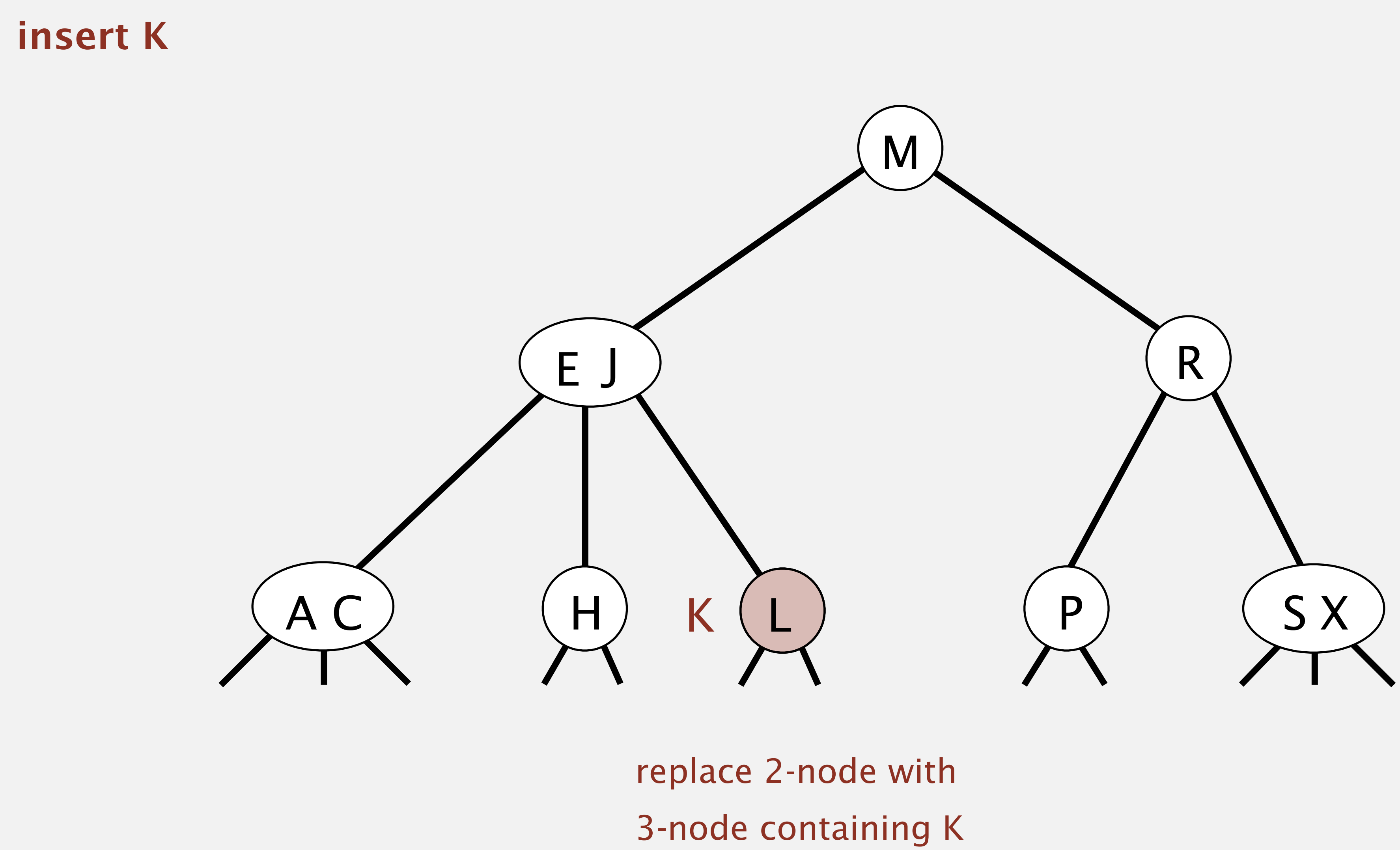

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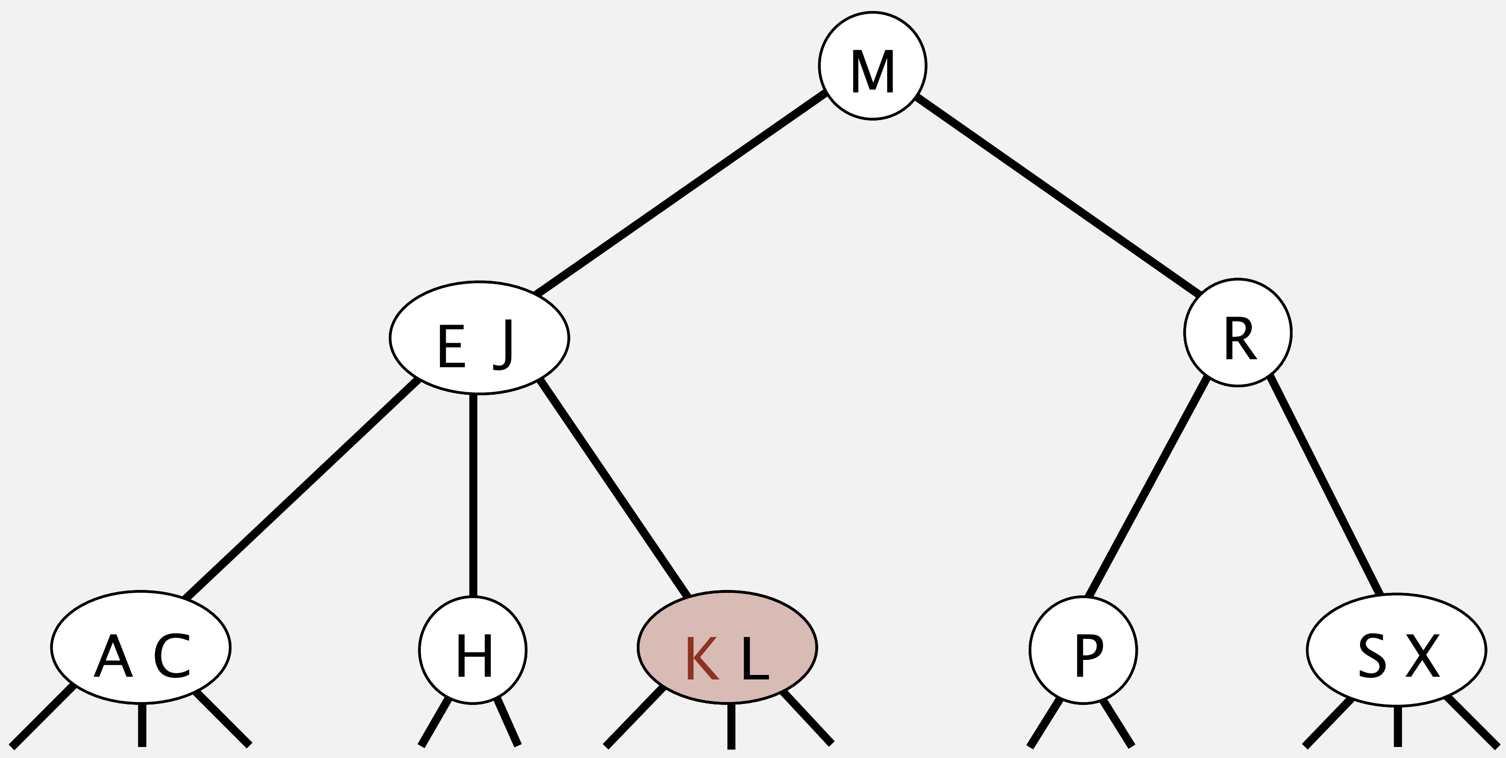
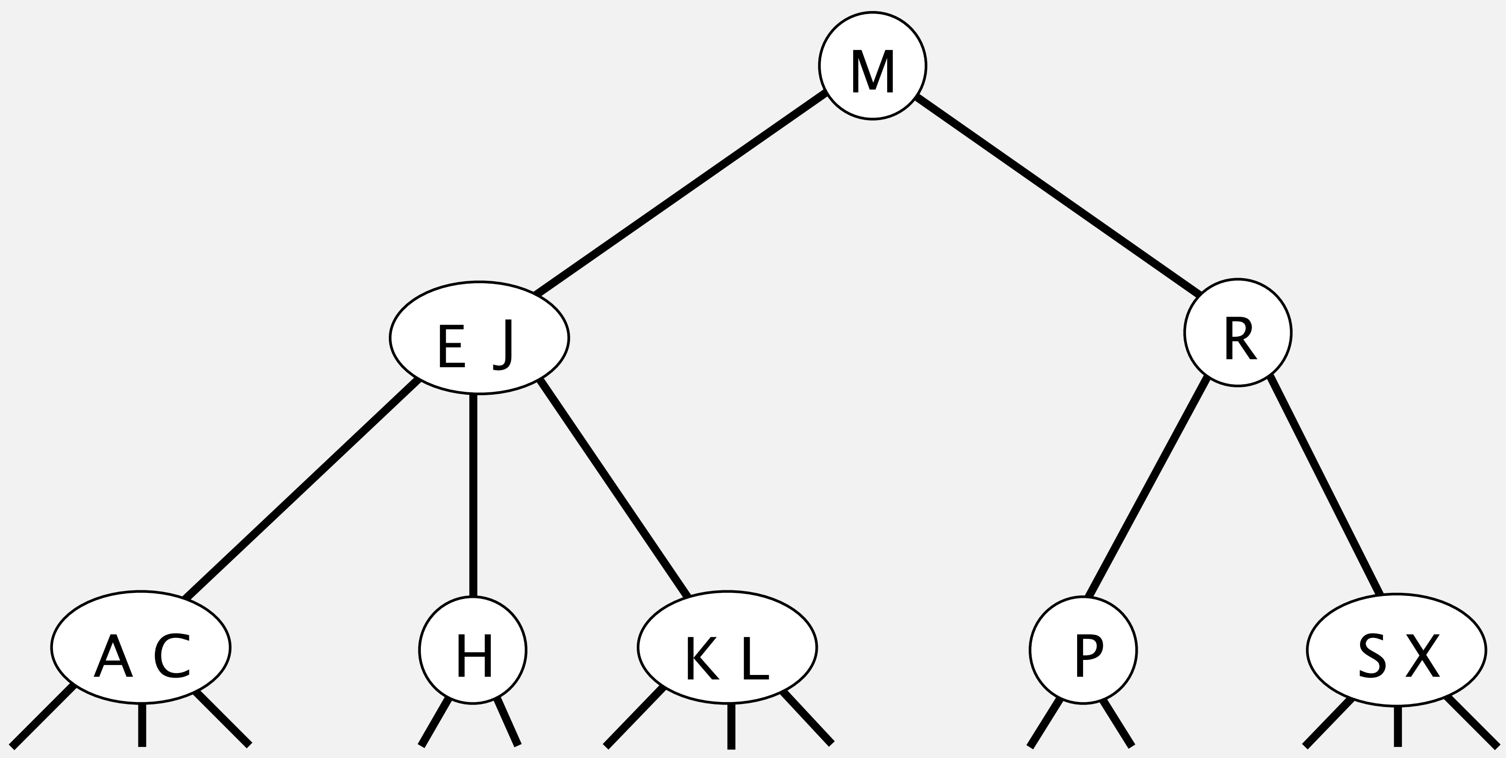


2-3 tree demo: insertion

Insert into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.

insert Z



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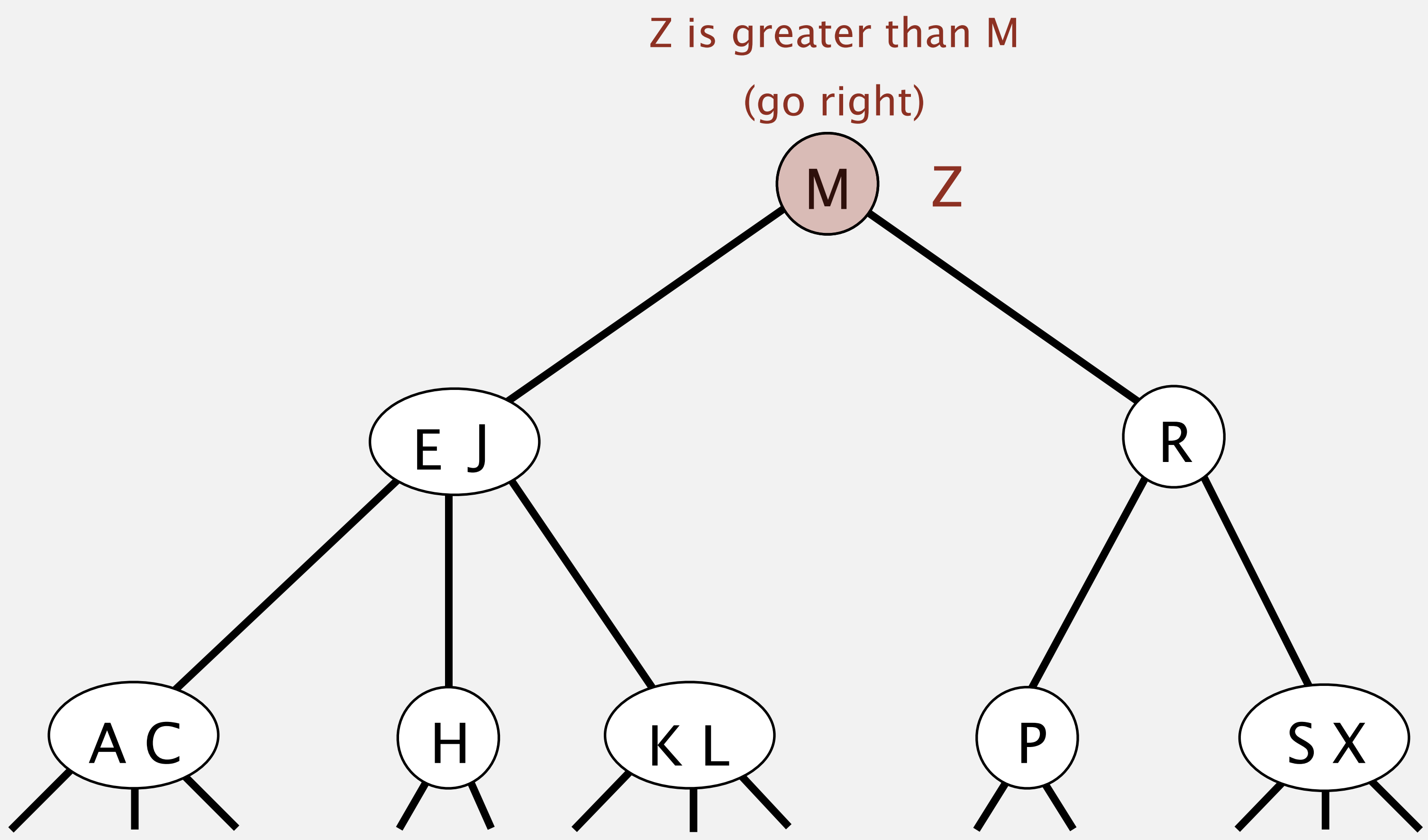

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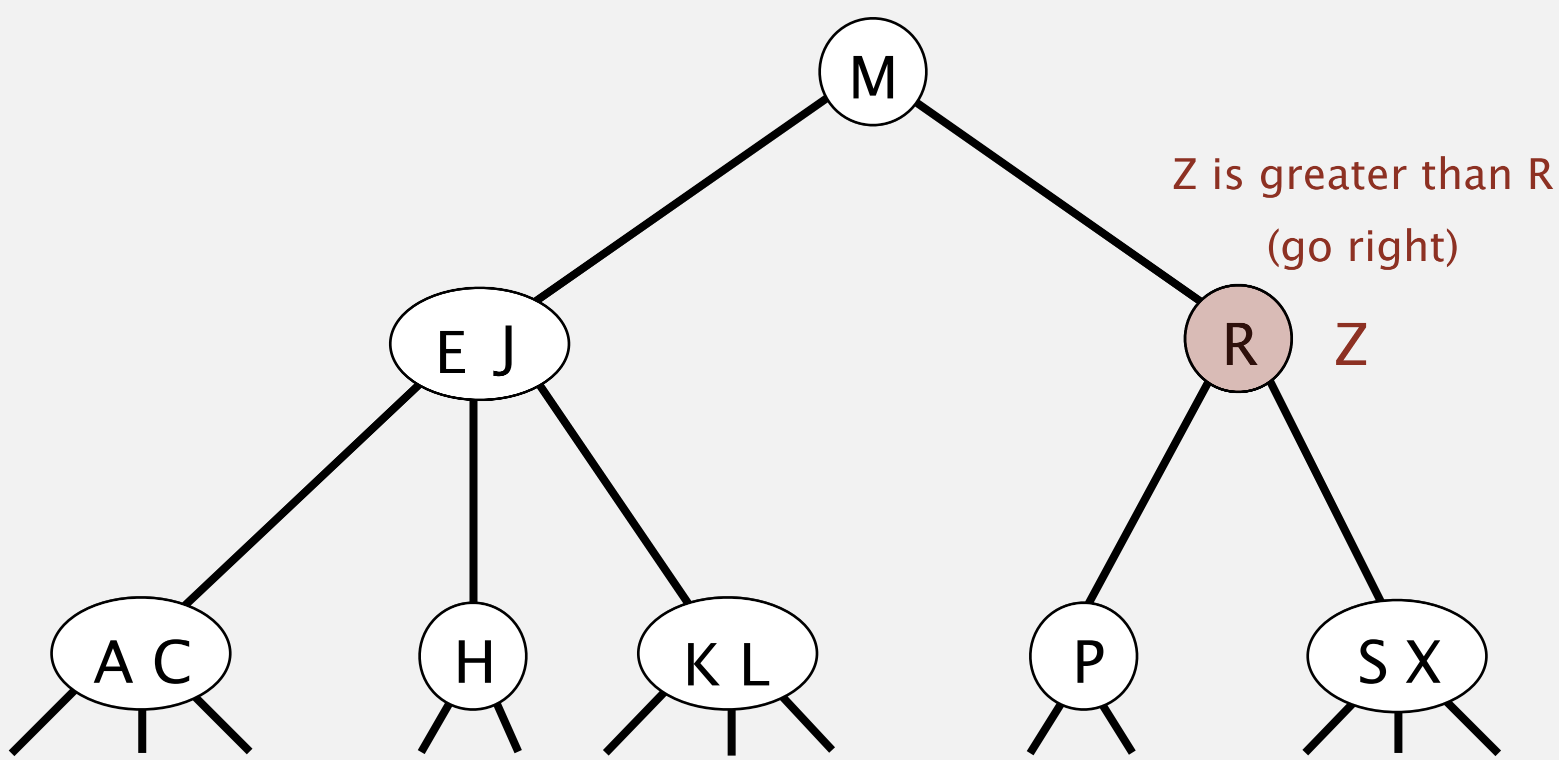

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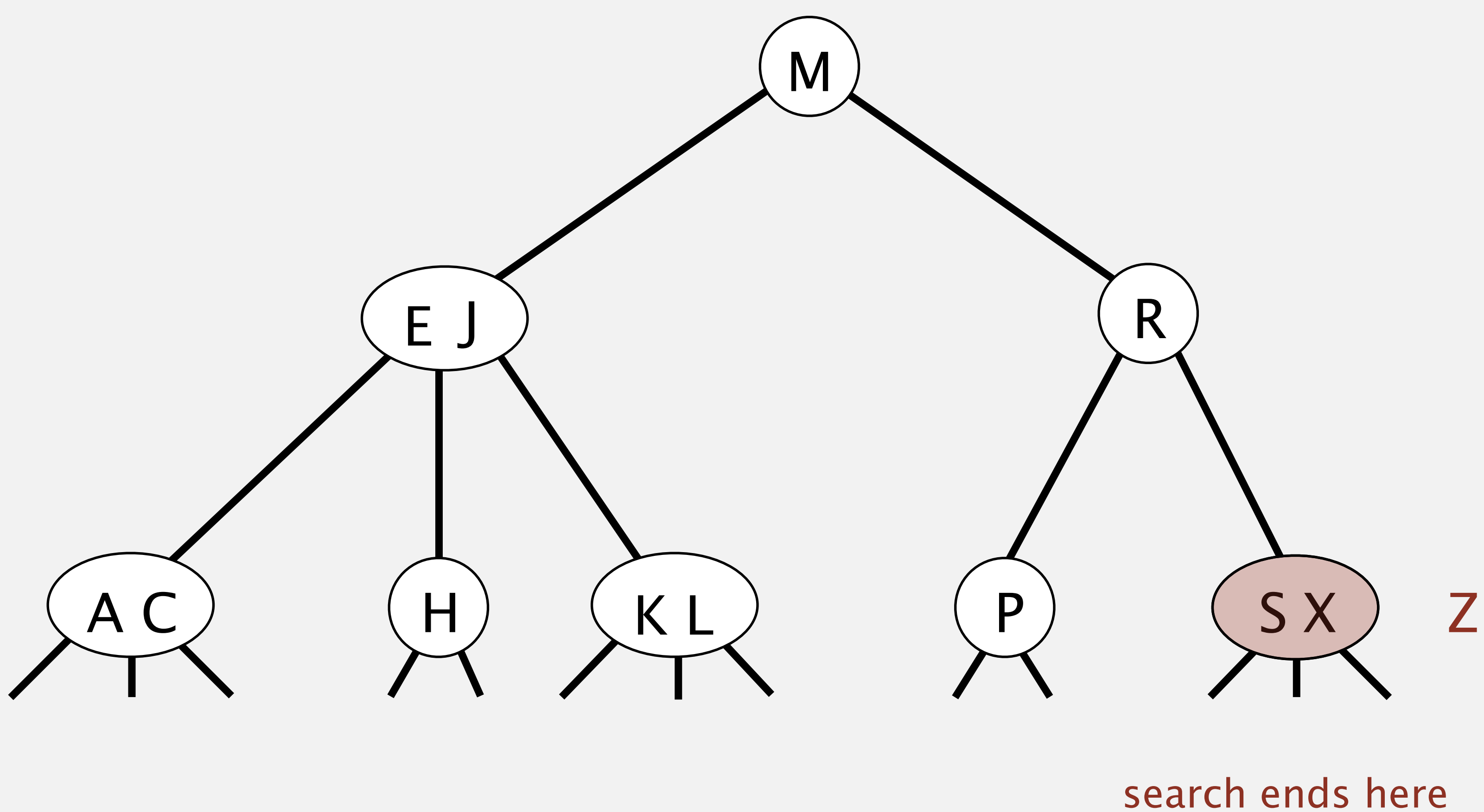

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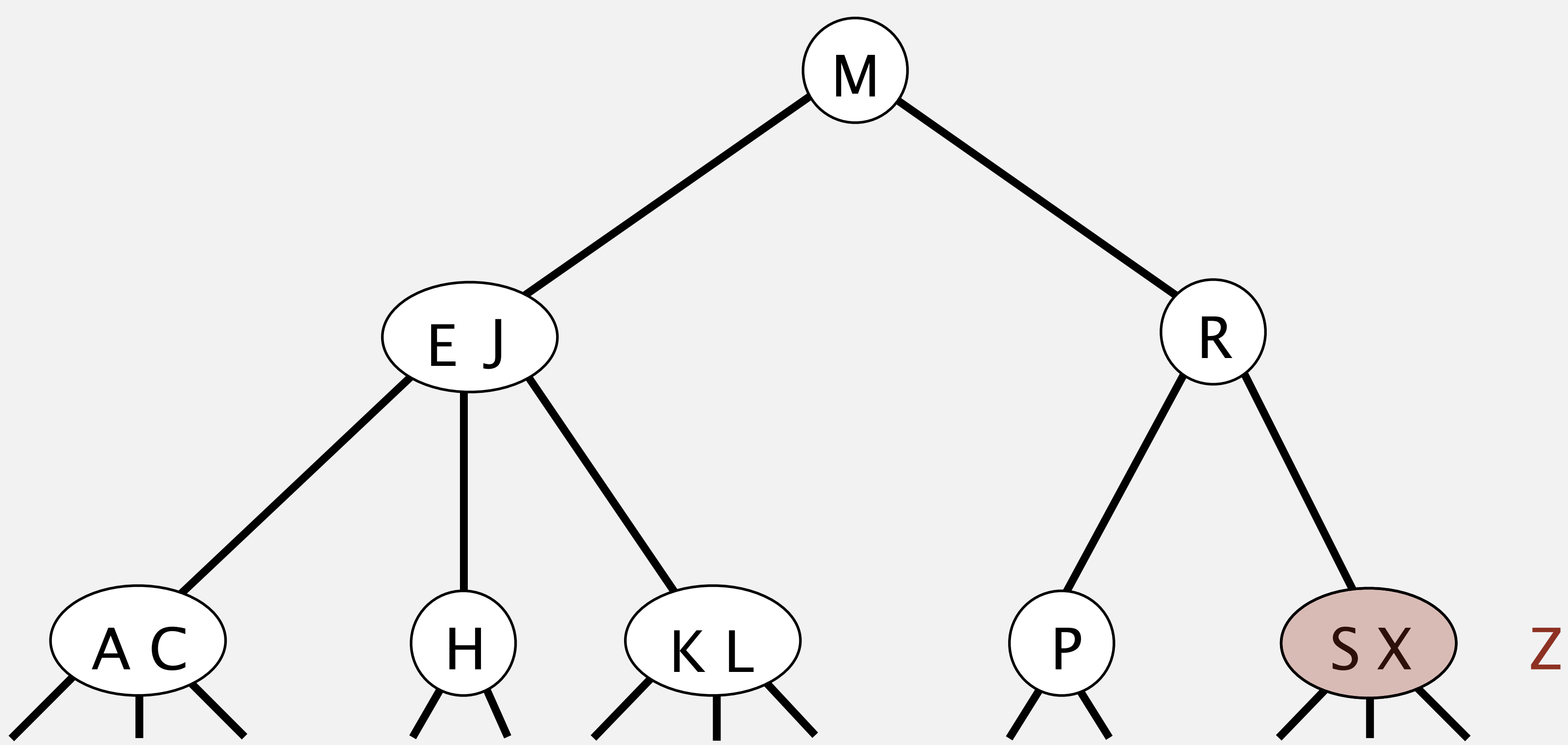

replace 3-node with
temporary 4-node containing Z

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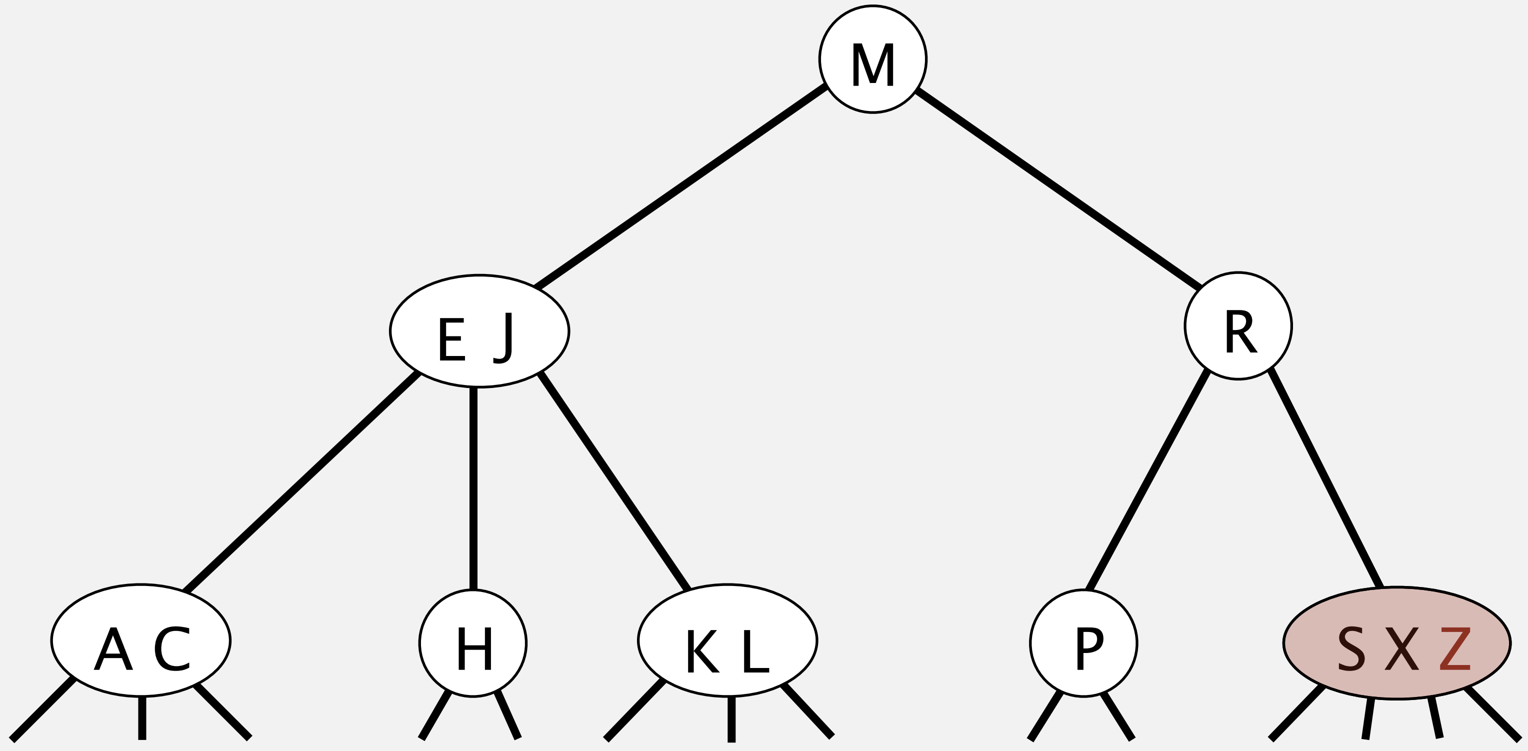

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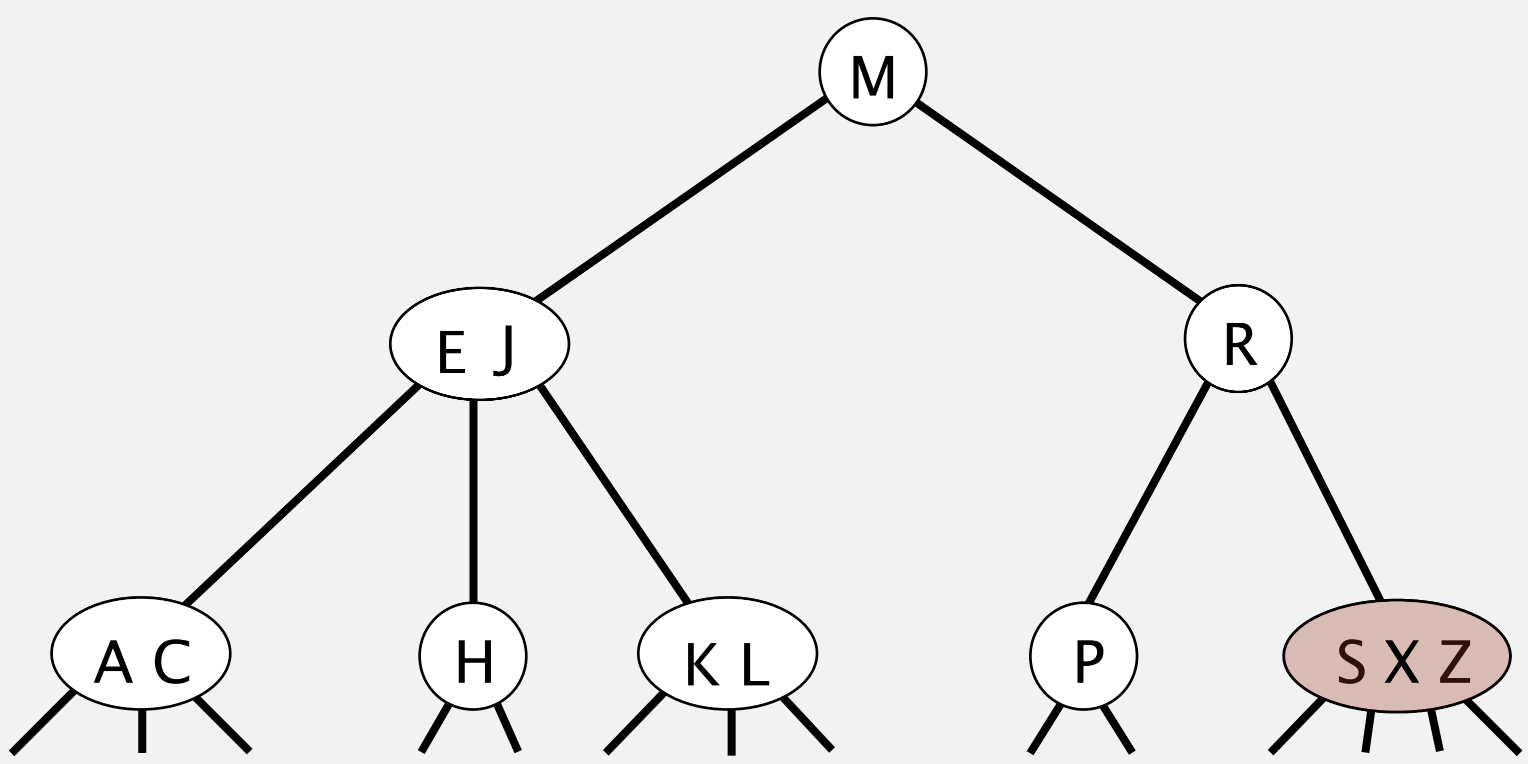

split 4-node into two 2-nodes
(pass middle key to parent)

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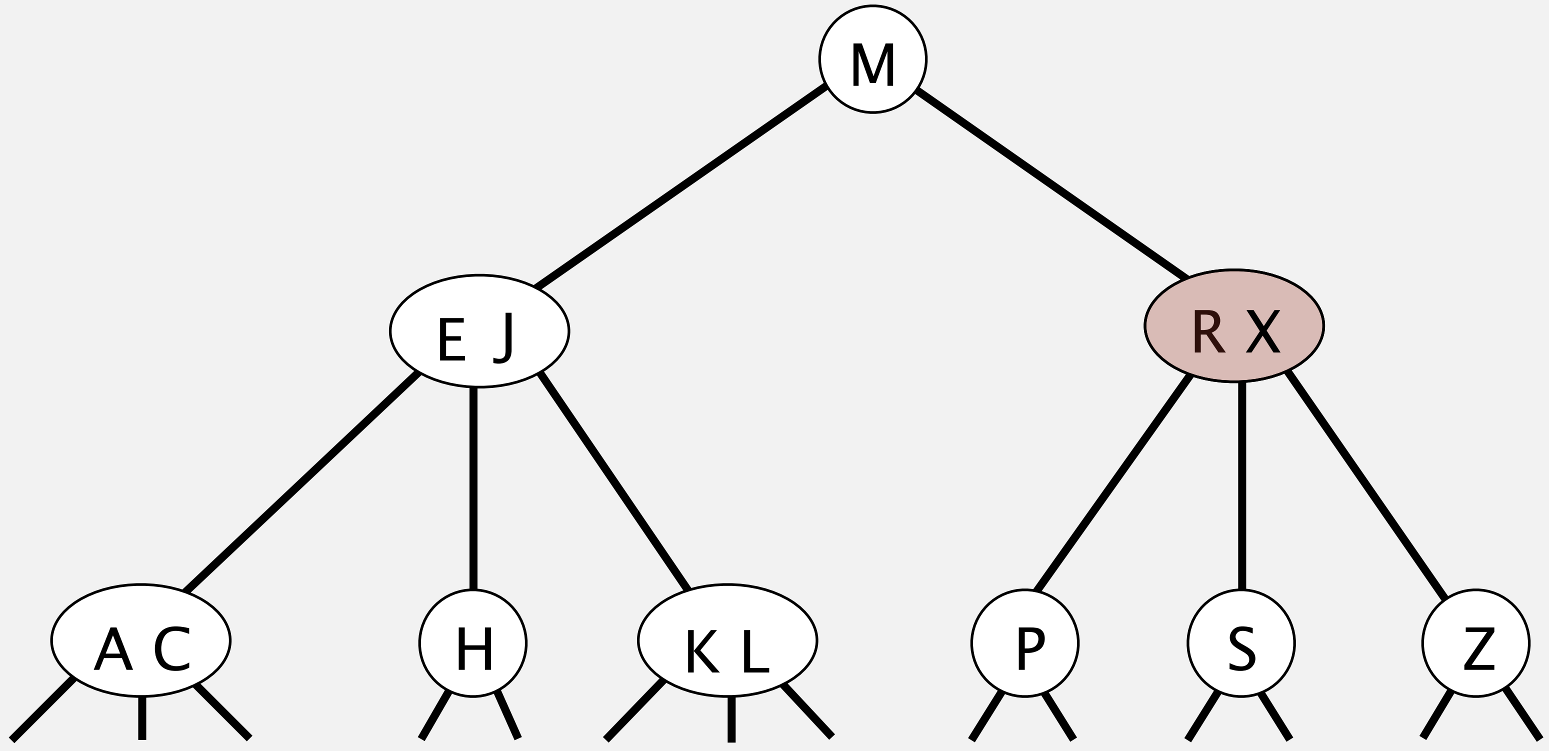

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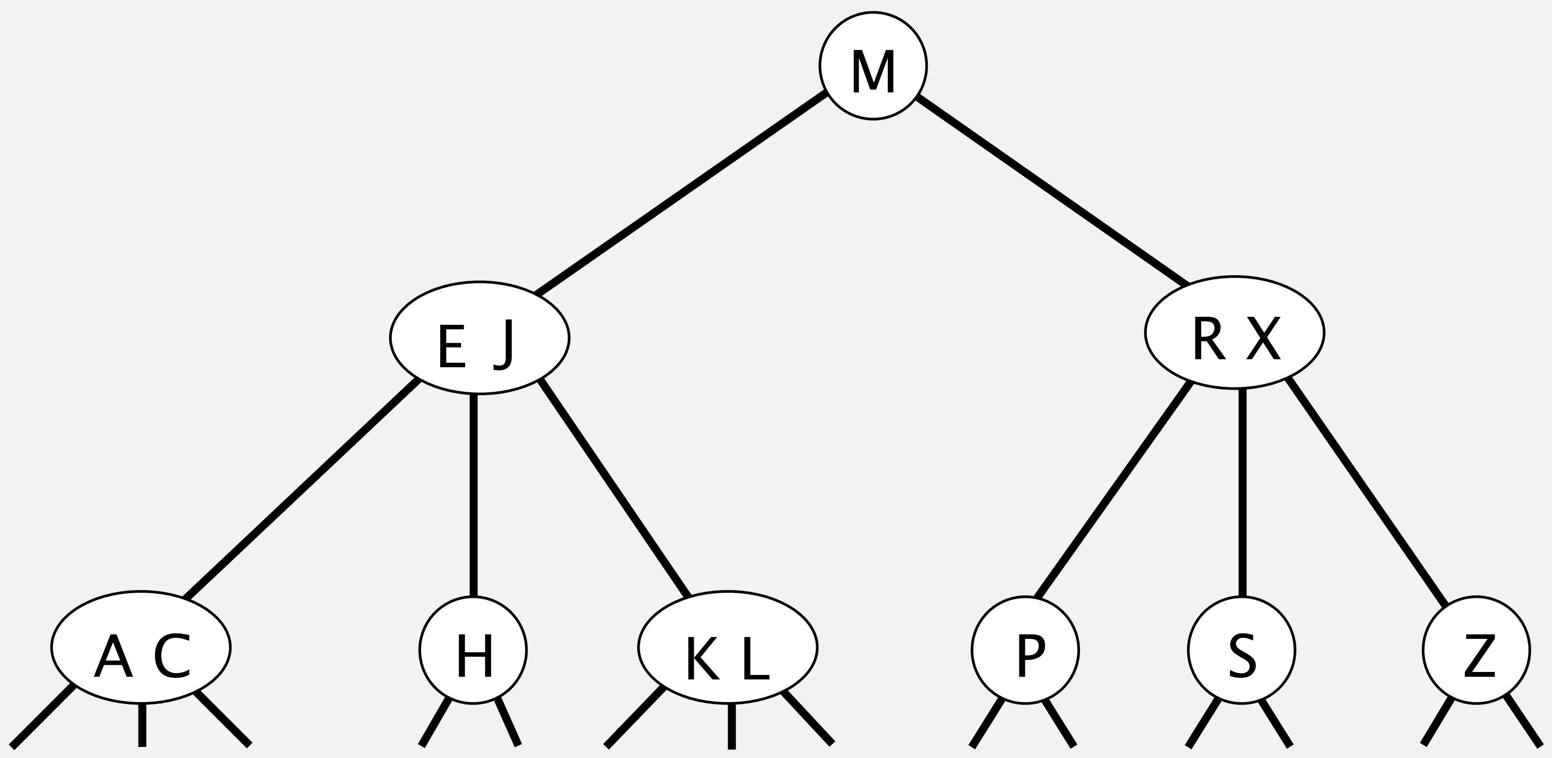

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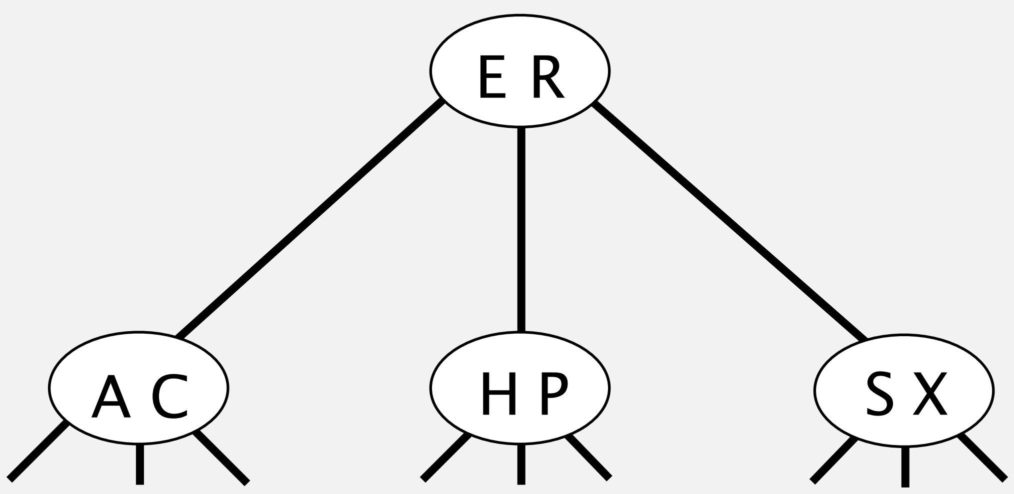
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

Text

insert L



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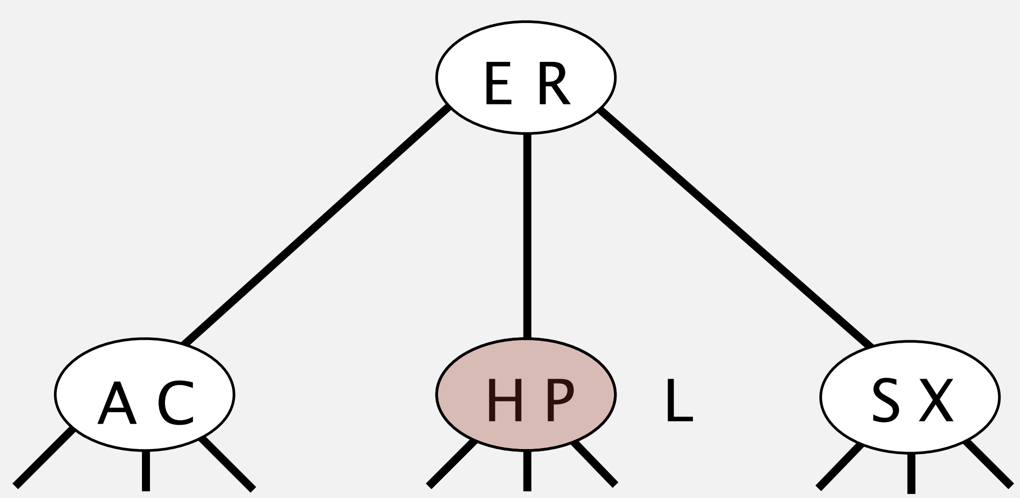

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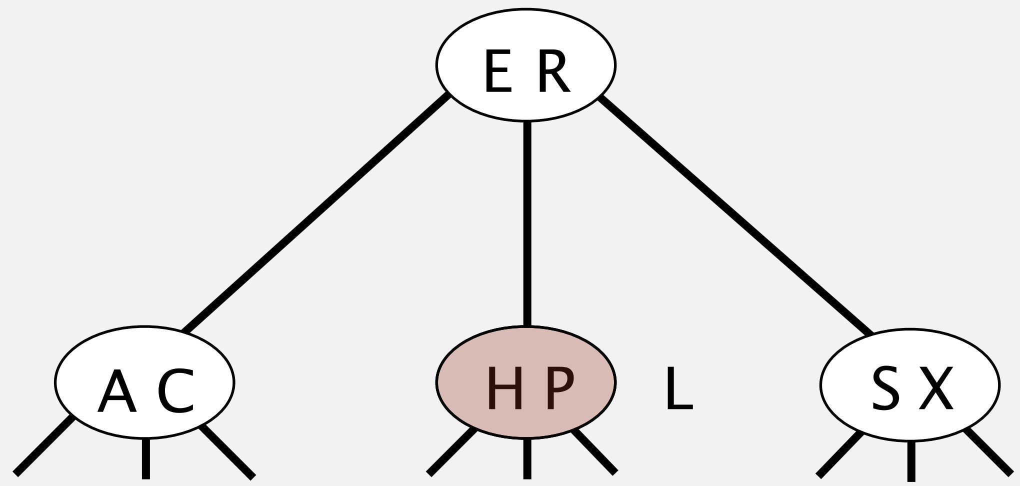

convert 3-node into 4-node

2-3 tree demo: insertion

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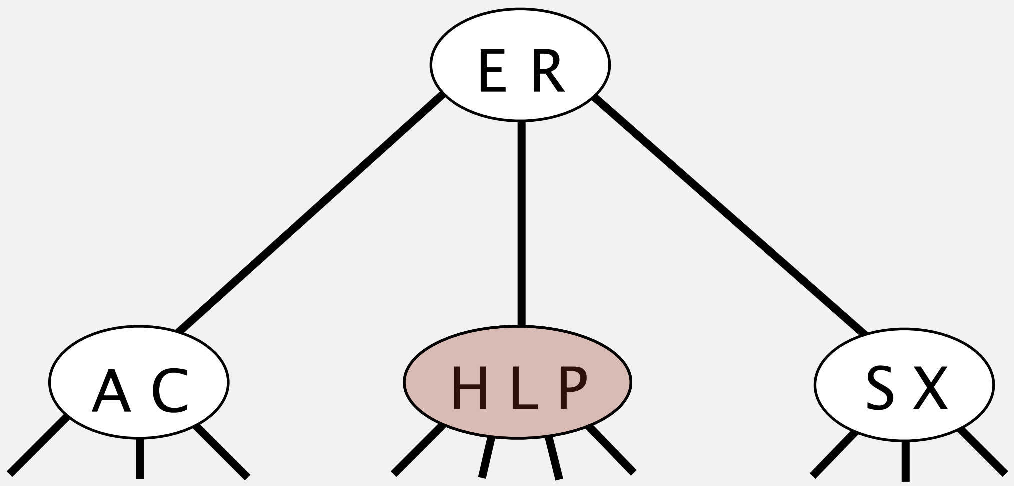

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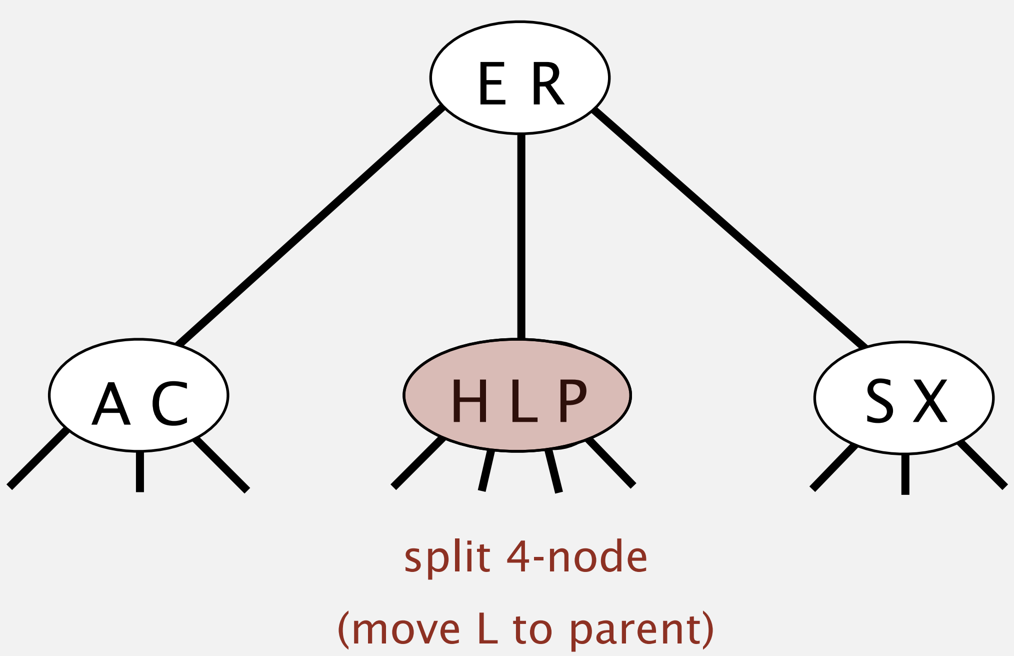

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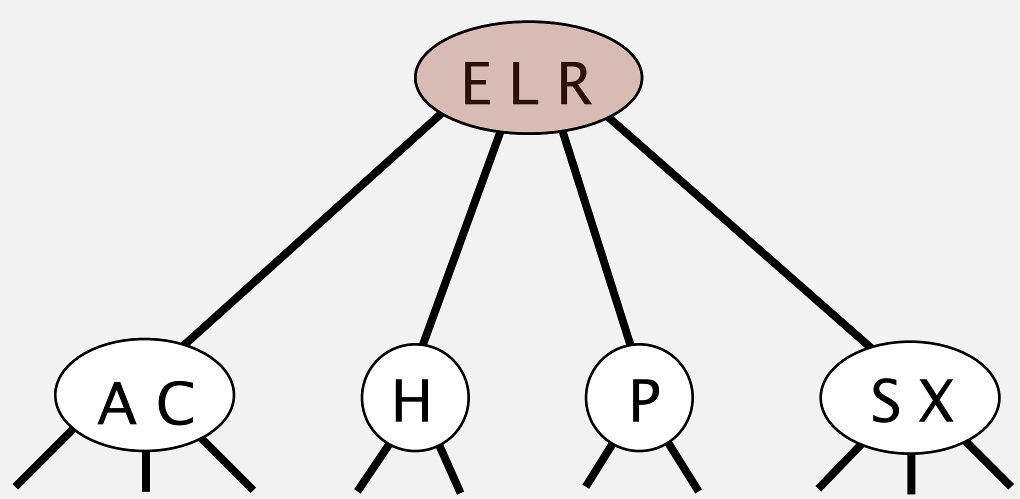

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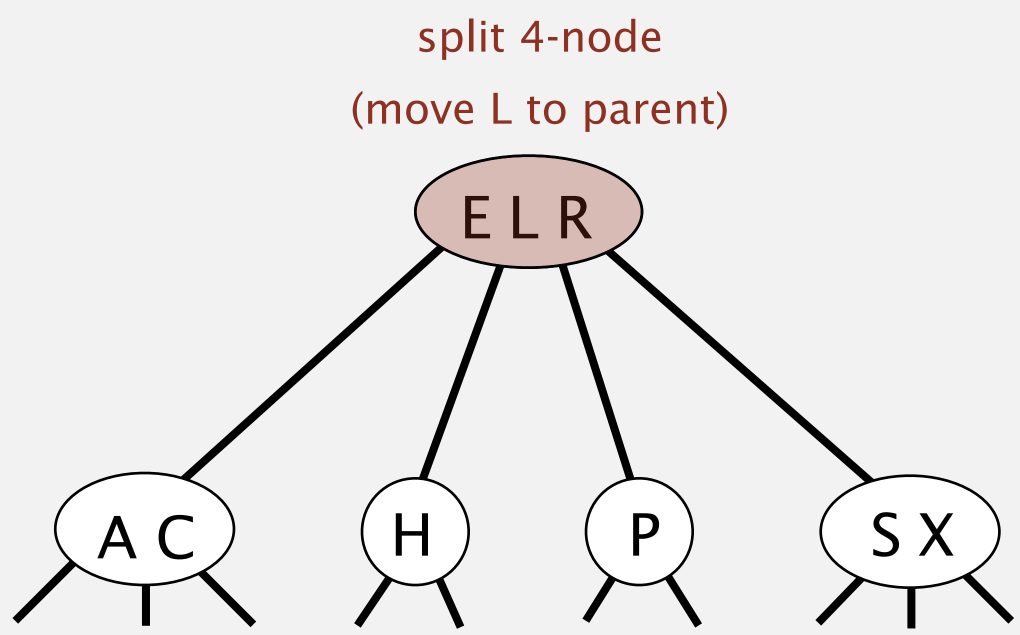

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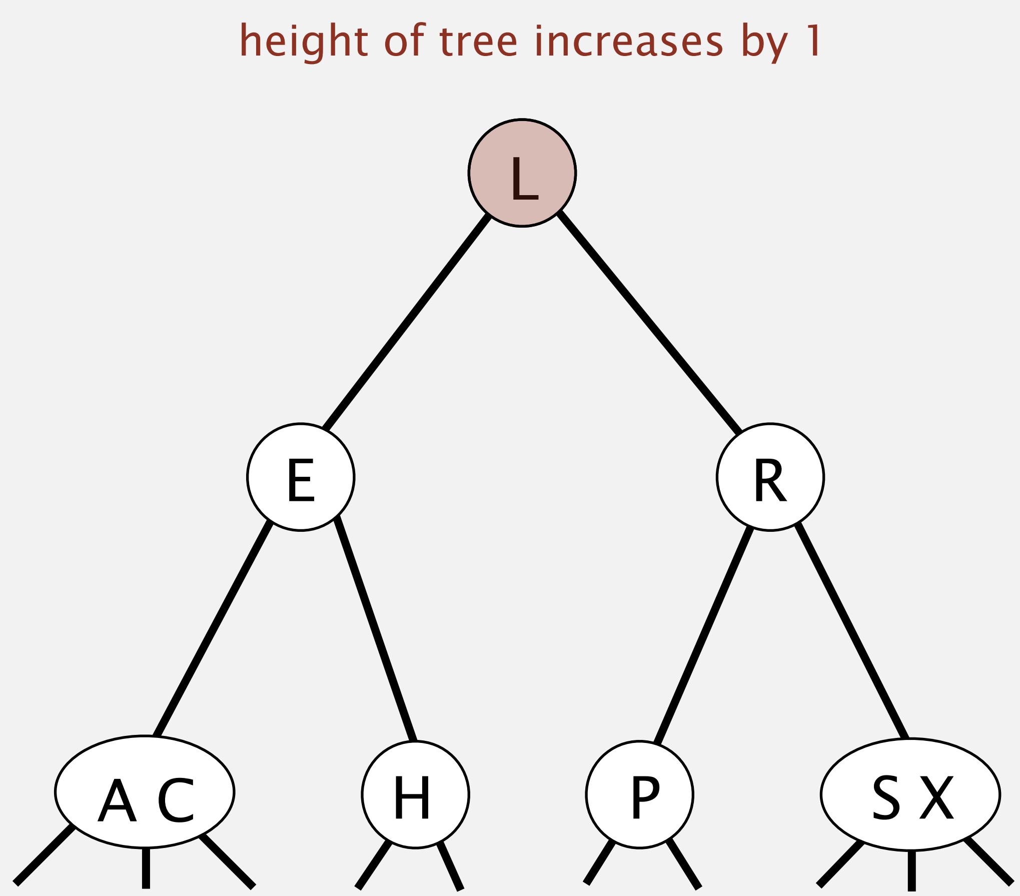

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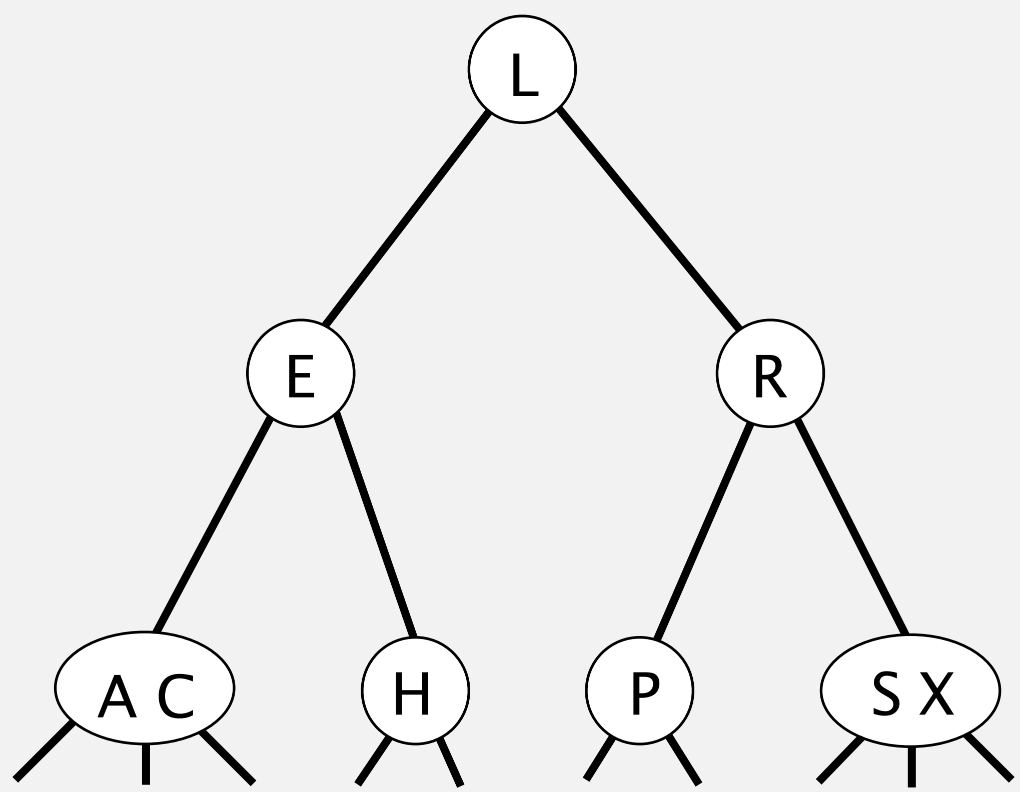
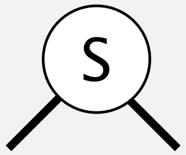


2-3 tree demo: construction

insert S

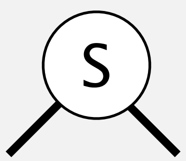
2-3 tree demo: construction

insert S



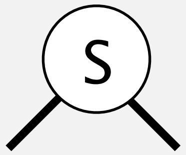
2-3 tree demo: construction

2-3 tree



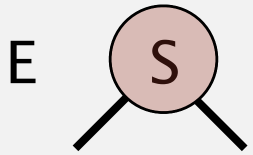
2-3 tree demo: construction

insert E



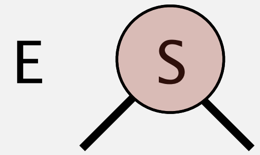
2-3 tree demo: construction

insert E



2-3 tree demo: construction

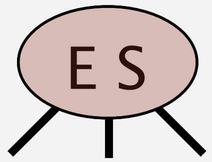
insert E



convert 2-node into 3-node

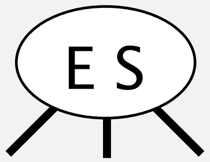
2-3 tree demo: construction

insert E



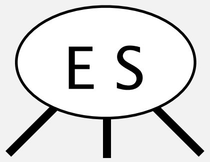
2-3 tree demo: construction

2-3 tree



2-3 tree demo: construction

insert A



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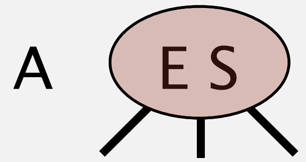
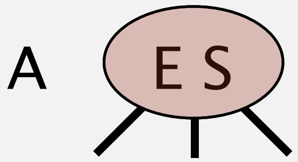


2-3 tree demo: construction

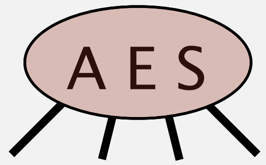
insert A



convert 3-node into 4-node

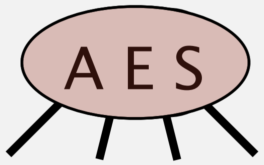
2-3 tree demo: construction

insert A



2-3 tree demo: construction

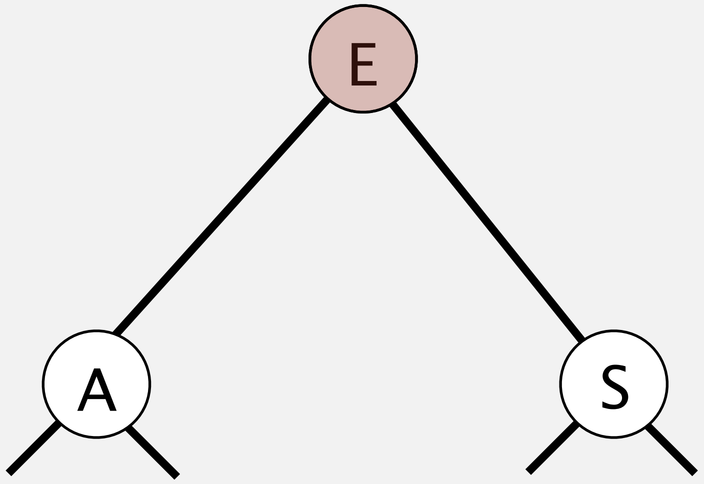
insert A



split 4-node
(move E to parent)

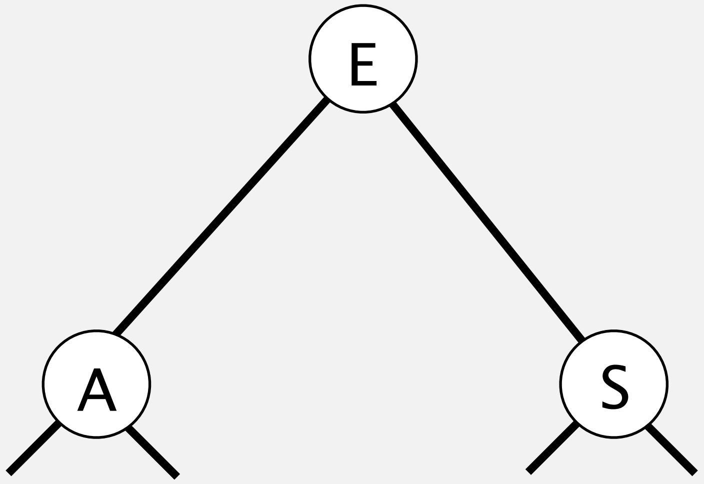
2-3 tree demo: construction

insert A



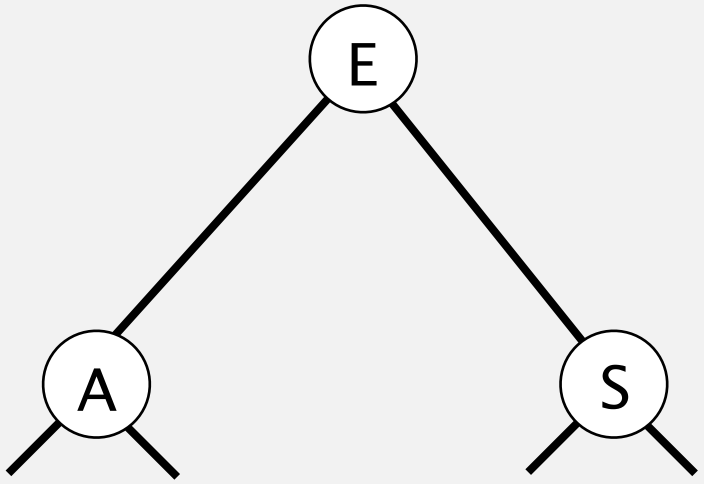
2-3 tree demo: construction

2–3 tree



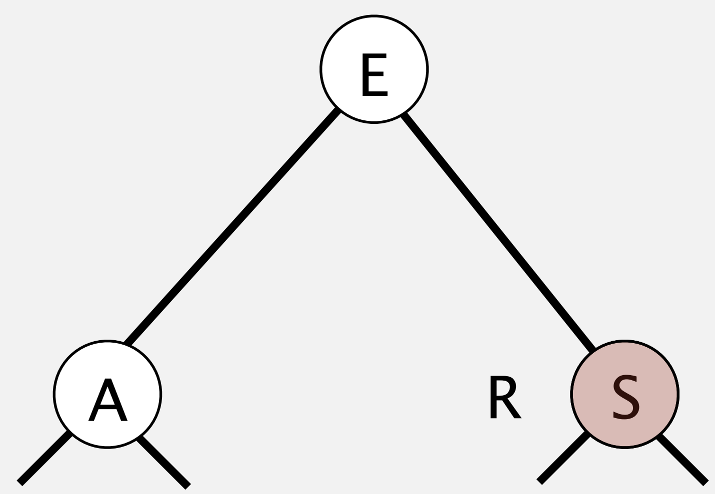
2-3 tree demo: construction

insert R



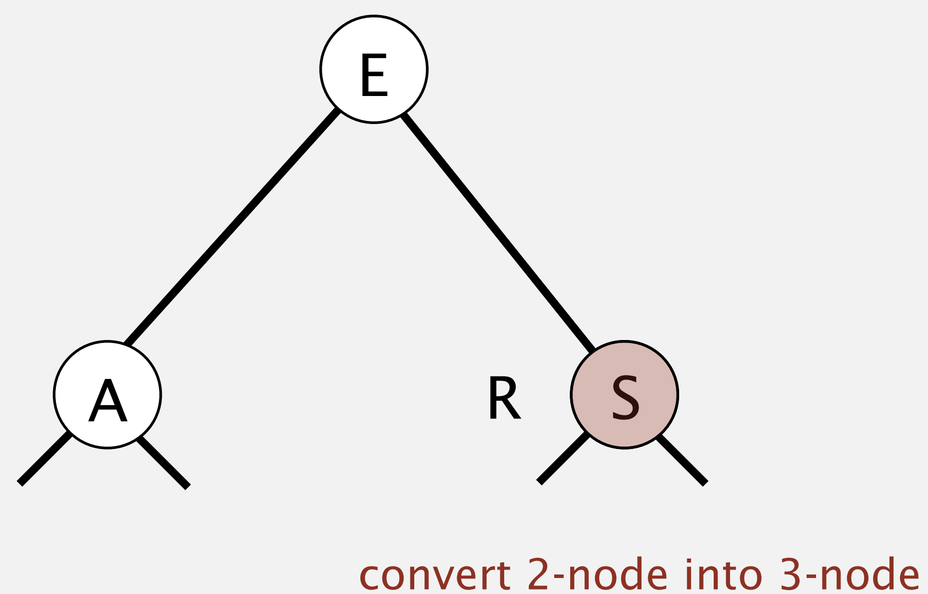
2-3 tree demo: construction

insert R



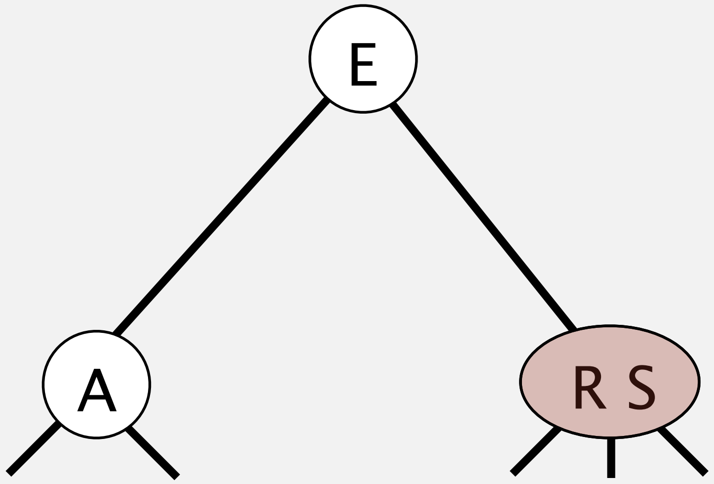
2-3 tree demo: construction

insert R



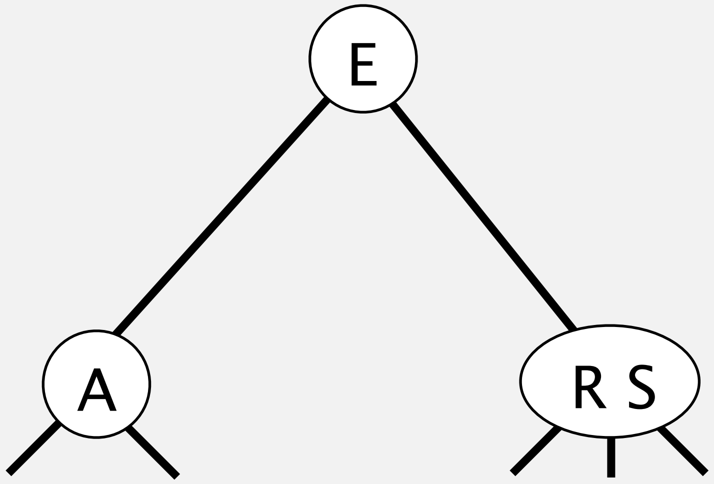
2-3 tree demo: construction

insert R



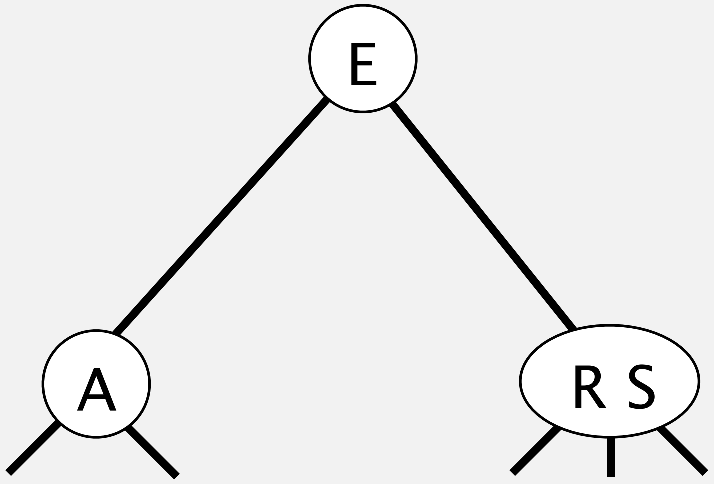
2-3 tree demo: construction

2-3 tree



2-3 tree demo: construction

insert C



2-3 tree demo: construction

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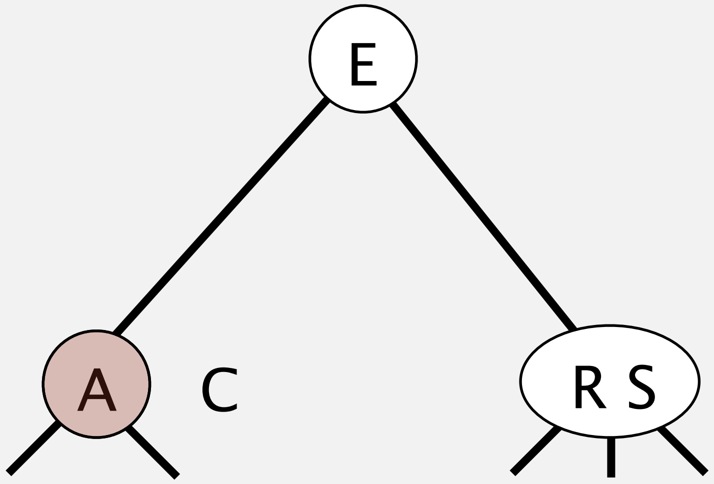
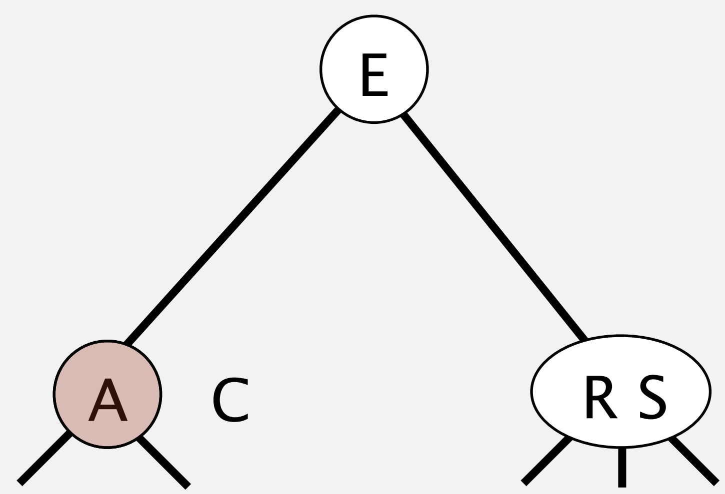


2-3 tree demo: construction

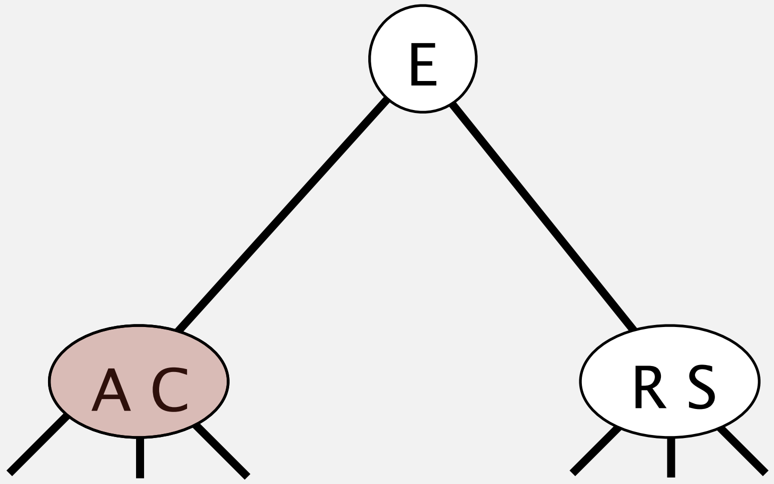
insert C



convert 2-node into 3-node

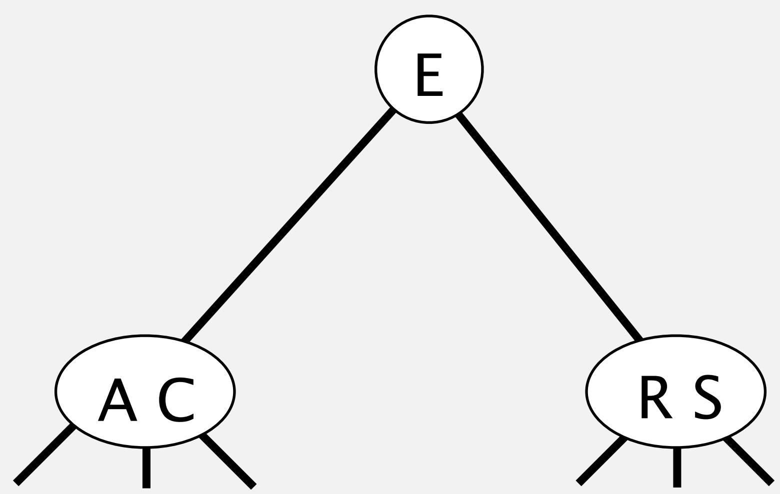
2-3 tree demo: construction

insert C



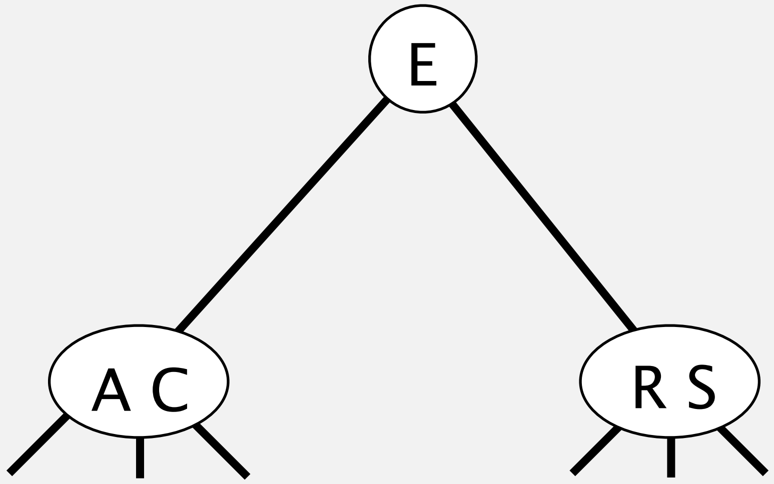
2-3 tree demo: construction

2-3 tree



2-3 tree demo: construction

insert H



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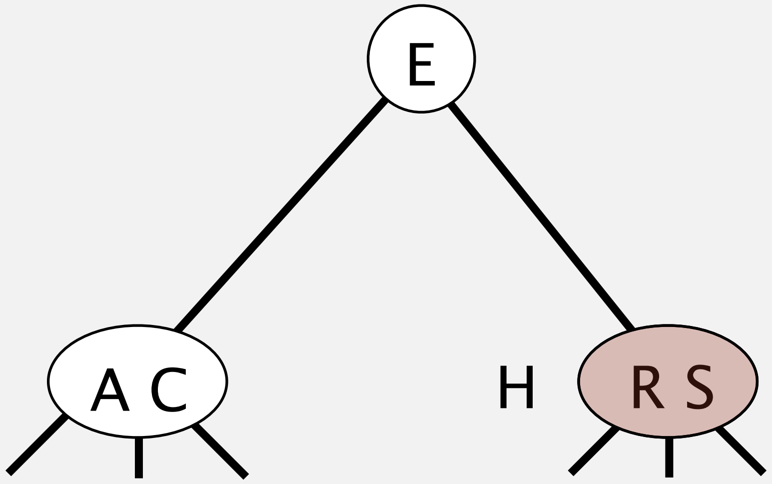

2-3 tree demo: construction

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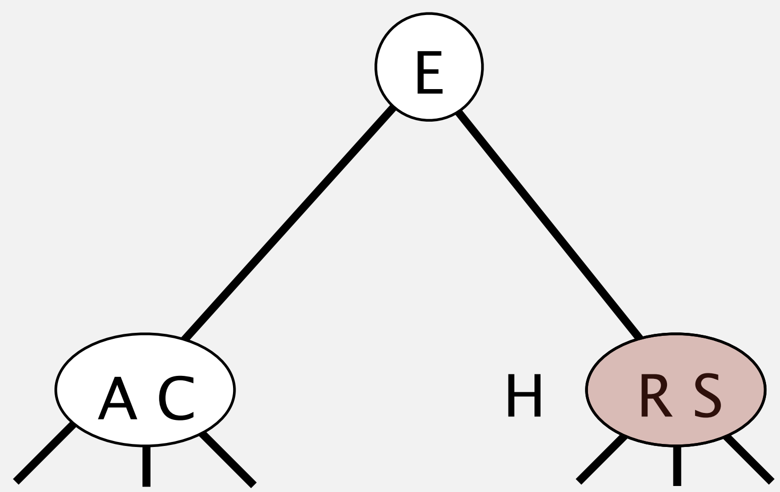
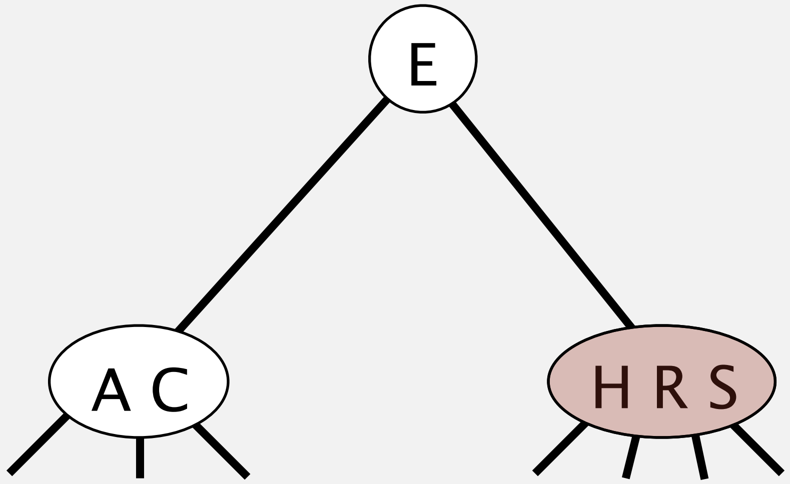


convert 3-node into 4-node

2-3 tree demo: construction

insert H



2-3 tree demo: construction

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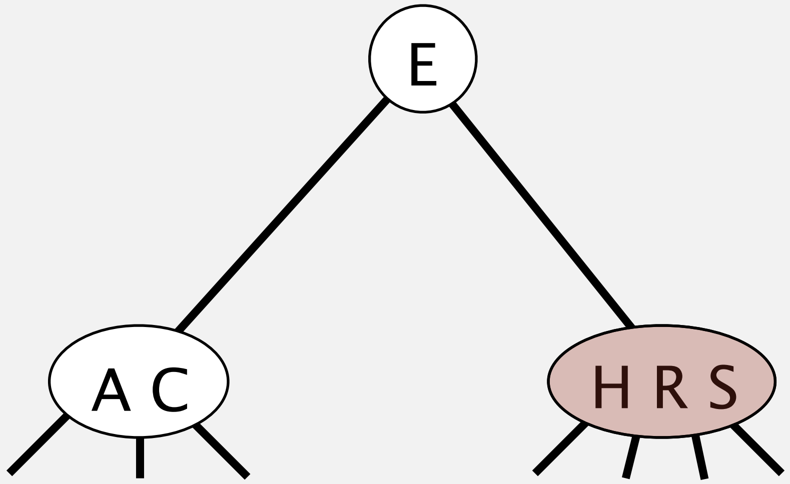

split 4-node
(move R to parent)

2-3 tree demo: construction

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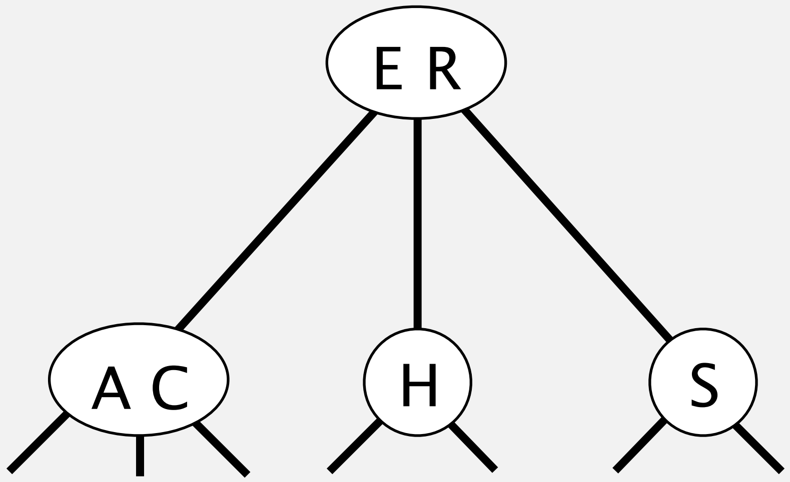
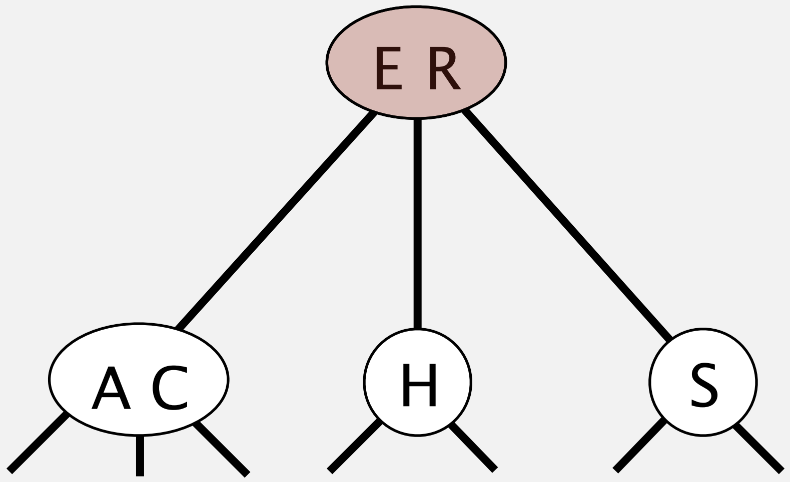


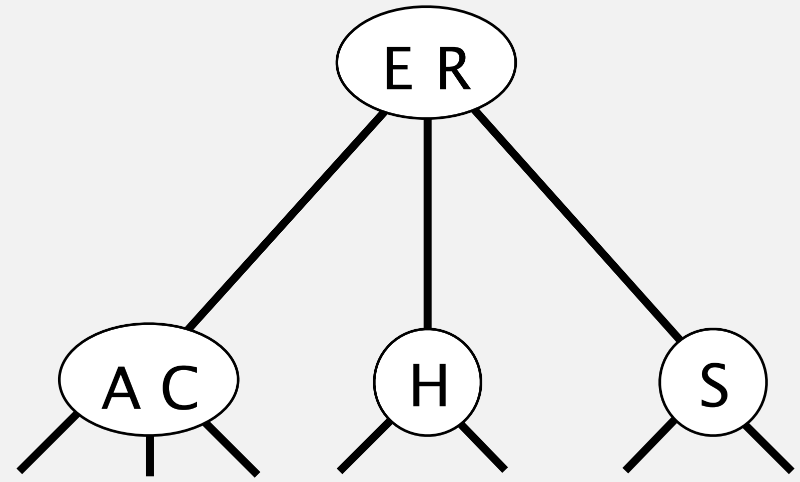
2-3 tree demo: construction

insert H



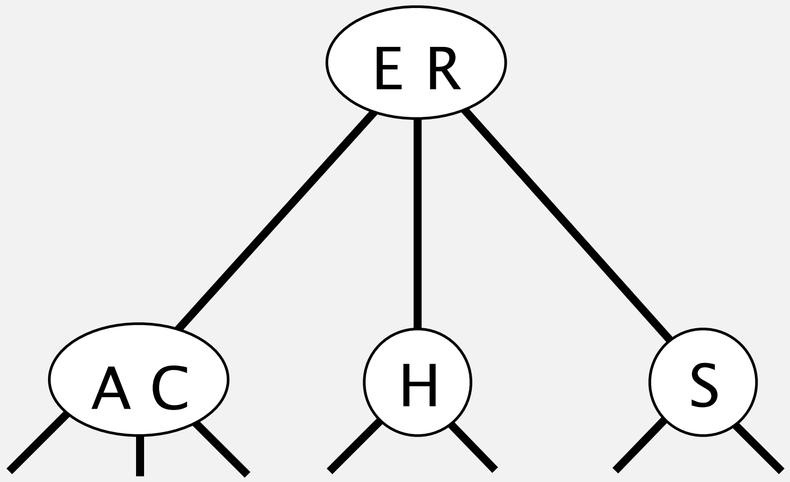
2-3 tree demo: construction

2–3 tree



2-3 tree demo: construction

insert X



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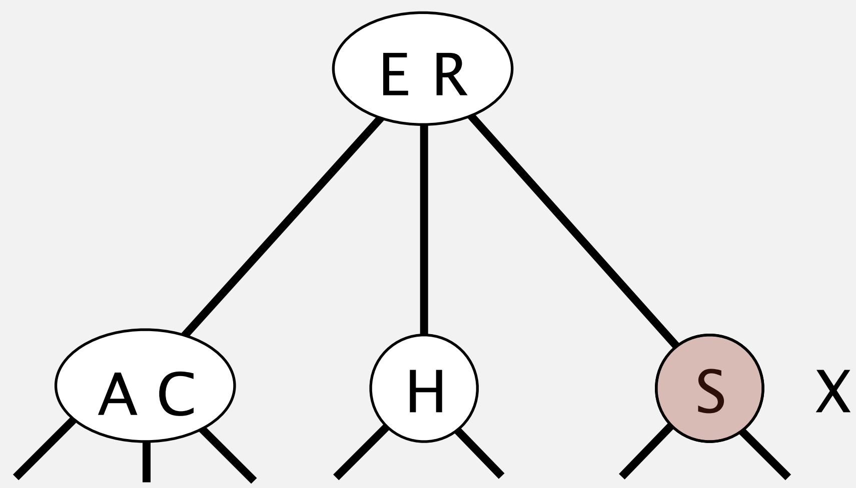
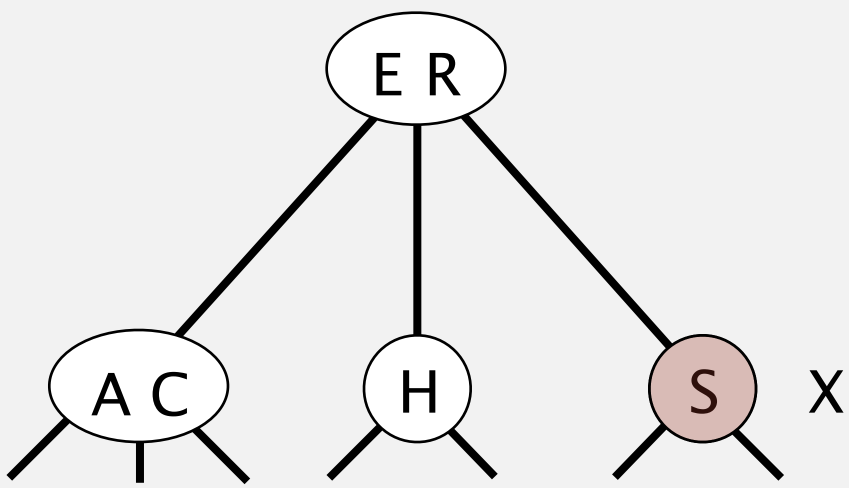


2-3 tree demo: construction

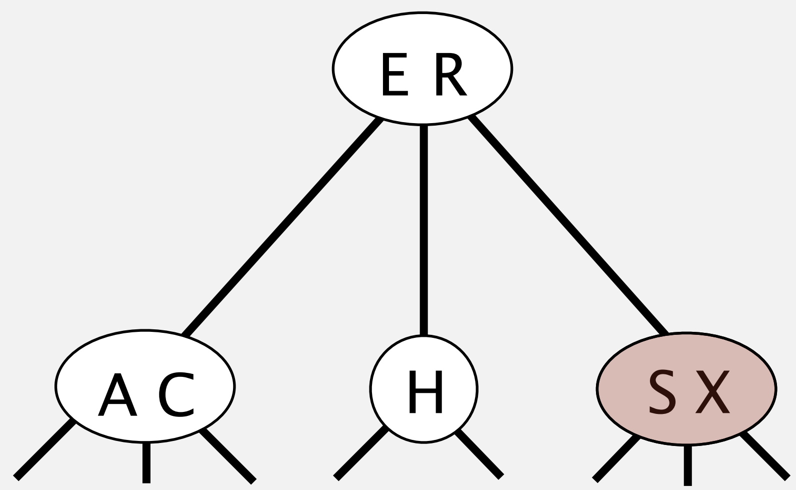
insert X



convert 2-node into 3-node

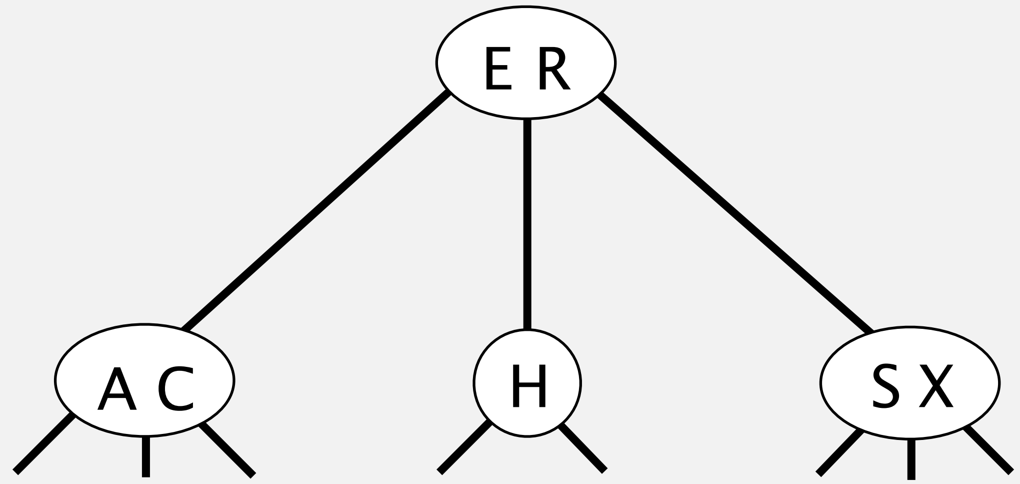
2-3 tree demo: construction

insert X



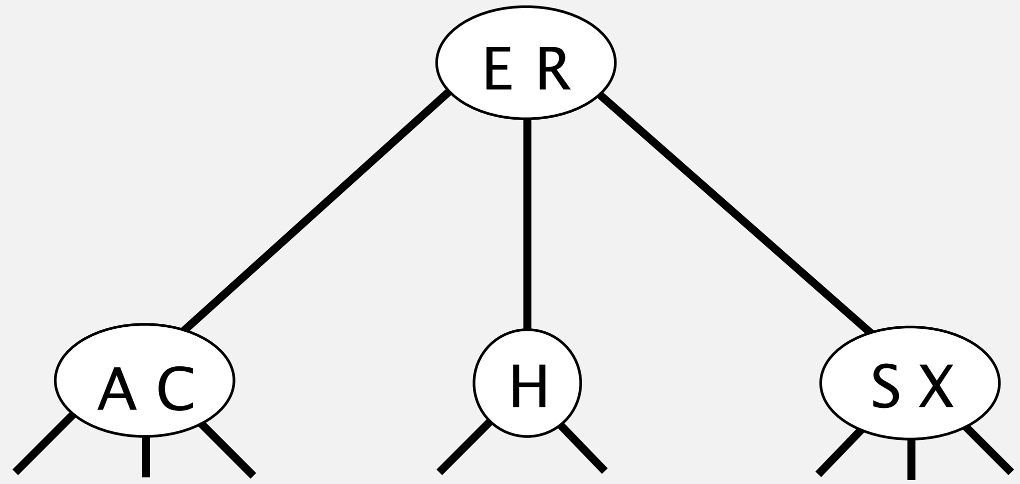
2-3 tree demo: construction

2-3 tree



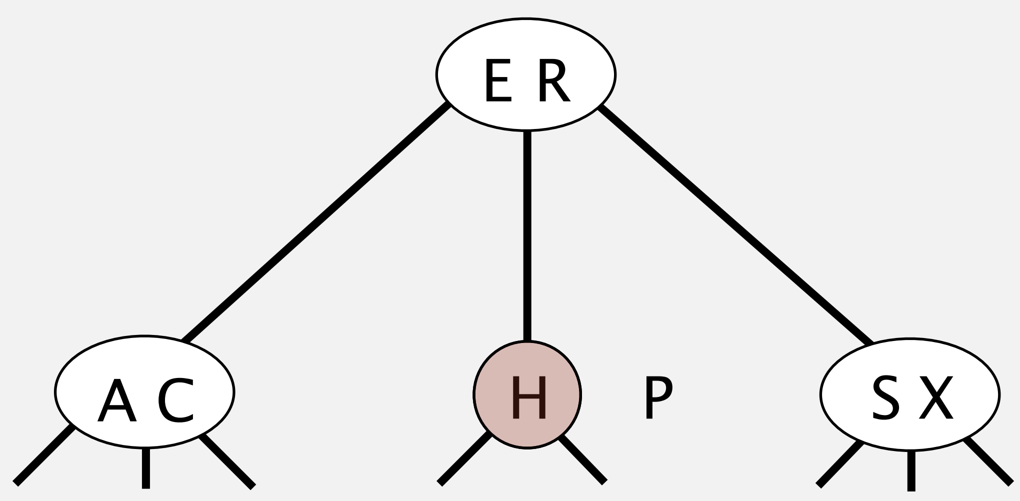
2-3 tree demo: construction

insert P



2-3 tree demo: construction

insert P



2-3 tree demo: construction

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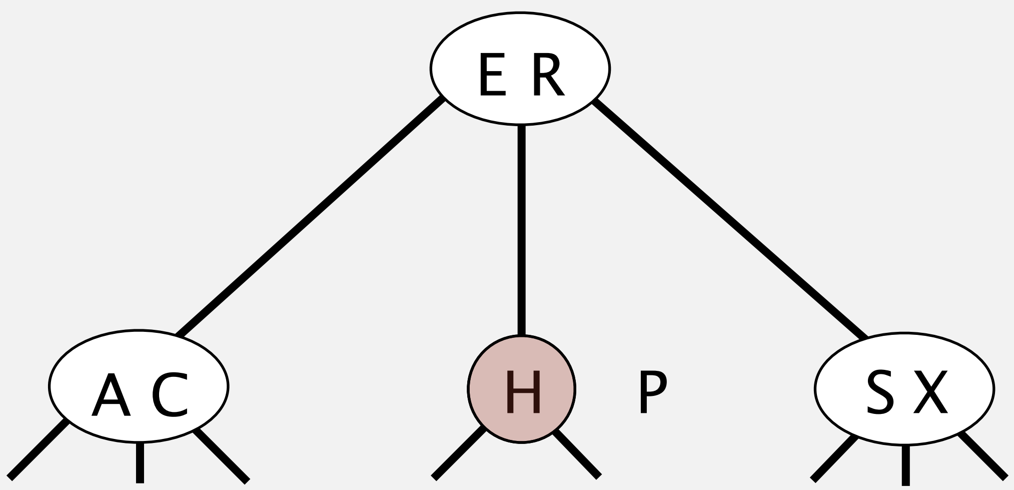

convert 2-node into 3-node

2-3 tree demo: construction

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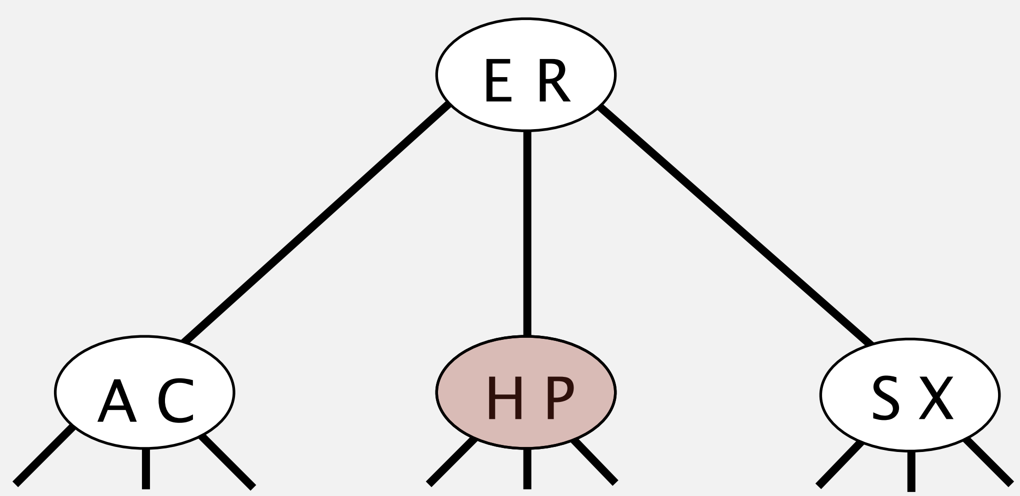
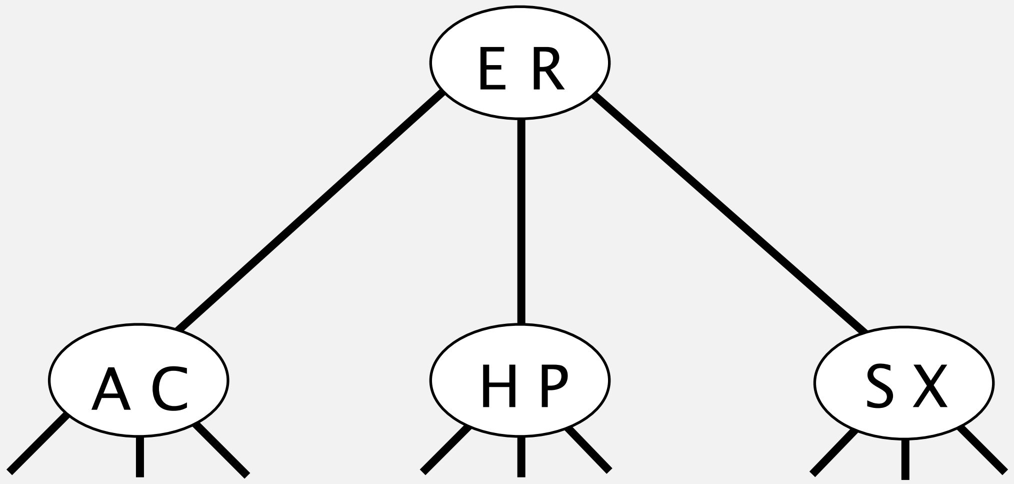


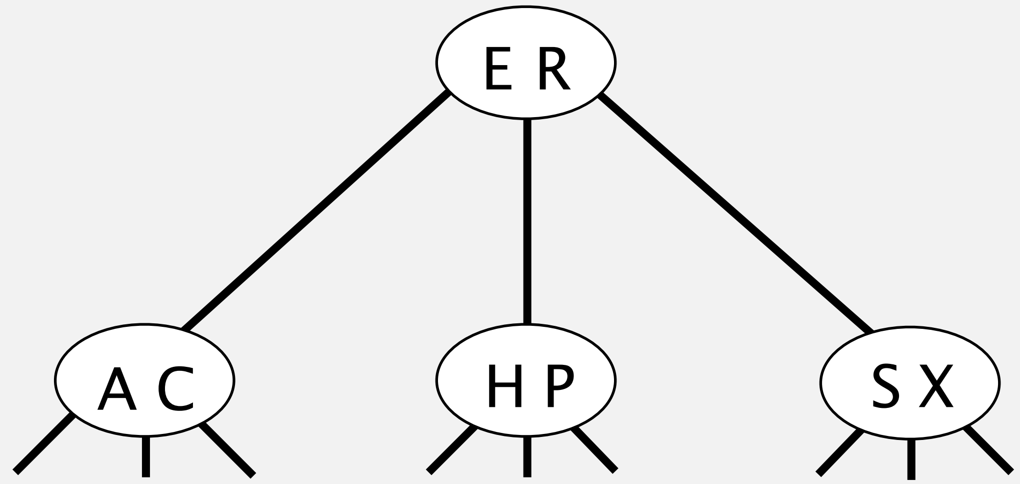
2-3 tree demo: construction

2-3 tree



2-3 tree demo: construction

insert L



2-3 tree demo: construction

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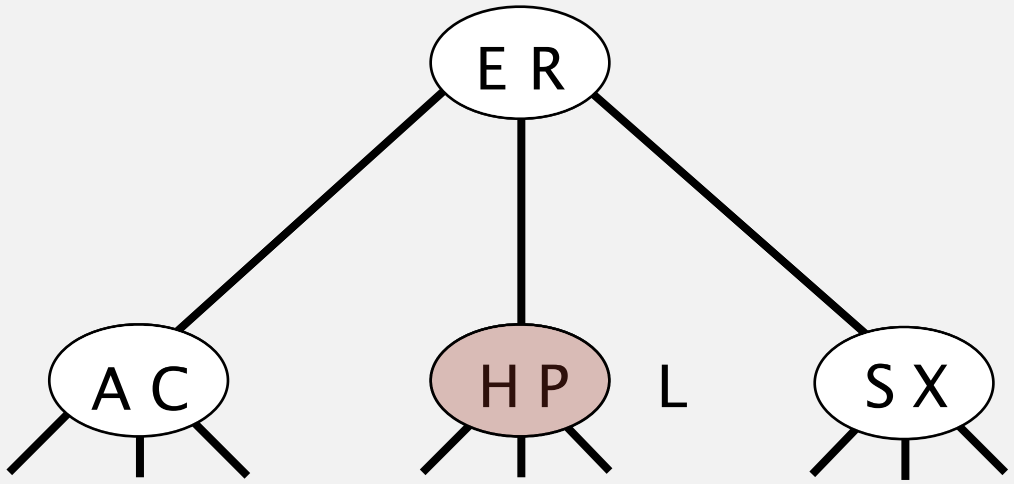

2-3 tree demo: construction

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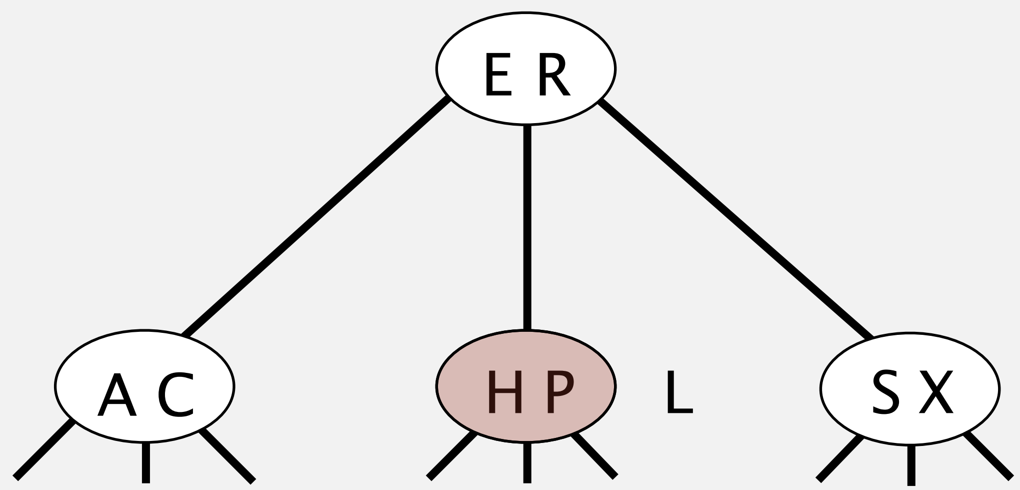
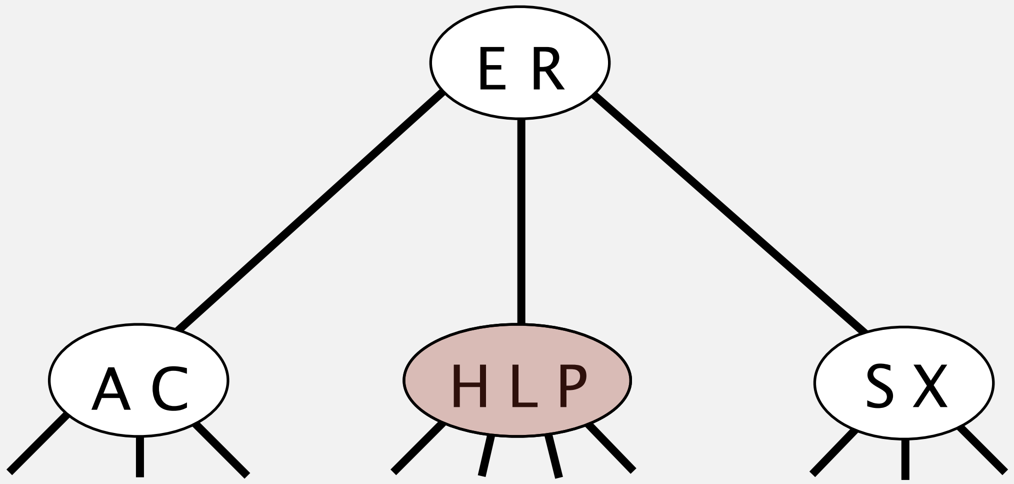


convert 3-node into 4-node

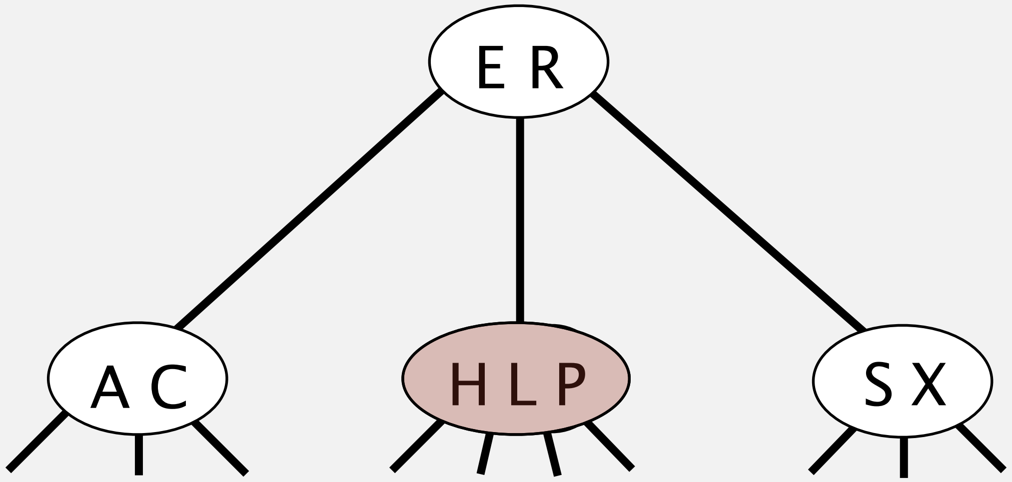
2-3 tree demo: construction

insert L



2-3 tree demo: construction

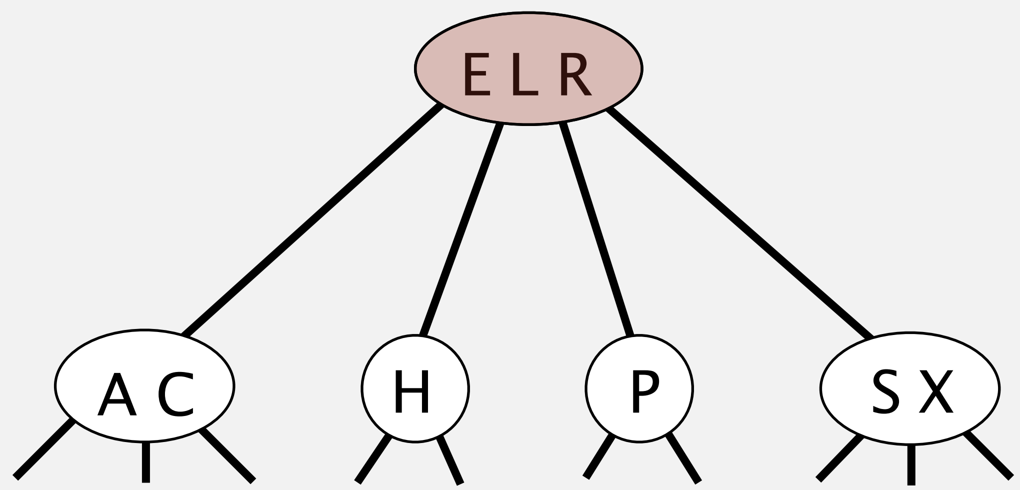
insert L



split 4-node
(move L to parent)

2-3 tree demo: construction

insert L



2-3 tree demo: construction

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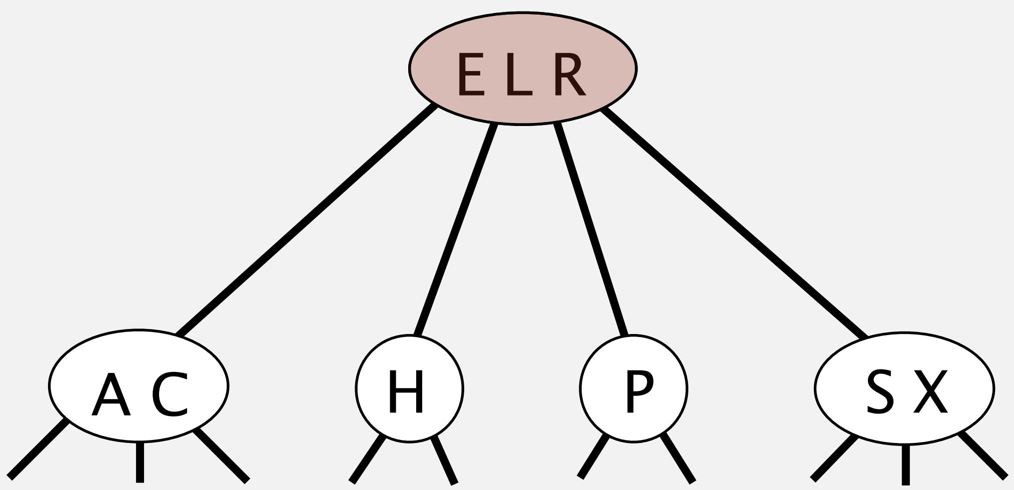
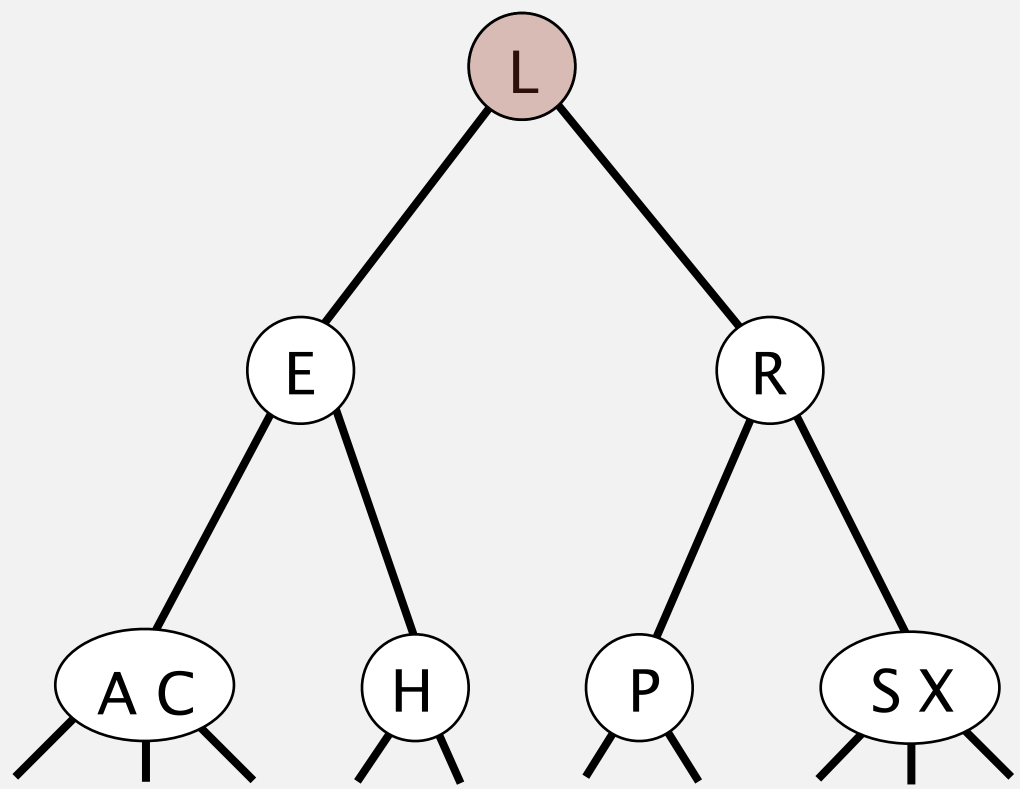


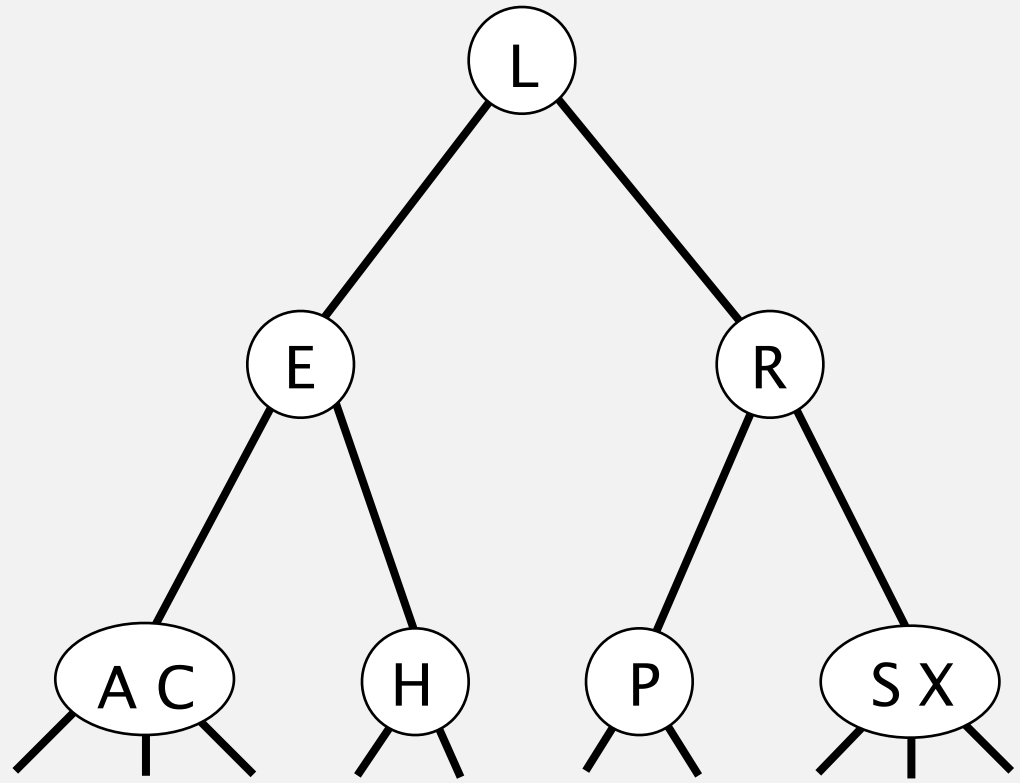
2-3 tree demo: construction

insert L



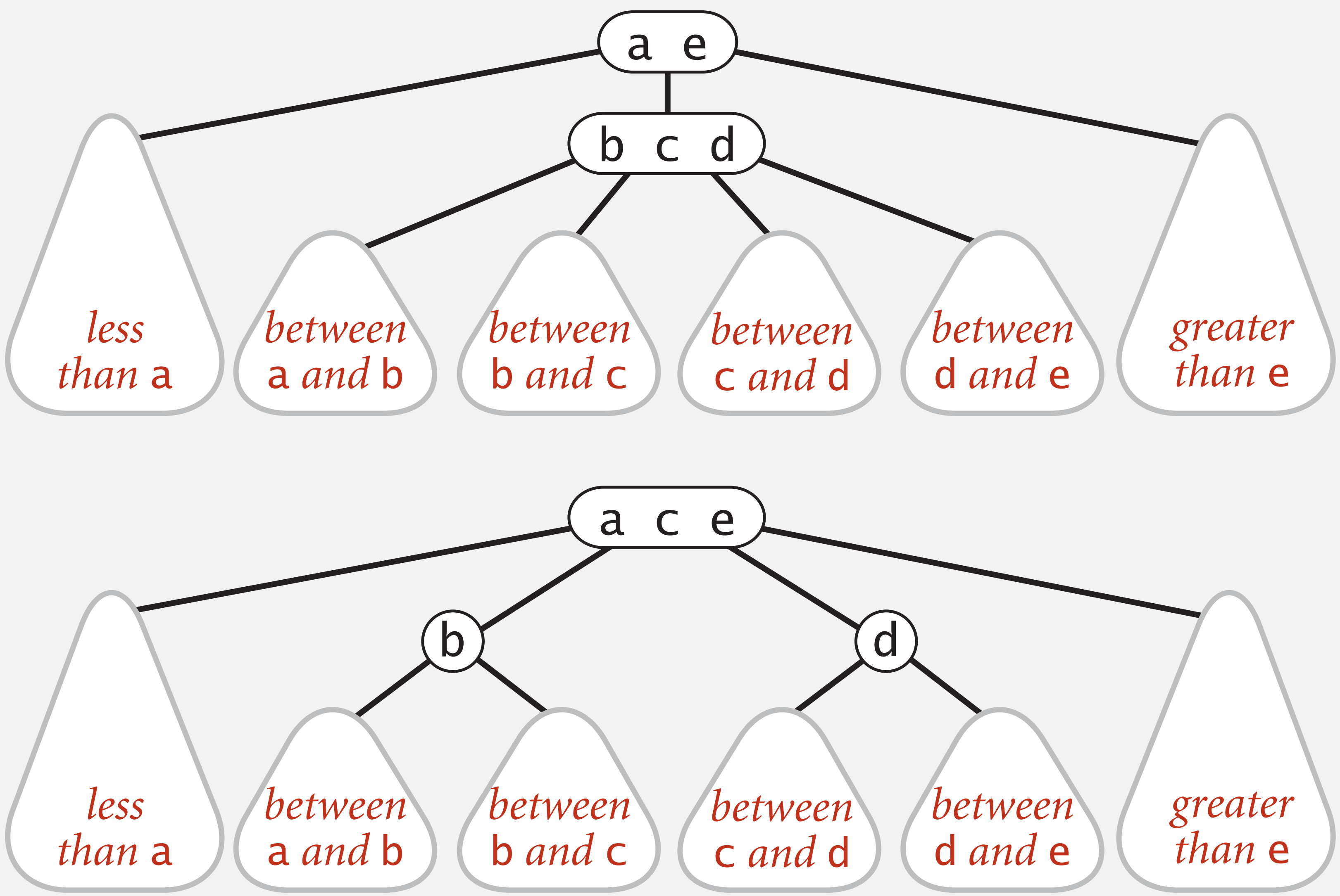
2-3 tree demo: construction

2–3 tree



Local transformations in a 2-3 tree

Splitting a 4-node is a local transformation: constant number of operations.



Global properties in a 2-3 tree

Invariants. Maintains symmetric order and perfect balance.

Pf. Each transformation maintains symmetric order and perfect balance.

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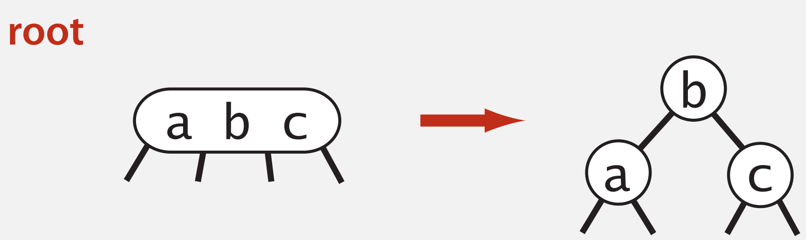

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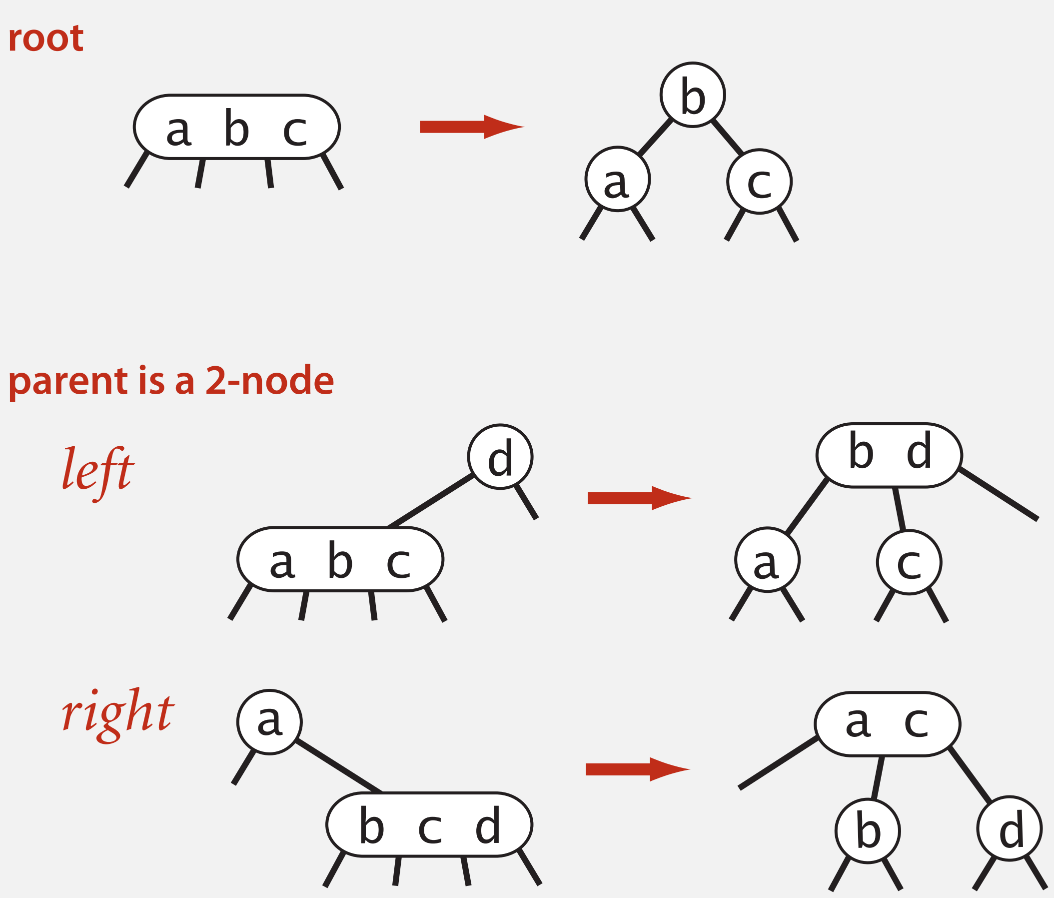

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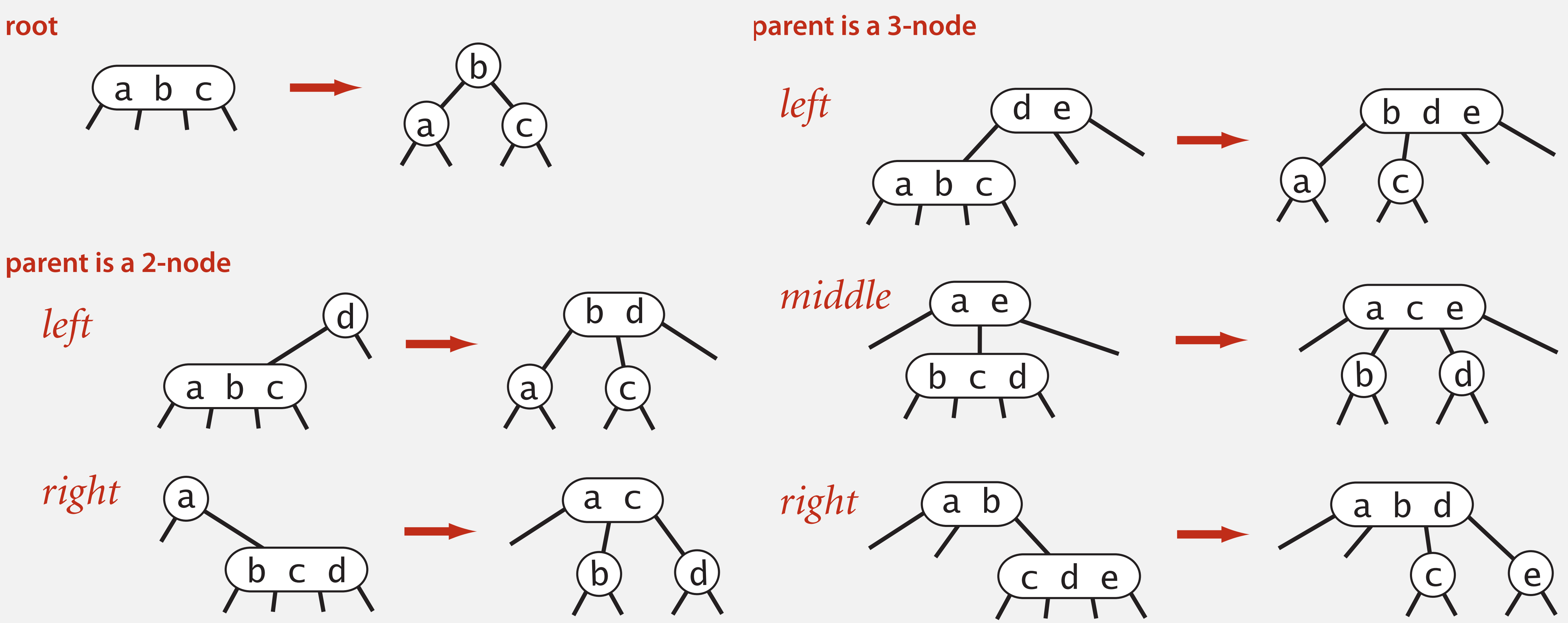
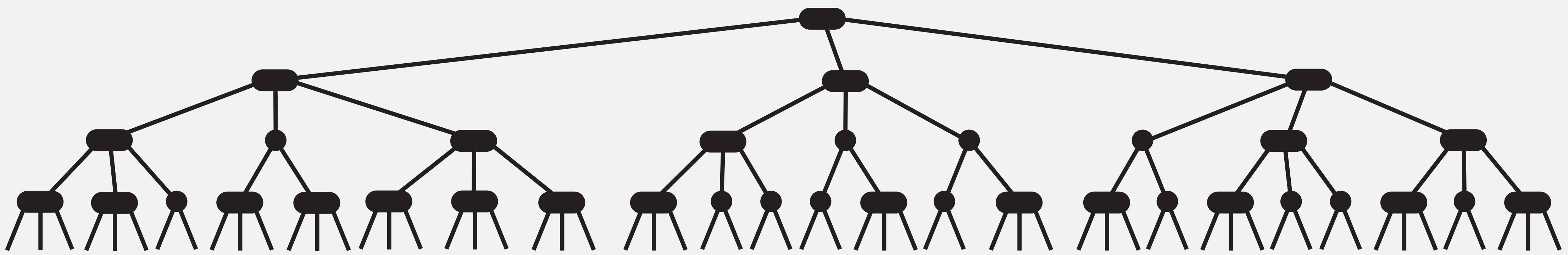


2-3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case:
- Best case:

2-3 tree: performance

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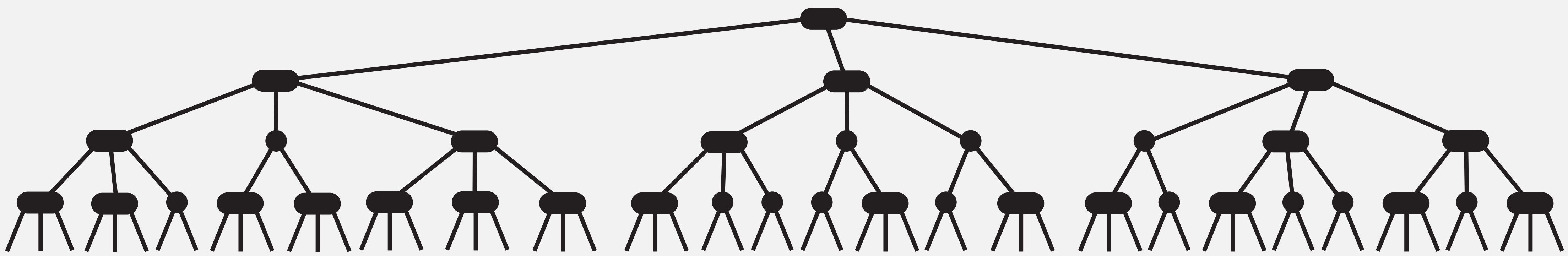

Tree height.

- Worst case: $\lg N$. [all 2-nodes]
- Best case: $\log_3 N \approx .631 \lg N$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Bottom line. Guaranteed logarithmic performance for search and insert.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	½ N	N	½ N		`equals()`
binary search (ordered array)	$\lg N$	N	N	$\lg N$	½ N	½ N	✔	`compareTo()`
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✔	`compareTo()`
2-3 tree	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	✔	`compareTo()`

constant c depend upon implementation

2-3 tree: implementation?

" Beautiful algorithms are not always the most useful. "

— Donald Knuth

2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

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fantasy code

```
public void put(Key key, Value val)
{
   Node x = root;
   while (x.getTheCorrectChild(key) != null)
   {
      x = x.getTheCorrectChildKey();
      if (x.is4Node()) x.split();
   }
   if      (x.is2Node()) x.make3Node(key, val);
   else if (x.is3Node()) x.make4Node(key, val);
}
```

2-3 tree: implementation?

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```

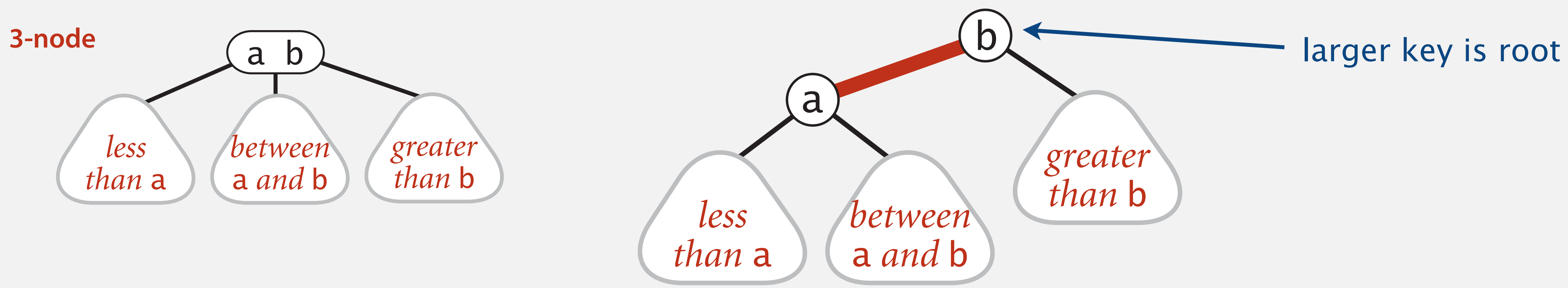
Bottom line. Could do it, but there's a better way.

3.3 BALANCED SEARCH TREES

- *2-3 search trees*
- *red-black BSTs*
- *B-trees*

Left-leaning red-black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3–nodes.



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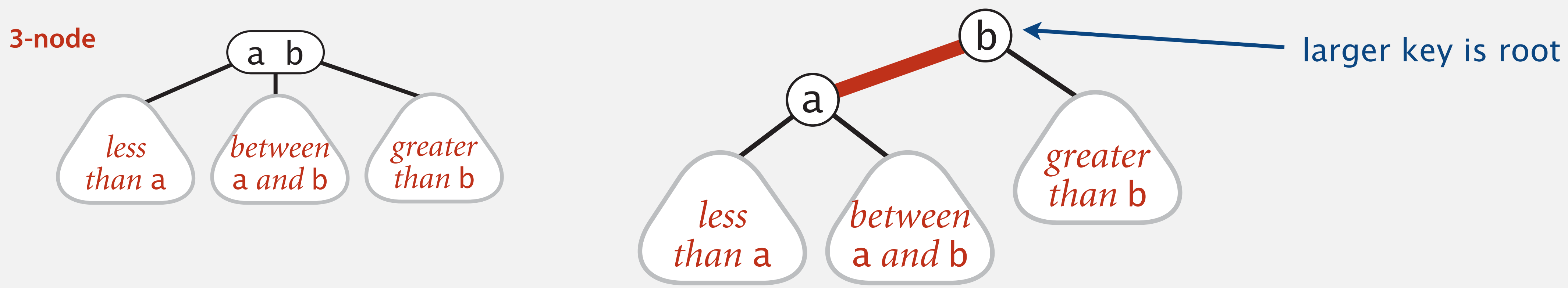
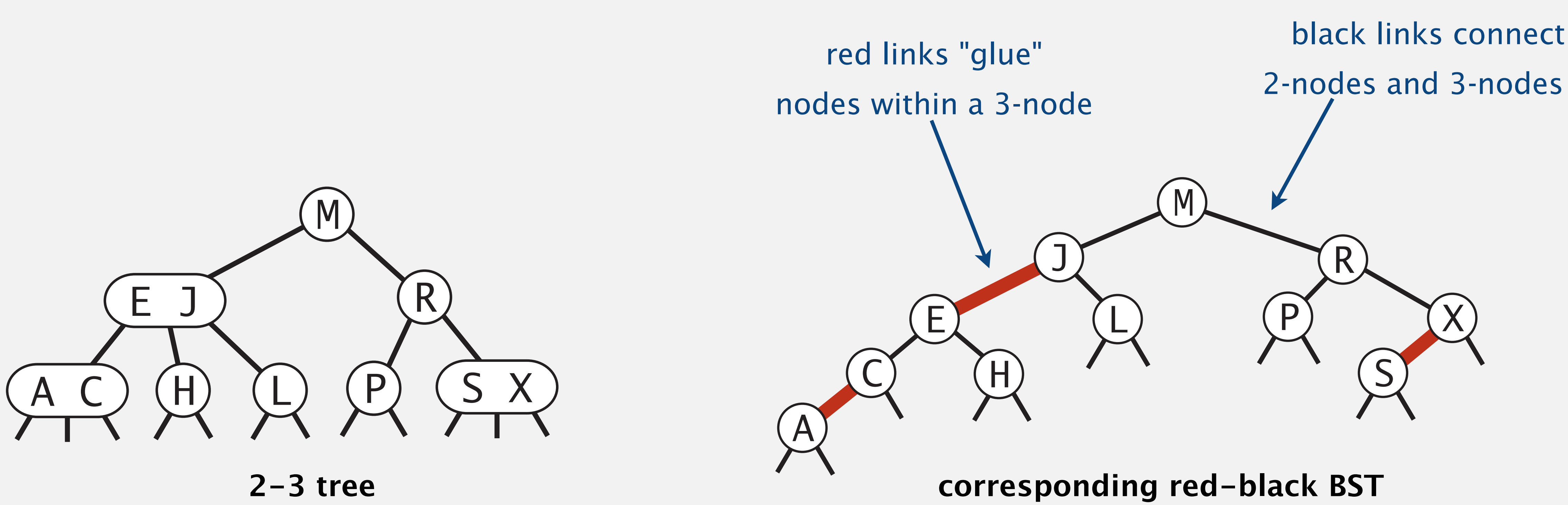


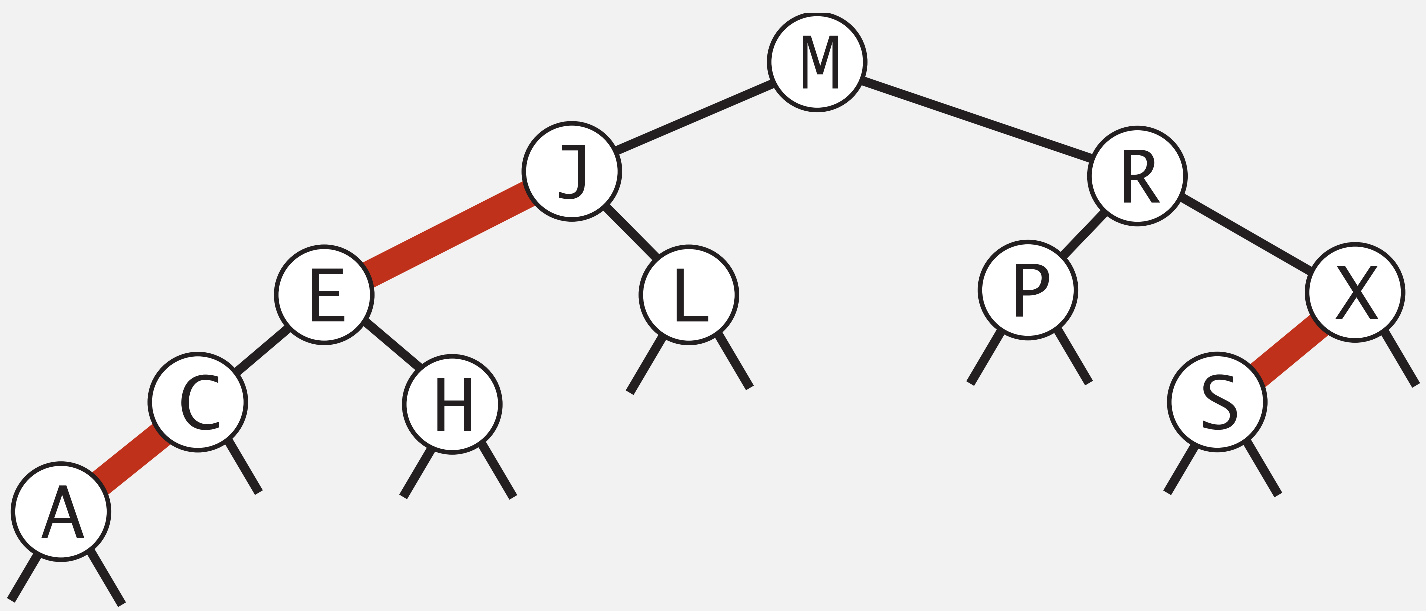


An equivalent definition

A BST such that:

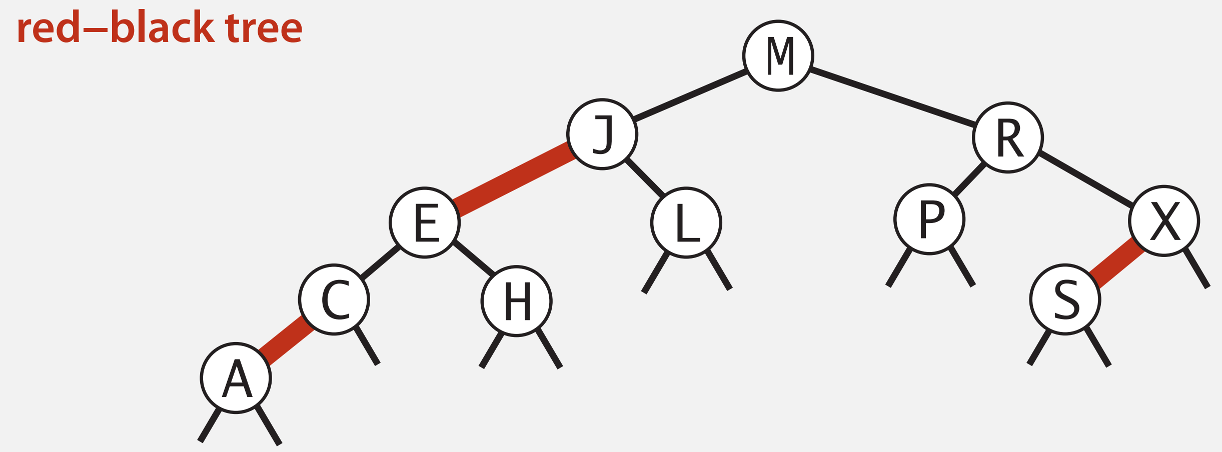
- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

"perfect black balance"



Left-leaning red-black BSTs: 1-1 correspondence with 2-3 trees

Key property. 1-1 correspondence between 2-3 and LLRB.



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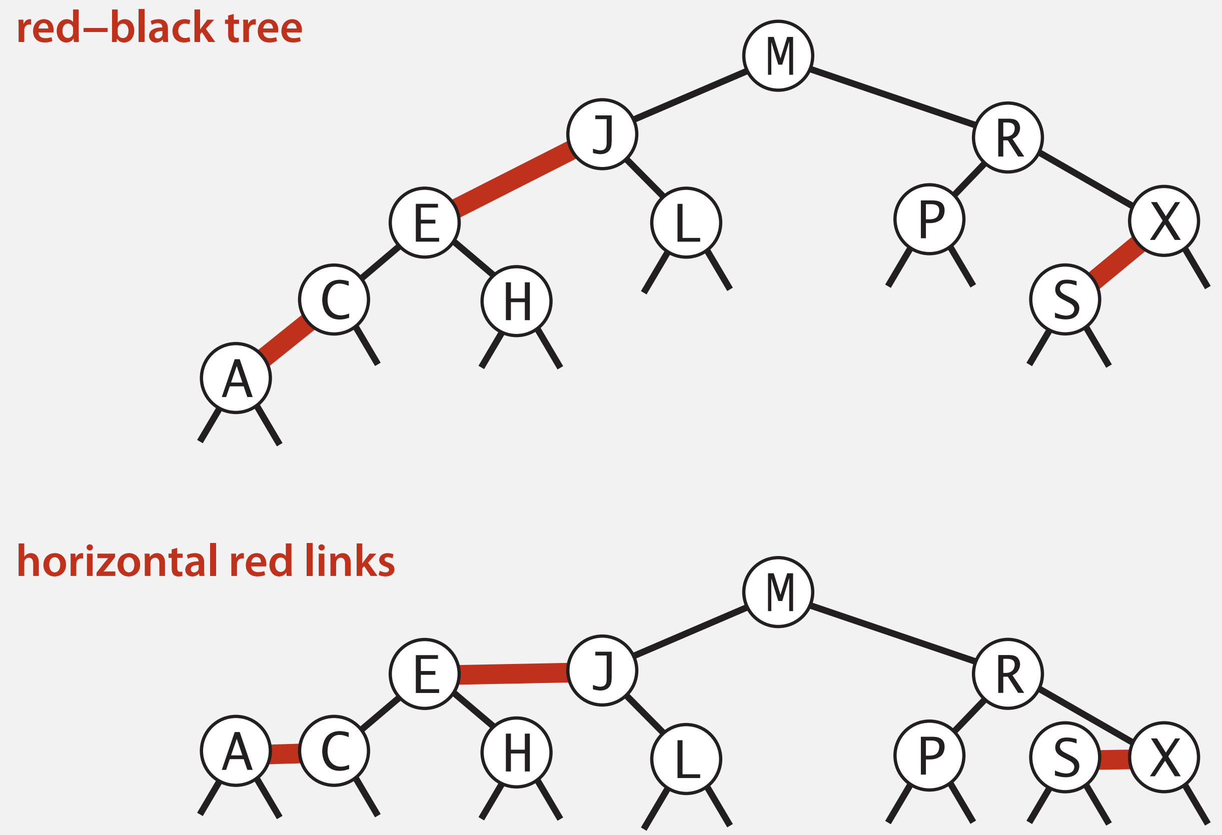

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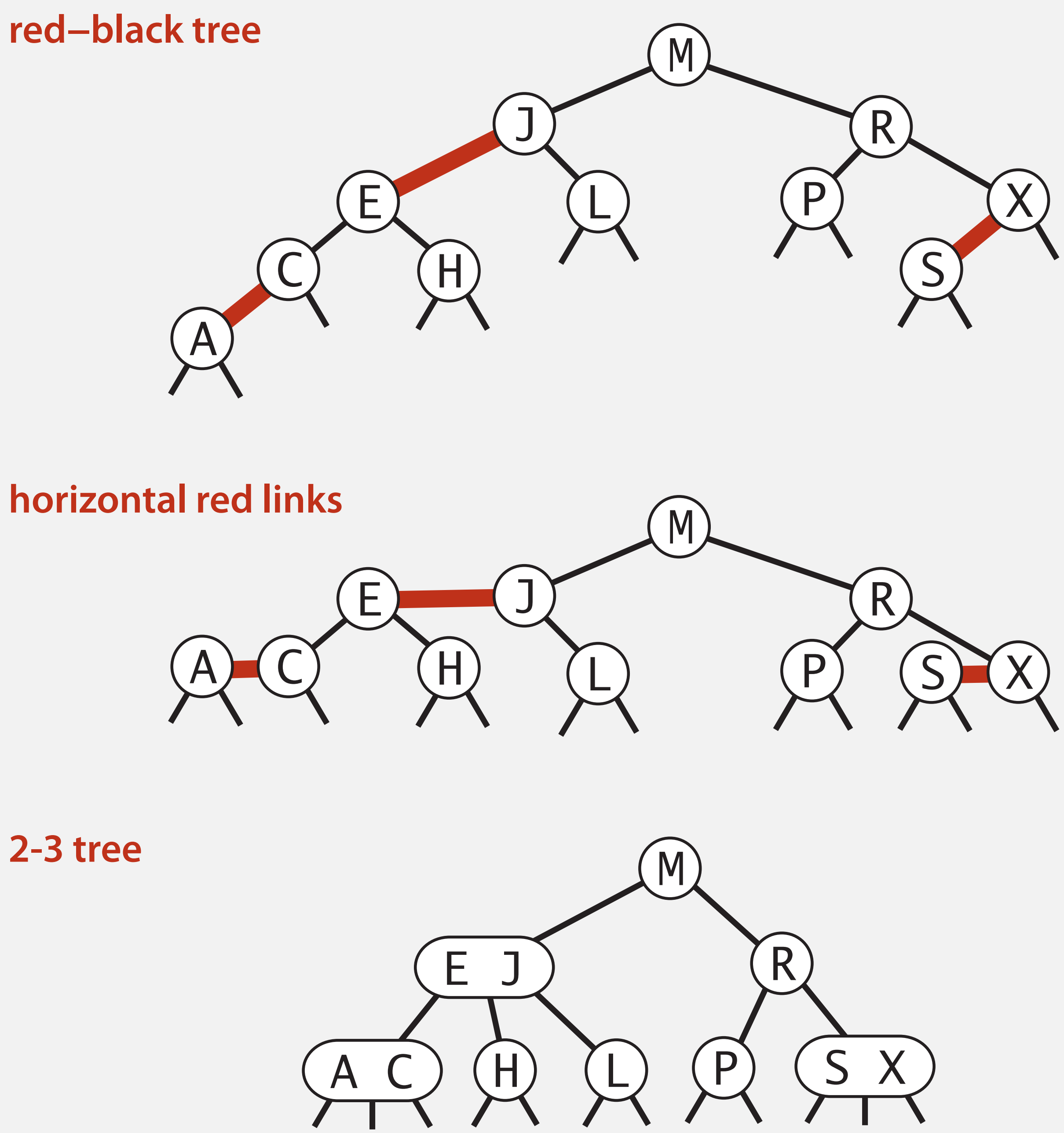
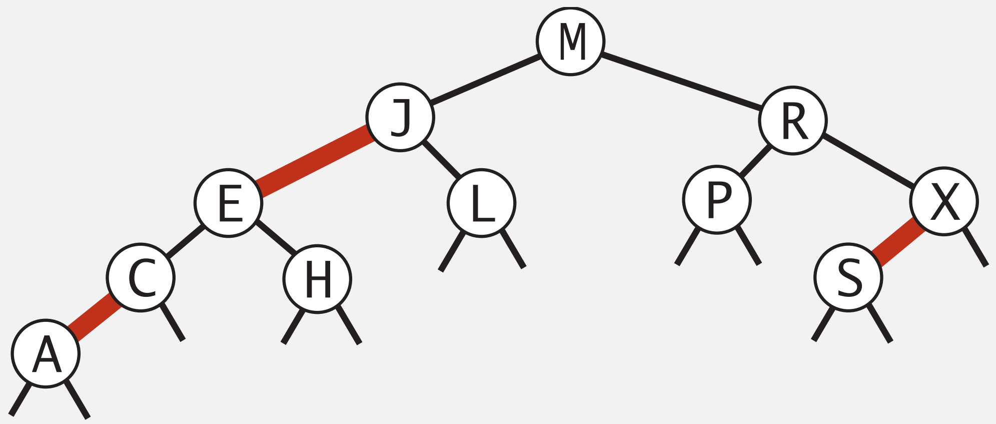


Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster
because of better balance

```
public Val get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



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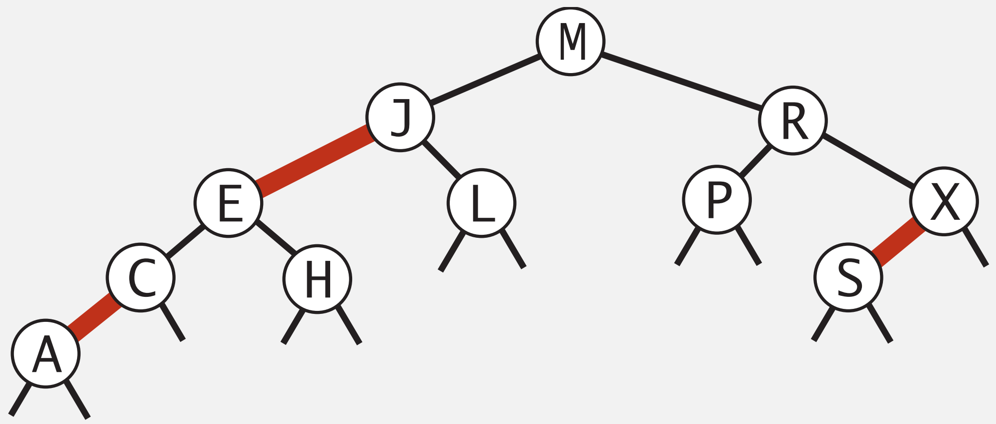
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



Remark. Most other ops (e.g., floor, iteration, selection) are also identical.

Red-black BST representation

Each node is pointed to by precisely one link (from its parent) $\Rightarrow$
can encode color of links in nodes.

Red-black BST representation

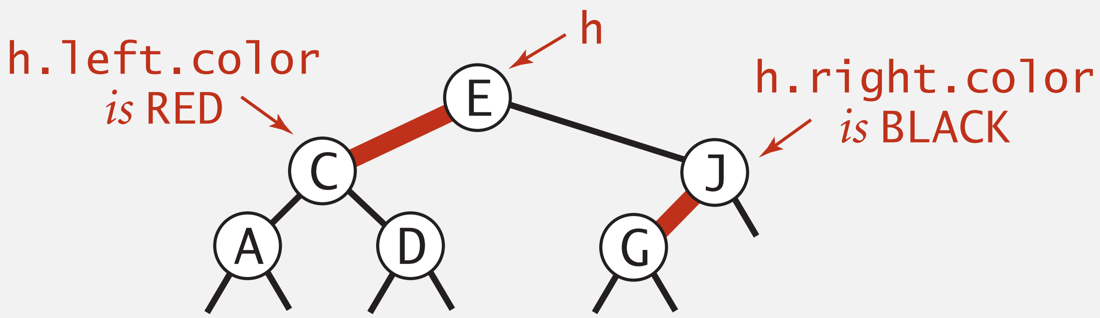
Each node is pointed to by precisely one link (from its parent) ⇒ can encode color of links in nodes.

```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;    // color of parent link
}

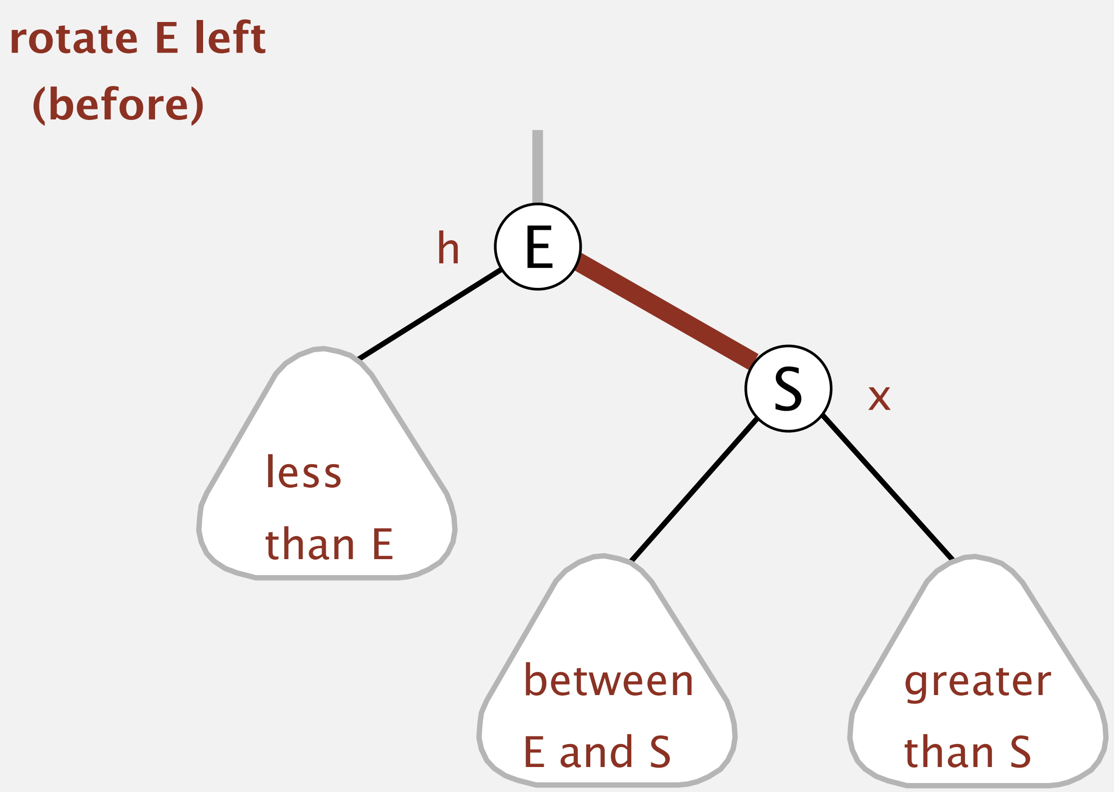
private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

null links are black



Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

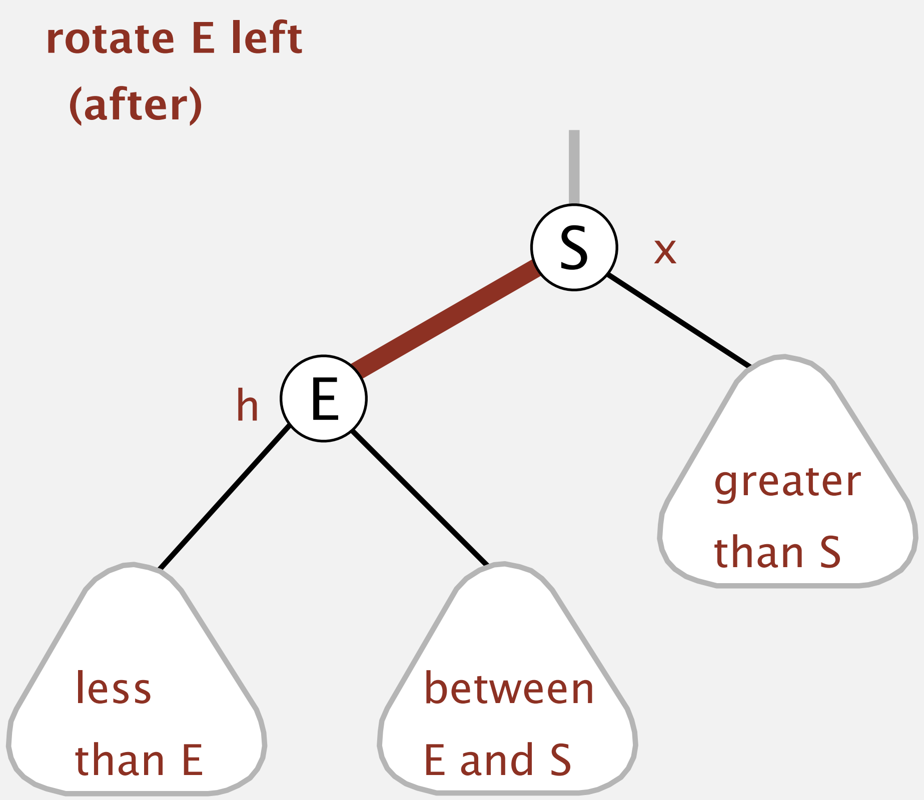



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.




```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

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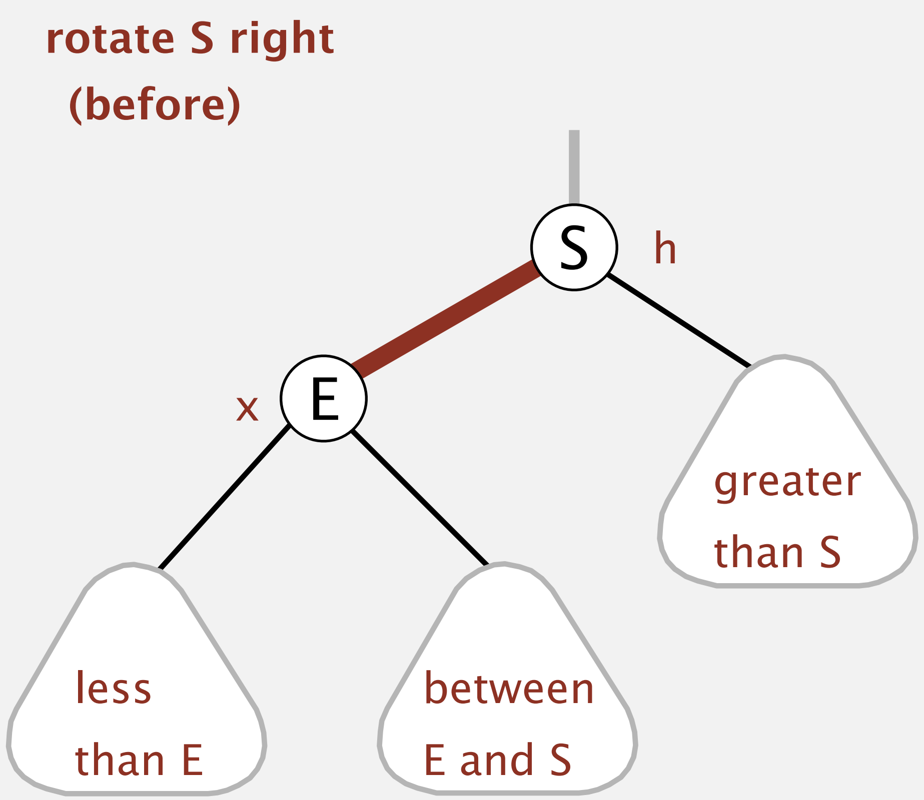
Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.




```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

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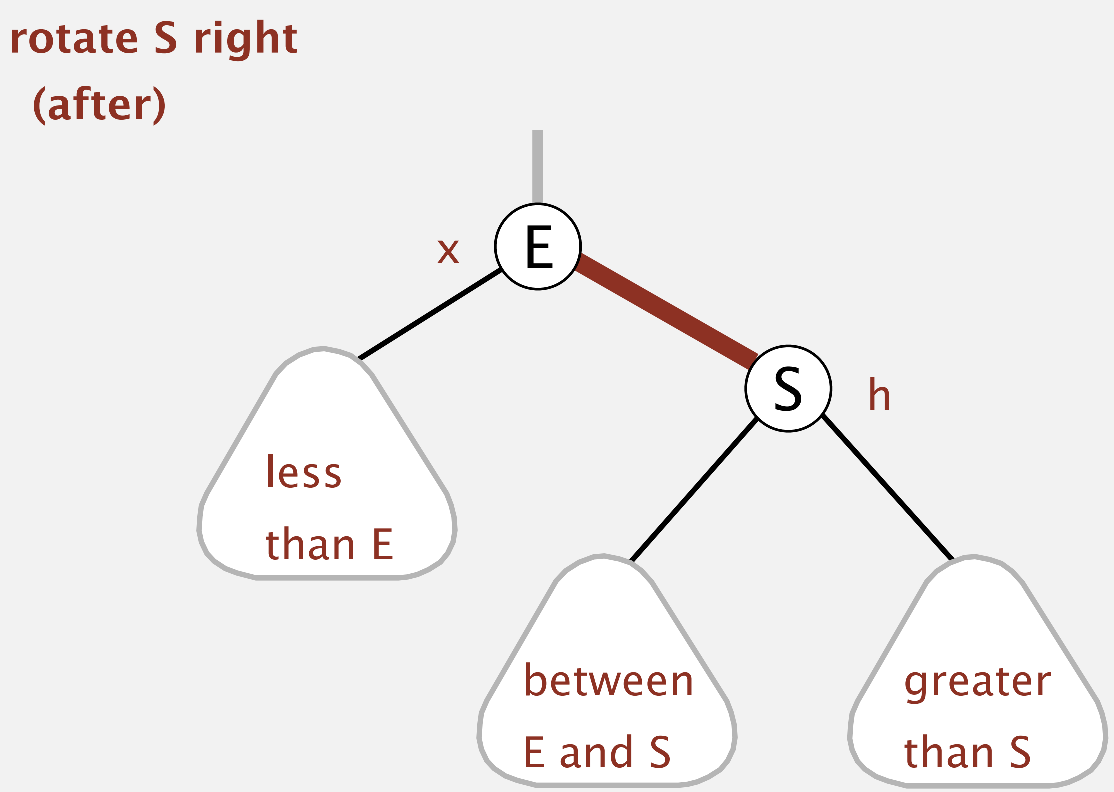
Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.




```
private Node rotateRight(Node h)
{
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   Node x = h.left;
   h.left = x.right;
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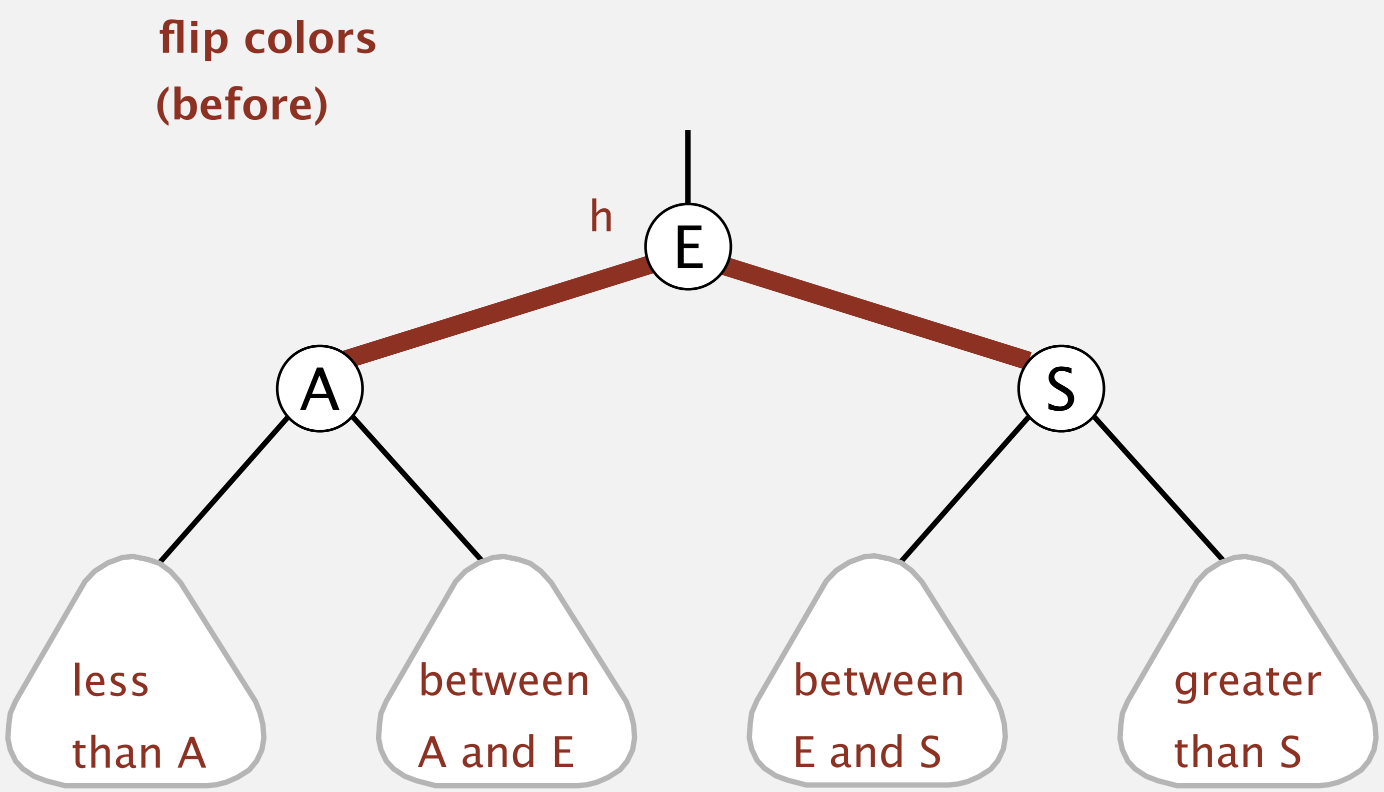
Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.




```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

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Elementary red-black BST operations

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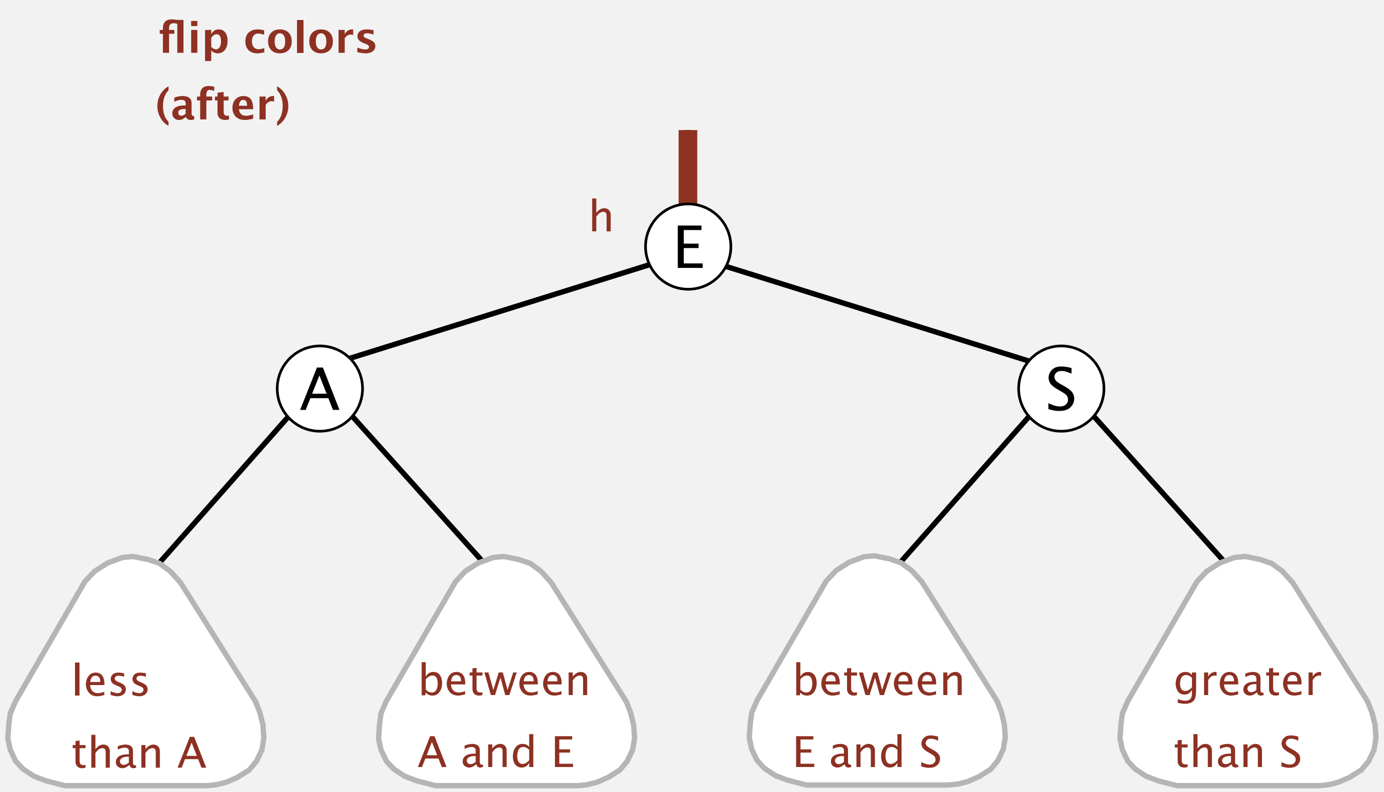


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Insertion in a LLRB tree: overview

Basic strategy. Maintain 1-1 correspondence with 2-3 trees.

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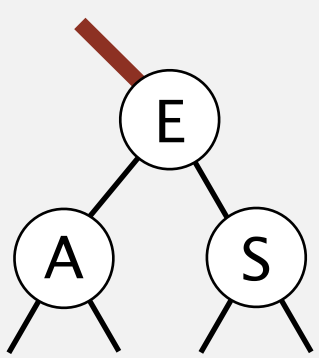

right-leaning red link

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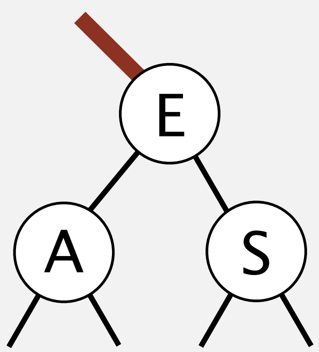

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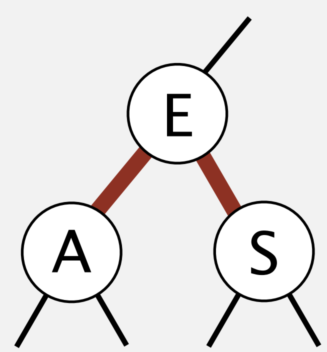

two red children
(a temporary 4-node)

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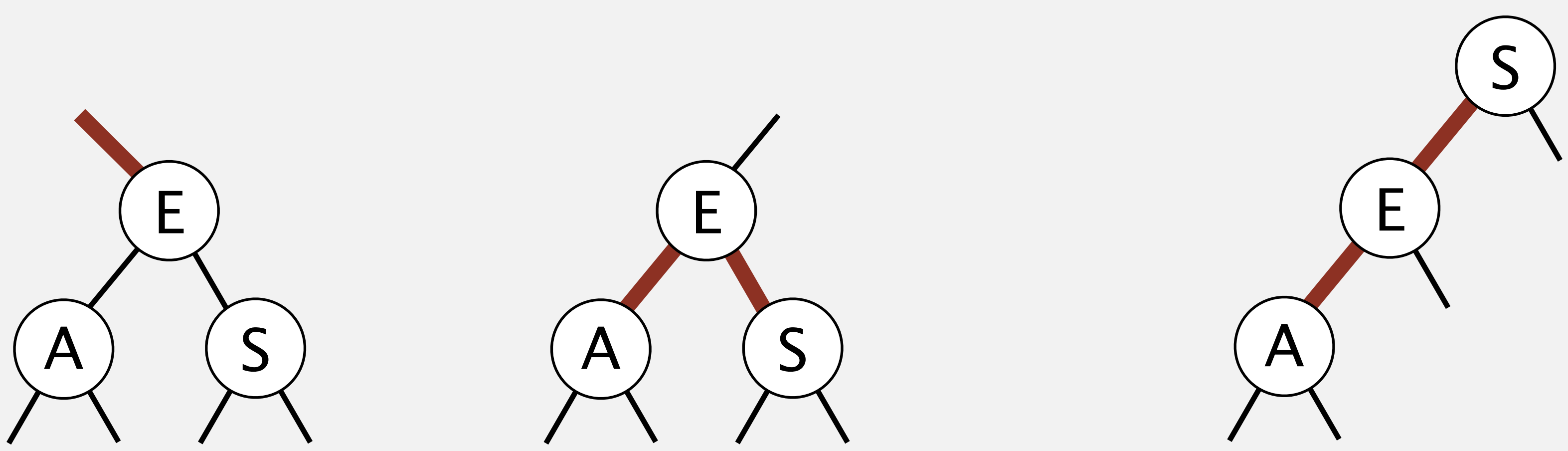

right–leaning red link

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left–left red (a temporary 4–node)

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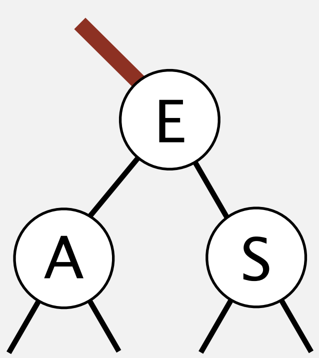
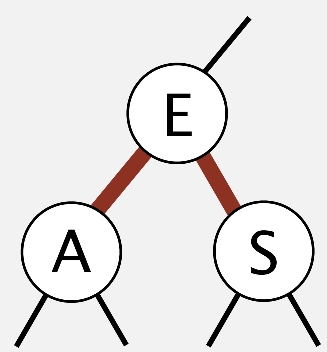


right–leaning red link



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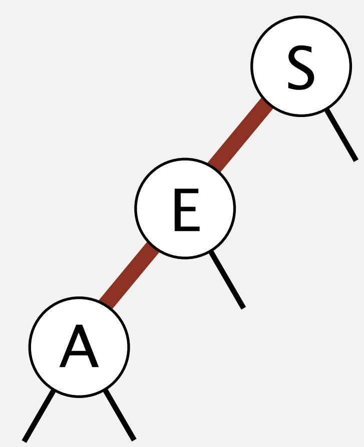
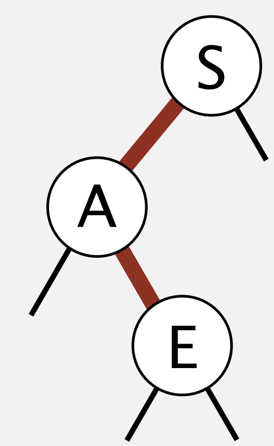


left–left red (a temporary 4–node)



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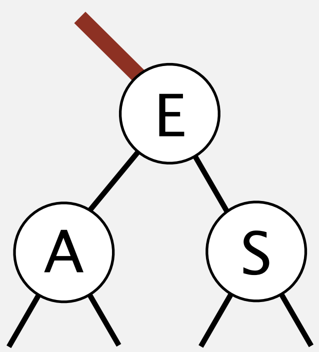

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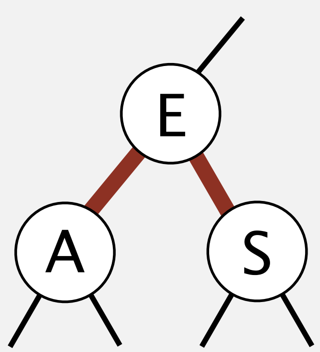

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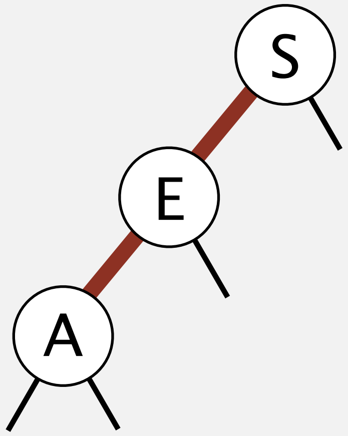

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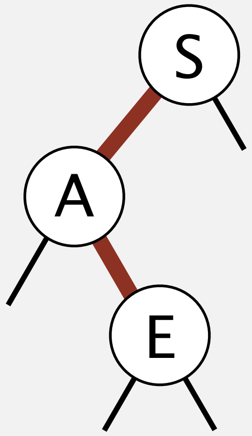

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How? Apply elementary red-black BST operations: rotation and color flip.

Insertion in a LLRB tree

Warmup 1. Insert into a tree with exactly 1 node.

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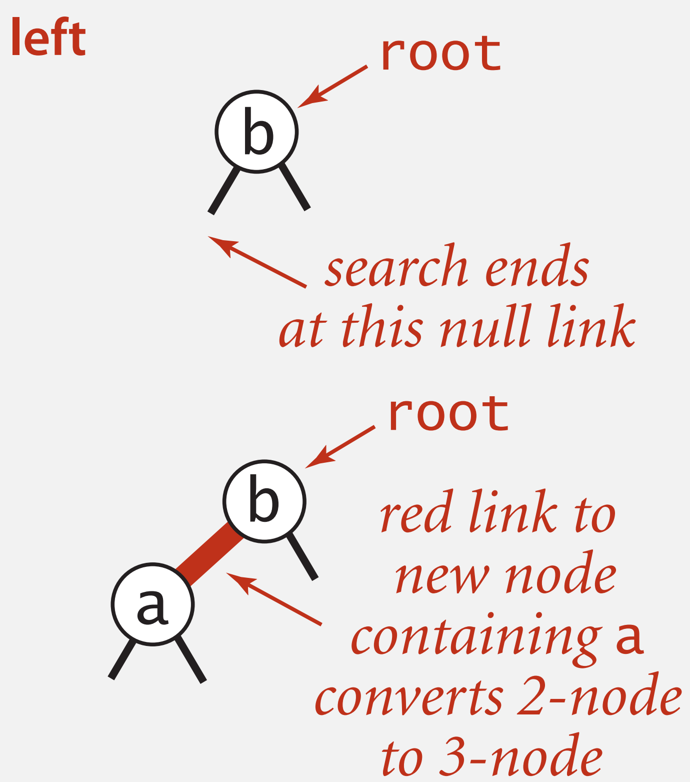

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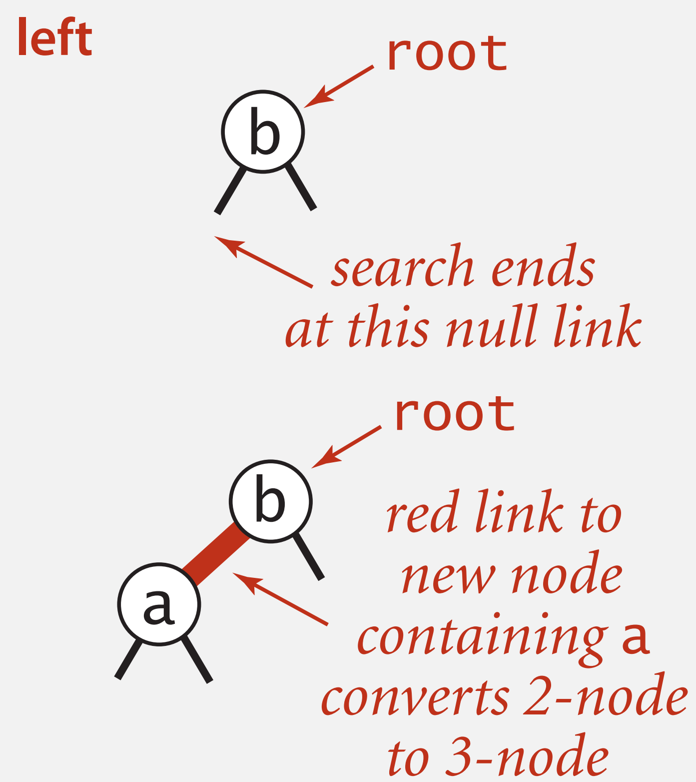

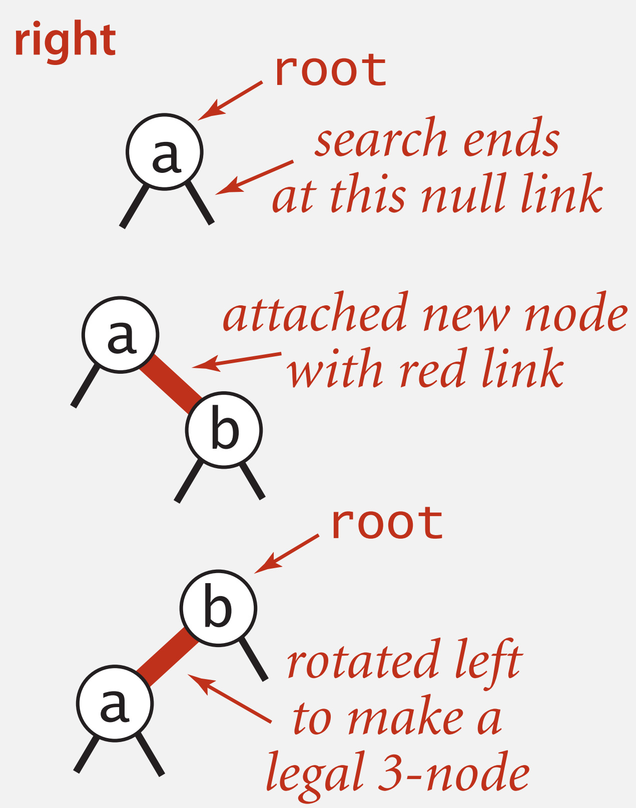

Insertion in a LLRB tree

Case 1. Insert into a 2-node at the bottom.

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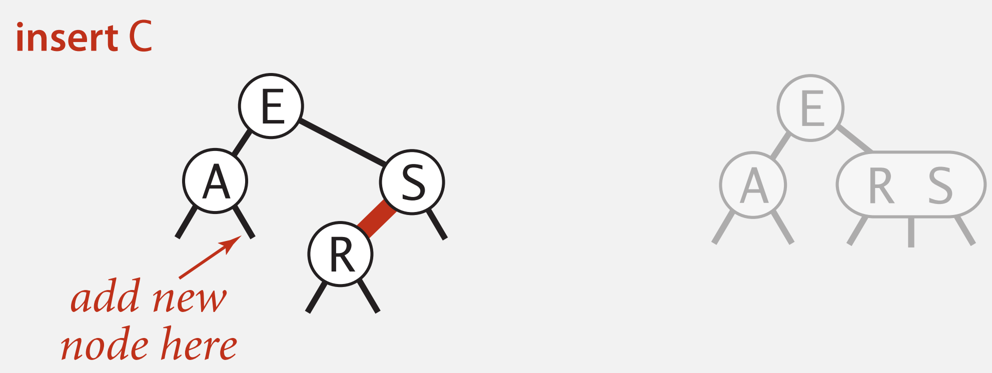

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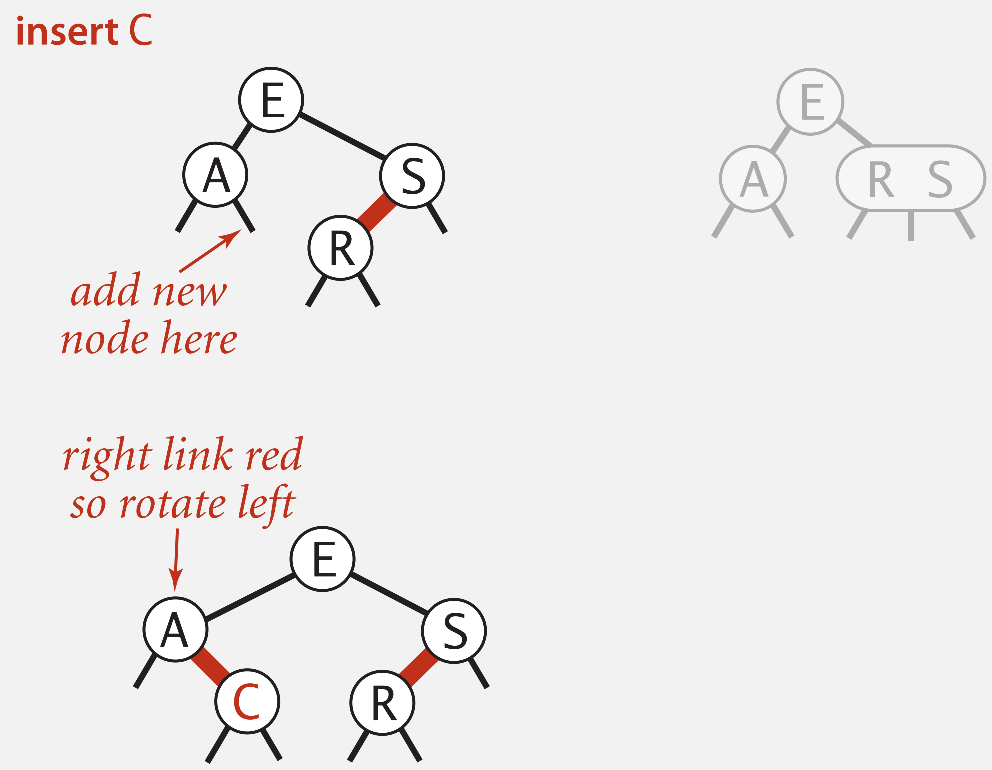

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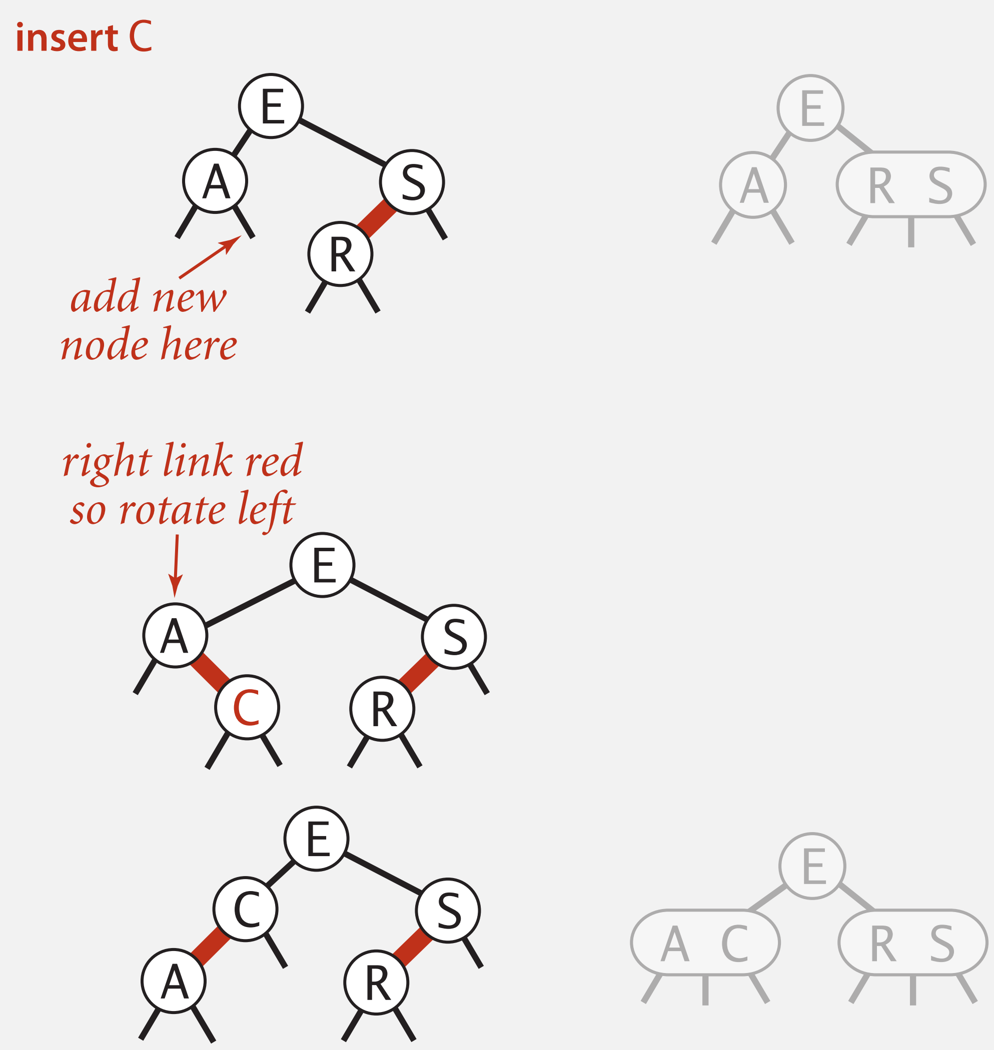

Insertion in a LLRB tree

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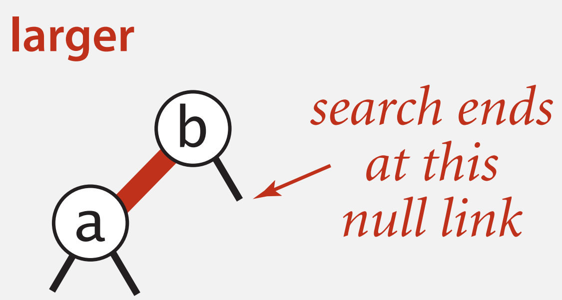
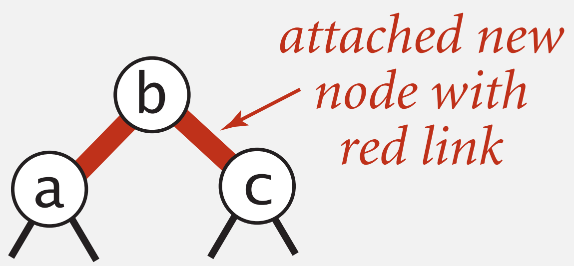
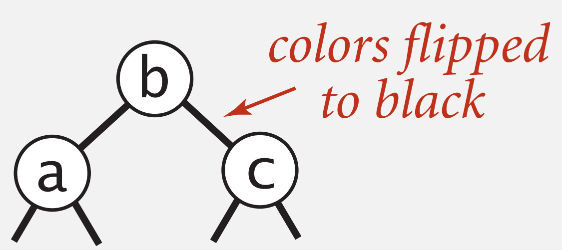


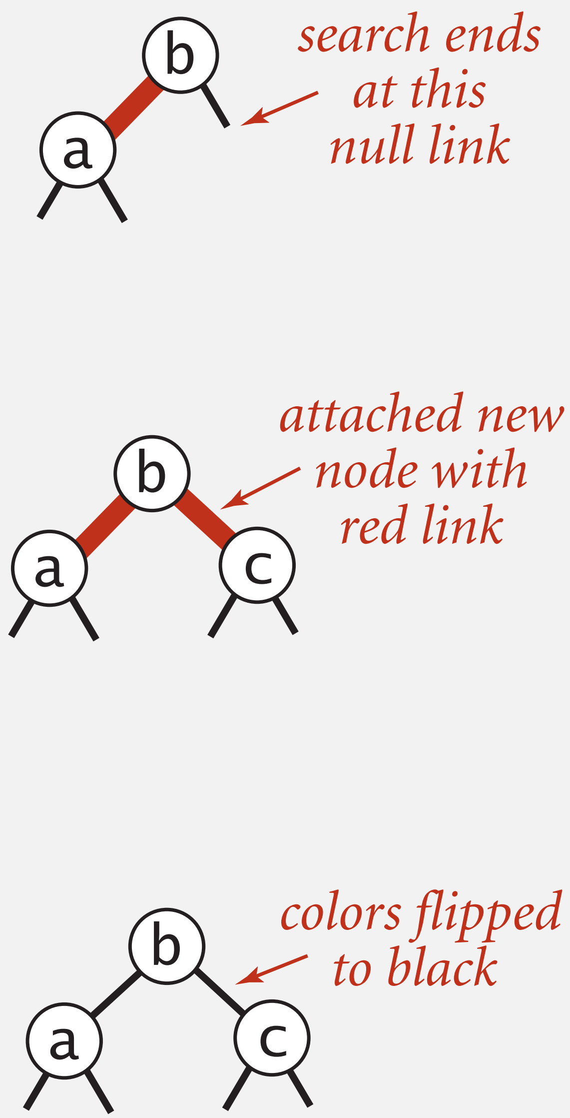




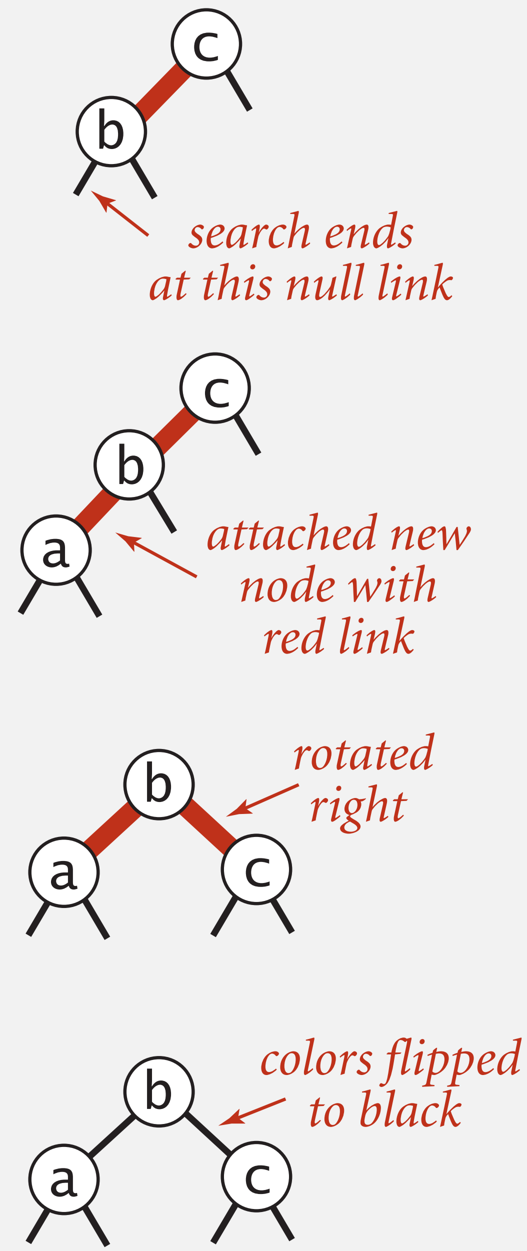
Insertion in a LLRB tree

Warmup 2. Insert into a tree with exactly 2 nodes.

larger



smaller



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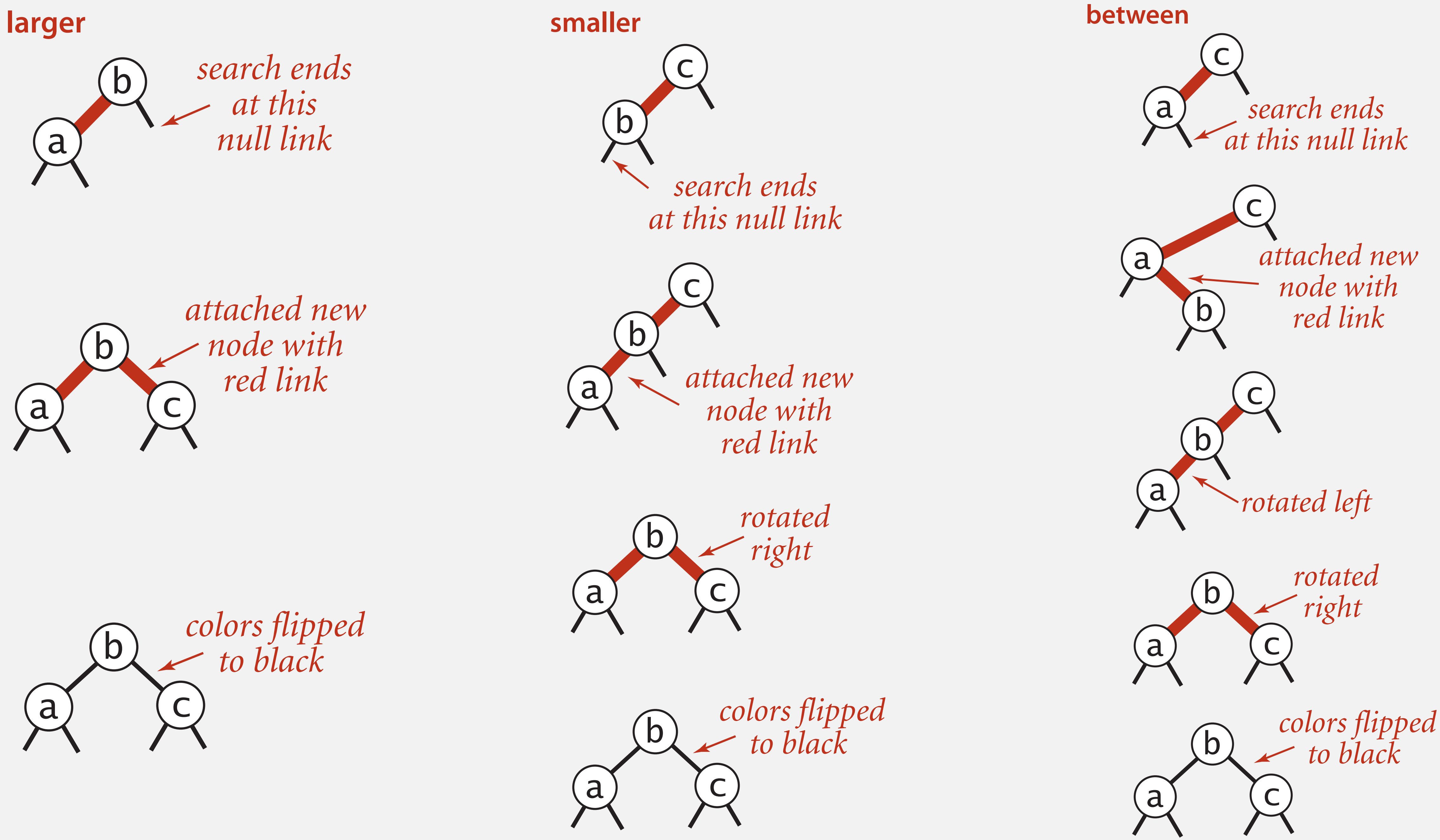

Insertion in a LLRB tree

Case 2. Insert into a 3-node at the bottom.

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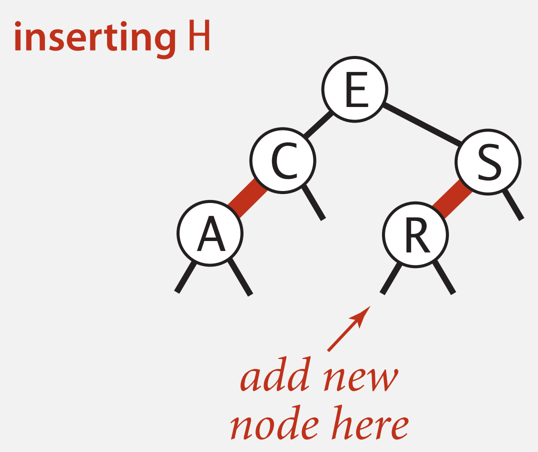
to maintain symmetric order and perfect black balance

to fix color invariants



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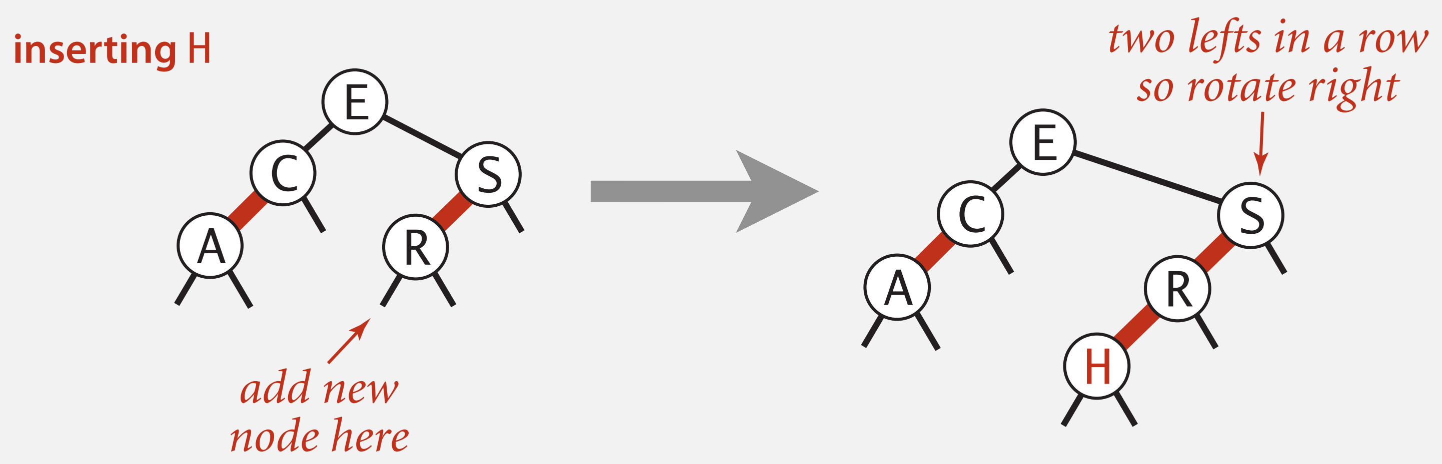

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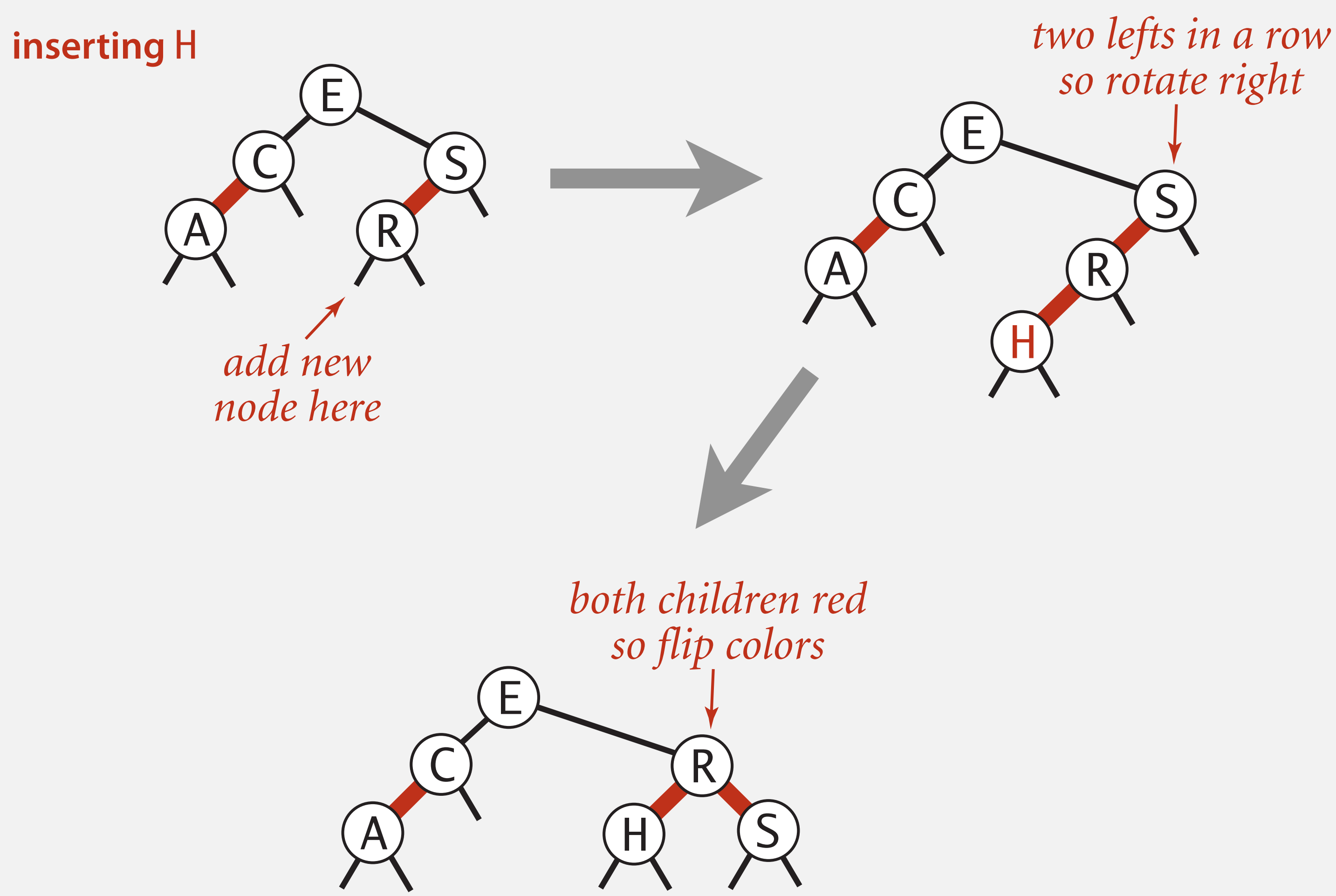

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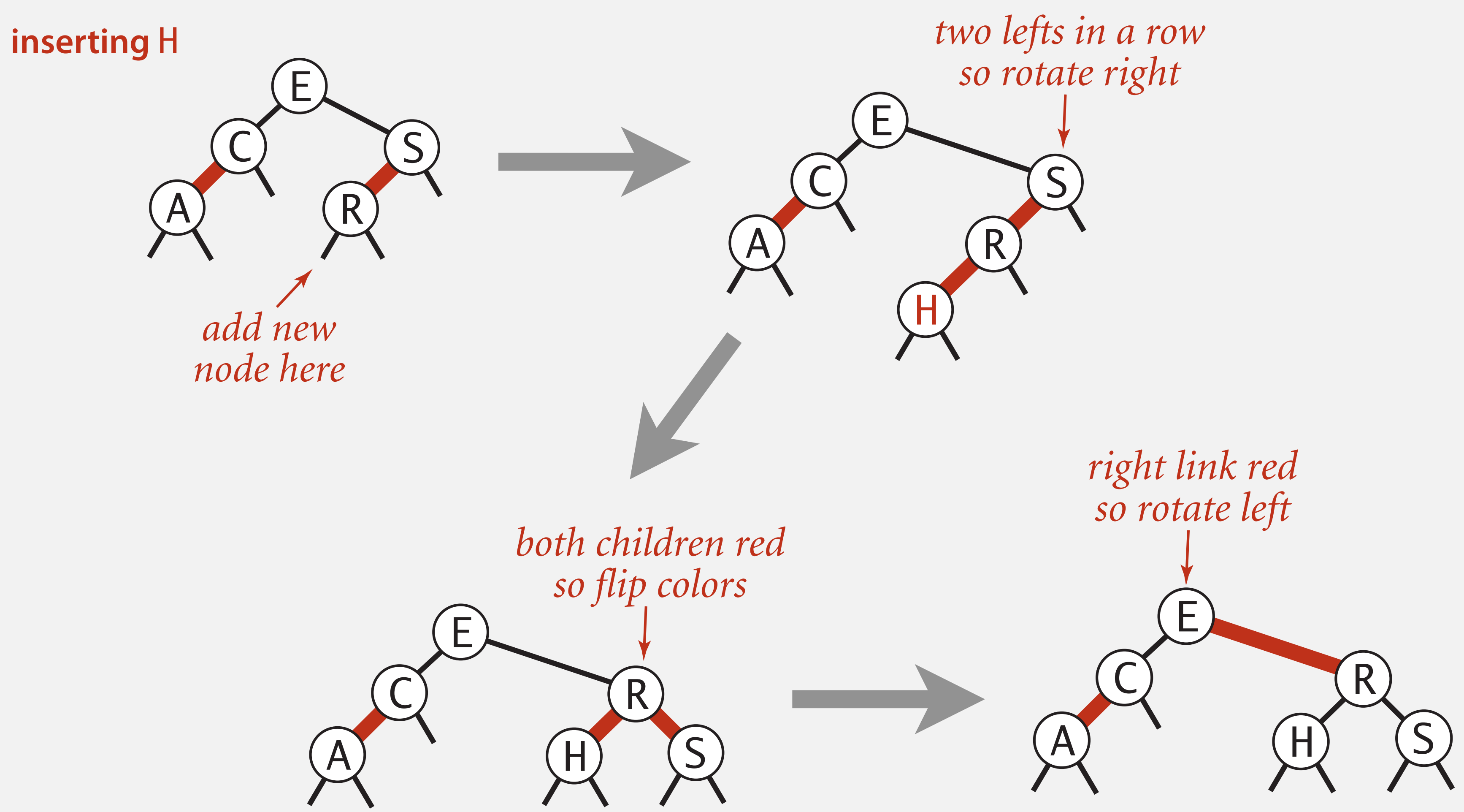

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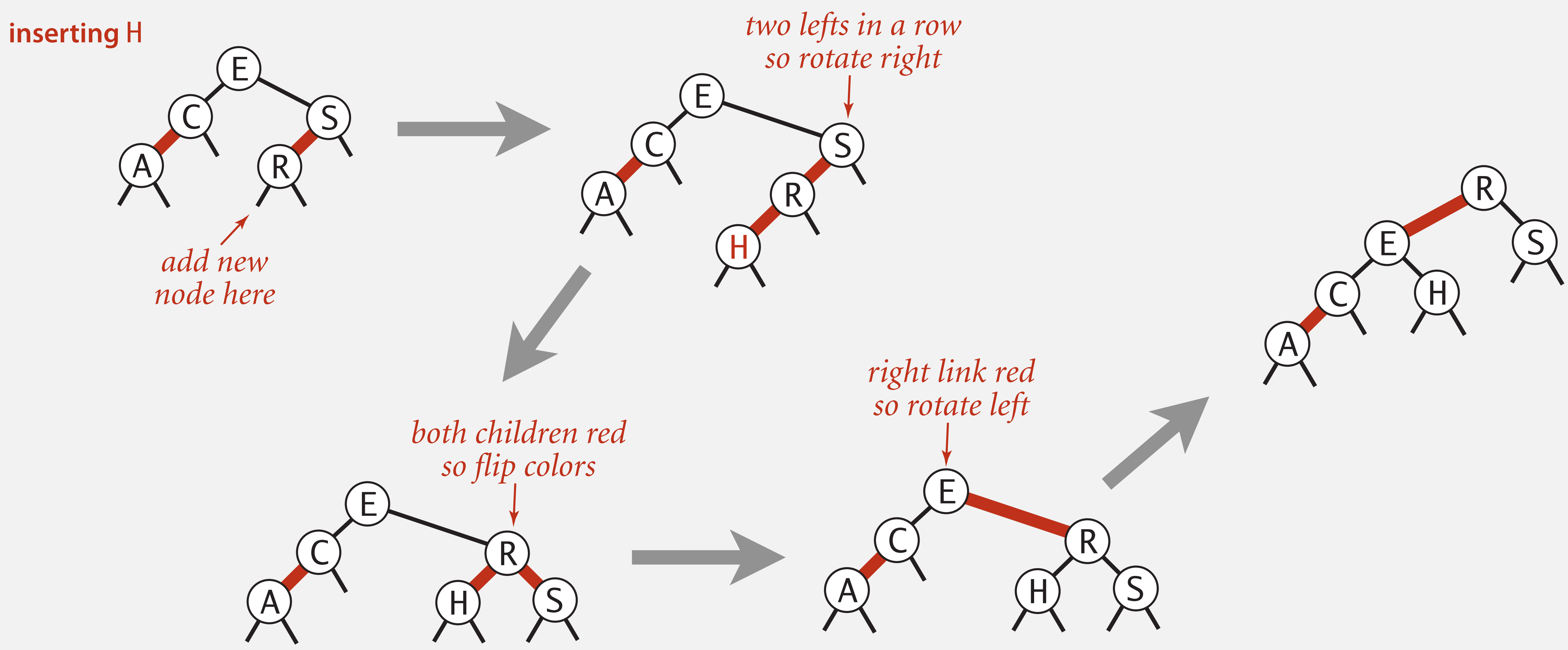

Insertion in a LLRB tree: passing red links up the tree

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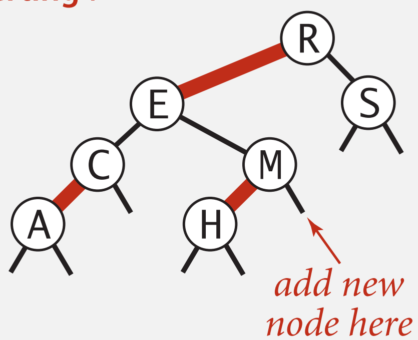
inserting P



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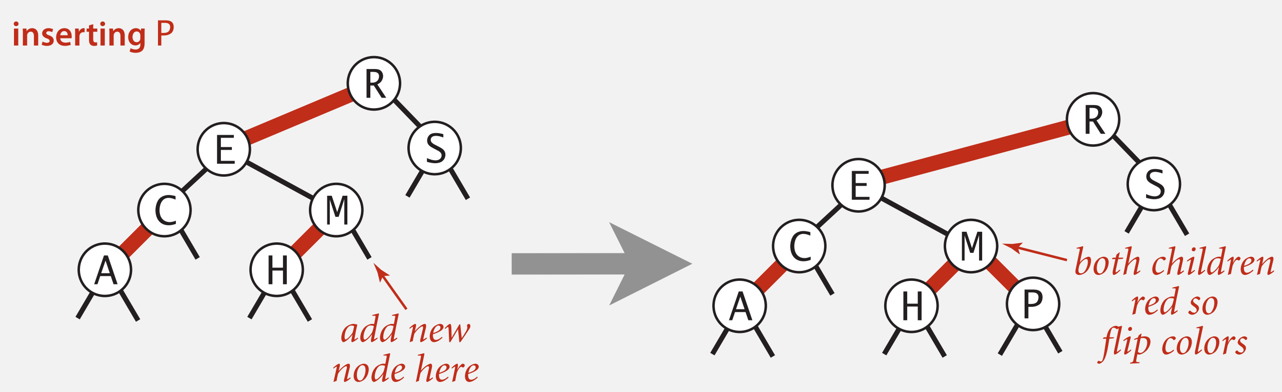

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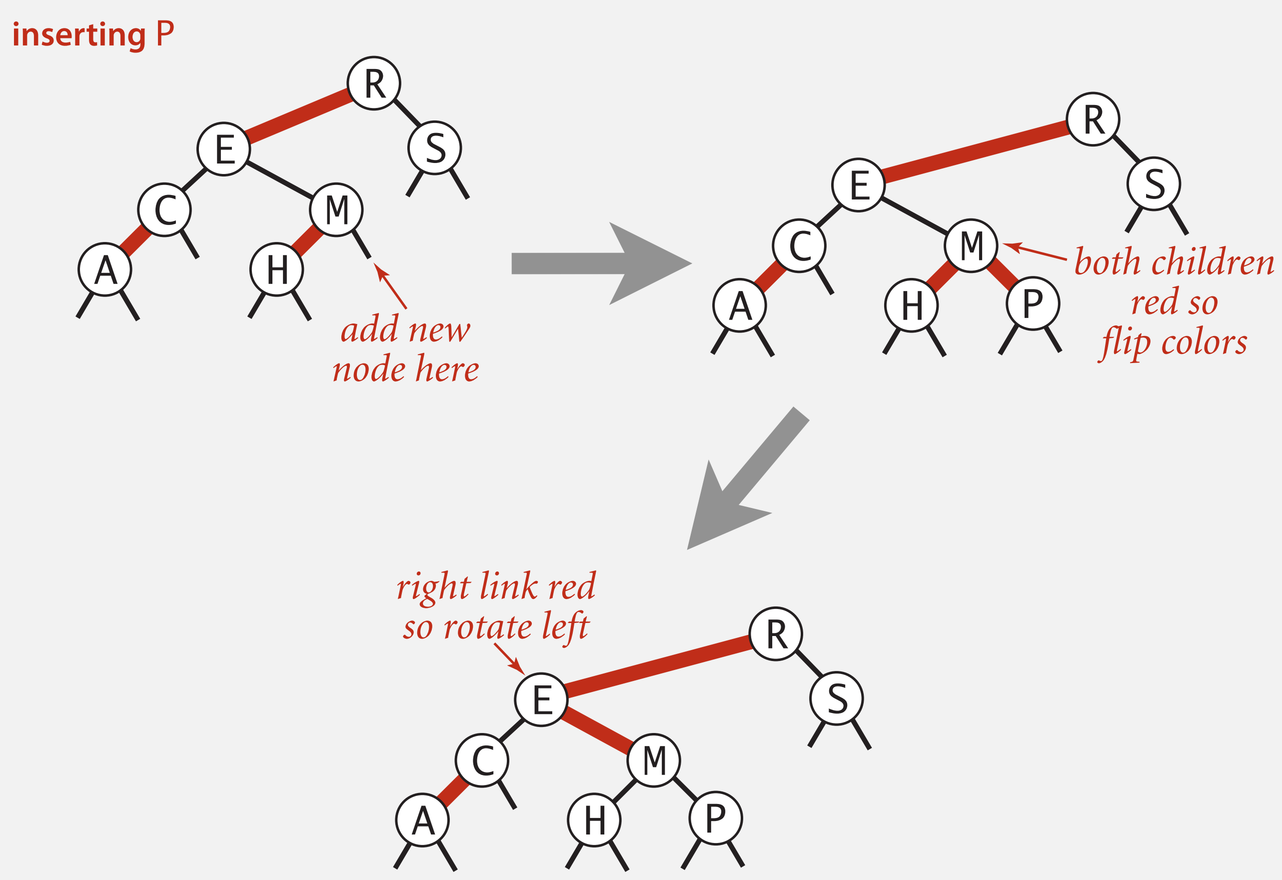

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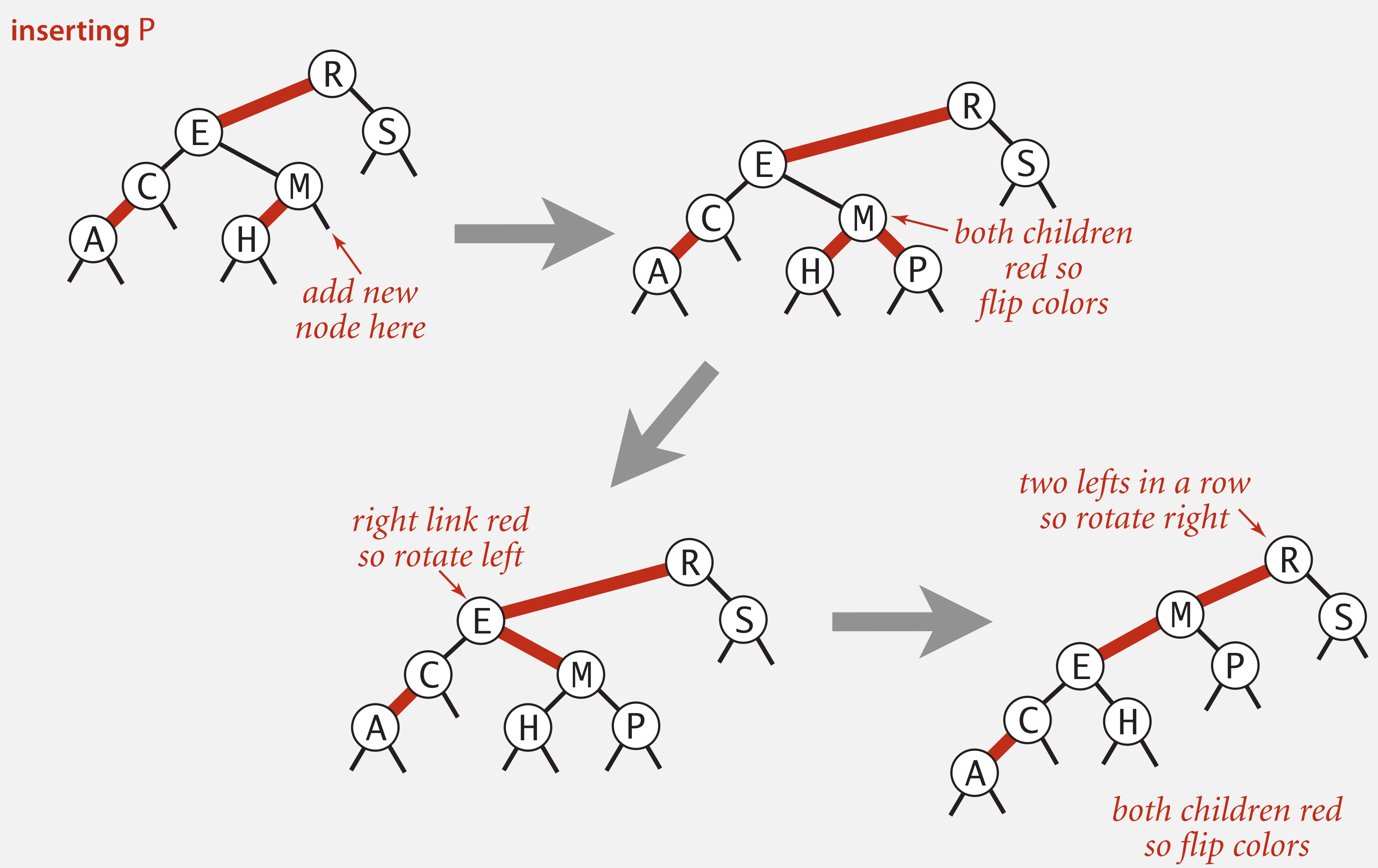

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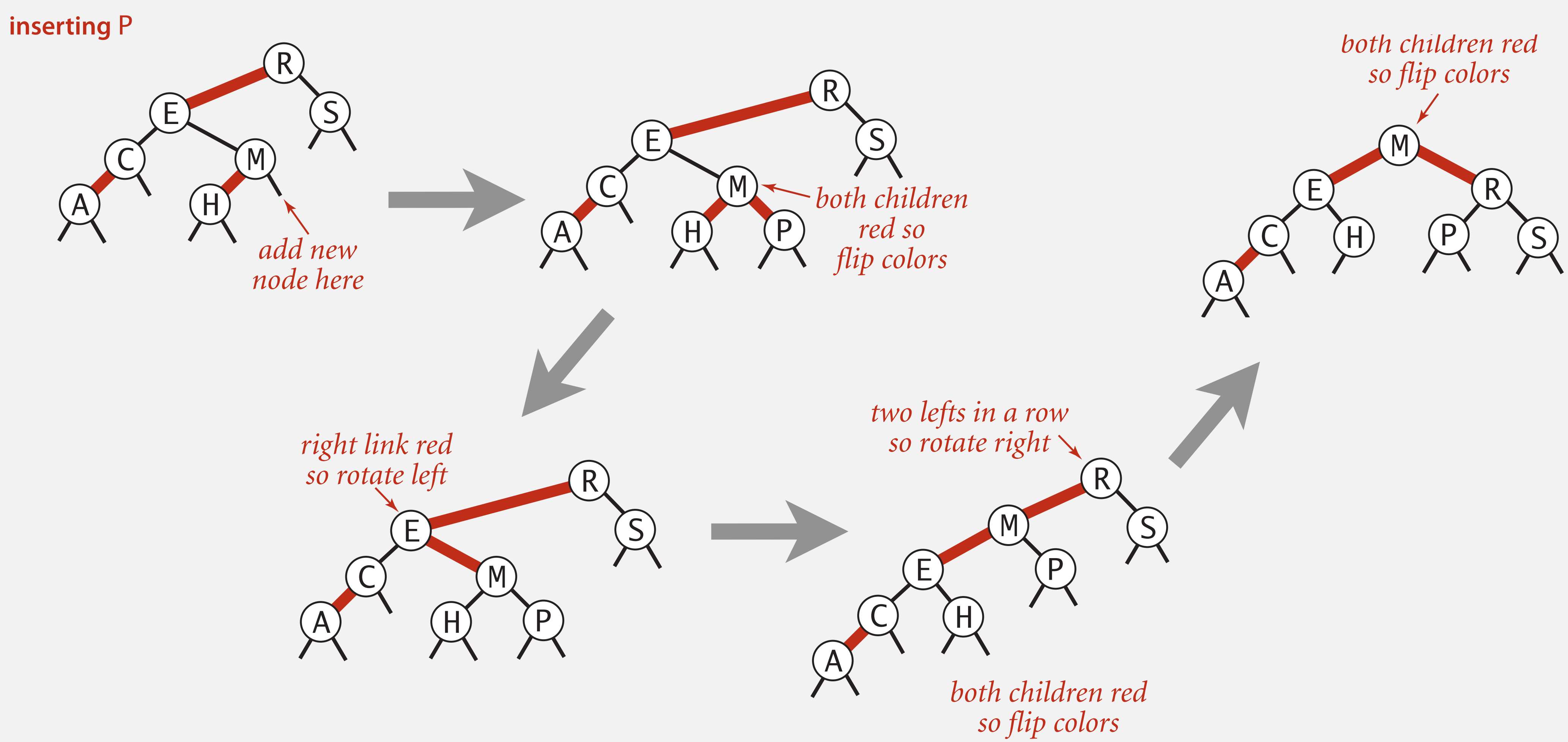

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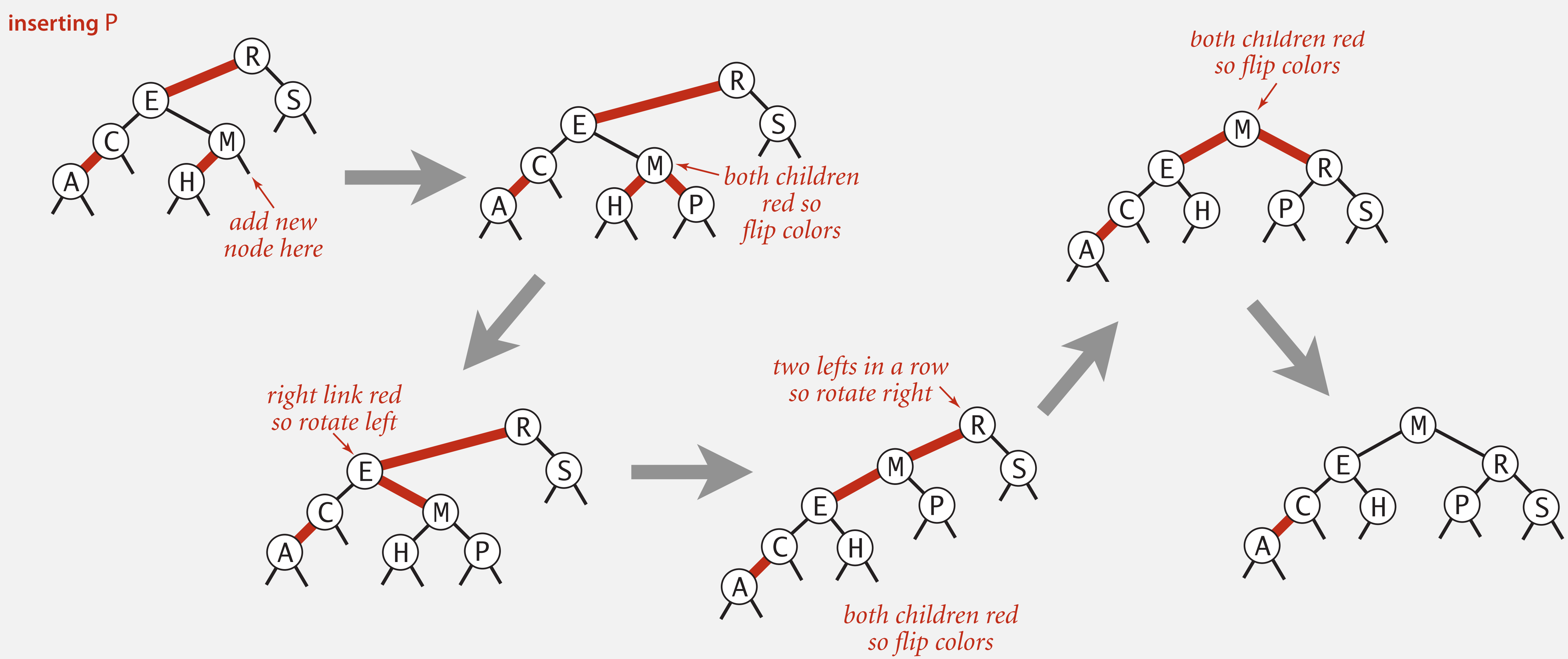

Red-black BST construction demo

insert S

Red-black BST construction demo

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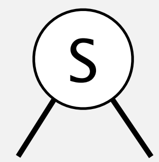
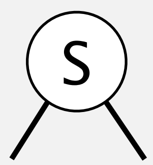


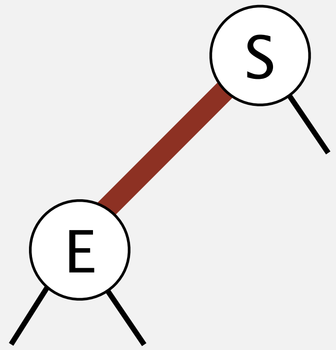
Red-black BST construction demo

insert E



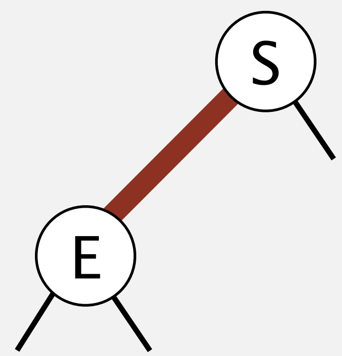
Red-black BST construction demo

insert E



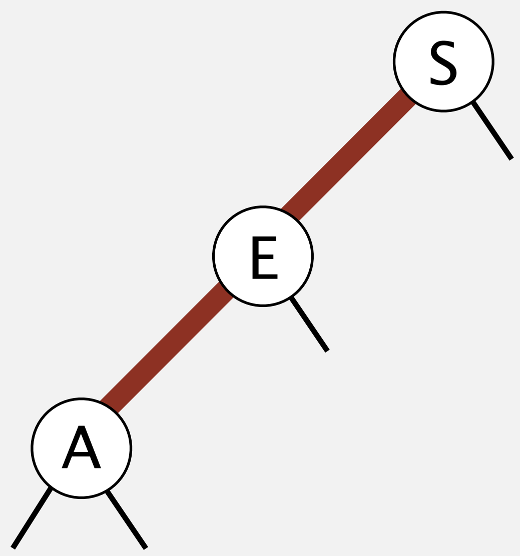
Red-black BST construction demo

insert A



Red-black BST construction demo

insert A

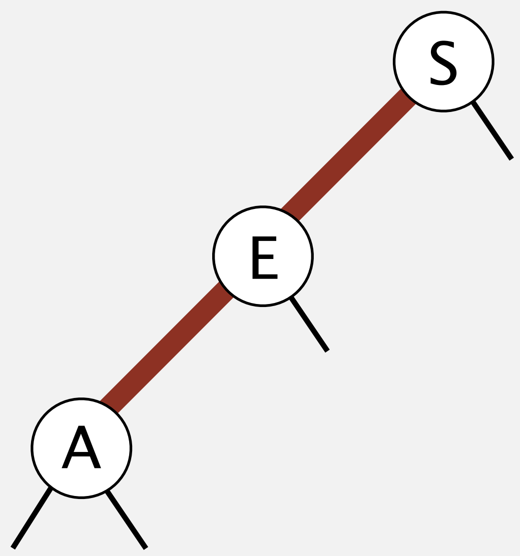


Red-black BST construction demo

insert A

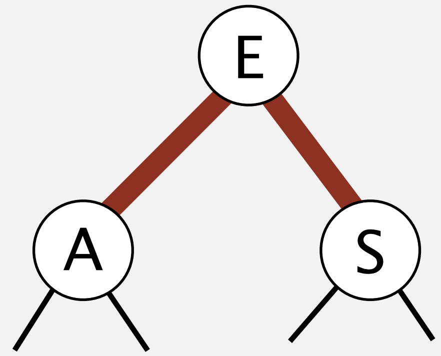
two left reds in a row

(rotate S right)



Red-black BST construction demo

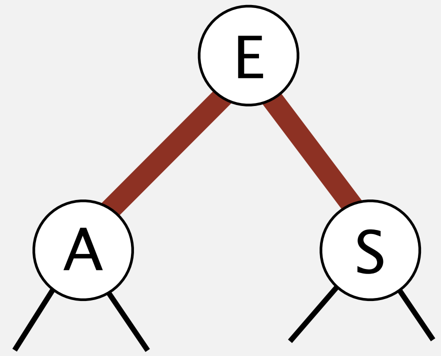
both children red
(flip colors)



Red-black BST construction demo

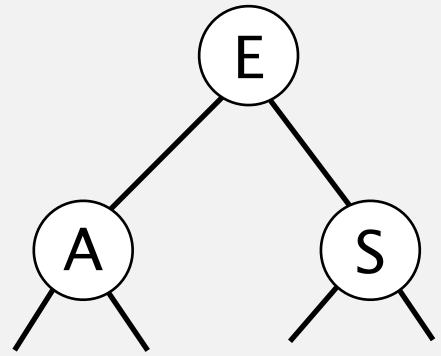
both children red

(flip colors)



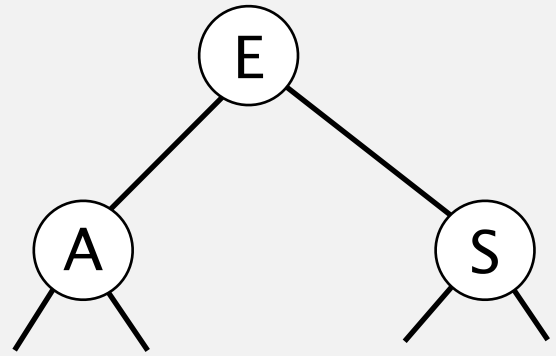
Red-black BST construction demo

red-black BST



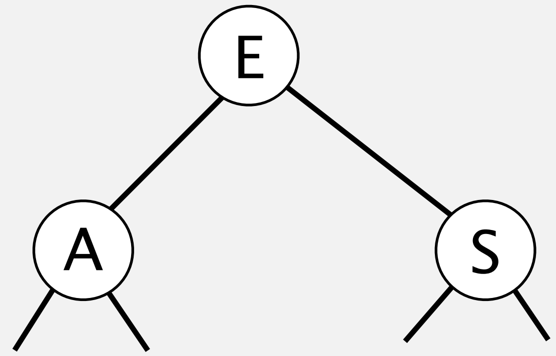
Red-black BST construction demo

red-black BST



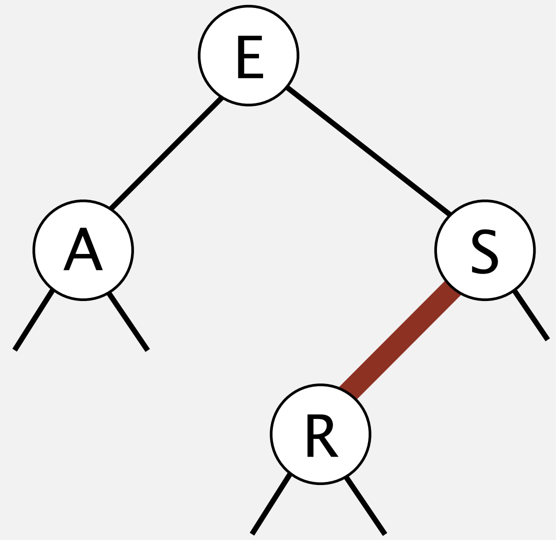
Red-black BST construction demo

insert R



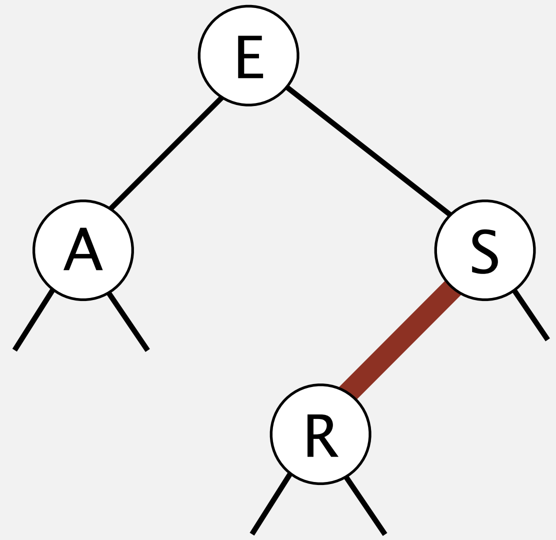
Red-black BST construction demo

insert R



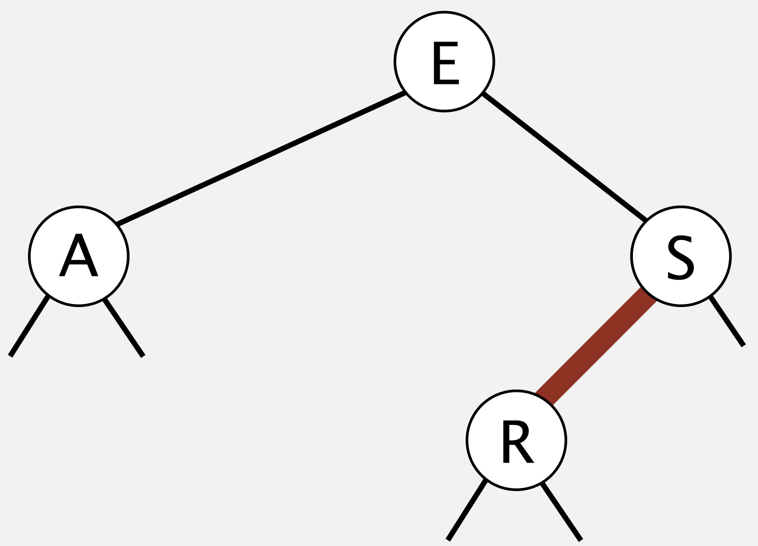
Red-black BST construction demo

red–black BST



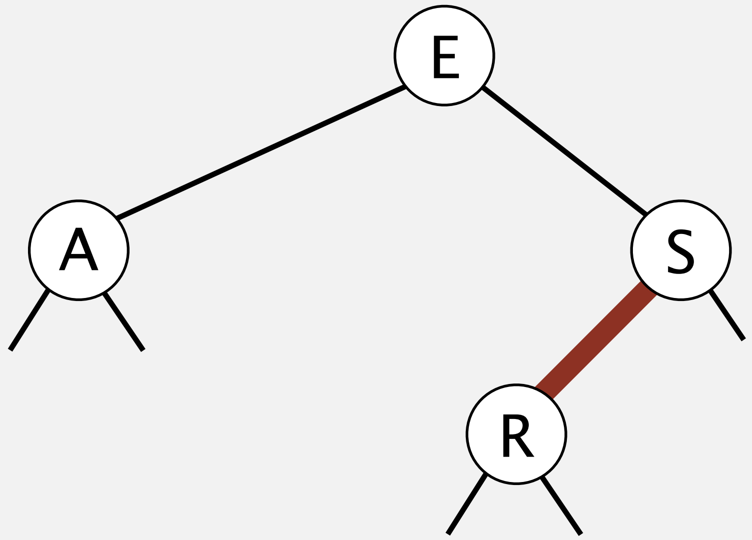
Red-black BST construction demo

red–black BST



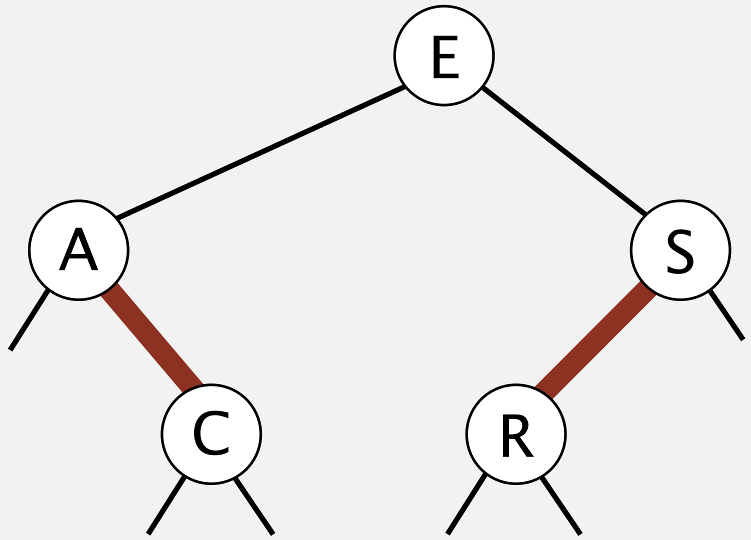
Red-black BST construction demo

insert C



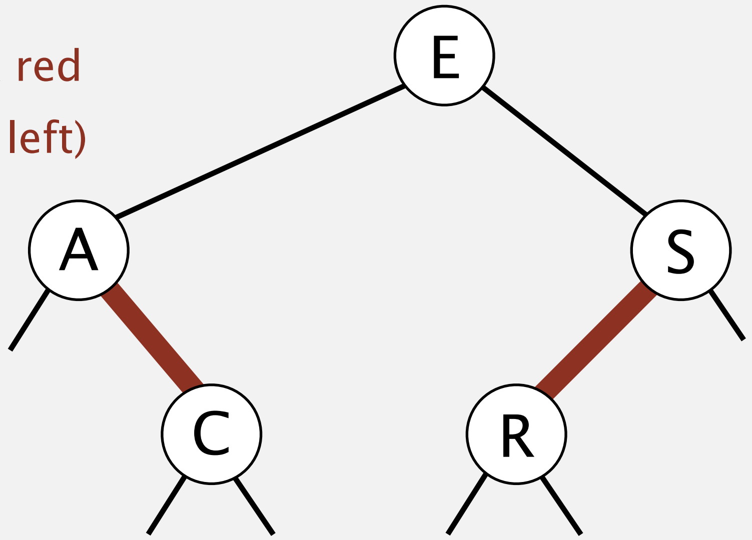
Red-black BST construction demo

insert C



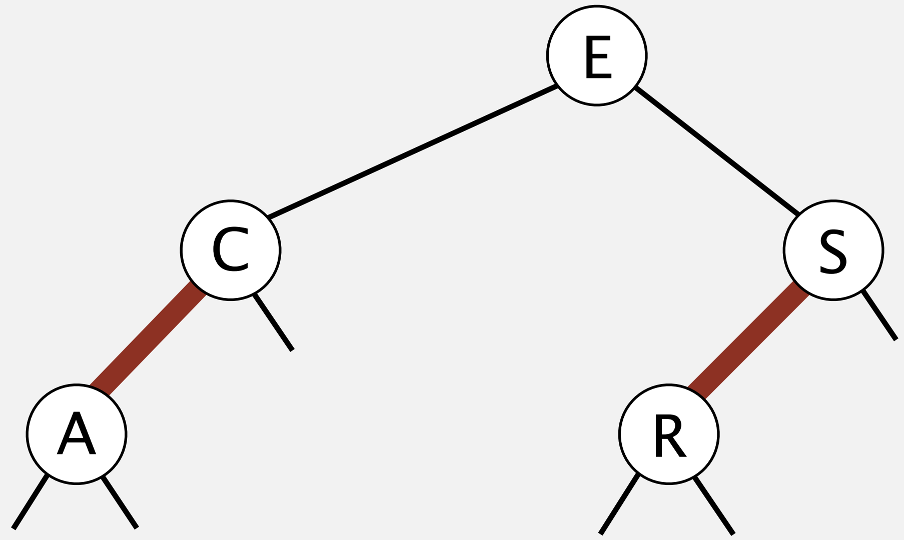
Red-black BST construction demo

right link red
(rotate A left)



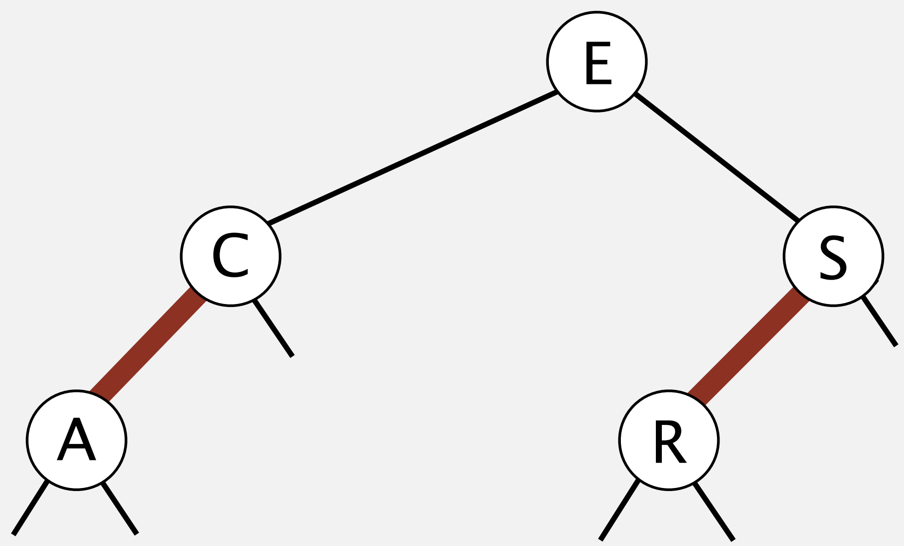
Red-black BST construction demo

red-black BST



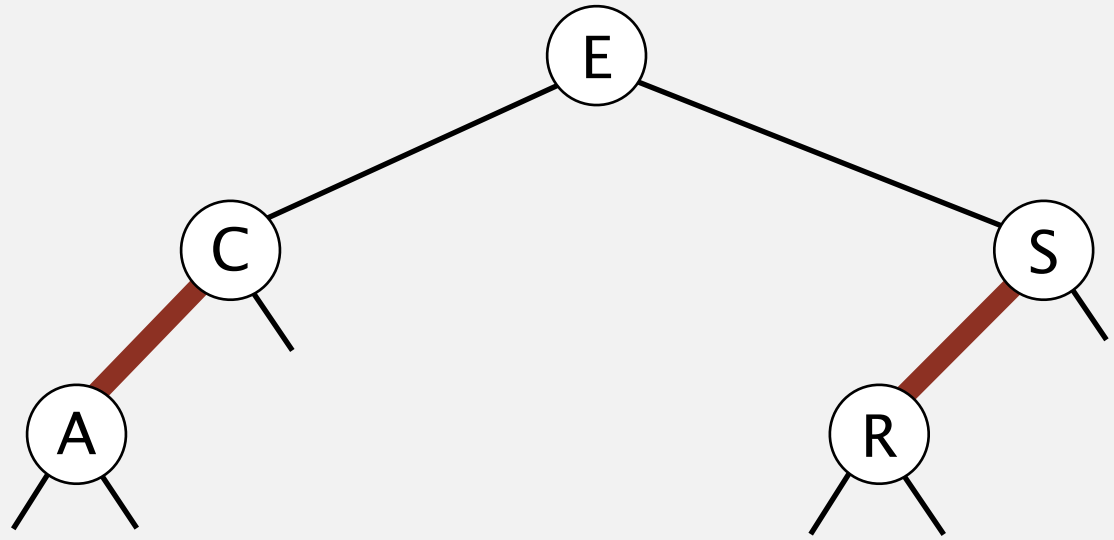
Red-black BST construction demo

red-black BST



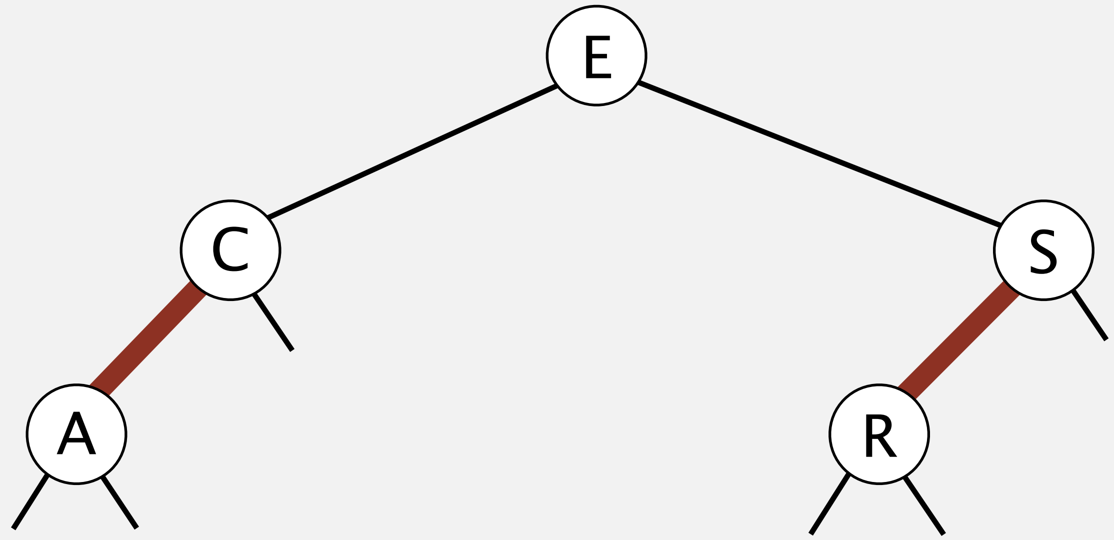
Red-black BST construction demo

red-black BST



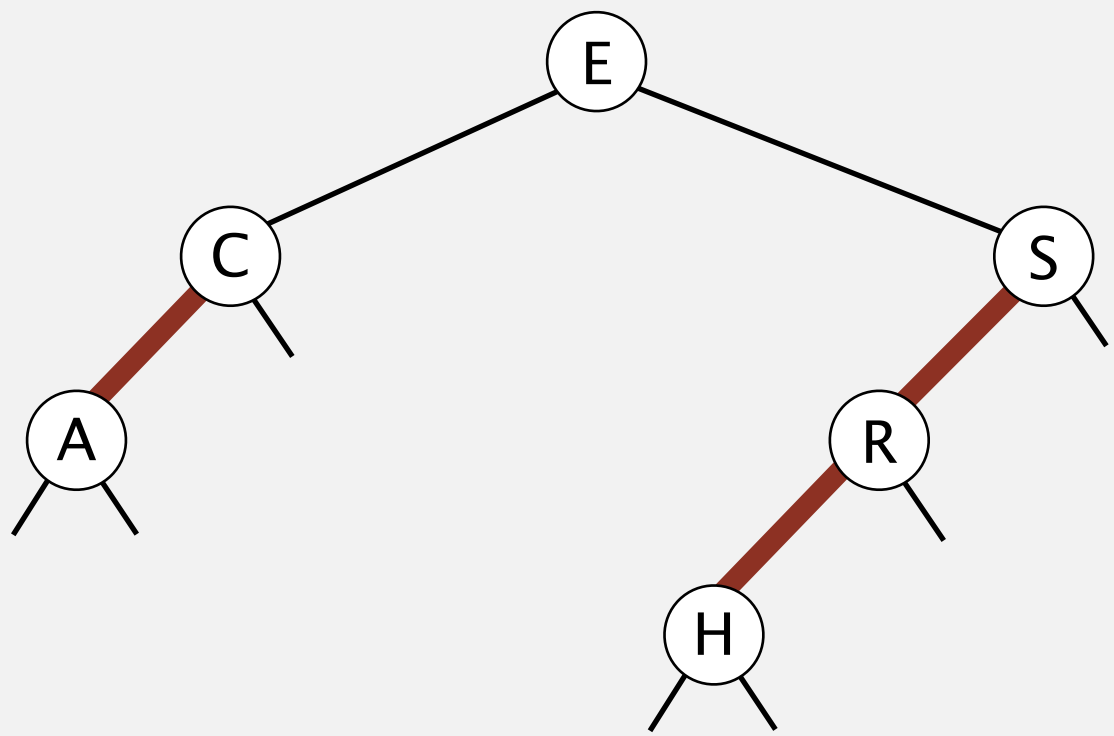
Red-black BST construction demo

insert H

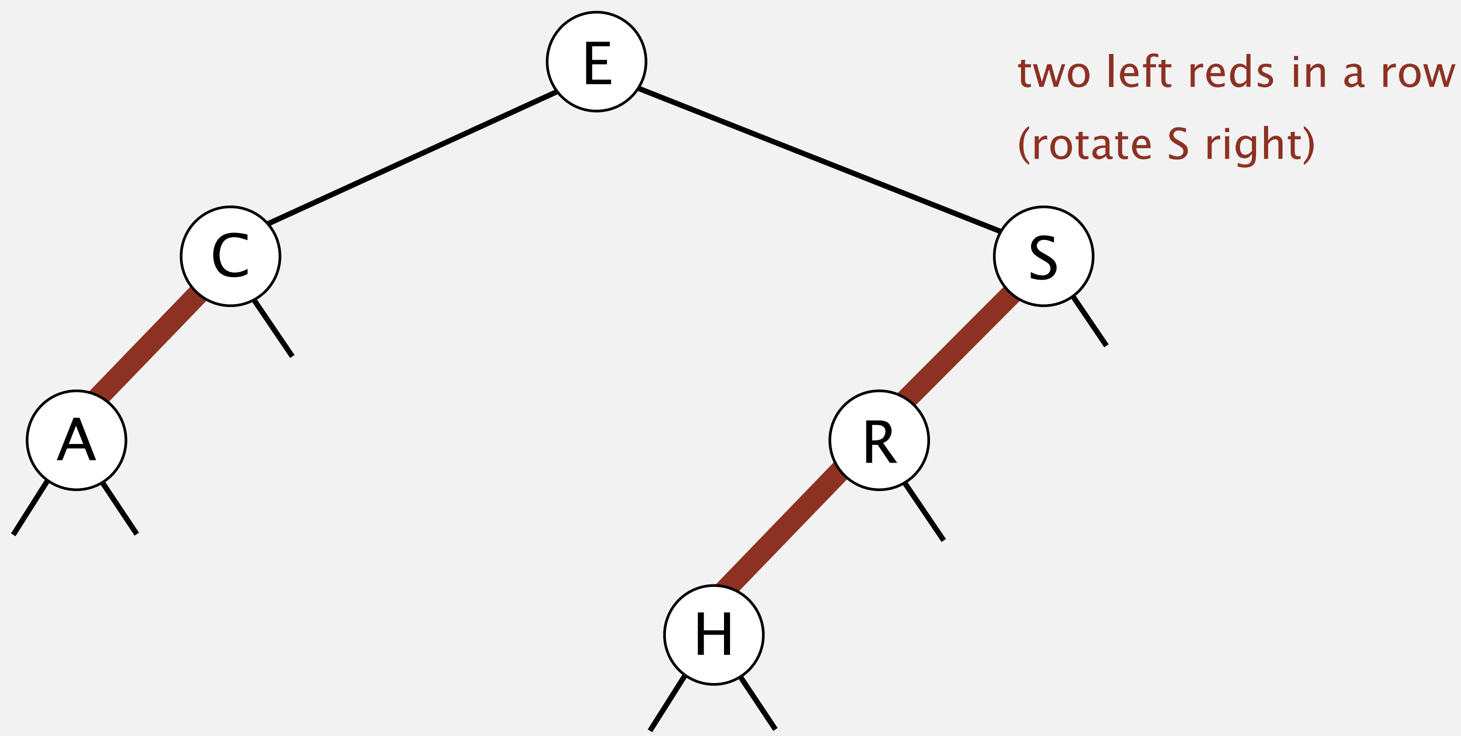


Red-black BST construction demo

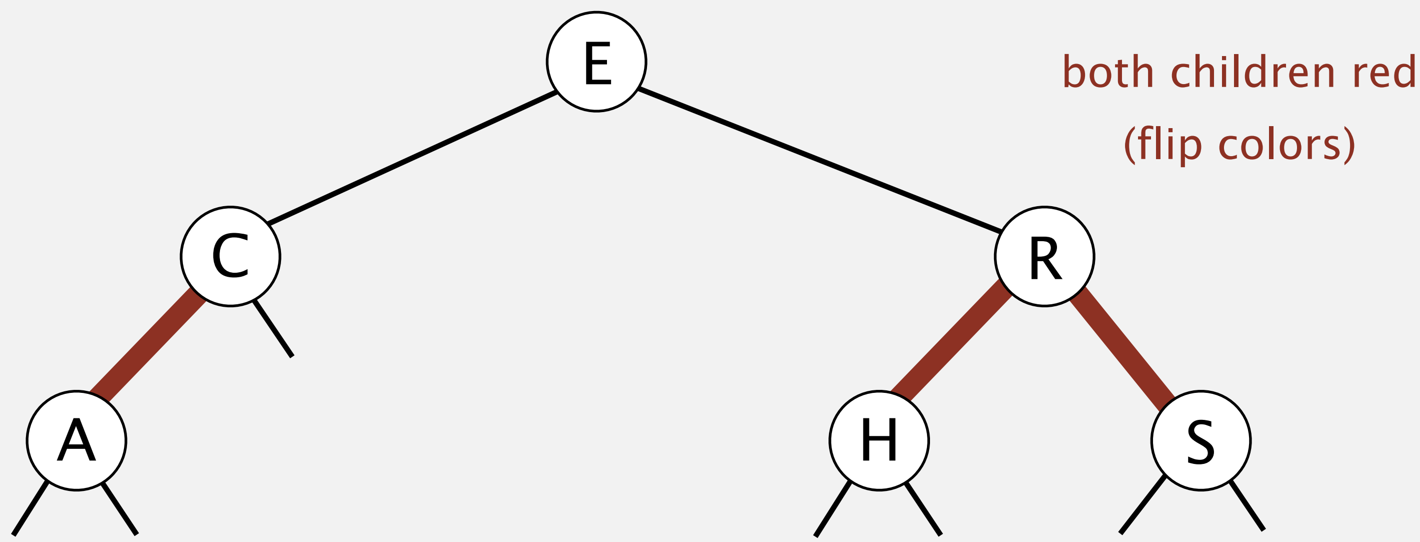
insert H



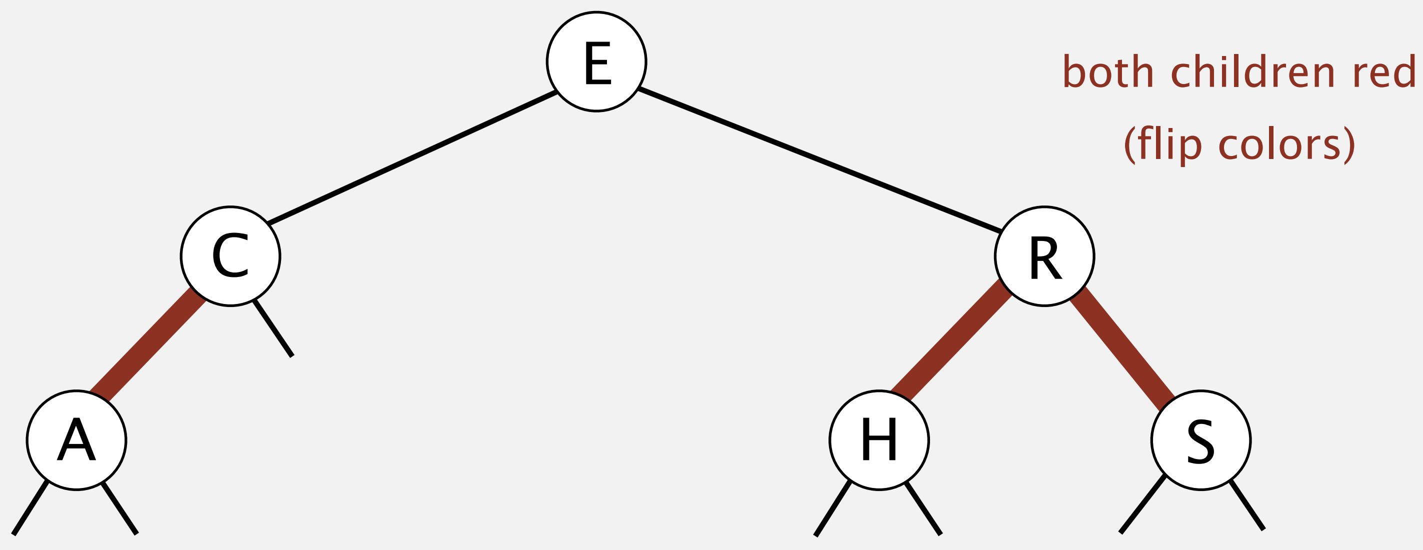
Red-black BST construction demo



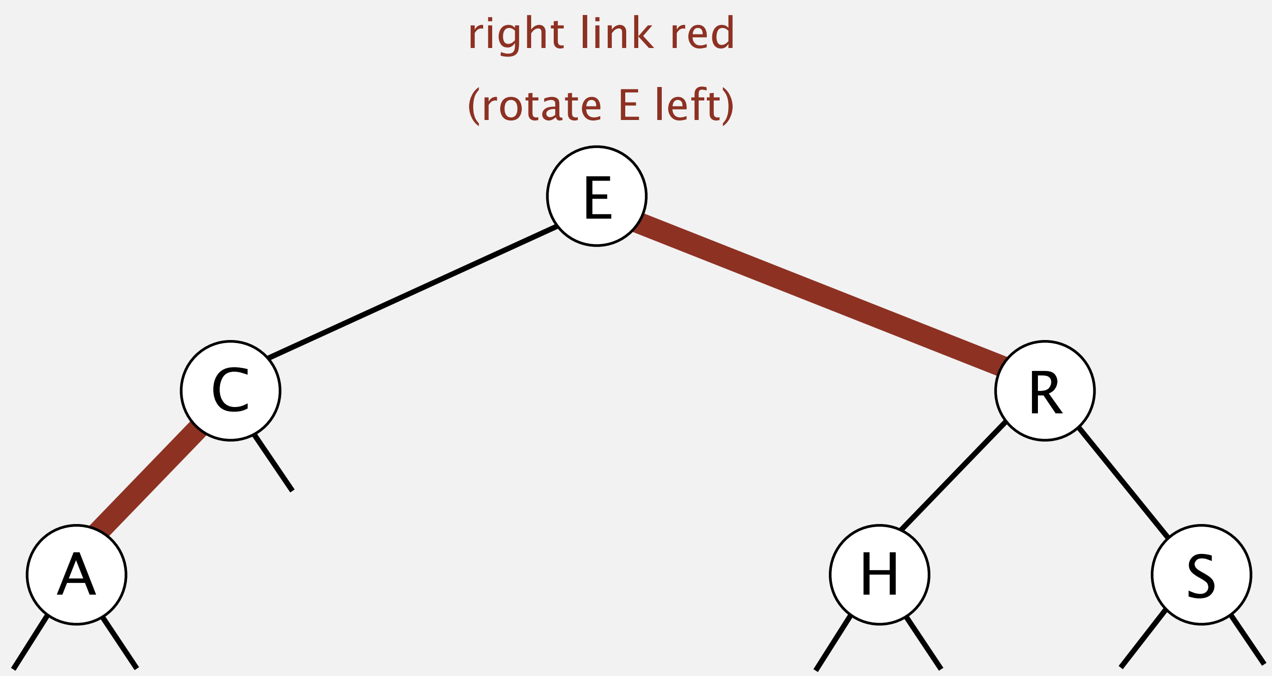
Red-black BST construction demo



Red-black BST construction demo

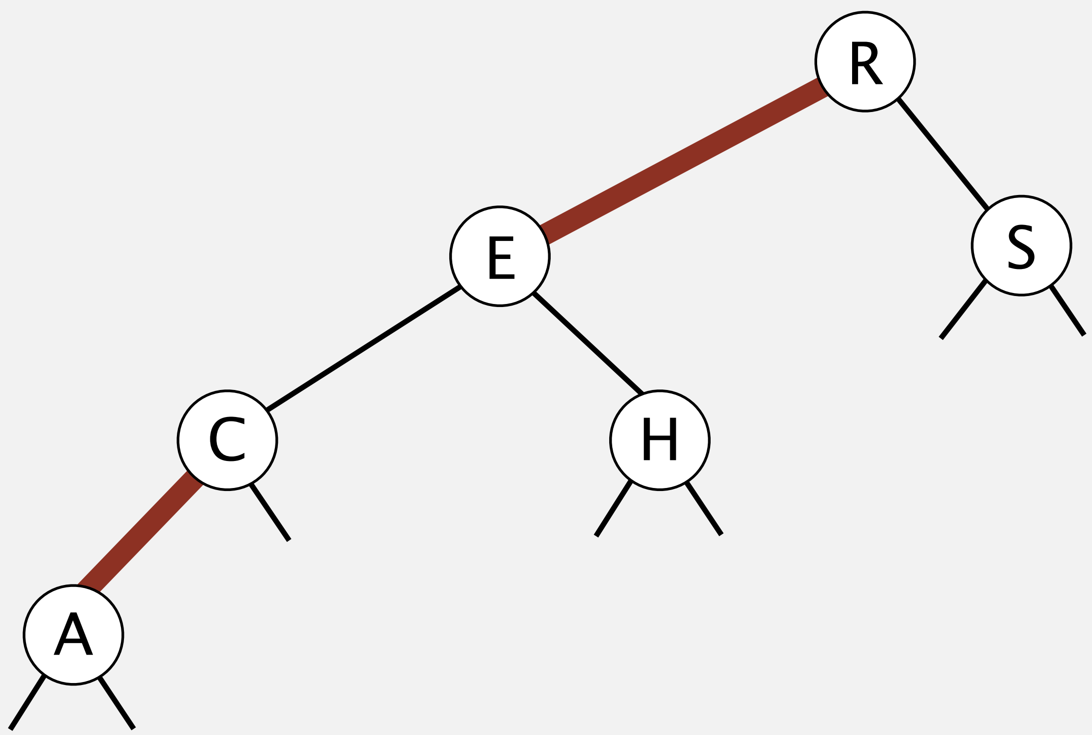


Red-black BST construction demo



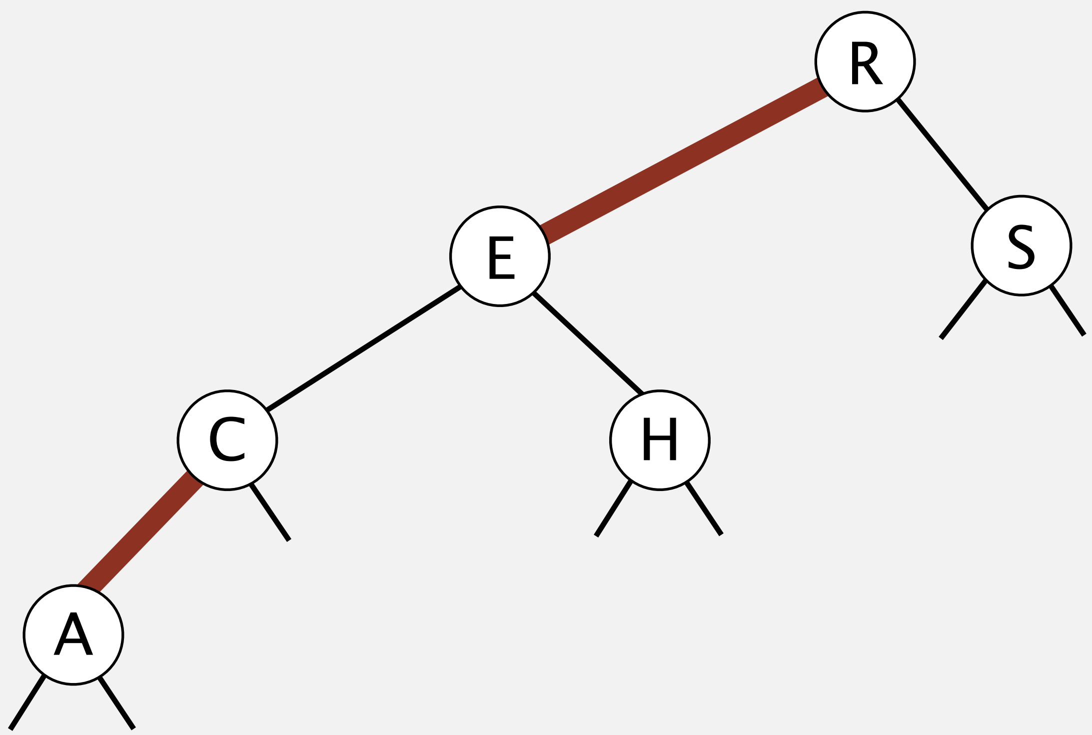
Red-black BST construction demo

red–black BST



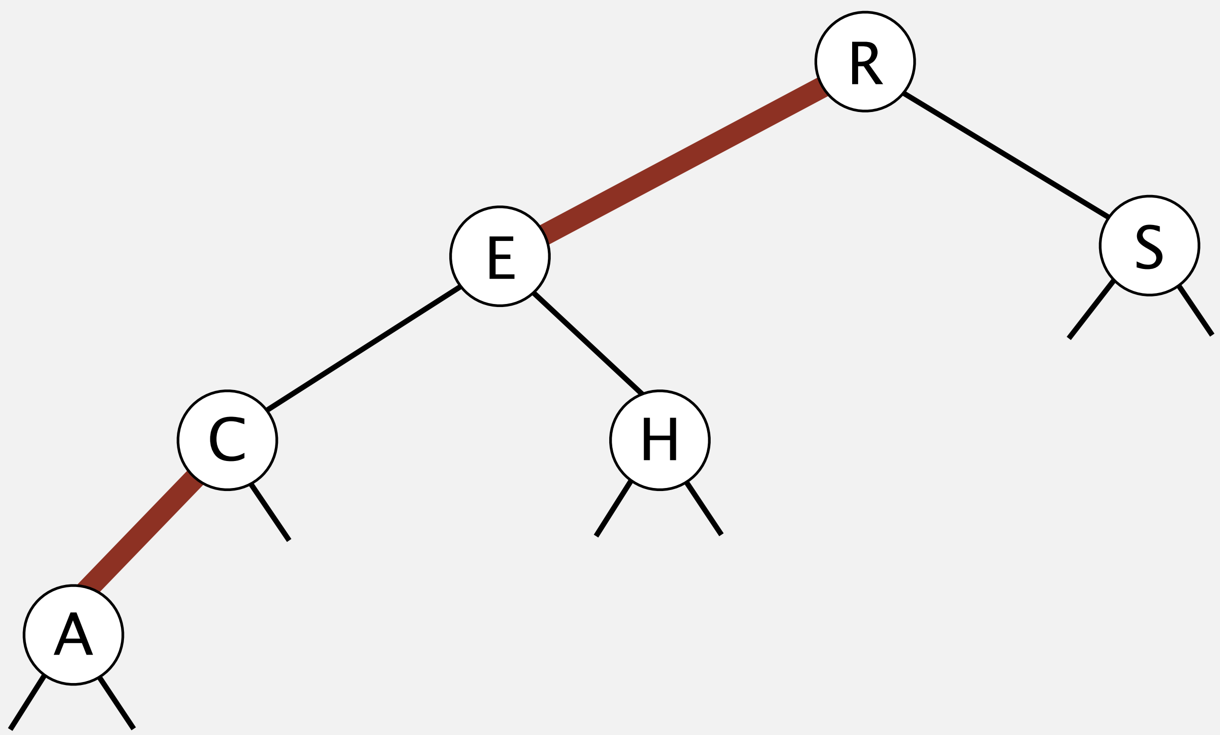
Red-black BST construction demo

red-black BST



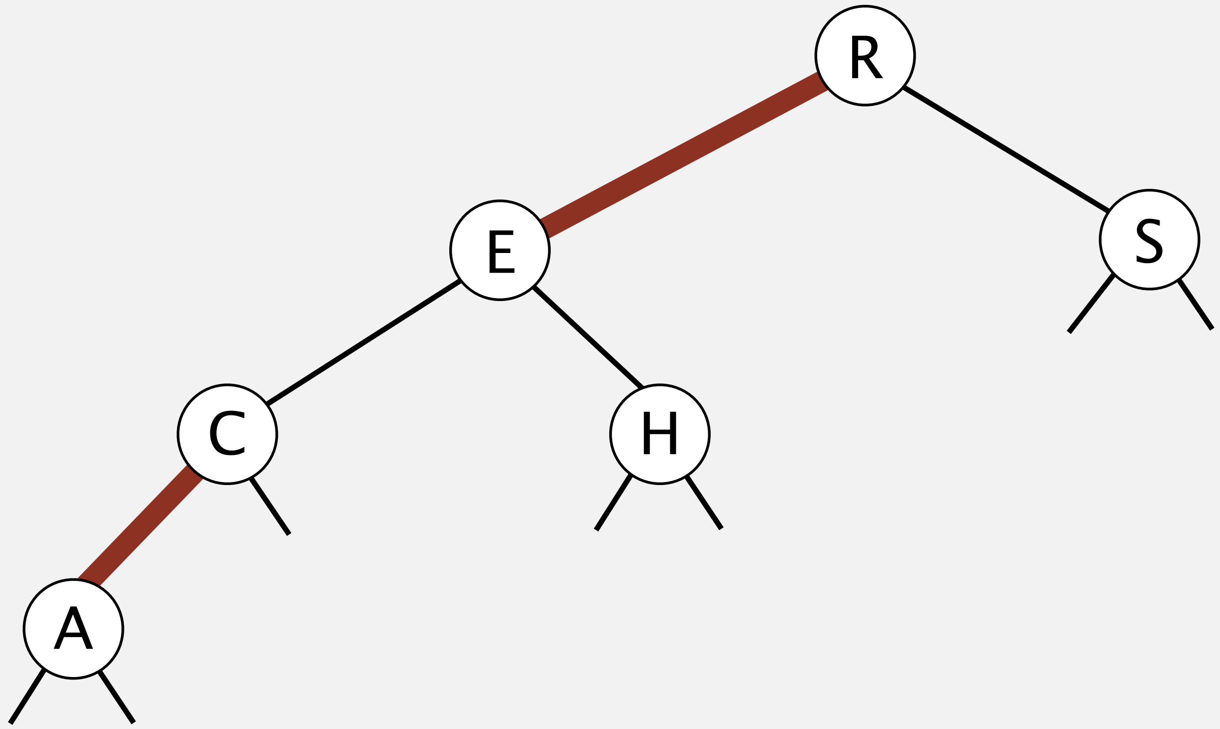
Red-black BST construction demo

red–black BST



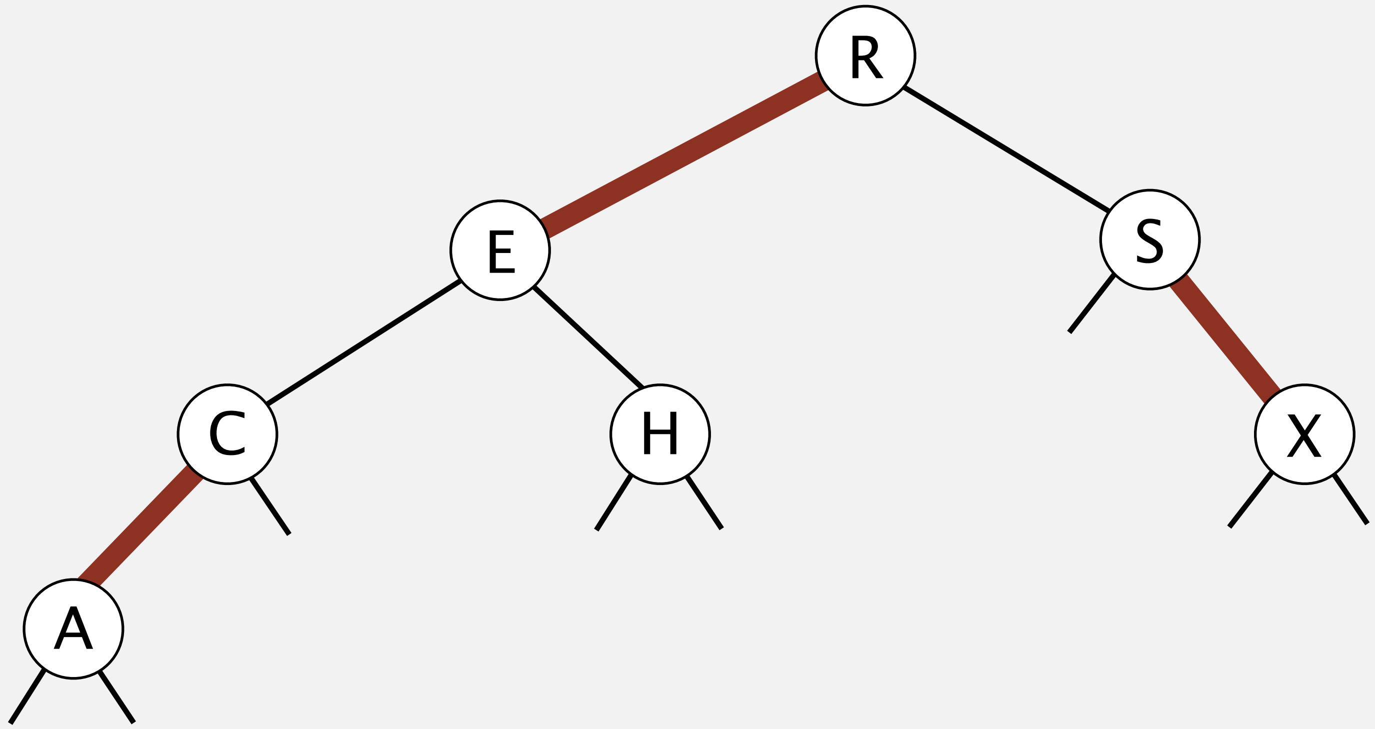
Red-black BST construction demo

insert X



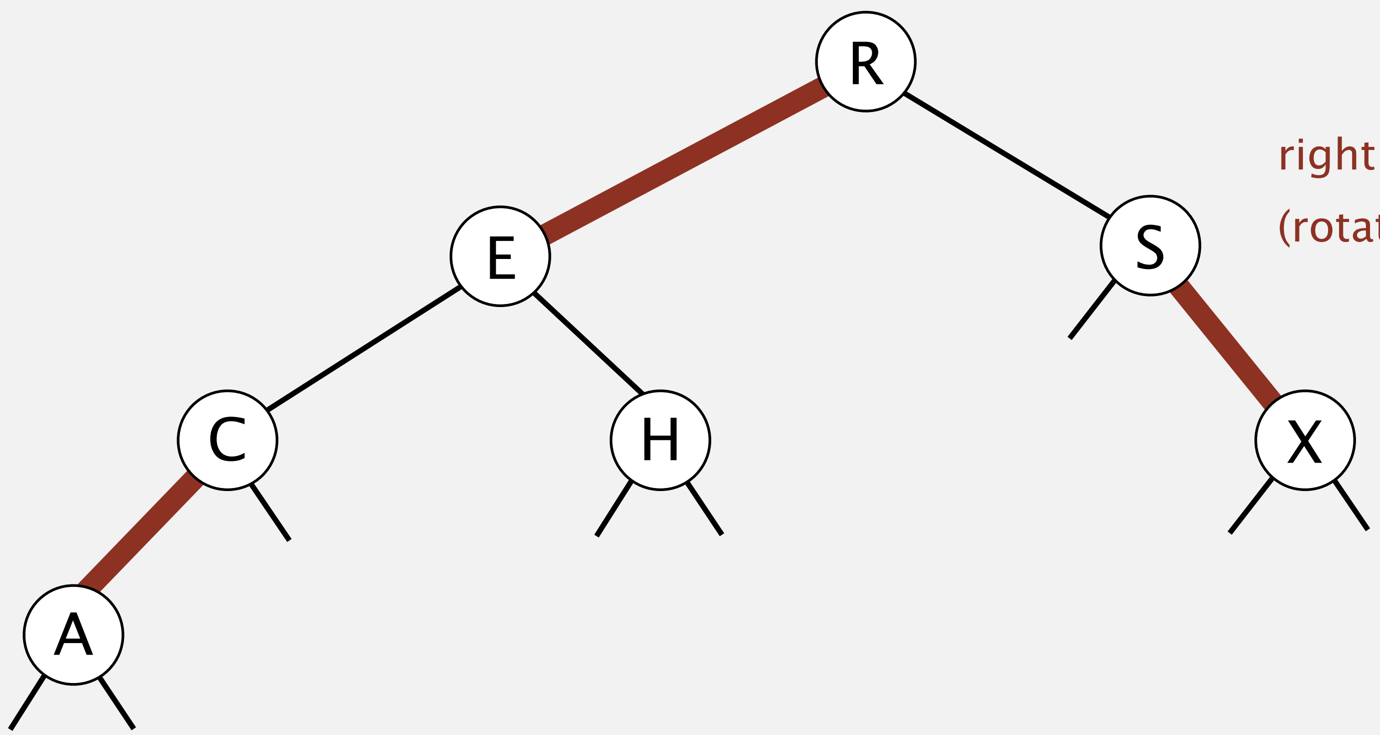
Red-black BST construction demo

insert X



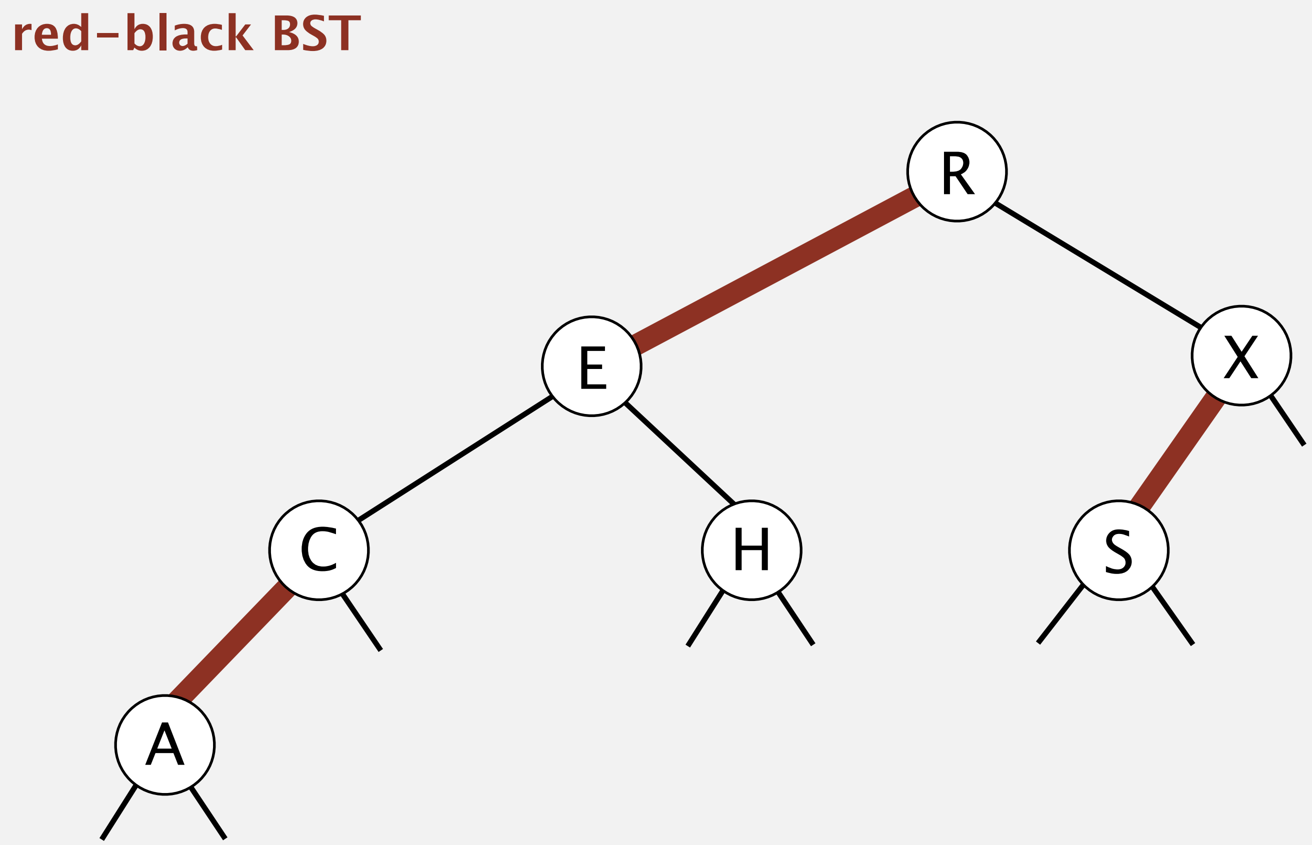
Red-black BST construction demo

insert X



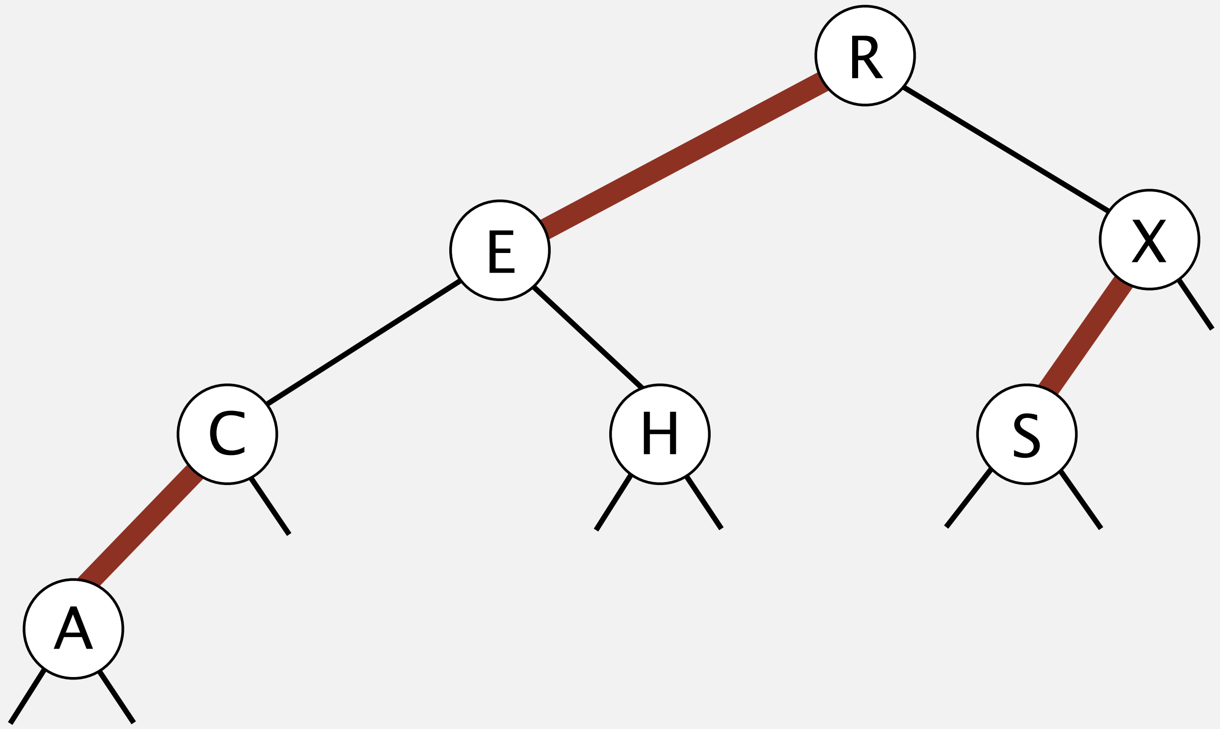
right link red
(rotate S left)

Red-black BST construction demo

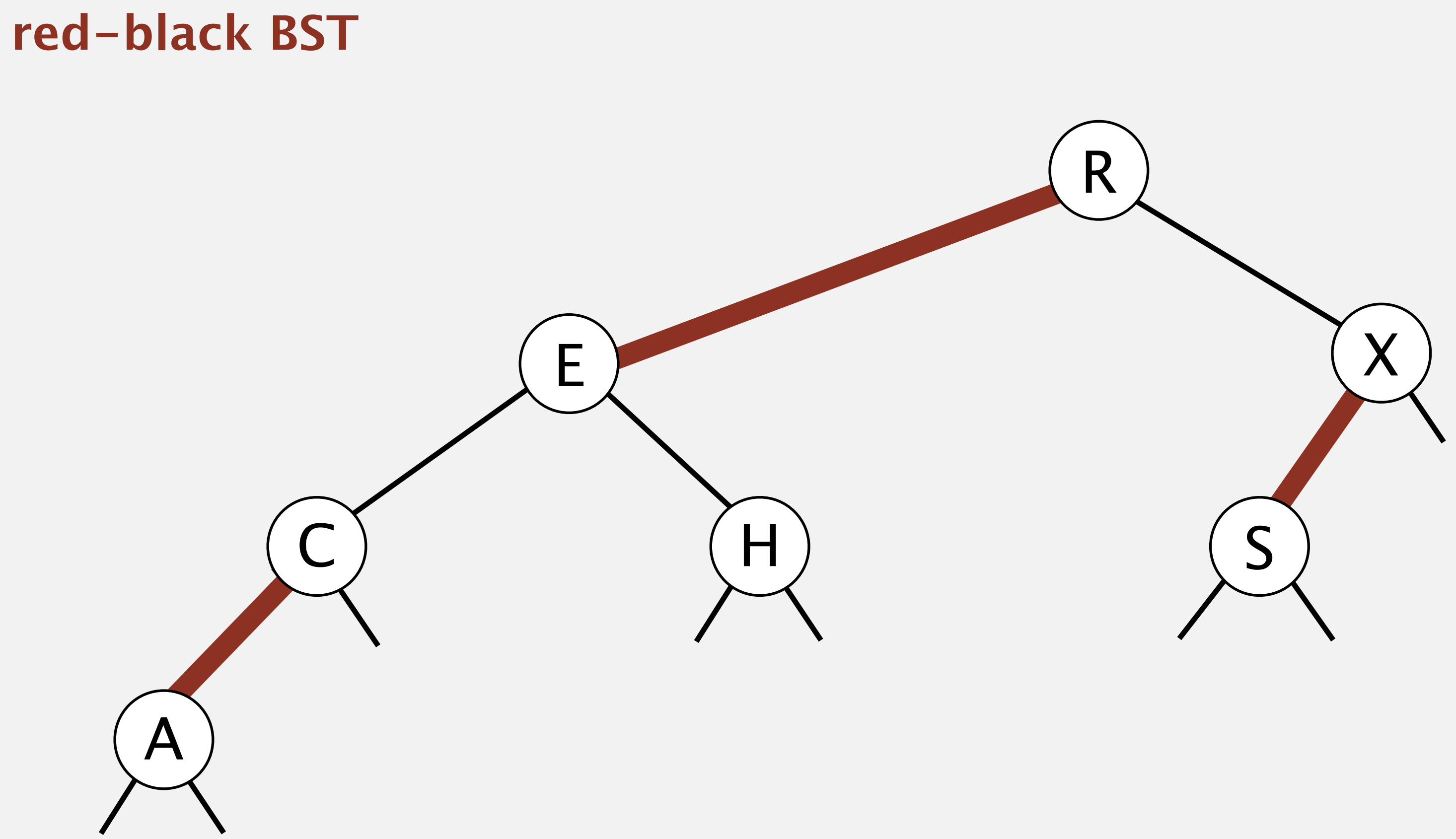


Red-black BST construction demo

red-black BST

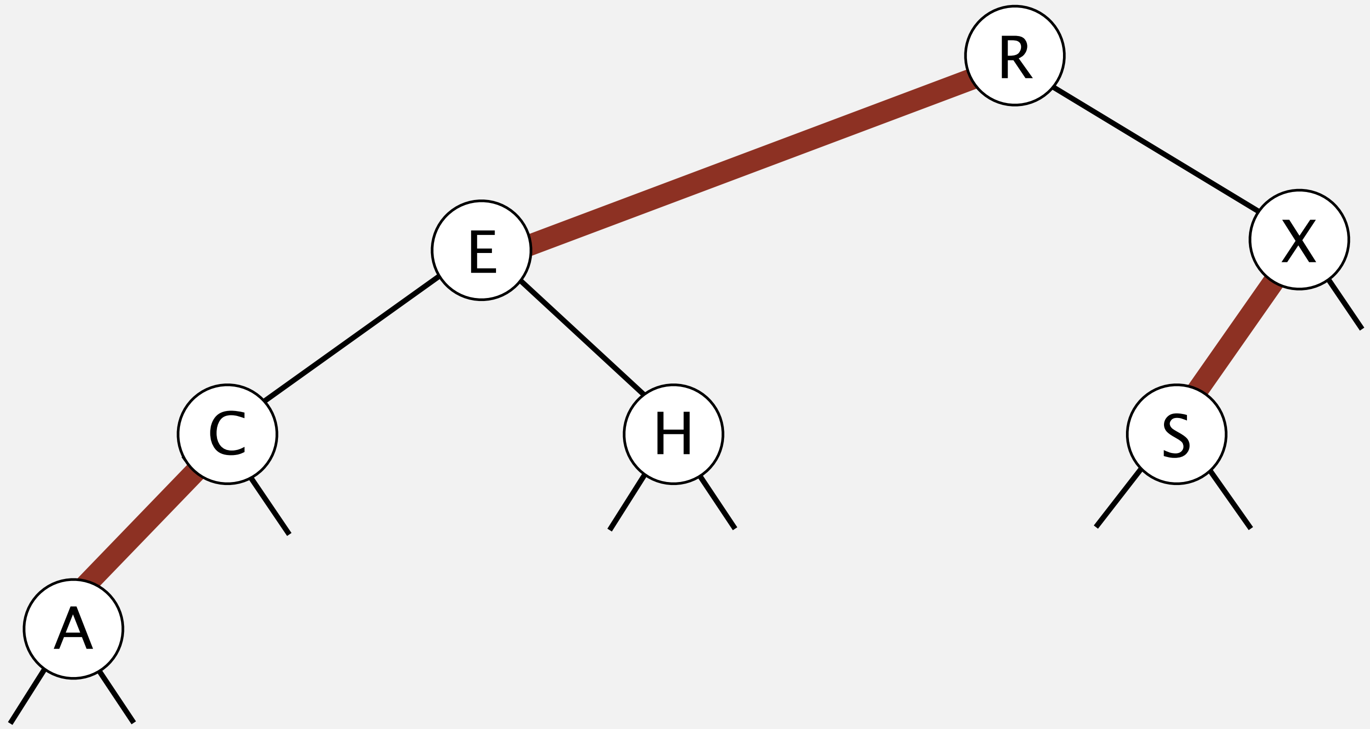


Red-black BST construction demo



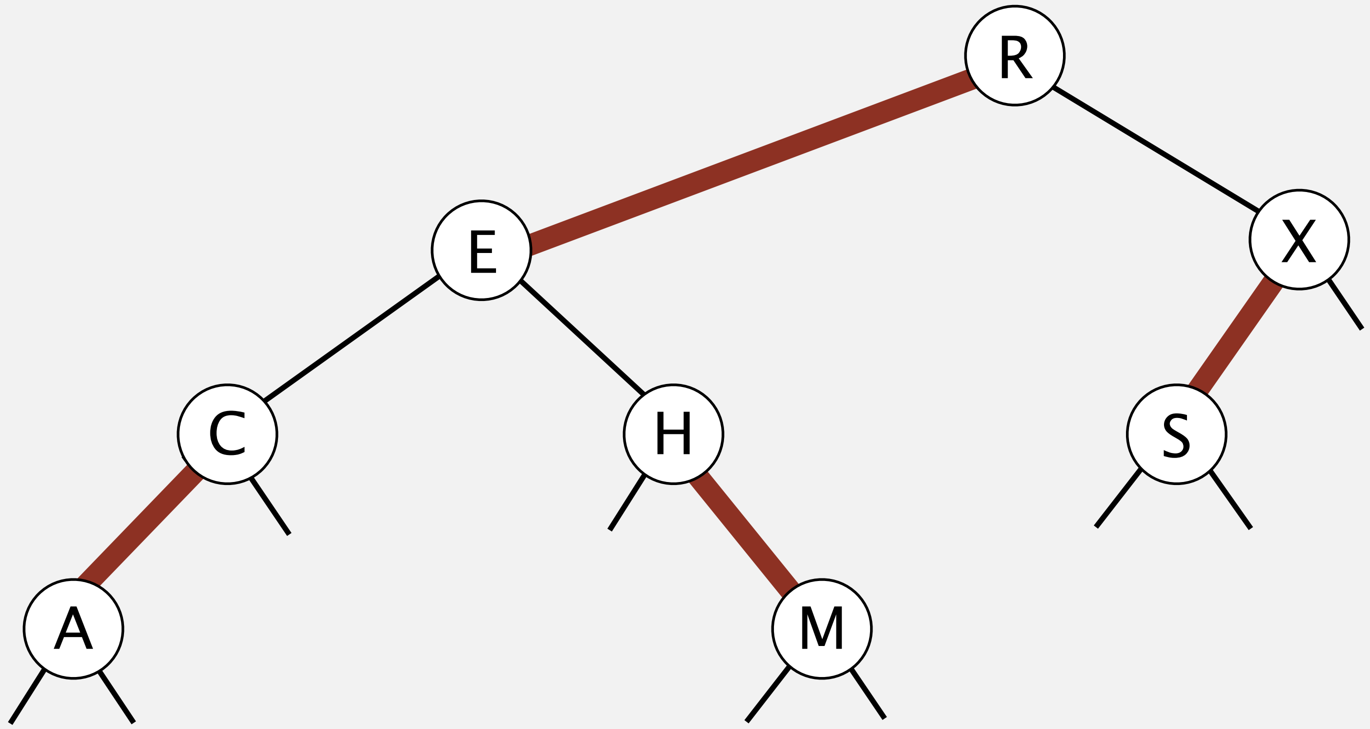
Red-black BST construction demo

insert M



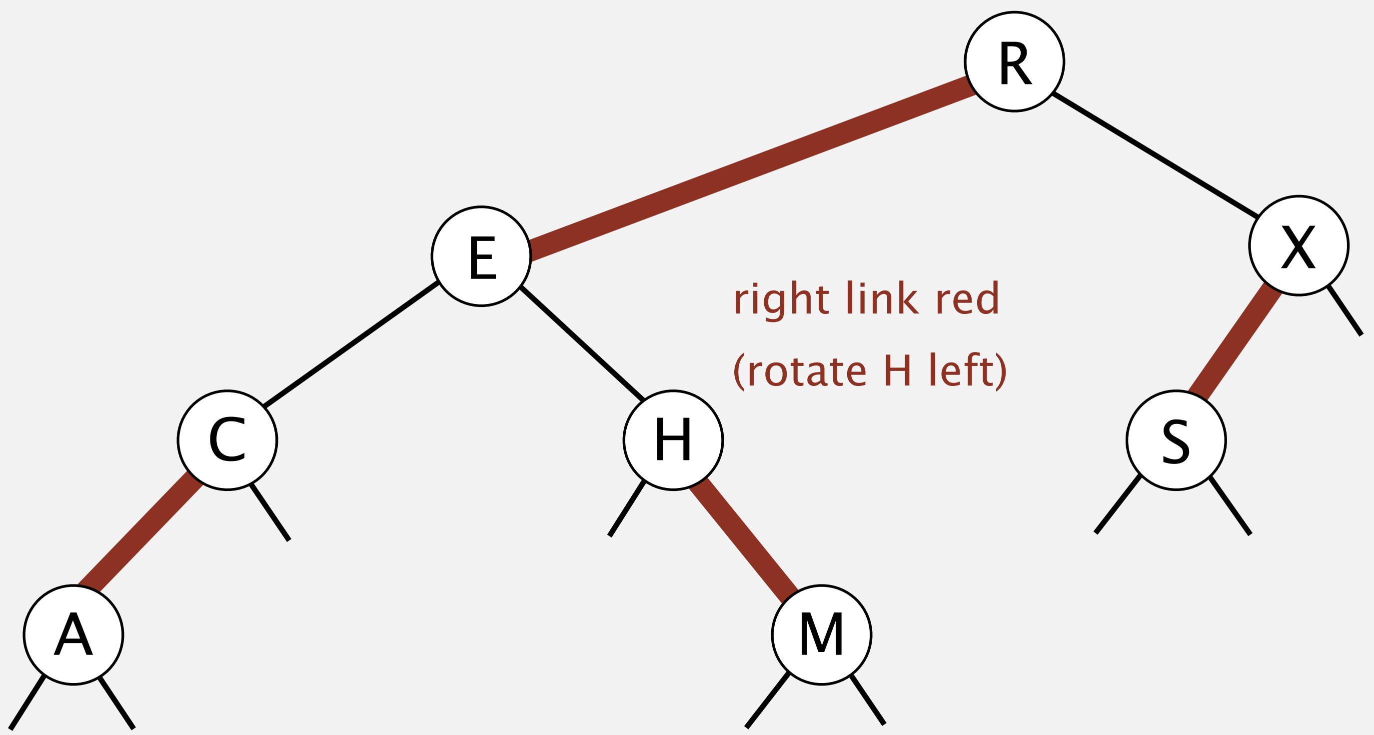
Red-black BST construction demo

insert M



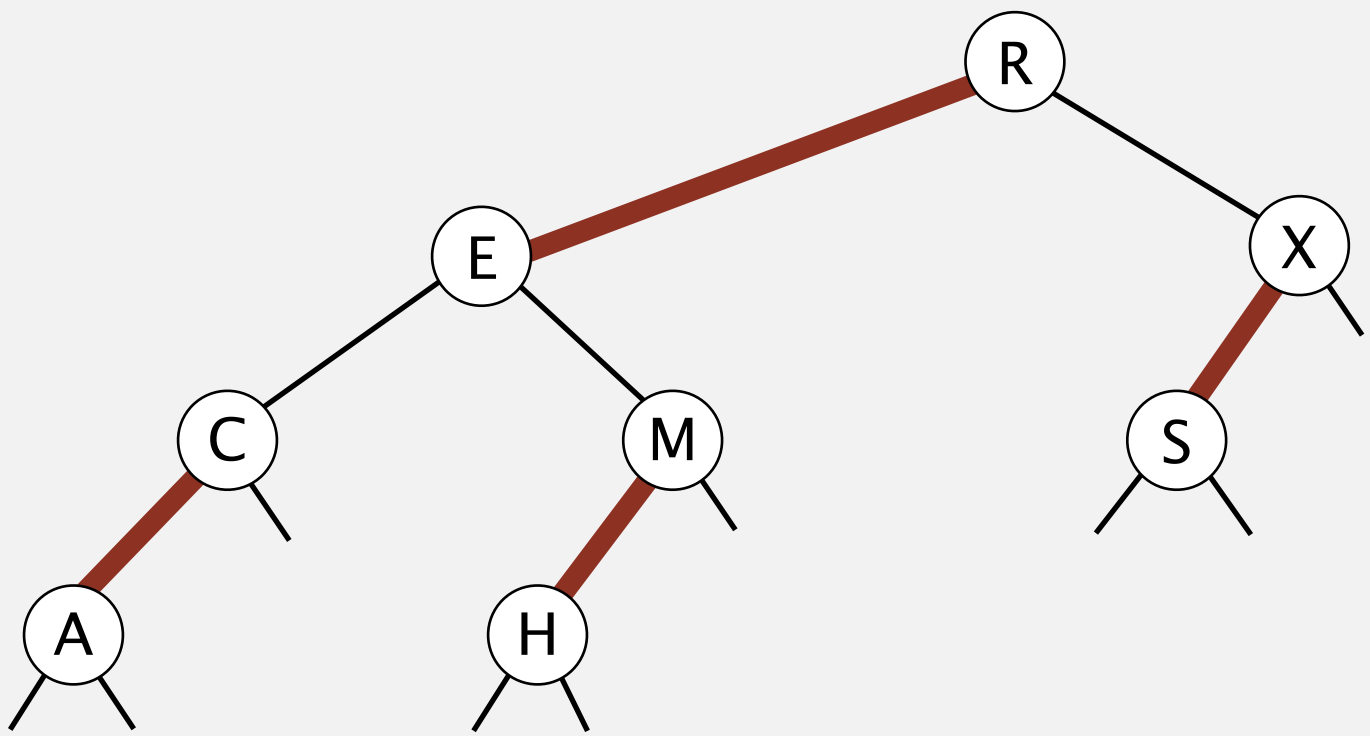
Red-black BST construction demo

insert M



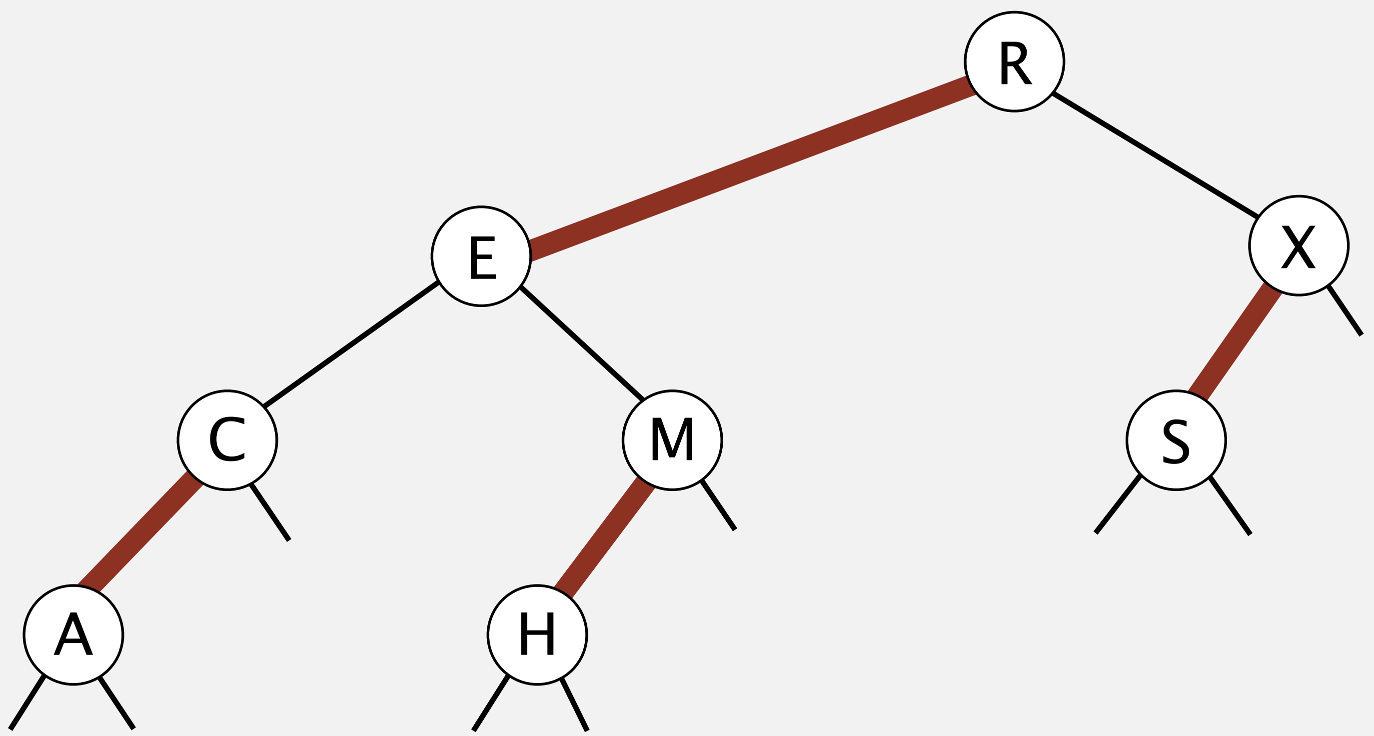
Red-black BST construction demo

red–black BST



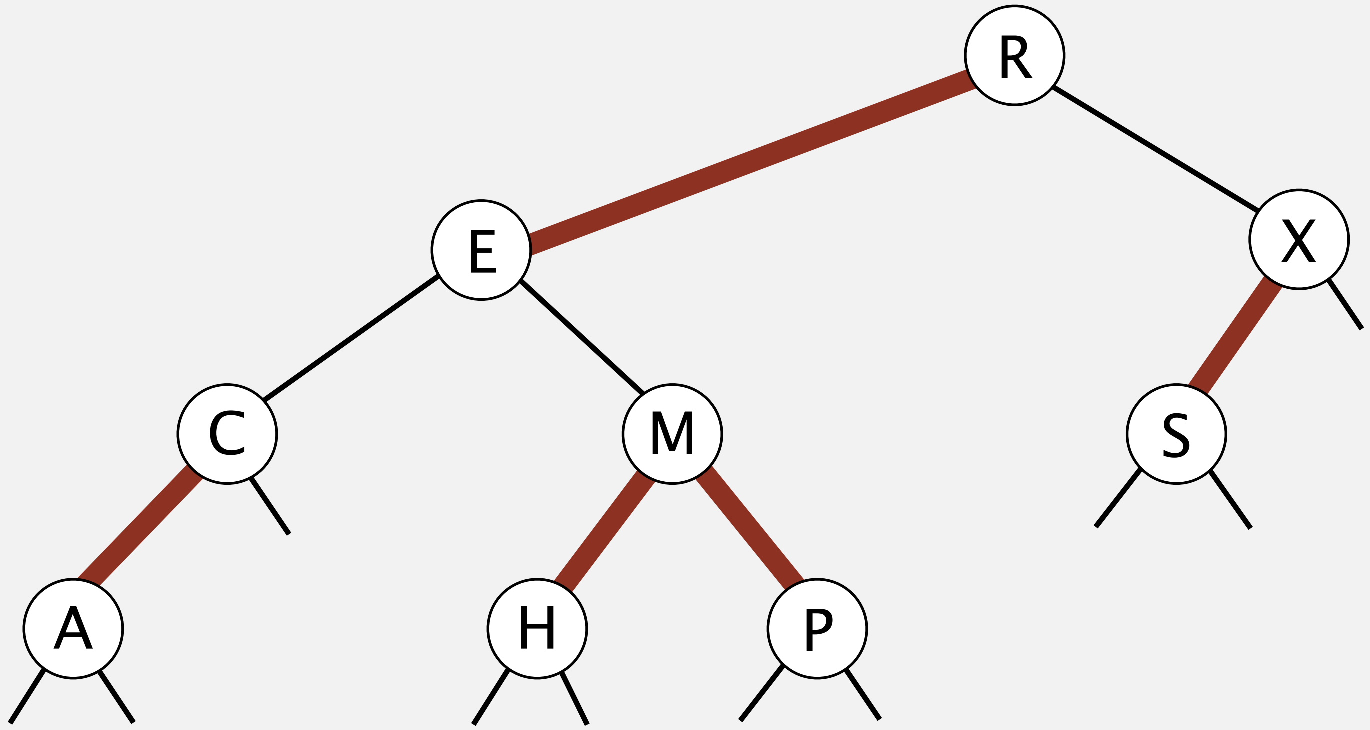
Red-black BST construction demo

insert P



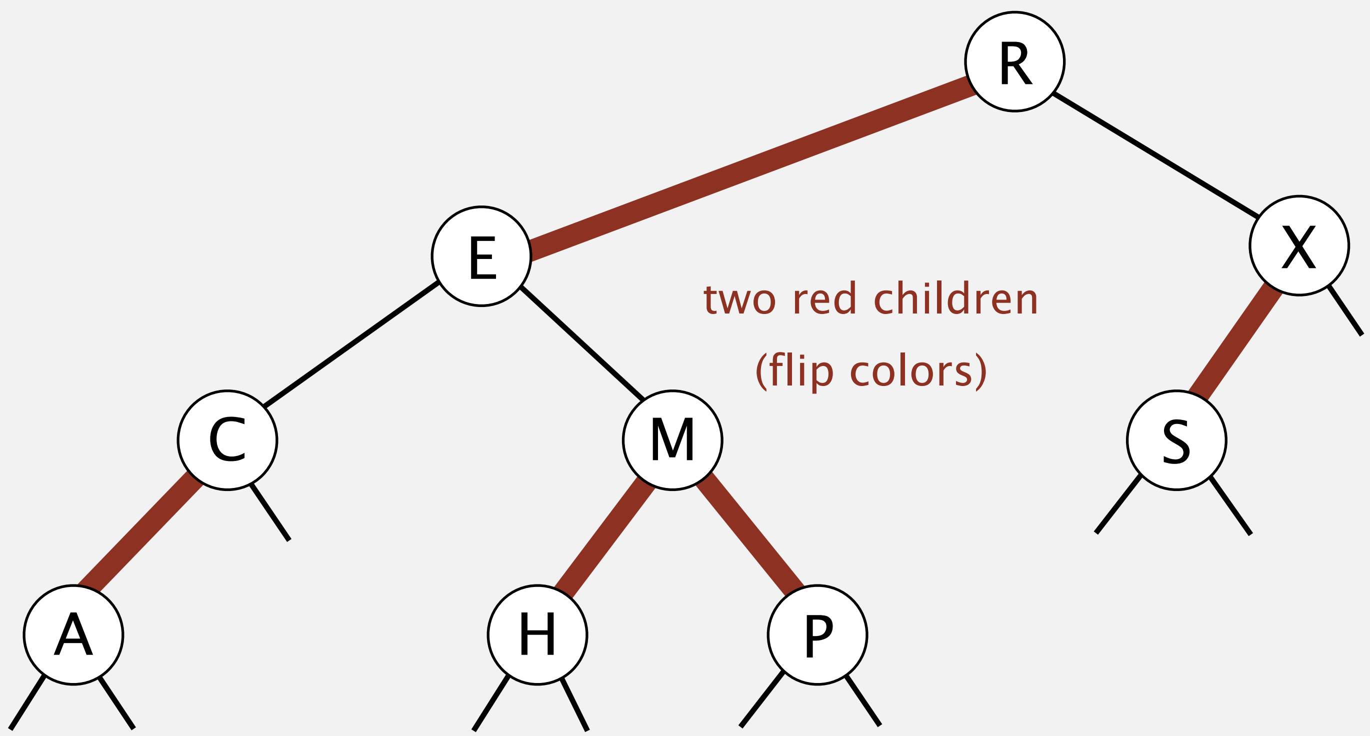
Red-black BST construction demo

insert P



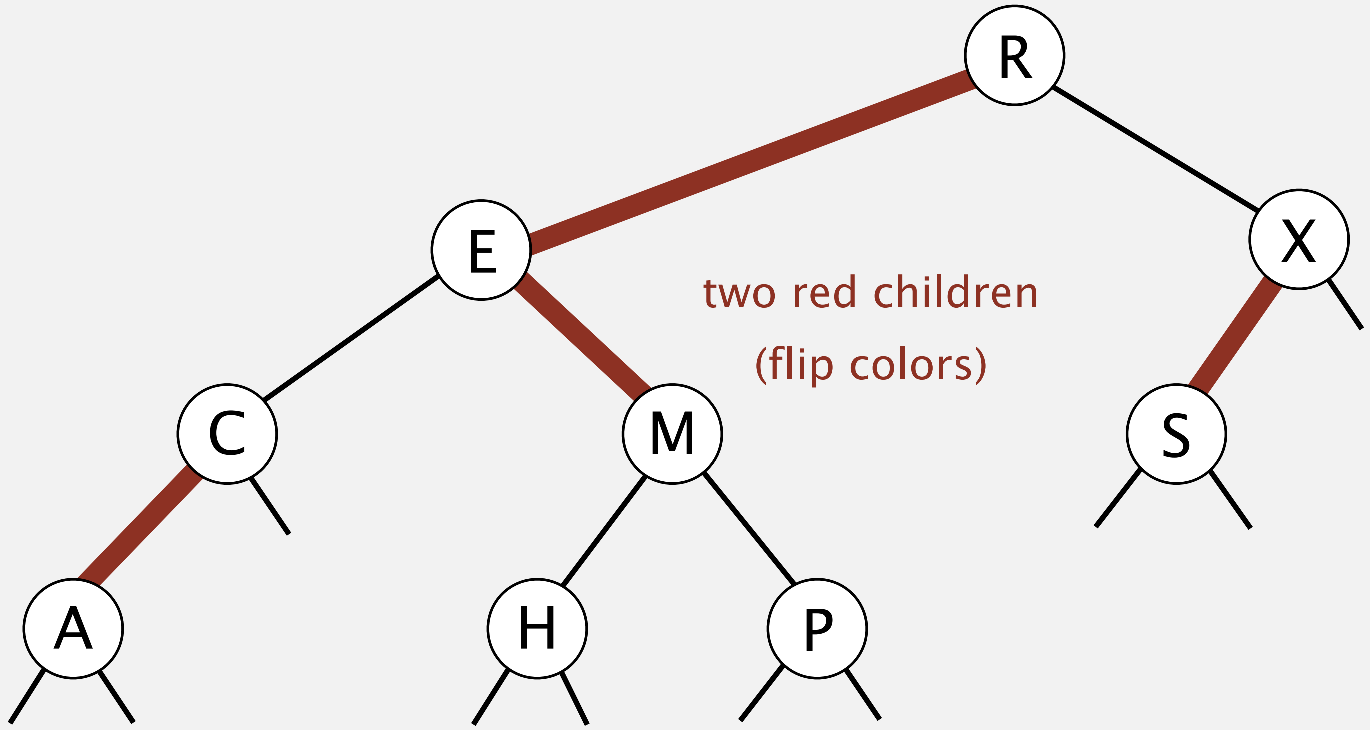
Red-black BST construction demo

insert P

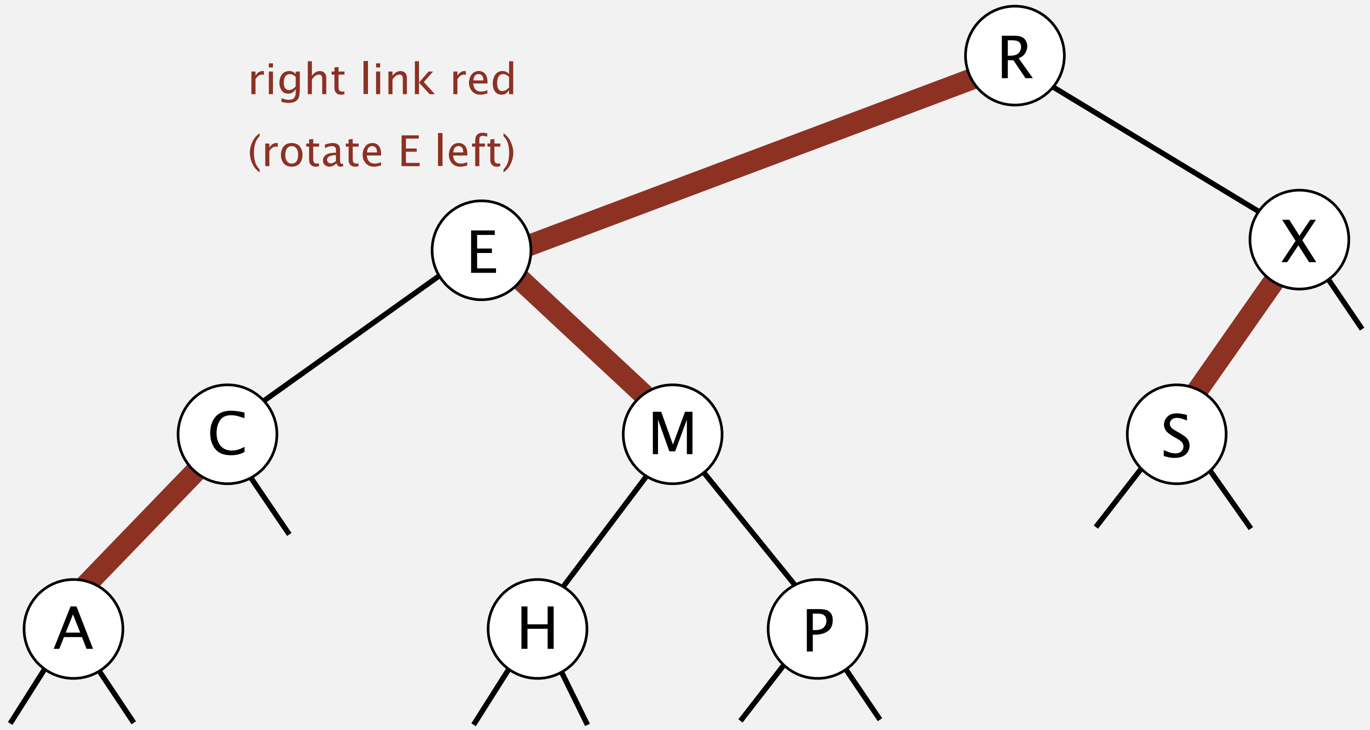


Red-black BST construction demo

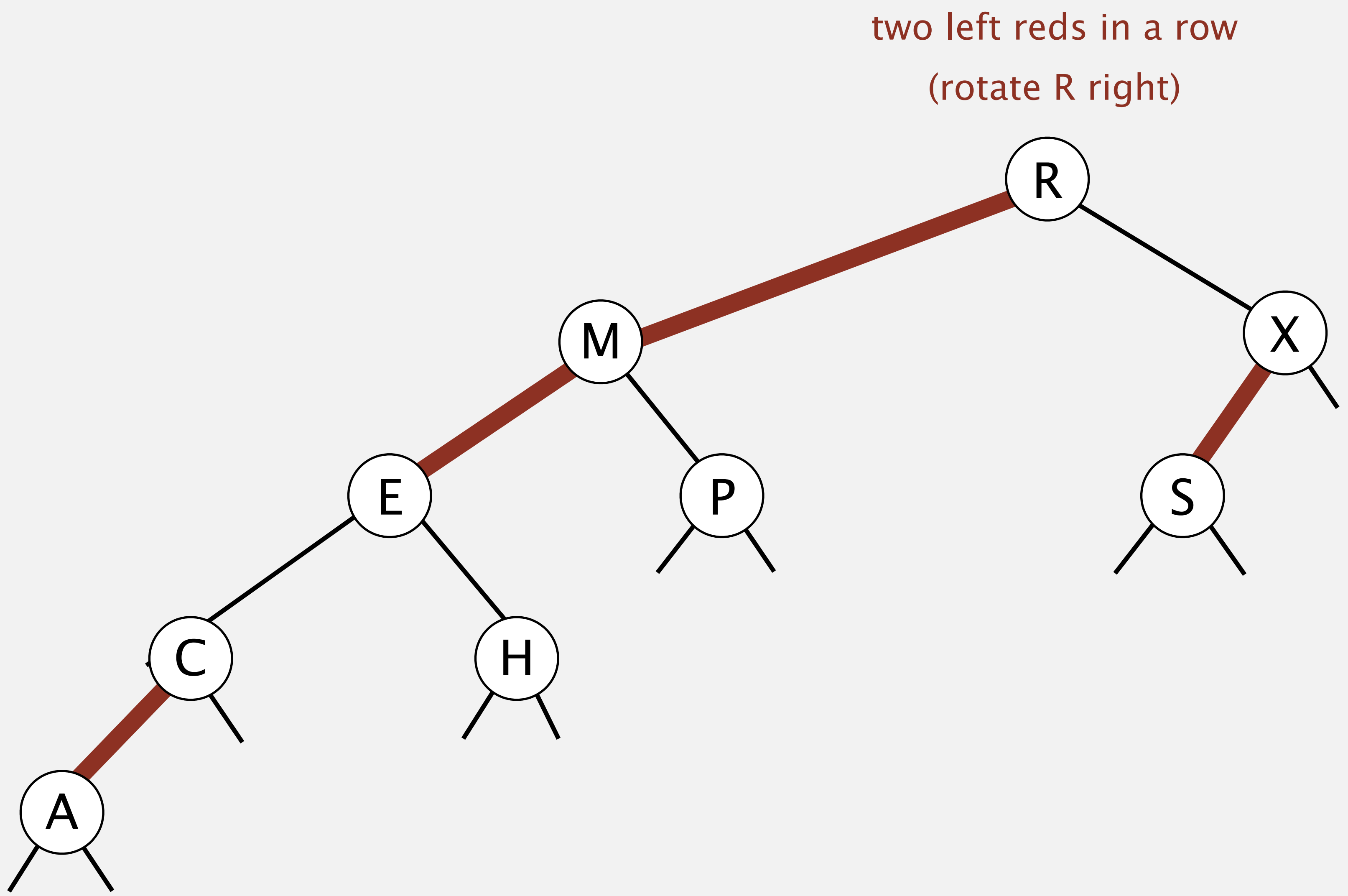
insert P



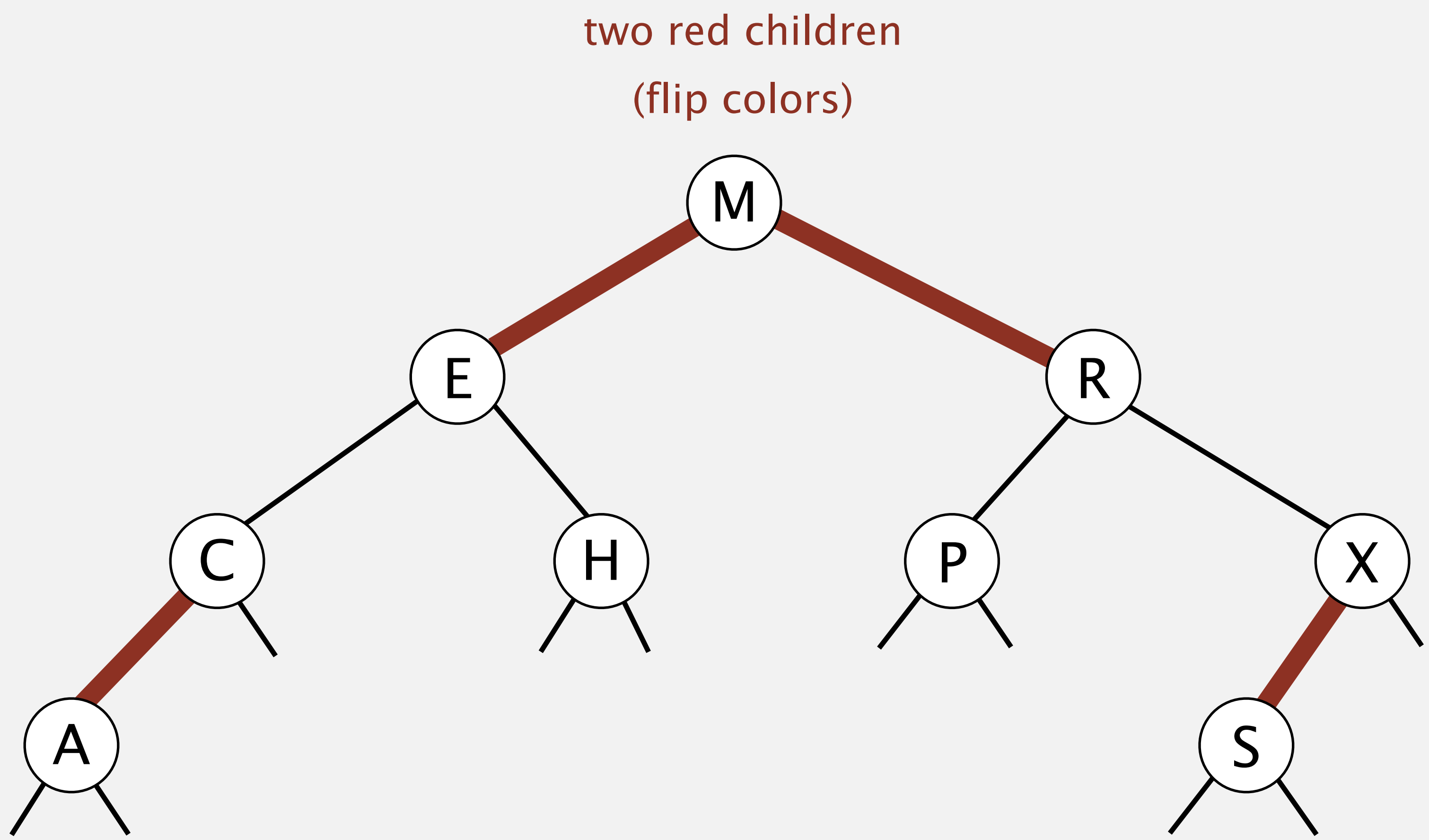
Red-black BST construction demo



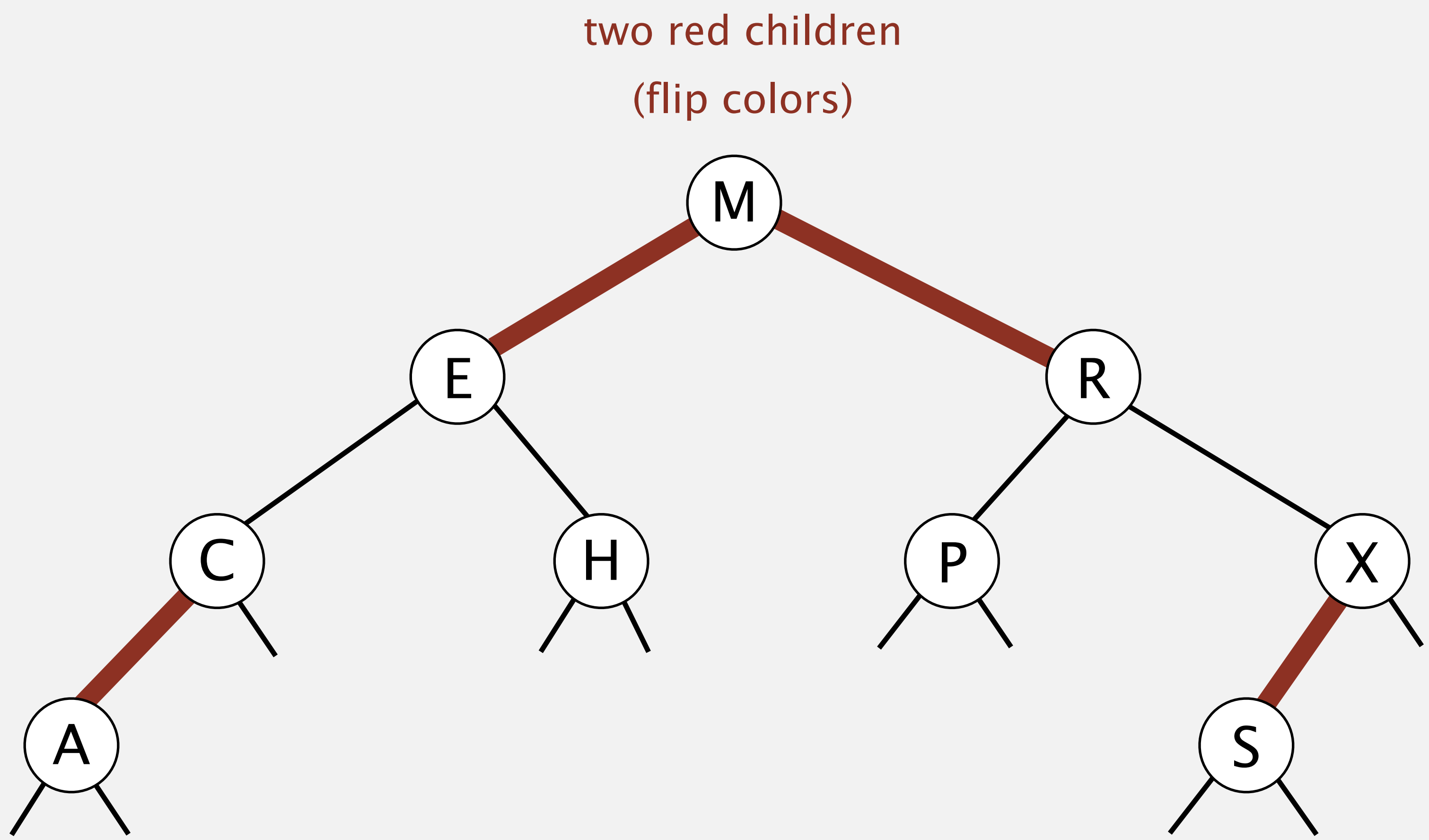
Red-black BST construction demo



Red-black BST construction demo

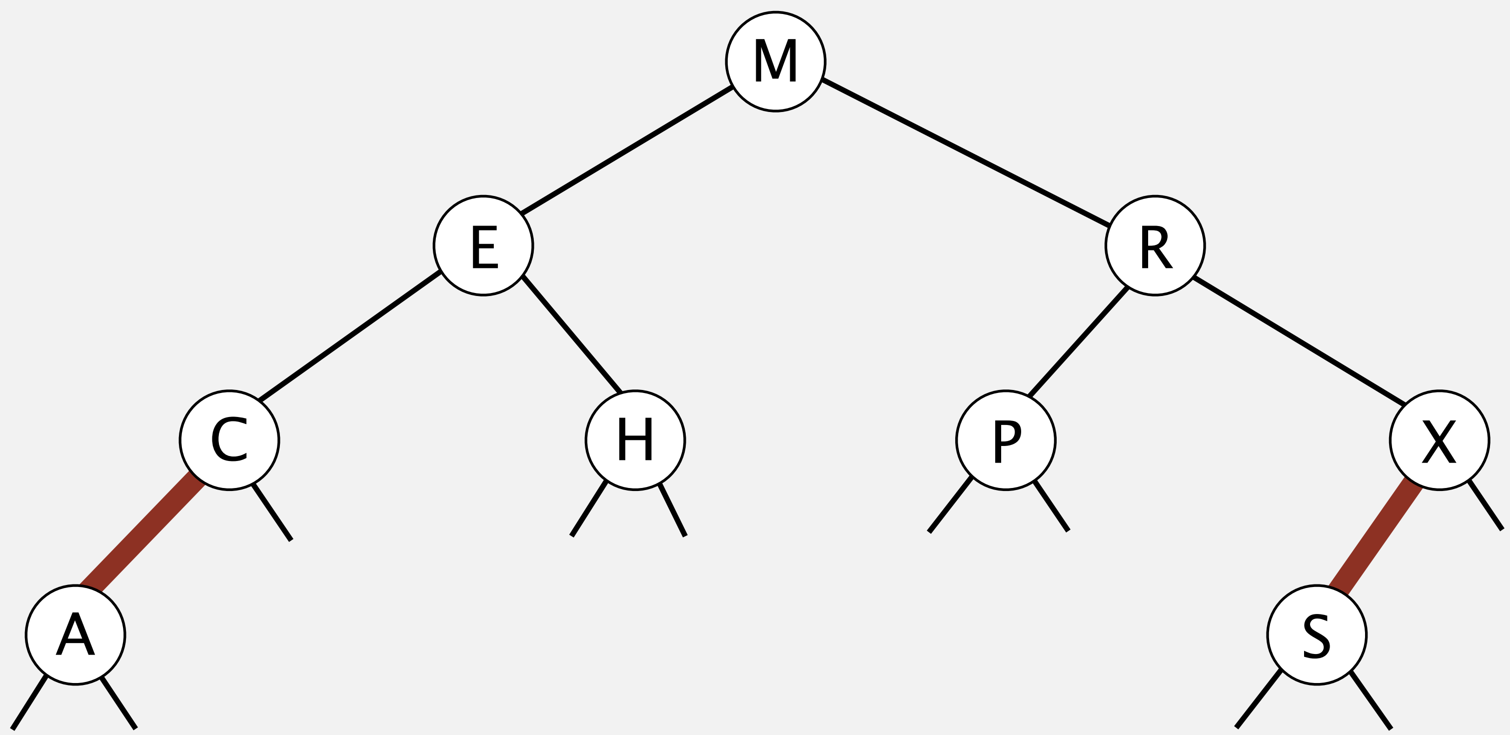


Red-black BST construction demo



Red-black BST construction demo

red-black BST



Red-black BST construction demo

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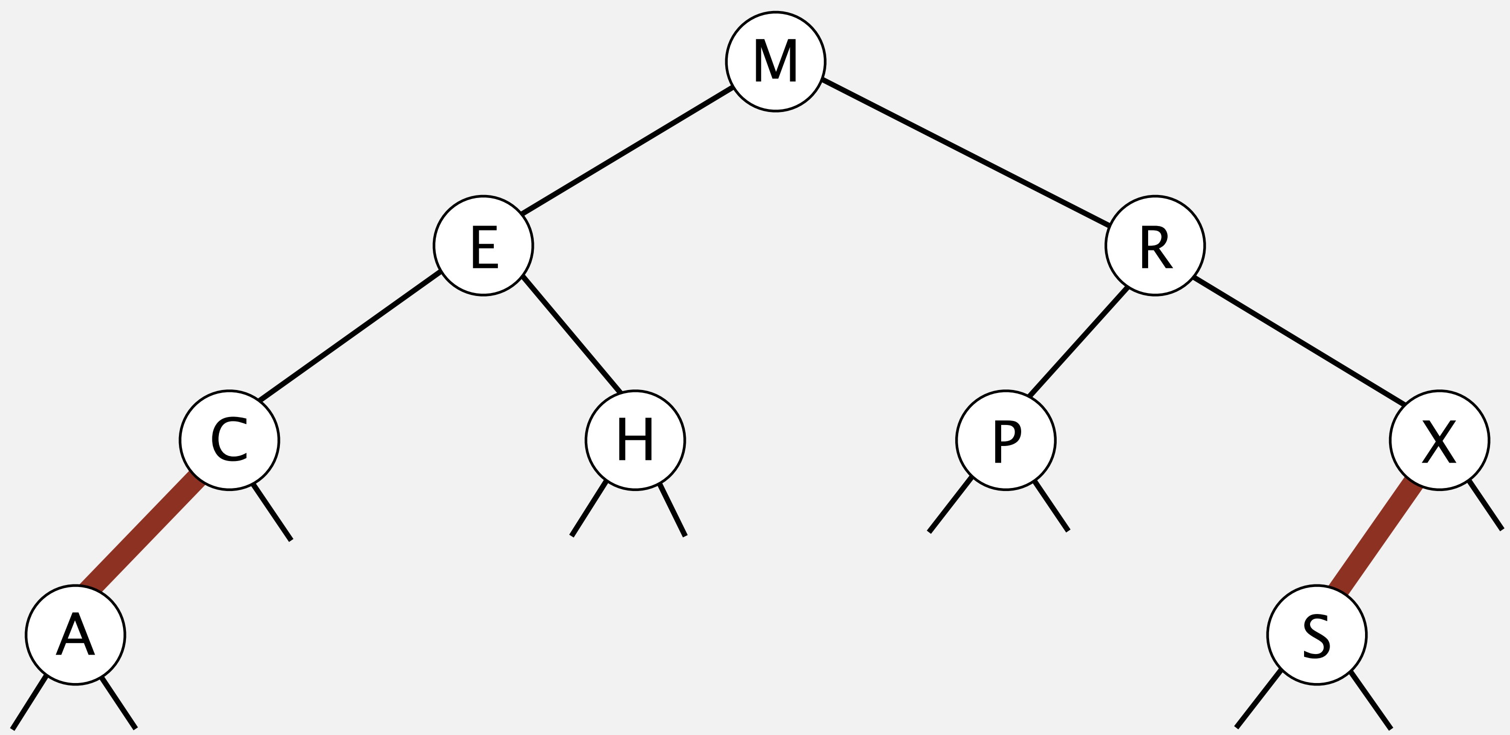
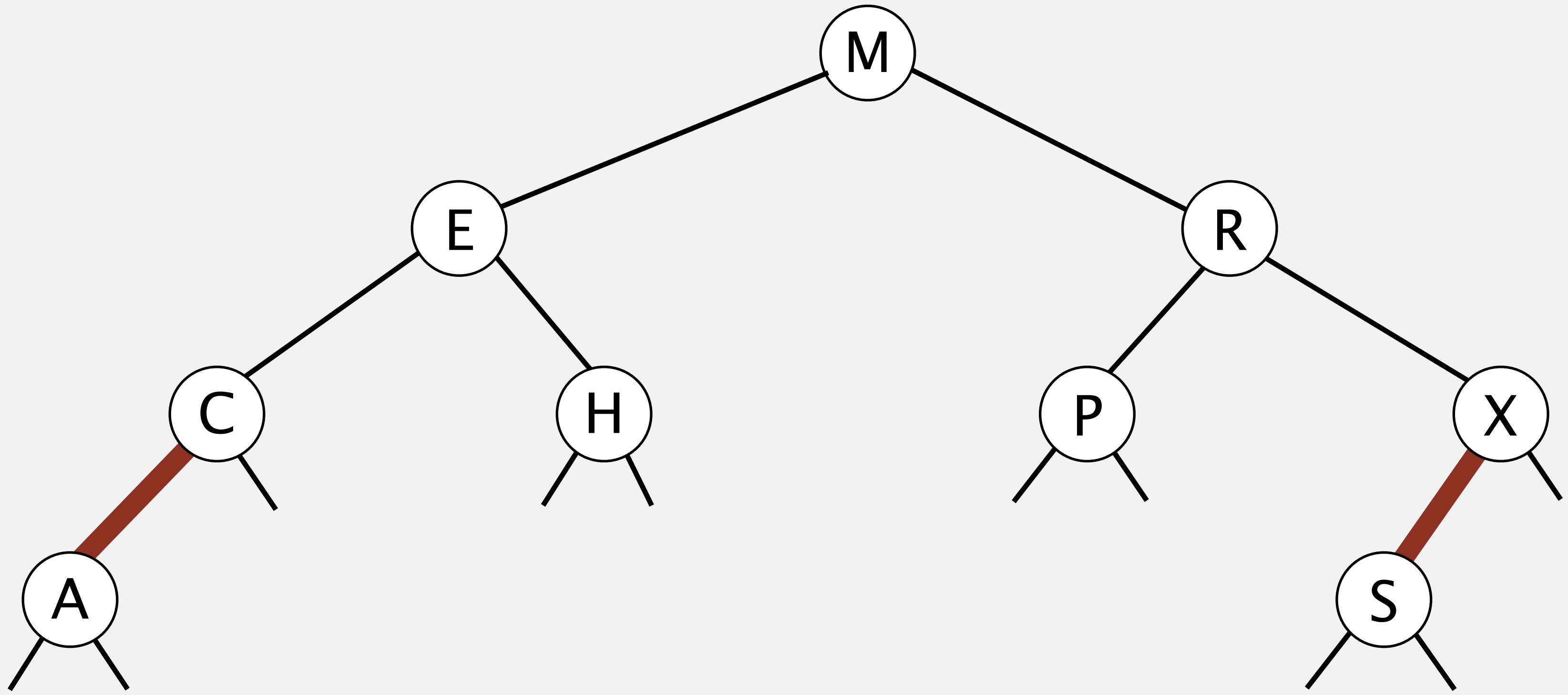


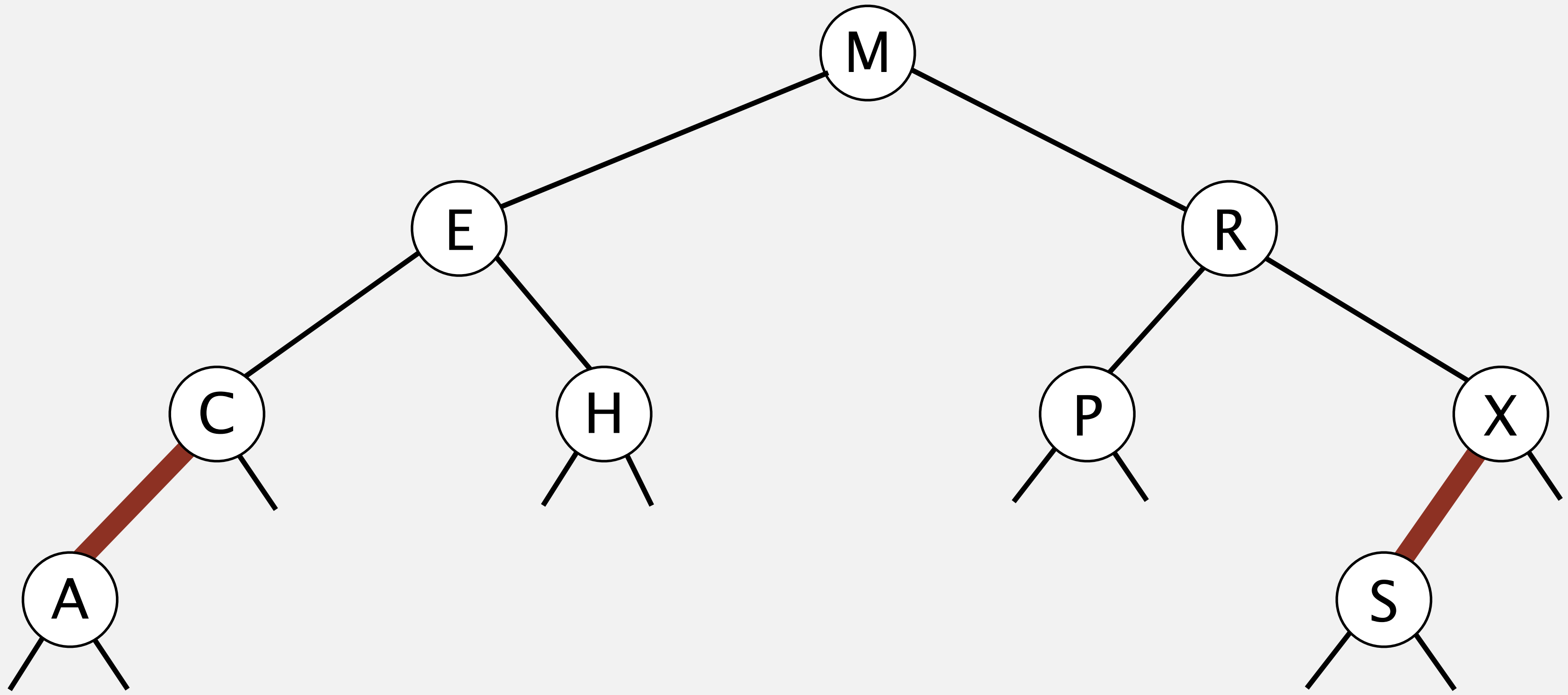
Red-black BST construction demo

red-black BST



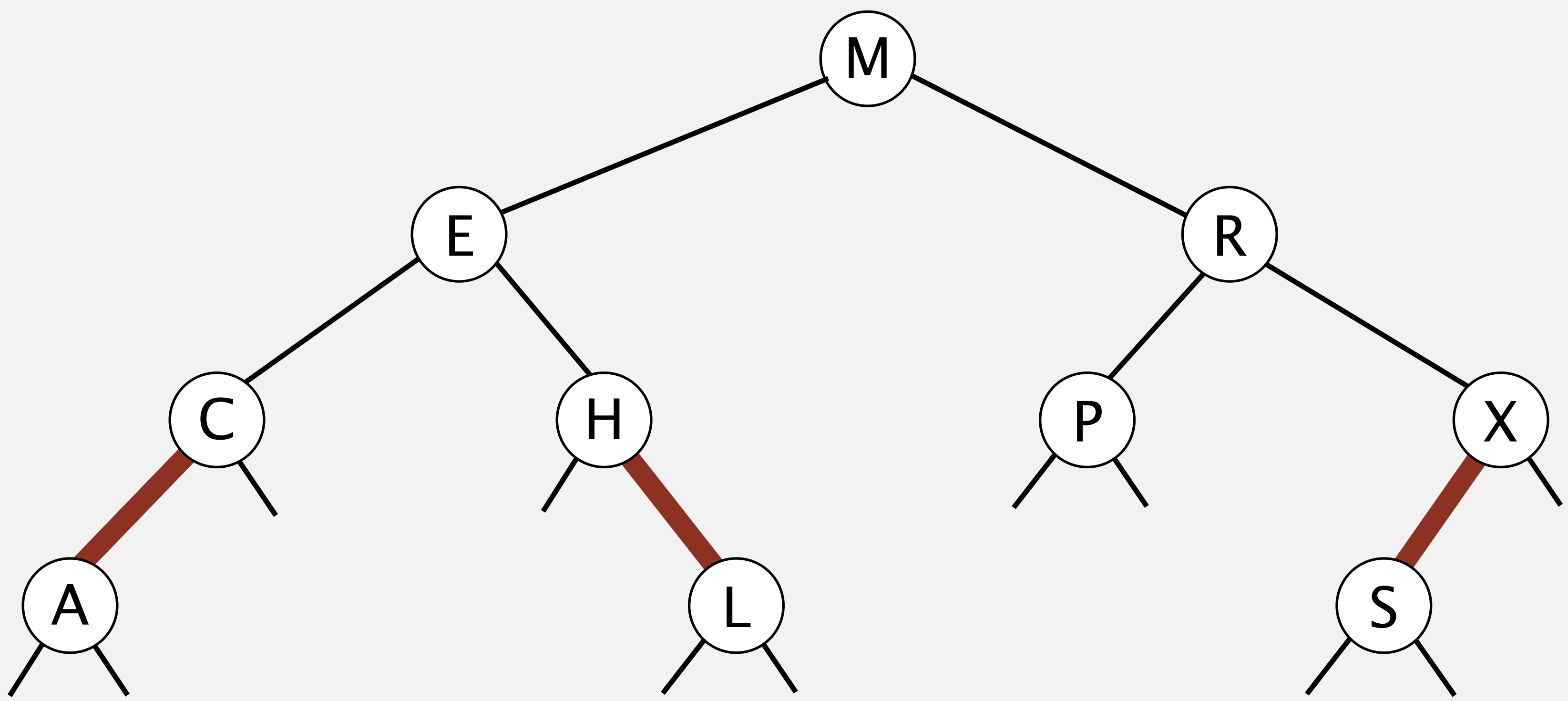
Red-black BST construction demo

insert L



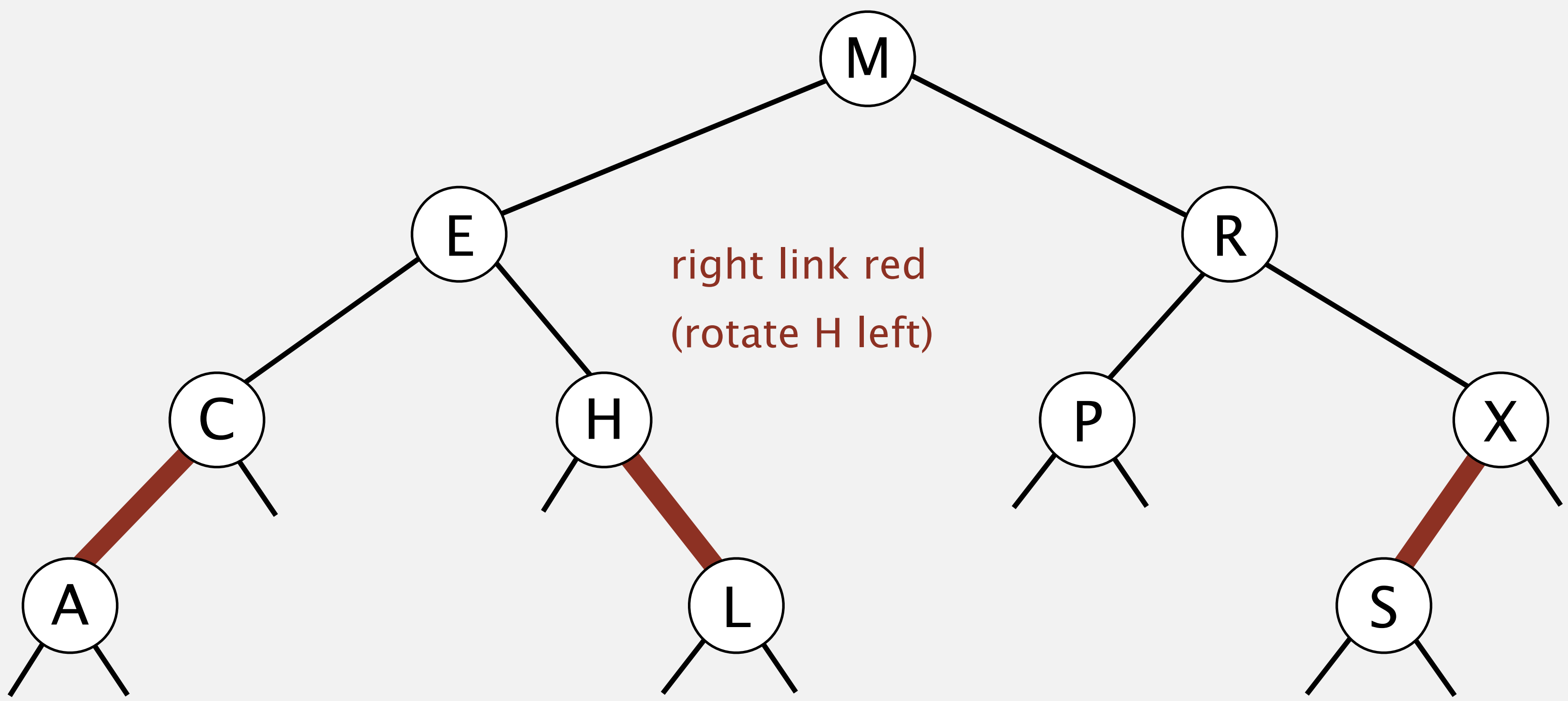
Red-black BST construction demo

insert L

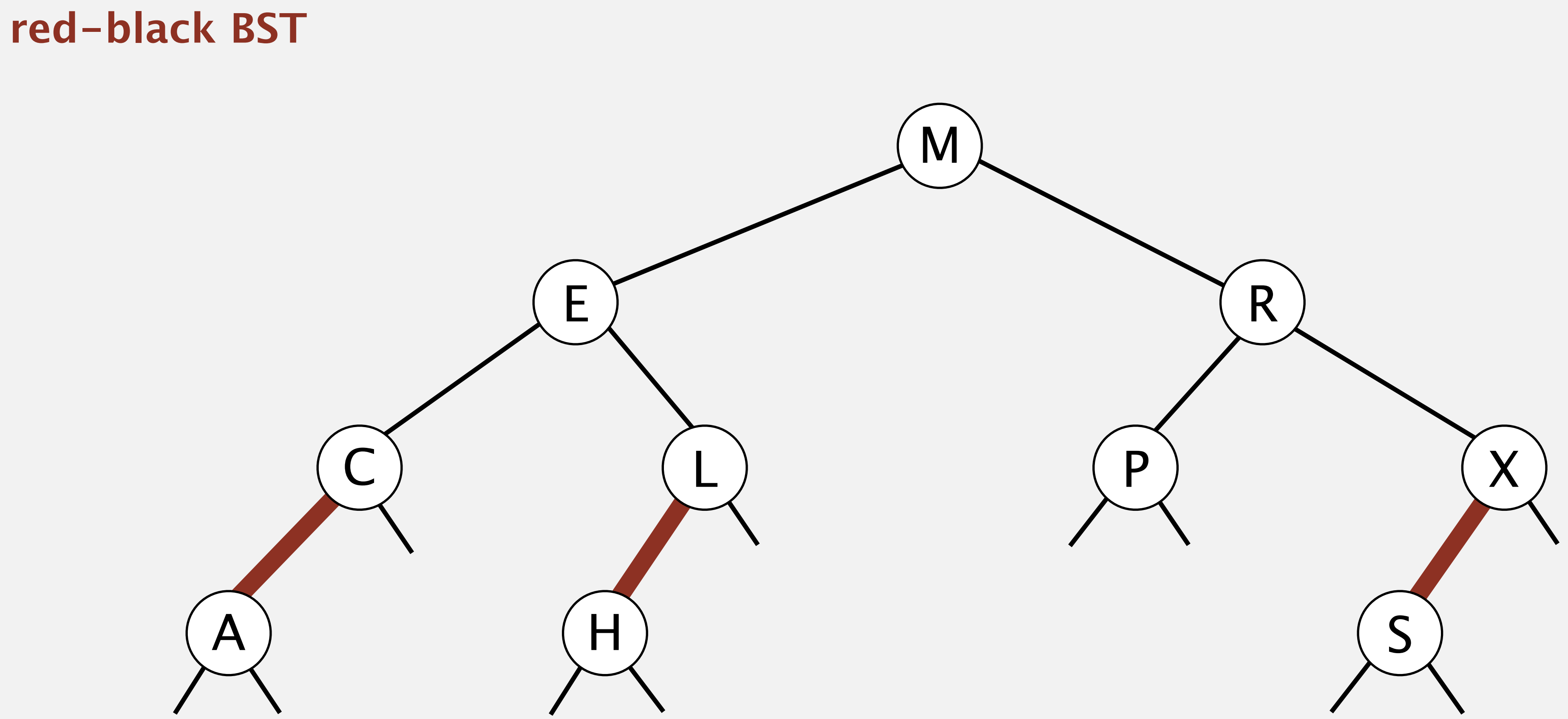


Red-black BST construction demo

insert L



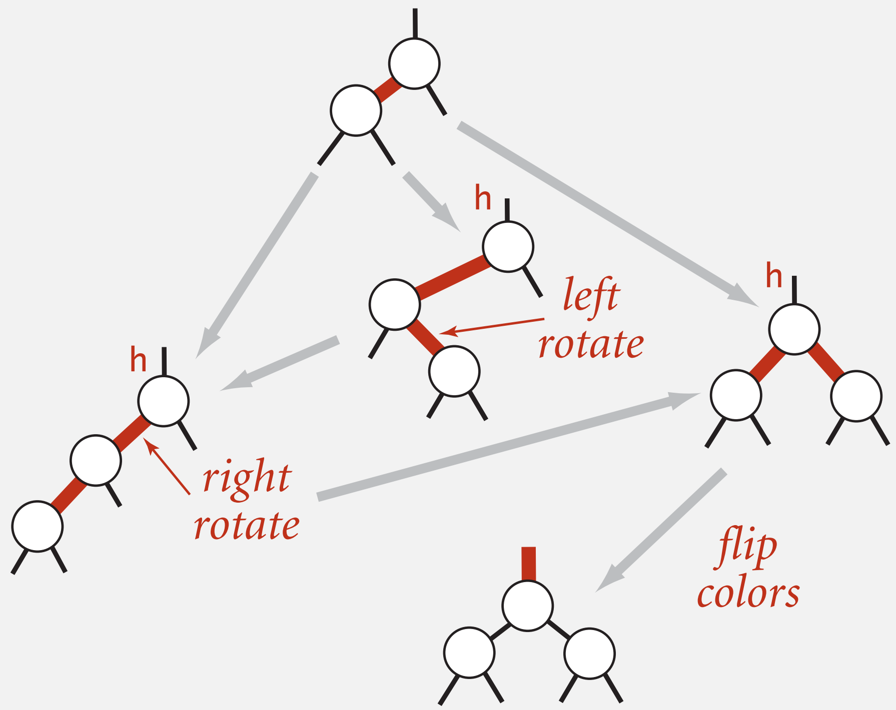
Red-black BST construction demo



Insertion in a LLRB tree: Java implementation

Same code for all cases.

- Right child red, left child black: rotate left.
- Left child, left-left grandchild red: rotate right.
- Both children red: flip colors.



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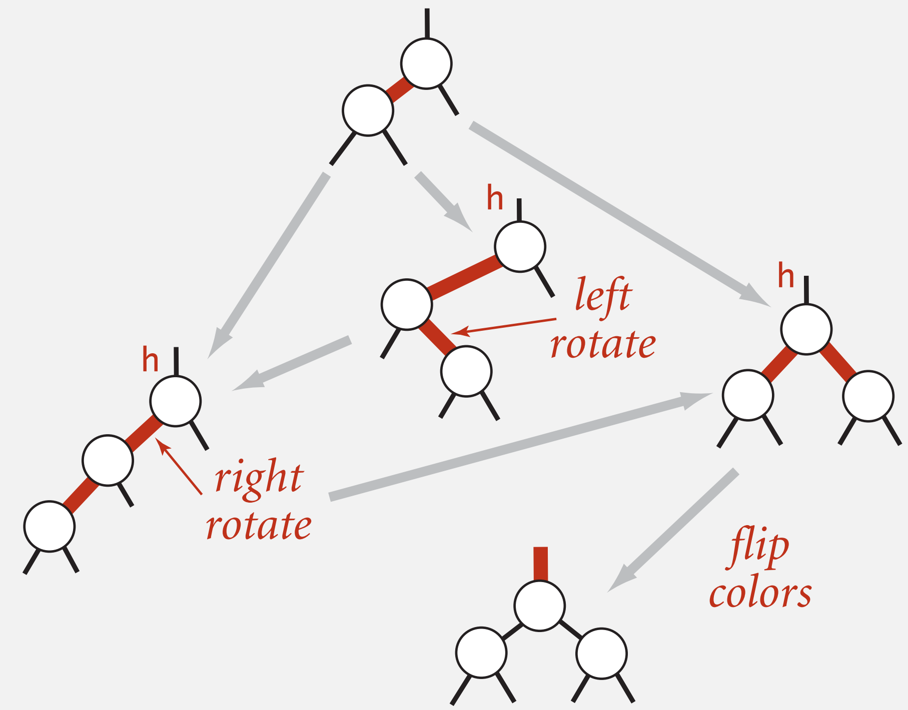



```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else if (cmp == 0) h.val = val;

   if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))     flipColors(h);

   return h;
}
```

insert at bottom (and color it red)

lean left

balance 4-node

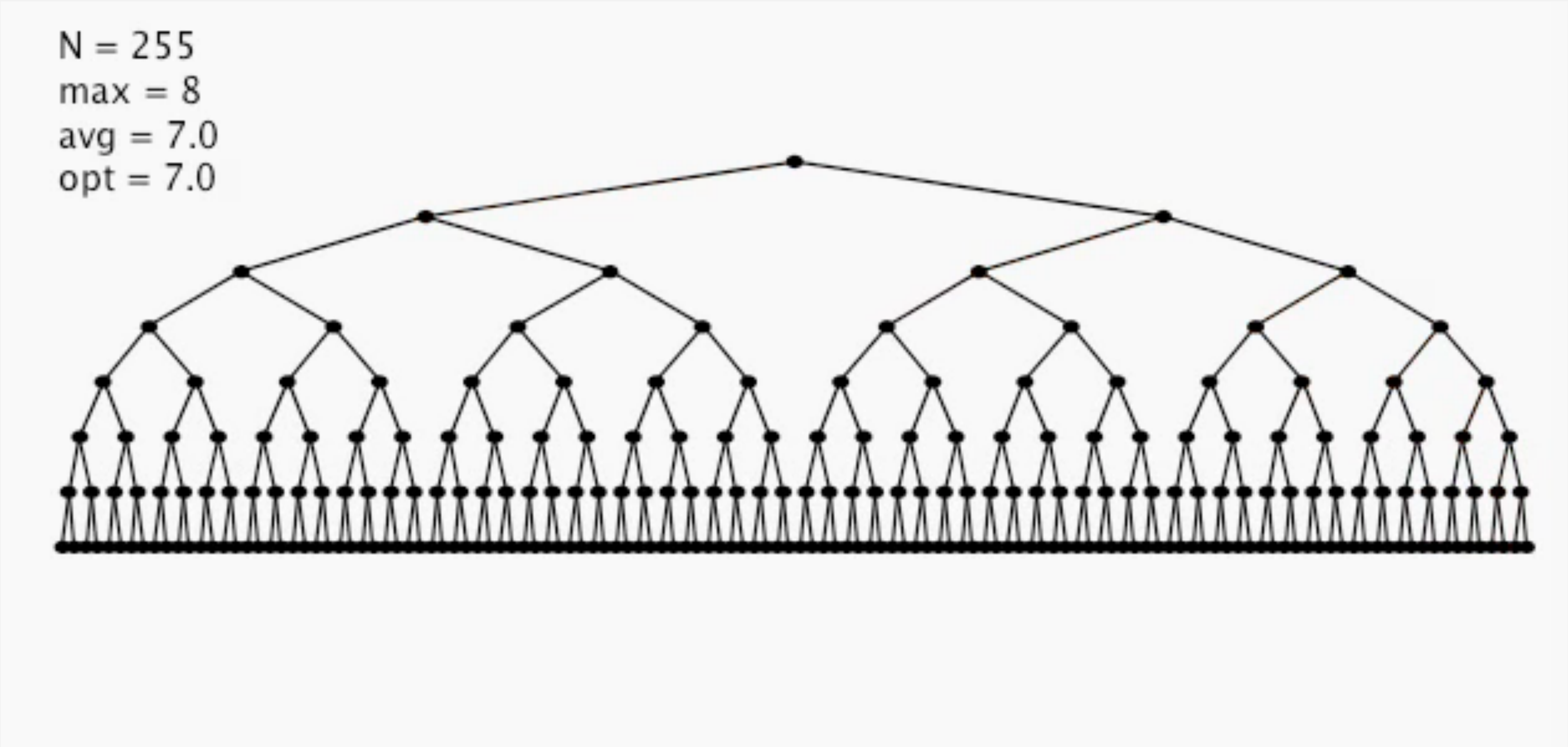
split 4-node

only a few extra lines of code provides near-perfect balance

Insertion in a LLRB tree: visualization

255 insertions in ascending order

Insertion in a LLRB tree: visualization

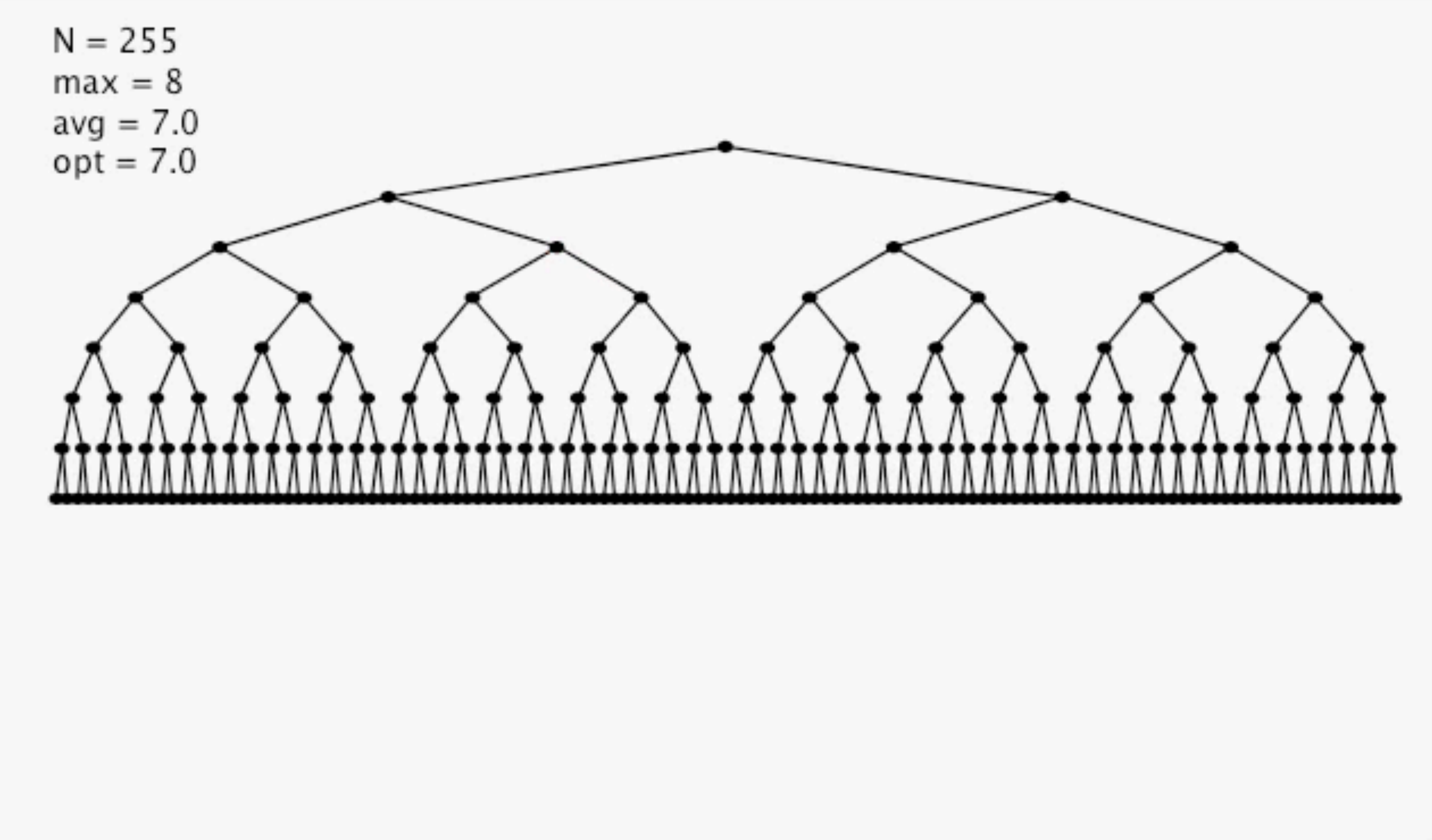


255 insertions in ascending order

Insertion in a LLRB tree: visualization

255 insertions in descending order

Insertion in a LLRB tree: visualization

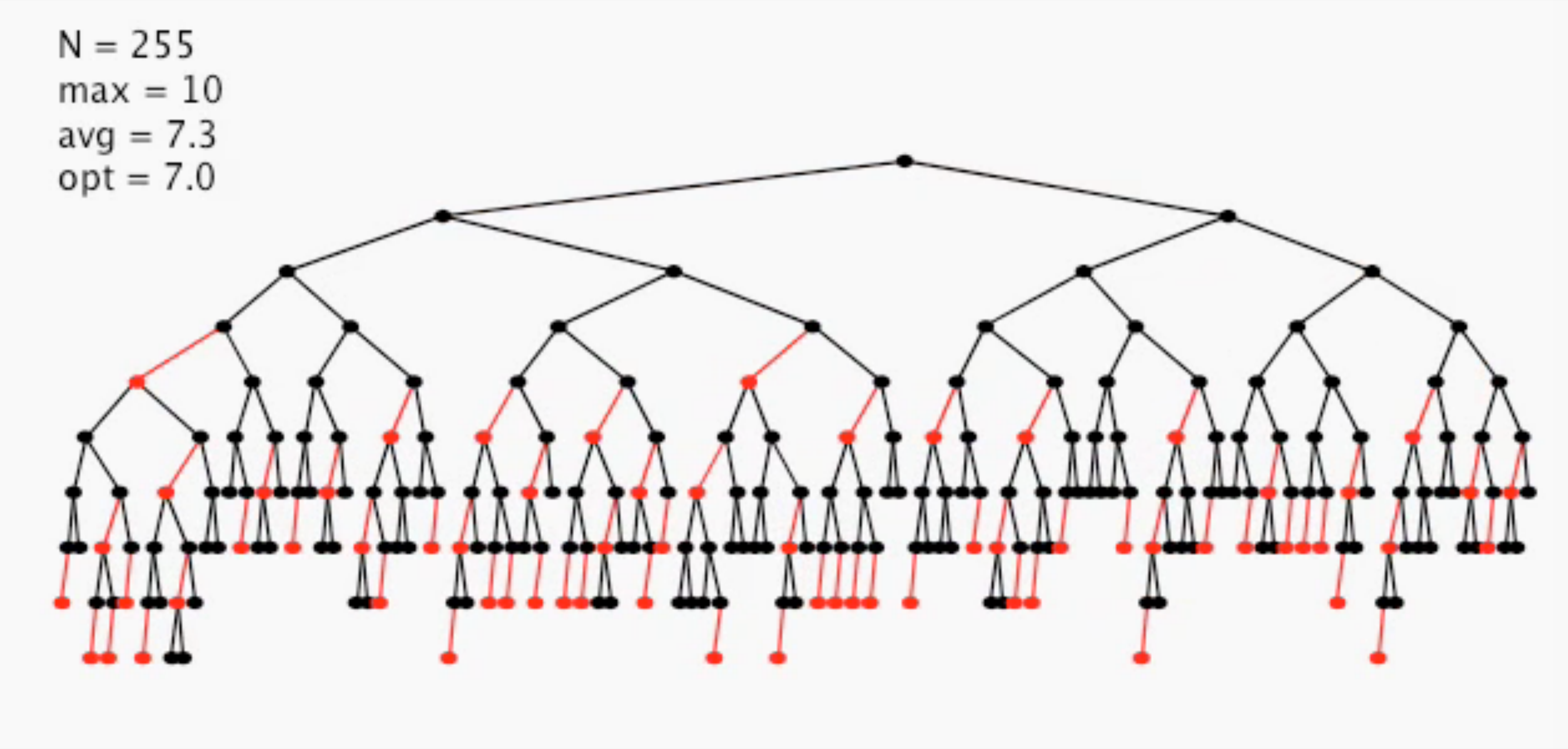


255 insertions in descending order

Insertion in a LLRB tree: visualization

255 random insertions

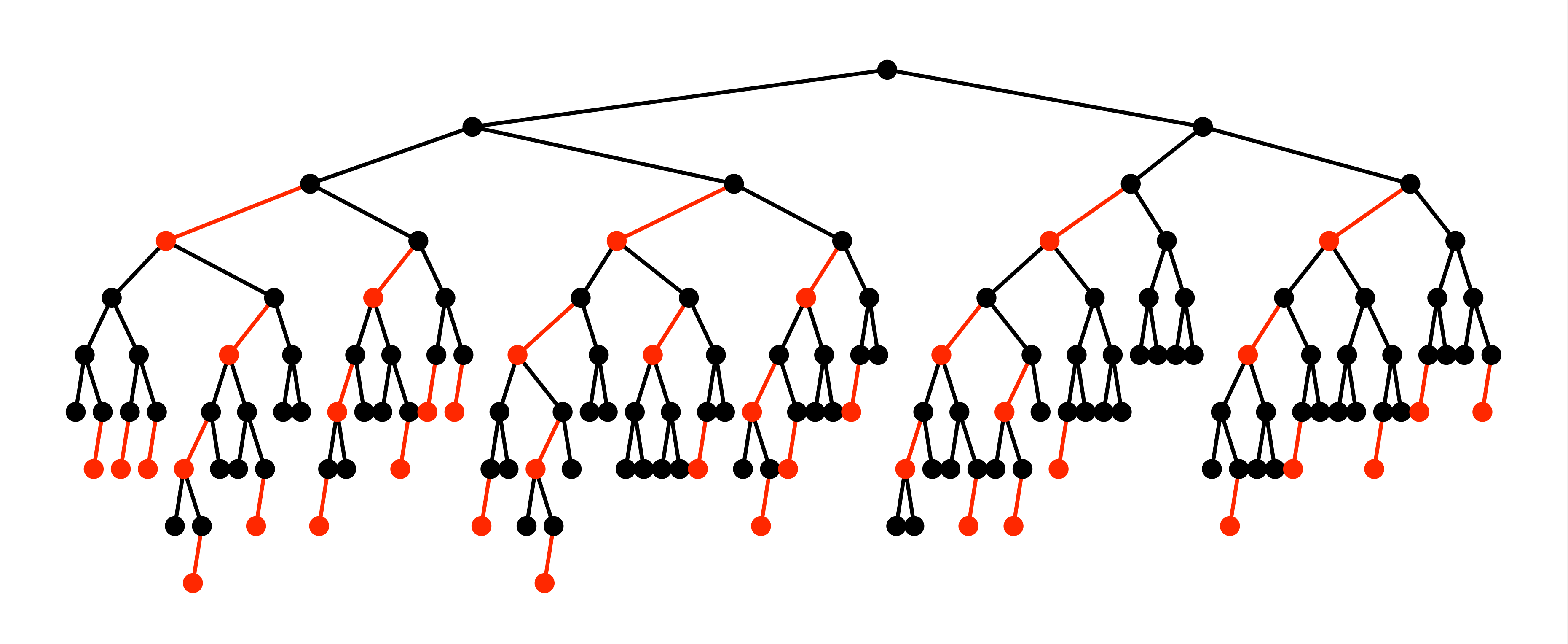
Insertion in a LLRB tree: visualization



255 random insertions

Balance in LLRB trees

Proposition. Height of tree is $\leq 2 \lg N$ in the worst case.

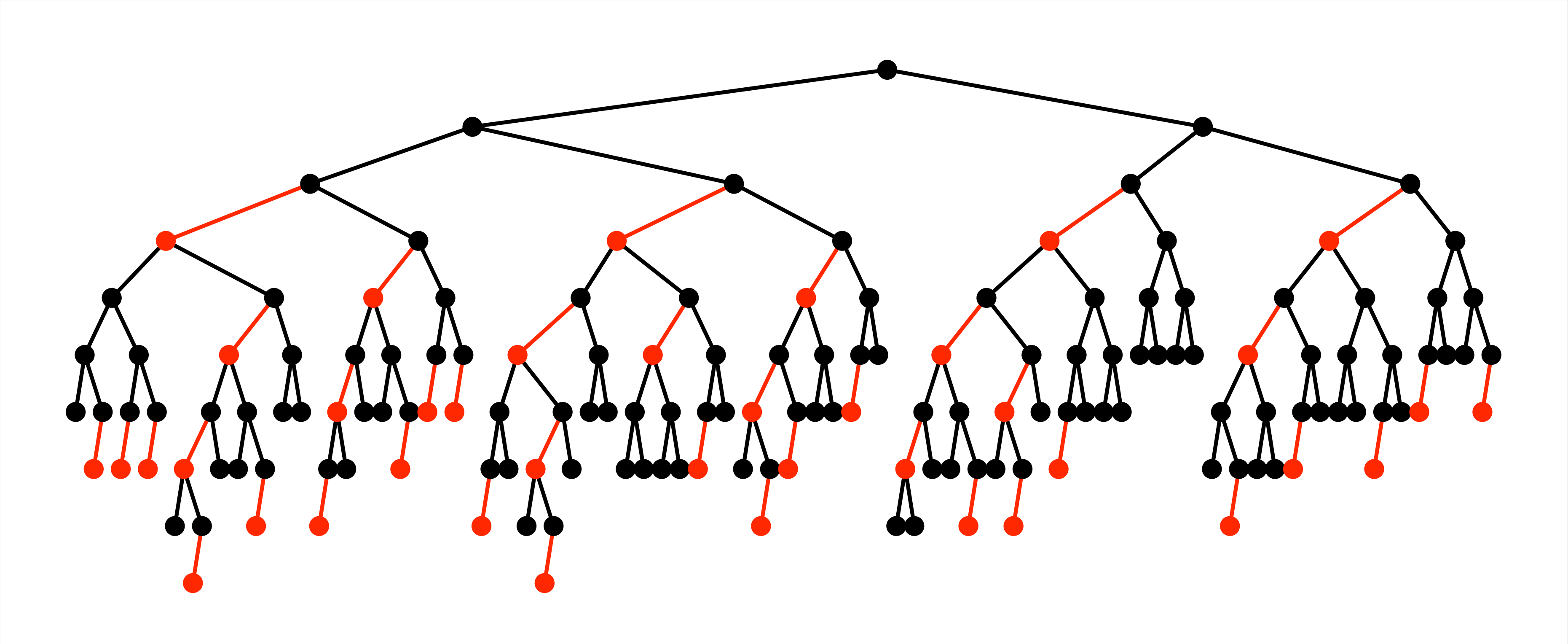


Balance in LLRB trees

Proposition. Height of tree is $\leq 2 \lg N$ in the worst case.

Pf.

- Every path from root to null link has same number of black links.
- Never two red links in-a-row.



Balance in LLRB trees

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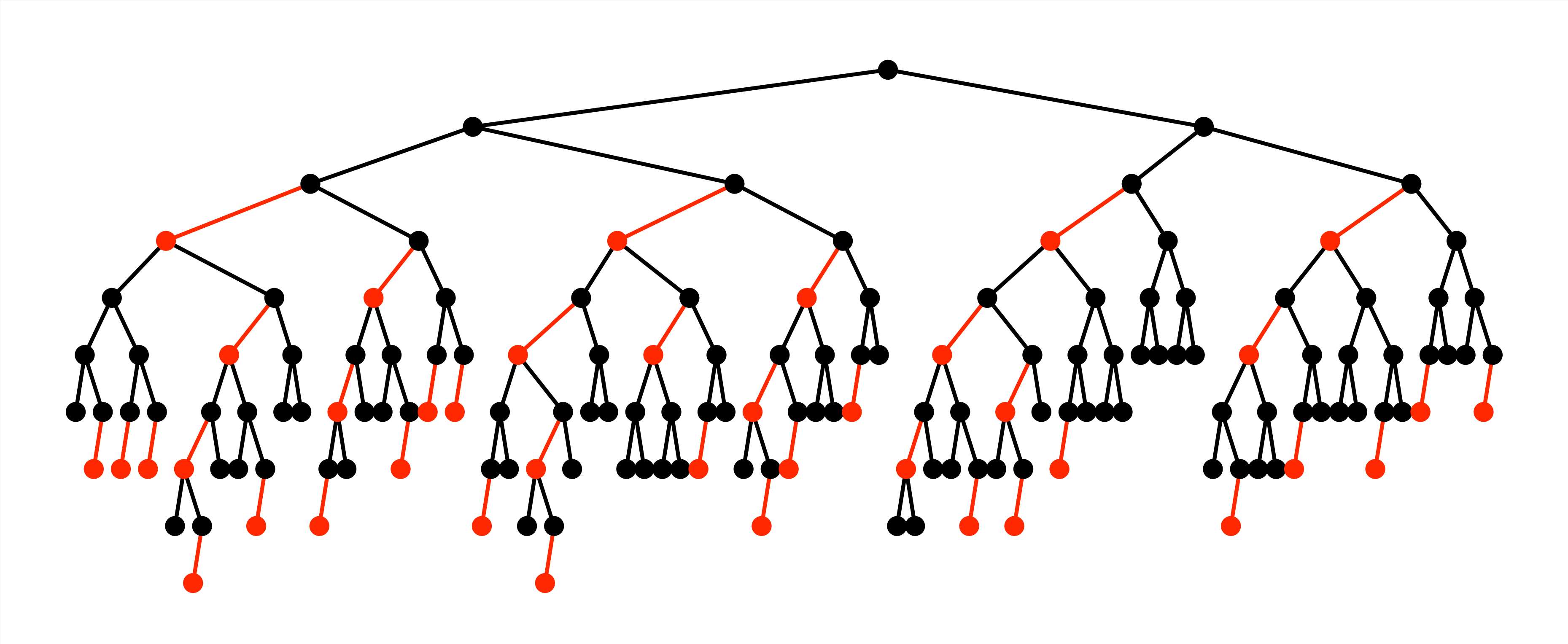

Property. Height of tree is $\sim 1.0 \lg N$ in typical applications.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	½ N	N	½ N		`equals()`
binary search (ordered array)	$\lg N$	N	N	$\lg N$	½ N	½ N	✔	`compareTo()`
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✔	`compareTo()`
2-3 tree	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	✔	`compareTo()`
red-black BST	$2 \lg N$	$2 \lg N$	$2 \lg N$	$1.0 \lg N$ *	$1.0 \lg N$ *	$1.0 \lg N$ *	✔	`compareTo()`

* exact value of coefficient unknown but extremely close to 1

War story: why red-black?

Xerox PARC innovations. [1970s]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- InterPress.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...

Xerox Alto

War story: why red-black?

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- WYSIWYG text editor.
- ...

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A DICHROMATIC FRAMEWORK FOR BALANCED TREES

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ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this

the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its

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3.3 Balanced Search Trees

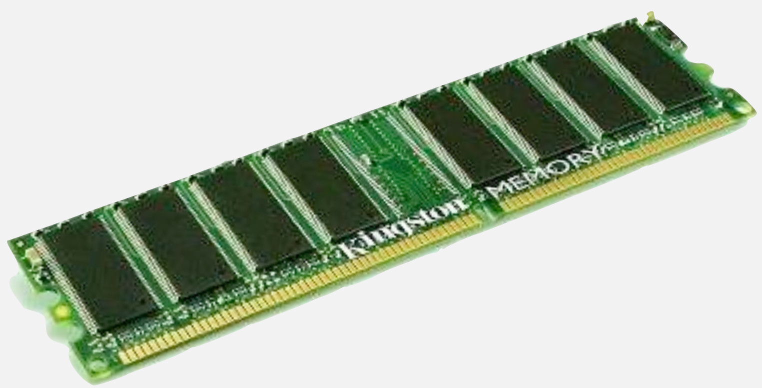
- 2-3 search trees
- red-black BSTs
- *B-trees*

File system model

Page. Contiguous block of data (e.g., a file or 4,096-byte chunk).

Probe. First access to a page (e.g., from disk to memory).

slow



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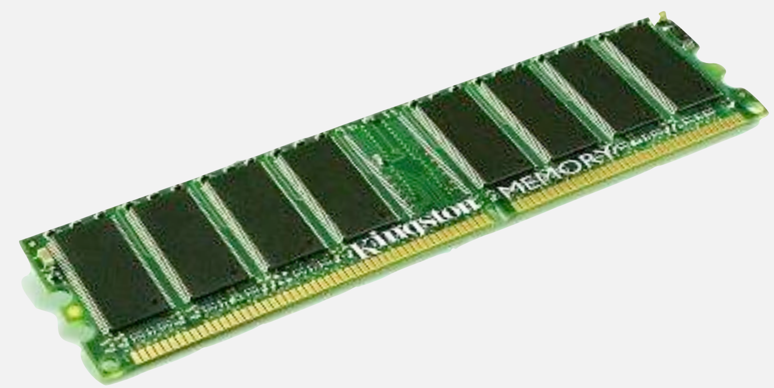

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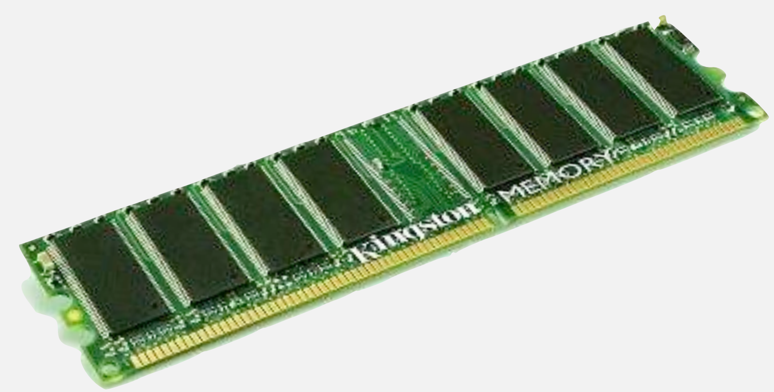

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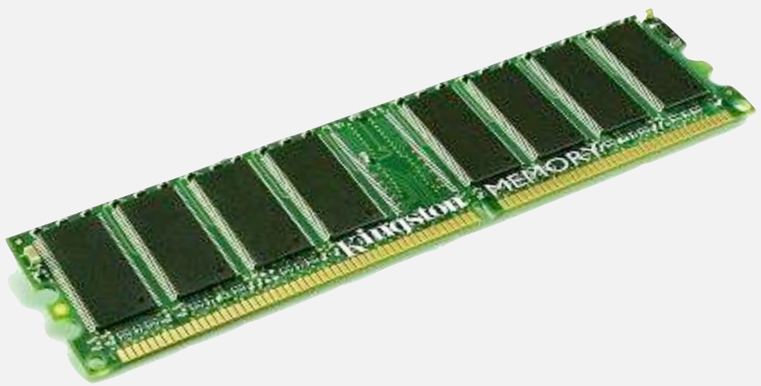

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Cost model. Number of probes.

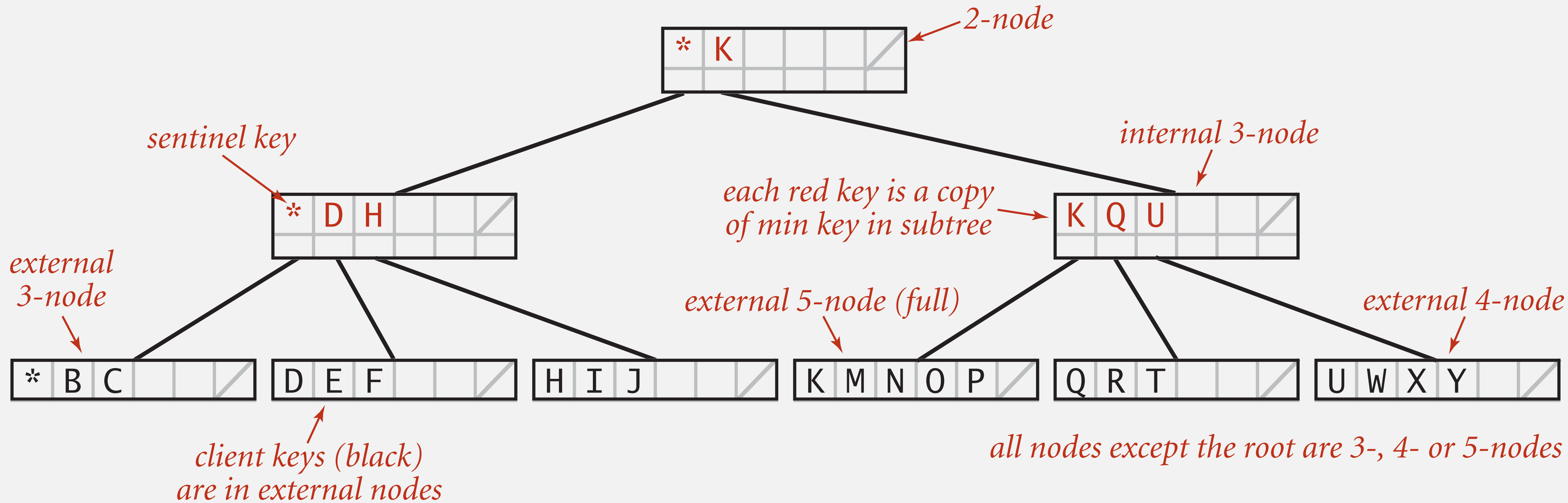
Goal. Access data using minimum number of probes.

B-trees (Bayer-McCreight, 1972)

B-tree. Generalize 2-3 trees by allowing up to $M - 1$ key-link pairs per node.

- At least 2 key-link pairs at root.
- At least $M / 2$ key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.

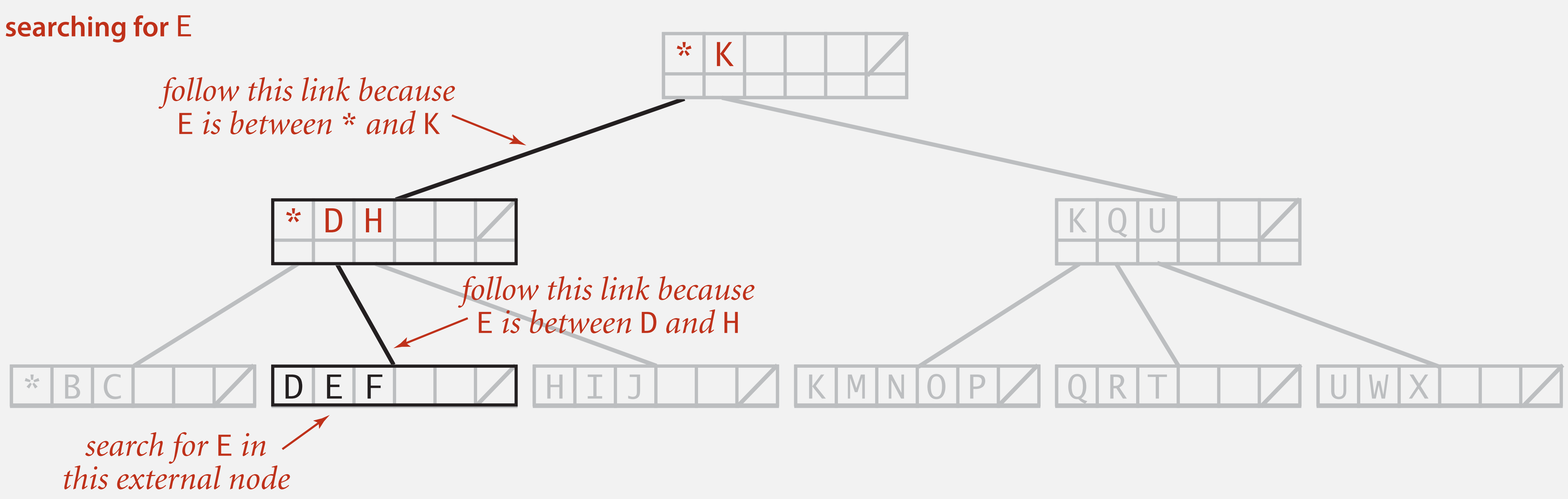
choose M as large as possible so that M links fit in a page, e.g., M = 1024



Anatomy of a B-tree set (M = 6)

Searching in a B-tree

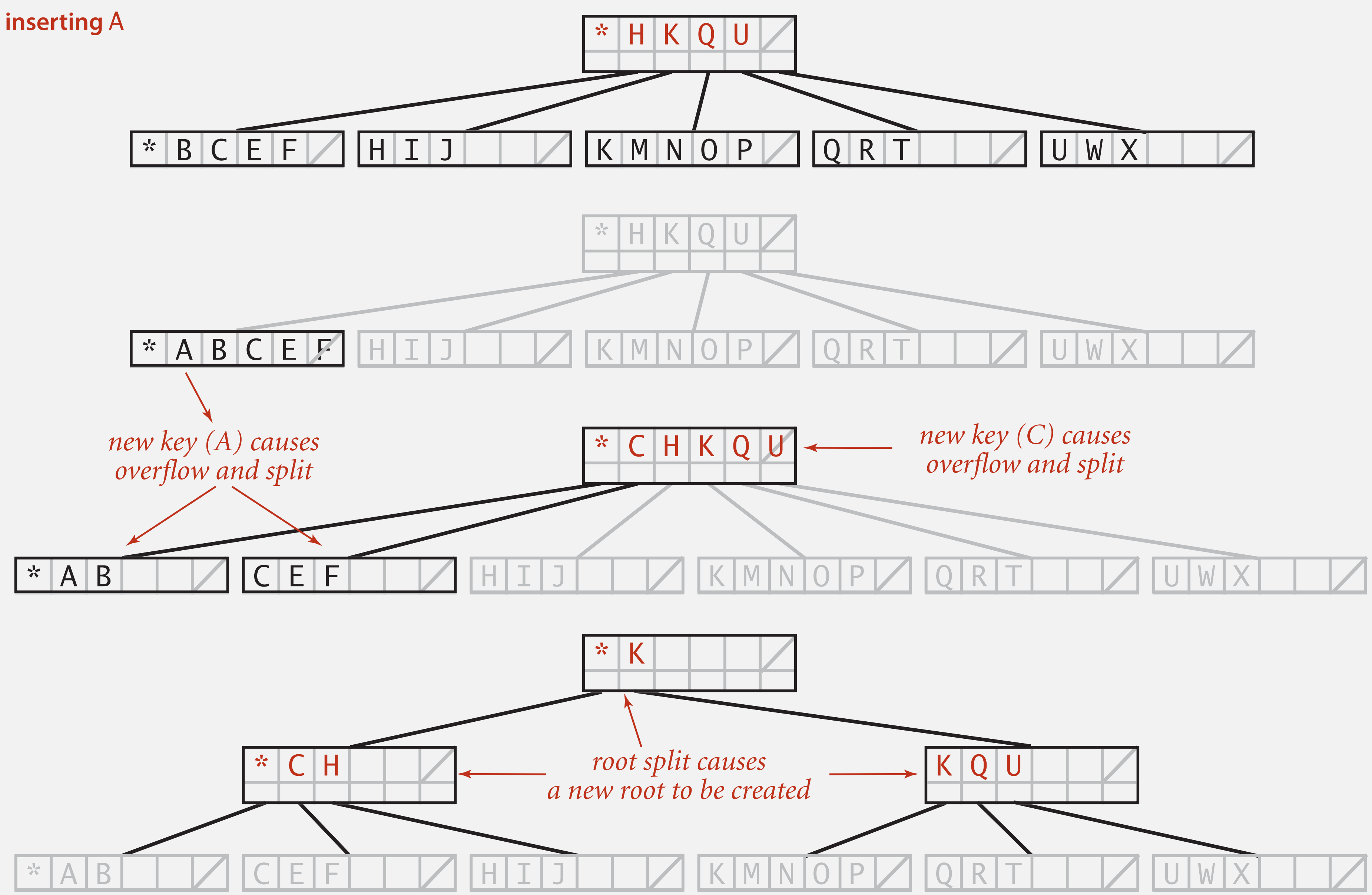
- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.



Searching in a B-tree set (M = 6)

Insertion in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with M key-link pairs on the way up the tree.



Inserting a new key into a B-tree set

Balance in B-tree

Proposition. A search or an insertion in a B-tree of order M with N keys requires between $\log_{M-1} N$ and $\log_{M/2} N$ probes.

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In practice. Number of probes is at most 4. ← M = 1024; N = 62 billion
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Optimization. Always keep root page in memory.

Building a large B tree

Balanced trees in the wild

Red-black trees are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.
- Emacs: conservative stack scanning.

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B-trees (and variants) are widely used for file systems and databases.

- Windows: NTFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.

Red-black BSTs in the wild

Common sense. Sixth sense.
Together they're the
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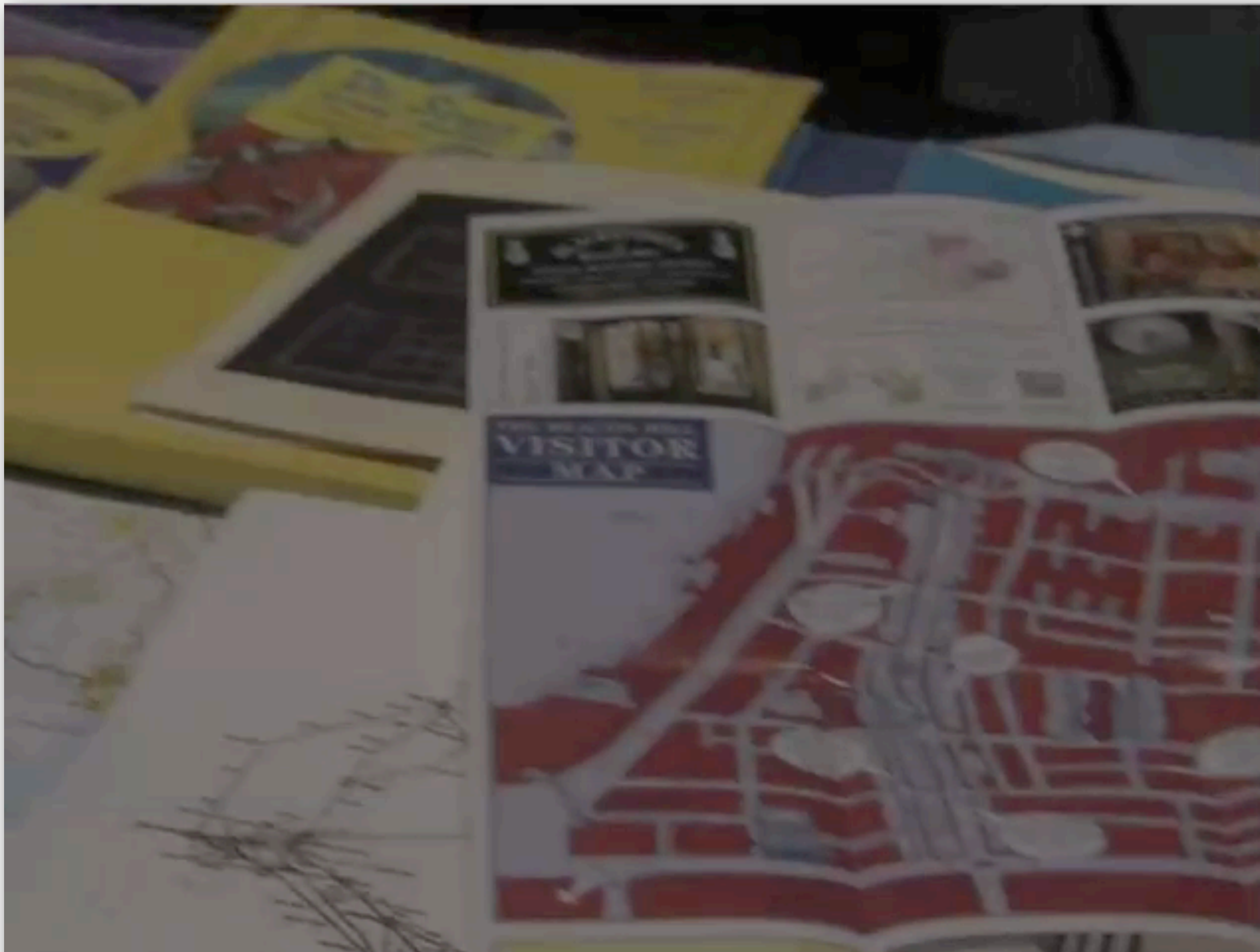

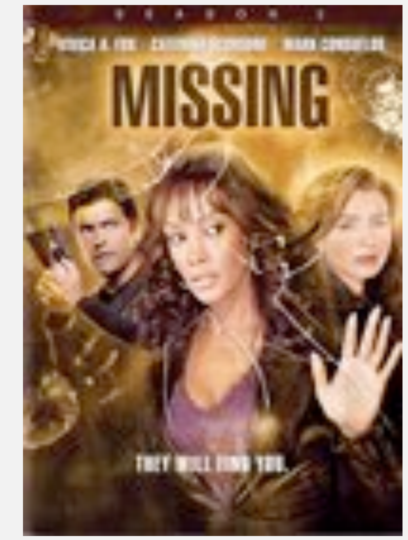

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Red-black BSTs in the wild

ACT FOUR

FADE IN:

48 INT. FBI HQ - NIGHT 48

Antonio is at THE COMPUTER as Jess explains herself to Nicole and Pollock. The CONFERENCE TABLE is covered with OPEN REFERENCE BOOKS, TOURIST GUIDES, MAPS and REAMS OF PRINTOUTS.

JESS
It was the red door again.

POLLOCK
I thought the red door was the storage container.

JESS
But it wasn't red anymore. It was black.

ANTONIO
So red turning to black means... what?

POLLOCK
Budget deficits? Red ink, black ink?

NICOLE
Yes. I'm sure that's what it is. But maybe we should come up with a couple other options, just in case.

Antonio refers to his COMPUTER SCREEN, which is filled with mathematical equations.

ANTONIO
It could be an algorithm from a binary search tree. A red-black tree tracks every simple path from a node to a descendant leaf with the same number of black nodes.

JESS
Does that help you with girls?