Week 9:
Eigenvalues and Eigenvectors

MSIN00180 Quantitative Methods for Business

In[640]:=

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$Post:=If[MatrixQ[#]||VectorQ[#],MatrixForm[#],#]&;
(*$Post:=If[MatrixQ[#],MatrixForm[#],#]&;*)
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Why Eigenvalues and Eigenvectors are Important

Dynamic Systems

- Solve large-scale dynamic systems such as marketplace & financial trading dynamics

Internet Search/Information Analysis

- Google PageRank - best way to solve using eigenvalues
- Document Search/Clustering - Latent Semantic Analysis

Network Analysis (eg Social Media)

- Network Eigenvalue Centrality - e.g. Identify Key “Influencers”
- Underpins important network clustering algorithms

AI/Machine Learning

- Used in some Facial Recognition algorithms
- Control Theory - important in Robotics and Autonomous Vehicles

“Big Data” Analysis

- Dimensionality Reduction - Principle Component Analysis

Diagonal matrices have a variety of benefits ...

It is easy to find the **determinant** of a diagonal matrix

- $\det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 2 \times 3 \times 4$

It is easy to find an arbitrary **power** of a diagonal matrix

- $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}^3 = \begin{pmatrix} 2^3 & 0 & 0 \\ 0 & 3^3 & 0 \\ 0 & 0 & 4^3 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 64 \end{pmatrix}$

Diagonalization

A square $n \times n$ matrix A is **diagonalizable** if (and only if) there exists an invertible matrix S such that

$$S^{-1} A S = B$$ where B is a diagonal matrix.

Or equivalently:

$$A = S B S^{-1}$$

We say that A is diagonalized as $S B S^{-1}$

Diagonalisation example

The diagonalisation of the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ is given by

$A = S\,B\,S^{-1}$ where

$$S = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

We will find out later how S and B are found.

Calculate A^{10} using diagonalization

$A^{10} = S\,B\,S^{-1}\,S\,B\,S^{-1}\,S\,B\,S^{-1}\ldots\ldots S\,B\,S^{-1}$

Each $S^{-1}\,S$ product is replaced by the identify matrix, which can then be ignored

$= S\,B\;I\,B\,I\,B\ldots\ldots B\,S^{-1}$

$= S\,B^{10}\,S^{-1}$

Calculate A^{10} using diagonalization

In[641]:=

```
S = (  0  0 -1
      -1  1  0
       1  0  1 );

S . ( 1^10   0     0
      0     2^10   0
      0      0    2^10 ) . Inverse[S] // MatrixForm
```

Out[642]//MatrixForm=

$$\begin{pmatrix} 1024 & 0 & 0 \\ 1023 & 1024 & 1023 \\ -1023 & 0 & 1 \end{pmatrix}$$

To raise a diagonal matrix to the n^{th} power we only need to raise each of the diagonal elements to the n^{th} power.

This calculation has only required 2 matrix products and 3 scalar power calculations.

Diagonalisation using the column vectors of S and the diagonal elements of B

Given $\boldsymbol{A = S\,B\,S^{-1}}$

When we represent S as a matrix of column vectors and B as a diagonal matrix with scalar diagonal elements $\lambda_1 \ldots \lambda_n$ thus

$$S = \begin{pmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{pmatrix} \qquad B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

It can be shown that the diagonal scalars λ_i of B and the corresponding column vectors of S, $\vec{v}_i$ must satisfy

$$A\,\vec{v}_i = \lambda_i\,\vec{v}_i \qquad \text{for all } i \in [1,\ n]$$

Eigenvalues and eigenvectors

Consider a square $n \times n$ matrix A.

A nonzero vector $\vec{v}$ is called an **eigenvector** of A if

$$A \vec{v} = \lambda \vec{v}$$

for some scalar λ, called the associated **eigenvalue** of eigenvector $\vec{v}$.

Eigenvalues and eigenvectors

Alternatively we can say:

When **diagonalising** a square matrix A to the form $A = S\,B\,S^{-1}$

- the **eigenvalues** of A are the scalar values λ_i that form the diagonal elements of B, and
- the **eigenvectors** of A are the corresponding column vectors $\vec{v}_i$ of S.

Finding eigenvalues when eigenvectors are known

We are told that $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix}$

What is the associated eigenvalue?

We are told that $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is an eigenvector of $A=\begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix}$

What is the associated eigenvalue?

If $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is an eigenvector then there must be an eigenvalue λ that satisfies

$$A\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

We can see that the **eigenvalue $\boldsymbol{\lambda = 2}$** satisfies this equation.

Finding eigenvectors when eigenvalues are known

Consider the matrix $A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

a. Given 2 and 4 are eigenvalues, find all the corresponding eigenvectors.

b. Diagonalise A.

Solution part (a)(i): $A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

For $\boldsymbol{\lambda = 2}$ to be an eigenvalue there must exist an eigenvector $\vec{v}$ such that

$$\boldsymbol{A\,\vec{v} = \lambda\,\vec{v} = 2\,\vec{v}}$$

We can re-express this using the identity matrix I thus:

$$A\,\vec{v} = 2\,I\,\vec{v} \quad \Longrightarrow \quad \boldsymbol{(A - 2\,I)\,\vec{v} = \vec{0}}$$

and $\quad A - 2\,I = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}$

So to find the eigenvector $\vec{v}$ we need to solve: $\quad \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \vec{v} = \vec{0}$

Solution part (a)(i)

So to find the eigenvector $\vec{v}$ we need to solve: $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{2} \end{pmatrix} \vec{\mathbf{v}} = \vec{\mathbf{0}}$ ...

We can use Gauss-Jordan elimination to solve this:

augmented matrix: $\begin{pmatrix} 0 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}$ swap rows $\longrightarrow \begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ divide row 1 by 3 $\longrightarrow \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

This RREF has the infinite solution $\vec{v} = \begin{pmatrix} -\frac{2}{3} t \\ t \end{pmatrix}$

Giving the general eigenvector solution $\vec{v} = t \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$ where t is any scalar

To find a specific eigenvector solution let $t = 1$ giving the eigenvector $\vec{v} = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$

Therefore $\lambda = 2$ is an eigenvalue of A with an associated eigenvector $\vec{v} = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$ (or any multiple of this vector)

Solution part (a) (ii): $A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

Now consider the candidate eigenvalue $\lambda = 4$:

$A\vec{v} = 4\vec{v} \quad \Longrightarrow \quad (A - 4I)\vec{v} = \vec{0}$

So to find the eigenvector $\vec{v}$ we need to solve $\begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix}\vec{v} = \vec{0}$

Now use Gauss-Jordan to find the RREF ...

In[643]:=

```
RowReduce[(-2 0 0
           3 0 0)]
```

Out[643]//MatrixForm=

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

This RREF represents the infinite solution $\vec{v} = \begin{pmatrix} 0 \\ t \end{pmatrix}$

Choose $t = 1$ for convenience gives the eigenvector $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Therefore $\boldsymbol{\lambda = 4}$ is an eigenvalue of A with an **associated eigenvector** $\boldsymbol{\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Diagonalise a matrix using its eigenvectors and eigenvalues

For the matrix $A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

b. Diagonalise A.

Solution part (b): Diagonalise *A*

The eigenvectors are $\begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We use these eigenvectors to form the columns of matrix S thus

$$S = \begin{pmatrix} -\frac{2}{3} & 0 \\ 1 & 1 \end{pmatrix}$$

The inverse of S is given by $S^{-1} = \frac{1}{\left(-\frac{2}{3}\times 1 - 1\times 0\right)} \begin{pmatrix} 1 & 0 \\ -1 & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$

B contains the **corresponding** eigenvalues as it diagonal elements

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

So A can be diagonalised thus: $A = S\,B\,S^{-1} = \begin{pmatrix} -\frac{2}{3} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$

Discrete Dynamic Systems

Solving dynamical systems using eigenvalues and eigenvectors

Dynamical systems provide a powerful application of eigenvectors and eigenvalues.

We will start by considering the solution of **discrete** dynamical systems.

Discrete dynamical systems

We can represent a **discrete dynamical system** as a linear system thus

$$\vec{x}(t+1) = A\,\vec{x}(t)$$

with initial conditions $\vec{x}(0) = \vec{x}_0$

The mini-web case study we considered in Week 7 is an example of a discrete dynamic system.

Example: A population model of coyotes and roadrunners

$c(t)$ be the population of coyotes at year t and
$r(t)$ be the population of roadrunners at year t

The following equations model the transition of these population from year to year.

$c(t+1) = 0.86\, c(t) + 0.08\, r(t)$
$r(t+1) = -0.12\, c(t) + 1.14\, r(t)$

With $\vec{x}(t) = \begin{pmatrix} c(t) \\ r(t) \end{pmatrix}$ these equations can be represented as the following linear system

$\vec{x}(t+1) = A\, \vec{x}(t) \qquad A = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}$

Example: A population model of coyotes and roadrunners

Assume the following initial values: $\vec{x}_0 = \begin{pmatrix} c_0 \\ r_0 \end{pmatrix}$

Then we can calculate the populations for each subsequent year $\vec{x}(1)$, $\vec{x}(2)$, $\vec{x}(3)$ … thus

$\vec{x}(1) = A\,\vec{x}_0$

$\vec{x}(2) = A\,\vec{x}(1) = A^2\,\vec{x}_0$

$\vec{x}(3) = A\,\vec{x}(2) = A^3\,\vec{x}_0$

$\vdots$

$$\vec{x}(t) = A^t\,\vec{x}_0$$

This equation allows us to calculate the populations for any given year t

- BUT it provides **no insight** into how the system will **behave in the long-term** nor how the **behaviour depends on initial values**.

We will now consider some different initial value cases …

Initial Value Case 1: $\vec{x}_0 = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$

$$\vec{x}(1) = A\,\vec{x}_0 = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 100 \\ 300 \end{pmatrix} = \begin{pmatrix} 110 \\ 330 \end{pmatrix}$$

We observe that this result is a scalar multiple of the input vector $\begin{pmatrix} 100 \\ 300 \end{pmatrix}$

$$\vec{x}(1) = \begin{pmatrix} 110 \\ 330 \end{pmatrix} = 1.1 \begin{pmatrix} 100 \\ 300 \end{pmatrix} = 1.1\,\vec{x}_0$$

It follows that

$$\vec{x}(1) = A\,\vec{x}_0 = 1.1\,\vec{x}_0$$
$$\vec{x}(2) = A\,\vec{x}(1) = A\left(1.1\,\vec{x}_0\right) = 1.1\left(A\,\vec{x}_0\right) = 1.1\left(1.1\,\vec{x}_0\right) = 1.1^2\,\vec{x}_0$$
$$\vdots$$
$$\vec{x}(t) = 1.1^t\,\vec{x}_0$$

so $\begin{pmatrix} c(t) \\ r(t) \end{pmatrix} = 1.1^t \begin{pmatrix} 100 \\ 300 \end{pmatrix}$ or $\boxed{\begin{array}{l} c(t) = 100\,(1.1)^t \\ r(t) = 300\,(1.1)^t \end{array}}$

Initial Value Case 1: $\vec{x}_0 = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$

$$\boxed{\begin{aligned} c(t) &= 100\,(1.1)^t \\ r(t) &= 300\,(1.1)^t \end{aligned}}$$

System behaviour

We observe that with these initial values both **populations grow exponentially, by 10% each year.**

What is the scalar value 1.1 and what is the vector $\begin{pmatrix} 100 \\ 300 \end{pmatrix}$?

Initial Value Case 2: $c_0 = 200$ and $r_0 = 100$

$$\vec{x}(1) = A\,\vec{x}_0 = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 180 \\ 90 \end{pmatrix} = 0.9 \begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

Using the same reasoning as for Case 1 we can show that

$$\boxed{\begin{array}{l} c(t) = 200\,(0.9)^t \text{ and} \\ r(t) = 100\,(0.9)^t \end{array}}$$

This time the populations decline by 10% each year. The initial populations are mismatched: too many coyotes are chasing too few roadrunners.

Eigenbasis of A
(Eigenbasis = set of eigenvectors)

We observe that $\begin{pmatrix} 200 \\ 100 \end{pmatrix}$ is another **eigenvector** of matrix A with **eigenvalue** of 0.9.

As A is a 2×2 matrix we have found all the eigenvectors of A so

$$\text{eigenbasis}(A) = \left\{ \begin{pmatrix} 100 \\ 300 \end{pmatrix}, \begin{pmatrix} 200 \\ 100 \end{pmatrix} \right\}$$

Initial Value Case 3: $c_0 = 1000$ and $r_0 = 1000$

Consider an initial value case that does not happen to also be a eigenvector ...

$$\vec{x}(1) = A\,\vec{x}_0$$
$$= \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$$
$$= \begin{pmatrix} 940 \\ 1020 \end{pmatrix} \neq \lambda \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$$

This time the transformation does not lead to a scalar multiple of the original vector.

$\Longrightarrow \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$ is not an eigenvector of A.

We need a different approach to find closed formulas for $c(t)$ and $r(t)$ in this case …

Initial Value Case 3: $c_0 = 1000$ and $r_0 = 1000$

We can write **any vector in $\mathbb{R}^2$** as a **linear combination** of the two **eigenvectors** we have found.

So we can express any initial value vector $\vec{x}_0$ as a linear combination:

$\vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2$ where $\vec{v}_1 = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 200 \\ 100 \end{pmatrix}$.

For the initial values $\vec{x}_0 = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$:

$$\begin{pmatrix} 1000 \\ 1000 \end{pmatrix} = c_1 \begin{pmatrix} 100 \\ 300 \end{pmatrix} + c_2 \begin{pmatrix} 200 \\ 100 \end{pmatrix} \implies \begin{pmatrix} 100 & 200 \\ 300 & 100 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$$

We now need to solve this for c_1 and c_2 ...

Initial Value Case 3: $c_0 = 1000$ and $r_0 = 1000$

Find c_1 and c_2 via Gauss-Jordan elimination

In[644]:=

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RowReduce[(100 200 1000
           300 100 1000)] //MatrixForm
```

Out[644]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

i.e. $c_1 = 2$ and $c_2 = 4$

so $\vec{x}_0 = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix} = 2\,\vec{v}_1 + 4\,\vec{v}_2$ ....(1)

Initial Value Case 3: $c_0 = 1000$ and $r_0 = 1000$

Using $\vec{x}_0 = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix} = 2\,\vec{v}_1 + 4\,\vec{v}_2$ we get

$$\vec{x}(t) = A^t\,\vec{x}_0 \quad = A^t\left(2\,\vec{v}_1 + 4\,\vec{v}_2\right) \quad = 2\,A^t\,\vec{v}_1 + 4\,A^t\,\vec{v}_2$$

And recalling that: $A^t\,\vec{v}_1 = (1.1)^t\,\vec{v}_1$
$A^t\,\vec{v}_2 = (0.9)^t\,\vec{v}_2$

$$\vec{x}(t) = 2\,A^t\,\vec{v}_1 + 4\,A^t\,\vec{v}_2 \quad = 2\,(1.1)^t\,\vec{v}_1 + 4\,(0.9)^t\,\vec{v}_2$$

$$= 2\,(1.1)^t\begin{pmatrix} 100 \\ 300 \end{pmatrix} + 4\,(0.9)^t\begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

$$= (1.1)^t\begin{pmatrix} 200 \\ 600 \end{pmatrix} + (0.9)^t\begin{pmatrix} 800 \\ 400 \end{pmatrix}$$

$$\boxed{\begin{aligned} c\,(t) &= 200\,(1.1)^t + 800\,(0.9)^t \\ r(t) &= 600\,(1.1)^t + 400\,(0.9)^t \end{aligned}}$$

What is the long term behaviour of the populations?

Initial Value Case 3: $c_0 = 1000$ and $r_0 = 1000$

What is the long term behaviour of the populations?

when $c_0 = 1000$ and $r_0 = 1000 \longrightarrow$ $\boxed{\begin{array}{l} c(t) = 200\,(1.1)^t + 800\,(0.9)^t \\ r(t) = 600\,(1.1)^t + 400\,(0.9)^t \end{array}}$

Answer

As the $(0.9)^t$ term approaches zero as t increases, both populations eventually grow at the rate of 10% per year with a ratio $r(t)$ to $c(t)$ of 3:1.

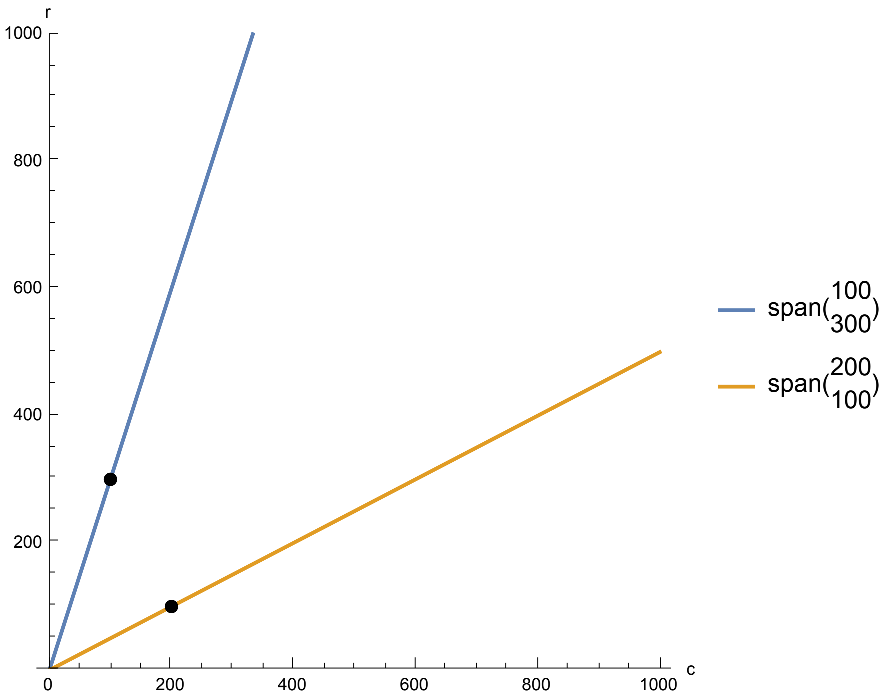
Eigenvectors and phase plots

The span of an eigenvector is a straight line trajectory on a phase plot

In[645]:=

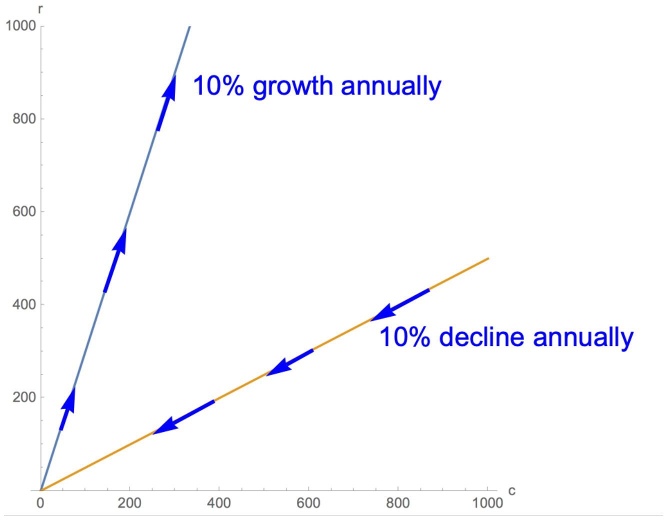
```
Plot[{3x,0.5x},{x,0,1000},PlotRange→{0,1000},Epilog→{PointSize[Large],Point[{{100,300}
```

Out[645]=



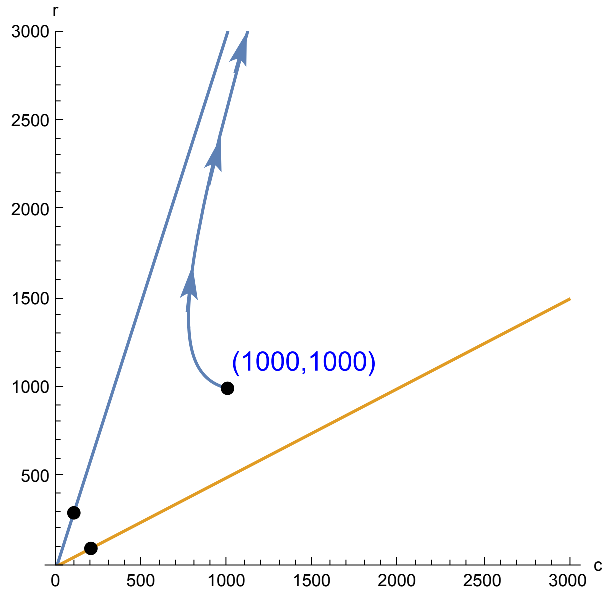
Eigenvectors and phase plots

The associated eigenvalues indicate trajectory direction (> 1 or < 1)



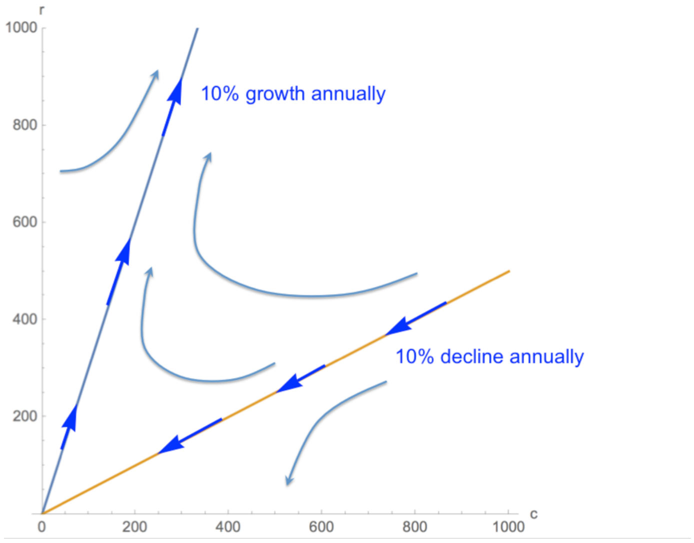
Eigenvectors and phase plots

Plotting the trajectory from (1000,1000) using the equations found for c(t) and r(t) in Case 3 helps provide an illustration of more general behaviour



Eigenvectors and phase plots

We can now sketch in the general direction of other trajectories



The general solution of a discrete linear dynamical system

The following two theorems formalise what we have already found in the previous example.

General solution in terms of A^t

Consider the dynamical system

$\vec{x}(t+1) = A\,\vec{x}(t)$ with $\vec{x}(0) = \vec{x}_0$ and $\vec{x}(t)$ in $\mathbb{R}^n$

Then

$$\vec{x}(t) = A^t\,\vec{x}_0$$

The general solution of a discrete linear dynamical system

Suppose we can find the eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$ for A, with associated eigenvalues given by $A\,\vec{v}_i = \lambda_i\,\vec{v}_i$.

Find the coefficients $c_1, \ldots, c_n$ that form a linear combination of the vector $\vec{x}_0$ with respect the eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$:

$$\vec{x}_0 = c_1\,\vec{v}_1 + \ldots + c_n\,\vec{v}_n$$

Then

$$\vec{x}(t) = A^t\,\vec{x}_0 = c_1\,A^t\,\vec{v}_1 + \ldots + c_n\,A^t\,\vec{v}_n$$

$$\boldsymbol{\vec{x}(t) = c_1\,\lambda_1{}^t\,\vec{v}_1 + \ldots + c_n\,\lambda_n{}^t\,\vec{v}_n}$$

Finding the eigenvalues of a matrix

Consider the definitional equation:

$A\vec{v} = \lambda\vec{v}$ where A is a square $n \times n$ matrix.

We can restate this thus using the **identity matrix** I_n:

$A\vec{v} = \lambda I_n \vec{v} \quad \Longrightarrow \quad (A - \lambda I_n)\vec{v} = \vec{0}$

$A\vec{v} = \lambda\vec{v}$ implies $\vec{v}$ must have an infinite solution since any multiple of $\vec{v}$ satisfies the same equation i.e.

$k(A\vec{v}) = k(\lambda\vec{v}) \;\Rightarrow\; A(k\vec{v}) = \lambda(k\vec{v})$

The solution to $(A - \lambda I_n)\vec{v} = \vec{0}$ must therefore be an infinite solution.

This implies the inverse of $(A - \lambda I_n)$ can not exist.

And if a matrix is non-invertible its **determinant is zero**, so

$$\boxed{\det(A - \lambda I_n) = 0}$$

This is an important result and the key to finding eigenvalues

The characteristic equation

Consider an $n \times n$ matrix A and a scalar λ.

Then λ is an **eigenvalue** of A if (and only if)

$$\det(A - \lambda I_n) = 0$$

This is called the **characteristic equation** matrix A.

The polynomial that results from calculating $\det(A - \lambda I_n)$ is called the **characteristic polynomial**.

Example 1: find the eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

Using the characteristic equation

$\det(A - \lambda I_2) = 0$

$\det\left(\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$

$\det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0$

What is the determinant and what are the values of λ?

Example 1: find the eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

What is determinant and what are the values of λ?

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 4 \times 2 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

Therefore the 2 eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$.

Example 2: find the eigenvalues of $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$

Using the characteristic equation $\det(A - \lambda I_3) = 0$:

$$\det\begin{pmatrix} 2-\lambda & 3 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0$$

What is the determinant and what are the values of λ?

Example 2: find the eigenvalues of $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$

What is determinant and what are the values of λ?

$$\det\begin{pmatrix} 2-\lambda & 3 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0$$

$(2-\lambda)(3-\lambda)(4-\lambda) = 0$ $\longleftarrow$ **determinant of triangular matrix**

Matrix A therefore has 3 eigenvalues, $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 4$.

This highlights a useful theorem …

Eigenvalues of triangular and diagonal matrices

THEOREM:

The **eigenvalues** of a **triangular or a diagonal matrix** are its **diagonal entries**.

Example: find the eigenvalues of $\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & 5 & 4 & 3 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$

Form the characteristic equation:

$$\det\begin{pmatrix} 5-\lambda & 4 & 3 & 2 & 1 \\ 0 & 4-\lambda & 3 & 2 & 1 \\ 0 & 0 & 5-\lambda & 4 & 3 \\ 0 & 0 & 0 & 4-\lambda & 3 \\ 0 & 0 & 0 & 0 & 5-\lambda \end{pmatrix} = 0$$

$(5-\lambda)^3 (4-\lambda)^2 = 0 \quad \longrightarrow$ Giving the eigenvalues 5 and 4.

Note that we only have 2 distinct eigenvalues yet we have a 5×5 matrix.

This introduces the concept of **algebraic multiplicities** …

Recurring eigenvalues and algebraic multiplicity

An eigenvalue λ_0 has **algebraic multiplicity** k if λ_0 is a root of multiplicity k of the characteristic equation

$\mathbf{det}(A - \lambda I_n) = (\lambda_0 - \lambda)^k g(\lambda)$ where $g(\lambda)$ is some polynomial with $g(\lambda_0) \neq 0$

We write $\mathbf{almu}(\lambda_0) = k$.

Example: find the eigenvalues of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(1-\lambda)(1-\lambda) + 1 + 1 - (1-\lambda) - (1-\lambda) - (1-\lambda) = 0$$

$$3\lambda^2 - \lambda^3 = 0$$
$$\lambda^2(3-\lambda) = 0$$

which is the same as $(\lambda - 0)^2 (3-\lambda) = 0$

We can now conclude that *A* has the following eigenvalues:

- 0 with algebraic multiplicity 2
- 3 with algebraic multiplicity 1

Alternatively, we can state that *A* has the eigenvalues 0, 0, 3.

Eigenspaces

The set of all the eigenvector solutions to $(A - \lambda I_n)\vec{v} = \vec{0}$, is called the **eigenspace** associated with the eigenvalue λ and is denoted E_λ.

$$E_\lambda = \left\{\vec{v} \ : \ (A - \lambda I_n)\vec{v} = \vec{0}\right\}$$

As this is the set of all scalar multiples of any eigenvector solution we can also state this as:

$E_\lambda = \left\{k\vec{v} \ : \ k \in \mathbb{R}\right\}$

A set of formed from all multiples of a vector is known as the “span” of a vector:

$E_\lambda = \text{span}(\vec{v})$

Eigenspaces: coyote/roadrunner example

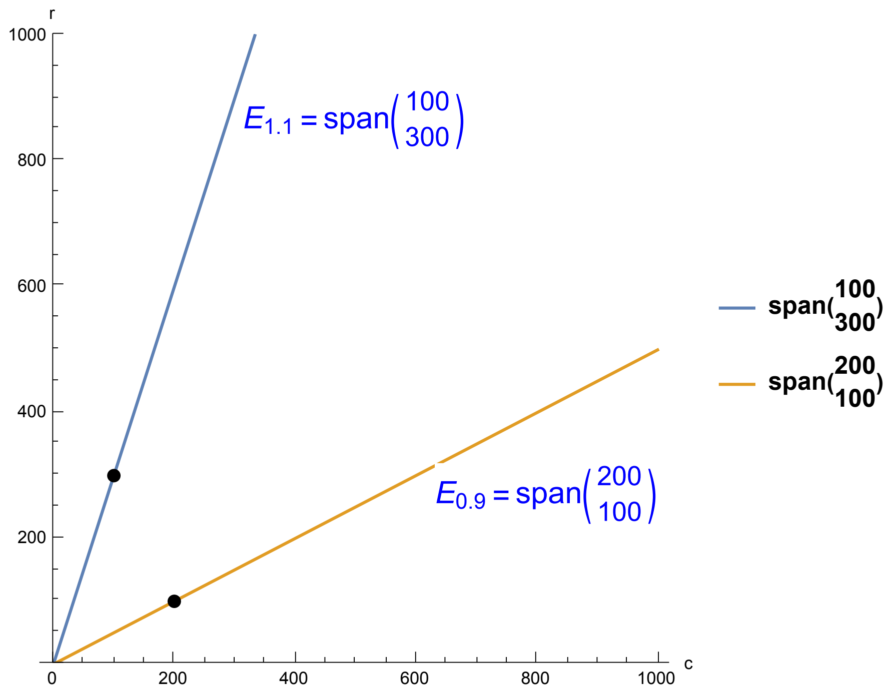
In this earlier problem we found the following eigenvalues and associated eigenvectors

- $\lambda = 1.1$ and $\vec{v} = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$ and $\lambda = 0.9$ and $\vec{v} = \begin{pmatrix} 200 \\ 100 \end{pmatrix}$

The **eigenspaces** are therefore

- $E_{1.1} = \text{span}\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ and $E_{0.9} = \text{span}\begin{pmatrix} 200 \\ 100 \end{pmatrix}$

These eigenspaces are the two straight line phase plot trajectories we plotted earlier …



Find the eigenspaces of $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

We found earlier that the eigenvalues of this matrix are $\lambda_1 = 5$ and $\lambda_2 = -1$.

Find E_5

$$(A - 5\,I) = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}$$

$$E_5 = \left\{ \vec{v} : \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \vec{v} = \vec{0} \right\}$$

In[646]:=

```
m= (-4  2
     4 -2);
m⟦1⟧ /= -4;
m⟦2⟧ -= 4m⟦1⟧; m
```

Out[648]//MatrixForm=

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

This has the infinite solution $\vec{v} = \begin{pmatrix} \frac{1}{2}\,t \\ t \end{pmatrix} = t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$

so $E_5 = \left\{ t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} = \text{span}\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$

Find the eigenspaces of $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

Find E_{-1}

$(A - (-1)\,I) = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$

$E_{-1} = \left\{ \vec{v} : \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \vec{v} = \vec{0} \right\}$

In[649]:=

```
m=(2 2
   4 4);
m[[1]] /= 2;
m[[2]] -= 4m[[1]]; m
```

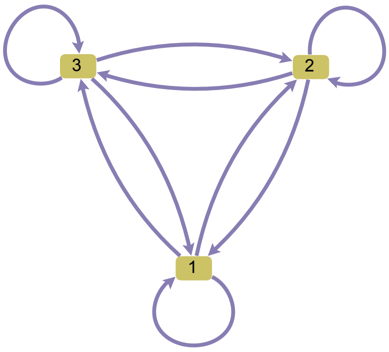
Out[651]//MatrixForm=

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

This has the infinite solution $\vec{v} = \begin{pmatrix} -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

so $E_{-1} = \left\{ t \begin{pmatrix} -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} = \text{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Mini-Web problems are examples of discrete dynamical systems



This web has the following transition equation

$$\vec{x}(t+1) = A\,\vec{x}\,(t) \qquad \text{where} \quad A = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{pmatrix}$$

and an initial distribution vector be $\vec{x}_0 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$

Distribution after *t* transitions

After *t* transitions the distribution vector is given by

$$\vec{x}\,(t) = A^t\,\vec{x}_0$$

Find the eigenvalues of *A*

Using *Mathematica* will can find the characteristic equation and solve it to find the eigenvalues of *A*

In[652]:=

```
A= (0.7 0.1 0.2
    0.2 0.4 0.2
    0.1 0.5 0.6);
Solve[Det[A-λ IdentityMatrix[3]]==0,λ]
```

Out[653]//MatrixForm=

$$\begin{pmatrix} \lambda \to 0.2 \\ \lambda \to 0.5 \\ \lambda \to 1. \end{pmatrix}$$

Find the eigenvectors of *A*, expressed as eigenspaces

Again, using *Mathematica*, we can find the eigenvectors by finding the RREF of $(A - \lambda I_3)$

In[654]:=

```
A= (0.7 0.1 0.2
    0.2 0.4 0.2
    0.1 0.5 0.6);
Print["λ=",#,"  RREF=",
MatrixForm[Chop[RowReduce[A-(# IdentityMatrix[3])]]]]& /@  {0.2,0.5,1};
```

λ=0.2   RREF= $\begin{pmatrix} 1 & 0 & 0.25 \\ 0 & 1 & 0.75 \\ 0 & 0 & 0 \end{pmatrix}$

λ=0.5   RREF= $\begin{pmatrix} 1 & 0 & 1. \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

λ=1   RREF= $\begin{pmatrix} 1 & 0 & -0.875 \\ 0 & 1 & -0.625 \\ 0 & 0 & 0 \end{pmatrix}$

From these RREF results we can determine the eigenspaces

$$E_{0.2} = \text{span}\begin{pmatrix} -0.25 \\ -0.75 \\ 1 \end{pmatrix} = \text{span}\begin{pmatrix} -1 \\ -3 \\ 4 \end{pmatrix}$$

$$E_{0.5} = \text{span}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$E_{1} = \text{span}\begin{pmatrix} 0.875 \\ 0.625 \\ 1 \end{pmatrix} = \text{span}\begin{pmatrix} 7 \\ 5 \\ 8 \end{pmatrix}$$

Find a closed formula for this mini-Web dynamical system

Remember to find a general formula:

- Find the coefficients $c_1, \ldots, c_n$ that form a linear combination of the vector $\vec{x}_0$ with respect the eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$:

  $\vec{x}_0 = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n$

- Whence we have the closed formula solution $\vec{x}(t) = c_1 \lambda_1{}^t \vec{v}_1 + \ldots + c_n \lambda_n{}^t \vec{v}_n$

$\vec{x}_0 = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n$

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ -3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 7 \\ 5 \\ 8 \end{pmatrix} \implies \begin{pmatrix} -1 & 1 & 7 \\ -3 & 0 & 5 \\ 4 & -1 & 8 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

Using *Mathematica* we can solve this matrix equation thus

In[656]:=

```
RowReduce[(-1  1  7  1/3
           -3  0  5  1/3
            4 -1  8  1/3)]//MatrixForm
```

Out[656]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{36} \\ 0 & 1 & 0 & -\frac{2}{45} \\ 0 & 0 & 1 & \frac{1}{20} \end{pmatrix}$$

So $c_1 = -\frac{1}{36}$, $c_2 = -\frac{2}{45}$, $c_3 = \frac{1}{20}$

So as the general solution is given by

$$\vec{x}(t) = c_1 \lambda_1{}^t \vec{v}_1 + \ldots + c_n \lambda_n{}^t \vec{v}_n$$

$$\vec{x}(t) = c_1 \lambda_1{}^t \vec{v}_1 + c_2 \lambda_2{}^t \vec{v}_1 + c_3 \lambda_3{}^t \vec{v}_n$$

$$\vec{x}(t) = -\frac{1}{36} \times 0.2^t \begin{pmatrix} -1 \\ -3 \\ 4 \end{pmatrix} - \frac{2}{45} \times 0.5^t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{20} \times 1^t \begin{pmatrix} 7 \\ 5 \\ 8 \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} x_{\text{Page1}}(t) \\ x_{\text{Page2}}(t) \\ x_{\text{Page3}}(t) \end{pmatrix} = \begin{pmatrix} \frac{7}{20} + \frac{0.2^t}{36} - \frac{2 \times 0.5^t}{45} \\ \frac{1}{4} + \frac{0.2^t}{12} \\ \frac{2}{5} - \frac{0.2^t}{9} + \frac{2 \times 0.5^t}{45} \end{pmatrix}$$

The equilibrium distribution is the $\lim_{t\to\infty} \vec{x}(t)$

In Week 7 we were told this results holds even though we did not prove it.

Now we can demonstrate that this is true for this problem.

$$\vec{x}(t) = \begin{pmatrix} \frac{7}{20} + \frac{0.2^t}{36} - \frac{2\times 0.5^t}{45} \\ \frac{1}{4} + \frac{0.2^t}{12} \\ \frac{2}{5} - \frac{0.2^t}{9} + \frac{2\times 0.5^t}{45} \end{pmatrix}$$

$$\vec{x}_{\text{equ}} = \lim_{t\to\infty} \vec{x}(t) = \begin{pmatrix} \frac{7}{20} \\ \frac{1}{4} \\ \frac{2}{5} \end{pmatrix}$$

A business application example

A truck rental company has 3 locations in London, where you can rent moving trucks. You can return them to any other location. Every customer returns their truck the next day.

Let $\vec{n}_t$be the vector whose entries x_t, y_t, z_t are the number of trucks in locations 1, 2, and 3, respectively.

Let A be the matrix whose i, j -entry is the probability that a customer renting a truck from location j returns it to location i.

For example, the matrix $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$

encodes a 30% probability that a customer renting from location 3 returns the truck to location 2, and a 40% probability that a truck rented from location 1 gets returned to location 3.

$A\,\vec{n}_t = \vec{n}_{t+1}$ therefore represents the number of trucks at each location the next day.

Find the general solution for $\vec{n}_t$ and the expected equilibrium distribution of trucks across the 3 rental locations.

Solution using Mathematica

In[657]:=

```
A= (.3 .4 .5
    .3 .4 .3
    .4 .2 .2);
λ=Eigenvalues[A];
v=Eigenvectors[A];
c={c1,c2,c3};
Print[(xt, yt, zt)//MatrixForm," = ",Row[Table[Row[{c〚i〛," ",λ〚i〛^t,v〚i〛//MatrixForm}],{i,3}]," + "]]
```

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = -0.413501\ 1.^t \begin{pmatrix} -0.667424 \\ -0.572078 \\ -0.476731 \end{pmatrix} + 0.00517039\ (-0.2)^t \begin{pmatrix} -0.707107 \\ 8.25063\times 10^{-17} \\ 0.707107 \end{pmatrix} + -0.0000224439\ 0.1^t \begin{pmatrix} 0.267261 \\ -0.801784 \\ 0.534522 \end{pmatrix}$$

From this we observe that the equilibrium distribution is given by

$$\lim_{t\to\infty} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = c1 \begin{pmatrix} -0.667424 \\ -0.572078 \\ -0.476731 \end{pmatrix}$$

As this is a distribution vector with elements adding to 1, we can find c1 and the equilibrium distribution thus

In[662]:=

```
c1 = 1/ Plus[−0.6674238124719145`,−0.5720775535473556`,−0.47673129462279623`]

c1{{−0.6674238124719145`},{−0.5720775535473556`},{−0.47673129462279623`}}//MatrixForm
```

Out[662]=

−0.582672

Out[663]//MatrixForm=

$$\begin{pmatrix} 0.388889 \\ 0.333333 \\ 0.277778 \end{pmatrix}$$

We expect 39% of trucks will be location 1, 33% at location 2, and 28% at location 3.

More than one eigenvector for repeated eigenvectors

It is possible (though not required) that when an eigenvalue is repeated (algebraic multiplicity >1) for it to have more than one associated non-parallel eigenvector.

Example

The eigenvalues for the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ are 2, 2, and -2.

To find the eigenvectors associated with the repeated eigenvalue $\lambda = 2$ we need to solve:

$(A - 2\,I)\,\vec{v} = \vec{0}$

$(A - 2\,I) = \begin{pmatrix} -2 & 1 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Forming the augmented matrix and row reducing yields the RREF:

In[670]:=

```
RowReduce[(-1  1 0 0
            3 -3 0 0
            0  0 0 0)]
```

Out[670]//MatrixForm=

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Which gives the infinite solution $\vec{v} = \begin{pmatrix} s \\ s \\ t \end{pmatrix}$

More than on eigenvector for repeated eigenvectors

The infinite solution $\vec{v} = \begin{pmatrix} s \\ s \\ t \end{pmatrix}$ can be alternatively expressed as linear combination of two non-parallel vectors:

$$\vec{v} = \begin{pmatrix} s \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Since we can choose any values of s, t two valid specific non-parallel eigenvectors are

$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (when $s = 1,\ t = 0$) and $\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (when $s = 0,\ t = 1$)

Here we see these multiple eigenvectors confirmed in Mathematica ...

```
λ=Eigenvalues[(1 1 0
               3 -1 0
               0 0 2)];  v=Eigenvectors[(1 1 0
                                          3 -1 0
                                          0 0 2)];

Do[Print["λ=",λ[[i]],":  v=",v[[i]]//MatrixForm], {i,3}]
```

$\lambda = -2$: $v = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$

$\lambda = 2$: $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 2$: $v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Finding eigenvectors via column relations

Remember that another way to multiply a matrix and a column vector is as a linear combination of the columns of the matrix, e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + c \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$$

Rather than using Gauss-Jordan Elimination to find eigenvectors (which can sometimes be time consuming in an exam!) it is sometimes possible to “spot” eigenvectors by identifying a linear combination of the columns of $(A - \lambda I)$ yield the zero vector.

Can you spot values for a, b, and c such that $a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + c \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

Finding eigenvectors via column relations

Can you spot values for a, b, and c such that $a\begin{pmatrix}1\\2\\3\end{pmatrix}+b\begin{pmatrix}2\\3\\4\end{pmatrix}+c\begin{pmatrix}3\\5\\7\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$?

Hopefully you spotted that $a = 1$, $b = 1$, $c = -1$ is one valid solution.

If this was imply that were we solving for an eigenvector where $A - \lambda I = \begin{pmatrix}1&2&3\\2&3&5\\3&4&7\end{pmatrix}$ a valid solution would be $\vec{v} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$.

If $(A-\lambda I) = \begin{pmatrix}1&3&-1\\2&2&2\\3&1&5\end{pmatrix}$ can you spot a valid eigenvector that solves $(A-\lambda I)\vec{v} = \vec{0}$?

Finding eigenvectors via column relations

If $(A - \lambda I) = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 5 \end{pmatrix}$ can you spot a valid eigenvector that solves $(A - \lambda I)\vec{v} = \vec{0}$?

You may have spotted the following solution, i.e. the first column is 2x the sum of the other two columns:

$$2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - 1\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Reading off the linear combination coefficients gives an eigenvector solution $\vec{v} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

While there is no guarantee that you can easily spot such solutions it is worth trying as it can save you time.

If you do use this method in an exam provide a clear explanation like above in your answer.

Remember though that using Gauss-Jordan Elimination will always get you the solution.