Week 7:
Matrix Algebra
Linear Transformations
State Transition Networks

MSIN00180 Quantitative Methods for Business

- Initialisation code (please ignore)

In[ ]:=
```
$Post:=If[MatrixQ[#]||VectorQ[#],MatrixForm[#],#]&;
(*$Post:=If[MatrixQ[#],MatrixForm[#],#]&;*)
```

Vector and Matrix Variables

Vector notation: $\vec{x}$

For increased clarity I am going to start using the standard overbar arrow to indicate vectors.

For Matrix variables use **capital letters**: A, B, ...

e.g.

$$A \vec{x} = \vec{b}$$

where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Sum of Matrices

If A, B are two equal sized matrices then there sum is defined as:

$$A + B = \begin{pmatrix} a_{11} & \dots & a_{1\,m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & \dots & b_{1\,m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} + b_{11} & \dots & a_{1\,m} + b_{1\,m} \\ \vdots & & \vdots \\ a_{n1} + b_{n1} & \dots & a_{nm} + b_{nm} \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 \\ 5 & 7 & 9 \end{pmatrix}$$

Scalar Multiples of Matrices

If A is an $n \times m$ matrix and k is a scalar then the scalar product is defined as:

$$k\,A = k \begin{pmatrix} a_{11} & \dots & a_{1\,m} \\ \vdots & & \vdots \\ a_{\text{n1}} & \dots & a_{\text{nm}} \end{pmatrix} = \begin{pmatrix} k\,a_{11} & \dots & k\,a_{1\,m} \\ \vdots & & \vdots \\ k\,a_{\text{n1}} & \dots & k\,a_{\text{nm}} \end{pmatrix}$$

Example

$$3 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

Product $A\vec{x}$

To formalise what we saw last week:

$$A\vec{x} = \begin{pmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{pmatrix} \vec{x} \quad = \begin{pmatrix} \vec{w}_1.\vec{x} \\ \vdots \\ \vec{w}_n.\vec{x} \end{pmatrix}$$

Where A is an $n \times m$ matrix with row vectors $\vec{w}_1, \ldots, \vec{w}_n$, and $\vec{x}$ is a vector in $\mathbb{R}^m$.

The number of columns of the matrix A must match the number of components of the vector $\vec{x}$ for the product $A\vec{x}$ to be defined.

Product $A\vec{x}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} x + 2\,y + 3\,z \\ 4\,x + 5\,y + 6\,z \end{pmatrix}$$

Product $A\vec{x}$ expressed in terms of Column Vectors

We can also express the product $A\vec{x}$ in terms as a **linear combination** of the **column vectors** of A:

$$A\vec{x} = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{pmatrix} \vec{x} =$$

$$x_1 \vec{v}_1 + \ldots + x_m \vec{v}_m$$

Where A is an $n \times m$ matrix with column vectors $\vec{v}_1, \ldots, \vec{v}_m$, and $\vec{x}$ is the vector $(x_1 \ldots x_m)$ in $\mathbb{R}^m$.

Product $A\,\vec{x}$: Column-based Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 4 \end{pmatrix} + y \begin{pmatrix} 2 \\ 5 \end{pmatrix} + z \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1\,x \\ 4\,x \end{pmatrix} + \begin{pmatrix} 2\,y \\ 5\,y \end{pmatrix} + \begin{pmatrix} 3\,z \\ 6\,z \end{pmatrix}$$

$$= \begin{pmatrix} x + 2\,y + 3\,z \\ 4\,x + 5\,y + 6\,z \end{pmatrix}$$

Algebraic Rules for $A \vec{x}$

If A is an $n \times m$ matrix, $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^m$, and k is a scalar then

$$A\left(\vec{x}+\vec{y}\right) = A\vec{x} + A\vec{y}$$

and

$$A\left(k\vec{x}\right) = k\left(A\vec{x}\right)$$

Example Problem

Consider the matrices

$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Find a 2×2 matrix C such that

$A(B\vec{x}) = C\vec{x}$

for all vectors $\vec{x}$ in $\mathbb{R}^2$

Solution

As $\vec{x}$ in $\mathbb{R}^2$ let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$B\,x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$A\,(B\,x) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} -x_2 \\ 2\,x_1 - x_2 \end{pmatrix}$$

by adding 0 and 1 coefficients we can re - express this result thus

$$\begin{pmatrix} -x_2 \\ 2\,x_1 - x_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0\,x_1 - 1\,x_2 \\ 2\,x_1 - 1\,x_2 \end{pmatrix}$$

and we can express this as a matrix product thus

$$\begin{pmatrix} 0\,x_1 - 1\,x_2 \\ 2\,x_1 - 1\,x_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

so $\boxed{C = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}}$

Linear Transformation

Consider the following problem ...

You have x_1 2p coins and x_2 5p coins

From the number of coins (x_1 and x_2) it is possible to easily calculate both the total value of the coins (y_1) and the total number of coins (y_2).

Question

What is the matrix A that performs this calculation when expressed thus?

$A\,\vec{x} = \vec{y}$ where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Linear Transformation

The equations for calculating y_1 (value of coins) and y_2 (number of coins) are:

$$2x_1 + 5x_2 = y_1$$
$$x_1 + x_2 = y_2$$

In matrix form this can be represented as

$$\begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

so $A = \begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix}$

For reasons that we will explore shortly this is an example of a **linear transformation**.

Multiplying the vector $\vec{x}$ by the matrix $A = \begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix}$ is said to **transform** the vector $\vec{x}$ into a vector $\vec{y}$.

Reversing a Linear Transformation

If you know the total value of the coins (y_1) and the total number of coins (y_2) can you uniquely determine the number of 2p and 5p coins (x_1 and x_2) ?

Another way of asking this question is can we find a matrix B such that

$B\vec{y} = \vec{x}$ given $A\vec{x} = \vec{y}$

How might we find B?

Reversing a Linear Transformation

It is possible to solve for $\vec{x}$ in terms of $\vec{y}$ (and thus B) using **Gauss-Jordan elimination**:

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \text{where } A = \begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix}$$

First form the augmented matrix

In[ ]:=
$$\mathtt{m} = \begin{pmatrix} 2 & 5 & y_1 \\ 1 & 1 & y_2 \end{pmatrix}$$

Out[ ]//MatrixForm=
$$\begin{pmatrix} 2 & 5 & y_1 \\ 1 & 1 & y_2 \end{pmatrix}$$

Then apply Gauss-Jordan elimination ...

The Inverse Linear Transformation

In[ ]:=
```
m=(2 5 y1
   1 1 y2);
m[[1]]=m[[1]]/2;    m
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{5}{2} & \frac{y_1}{2} \\ 1 & 1 & y_2 \end{pmatrix}$$

In[ ]:=
```
m[[2]]=m[[2]]-m[[1]];  m
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{5}{2} & \frac{y_1}{2} \\ 0 & -\frac{3}{2} & -\frac{y_1}{2}+y_2 \end{pmatrix}$$

In[ ]:=
```
m[[2]]=-2/3 m[[2]];  m
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{5}{2} & \frac{y_1}{2} \\ 0 & 1 & -\frac{2}{3}\left(-\frac{y_1}{2}+y_2\right) \end{pmatrix}$$

In[ ]:=
```
m[[1]]=m[[1]]-5/2 m[[2]];    m
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & \frac{y_1}{2}+\frac{5}{3}\left(-\frac{y_1}{2}+y_2\right) \\ 0 & 1 & -\frac{2}{3}\left(-\frac{y_1}{2}+y_2\right) \end{pmatrix}$$

In[ ]:=
```
m //Simplify
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & \frac{1}{3}\left(-y_1+5\,y_2\right) \\ 0 & 1 & \frac{1}{3}\left(y_1-2\,y_2\right) \end{pmatrix}$$

The Inverse Linear Transformation

Converting this augmented form back to normal matrix form gives:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{y_1}{3} + \frac{5\,y_2}{3} \\ \frac{y_1}{3} - \frac{2\,y_2}{3} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{5}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

We have therefore found B such that $B\,\vec{y} = \vec{x}$

$$B = \begin{pmatrix} -\frac{1}{3} & \frac{5}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

Confirm B reverses the transformation ...

First transform $\vec{x}$ to $\vec{y}$ using matrix A

Let: number of 2p coins = 5 coins
number of 5p coins = 10 coins

$$\vec{y} = A\,\vec{x} = \begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 10 + 50 \\ 5 + 10 \end{pmatrix} = \begin{pmatrix} 60 \\ 15 \end{pmatrix}$$

Then transform $\vec{y}$ to $\vec{x}$ using matrix B

$$\vec{x} = B\,\vec{y} = \begin{pmatrix} -\frac{1}{3} & \frac{5}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 60 \\ 15 \end{pmatrix} = \begin{pmatrix} -20 + 25 \\ 20 - 10 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

B has indeed reversed the process!

We say that the matrix *B* is the **inverse** of *A*.

Linear Transformations are functions expressed as a matrix multiplication

A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a linear transformation if there exists an $n \times m$ matrix A such that

$$T(\vec{x}) = A\,\vec{x}$$

for all $\vec{x}$ in the vector space $\mathbb{R}^m$.

Example

Given $T(\vec{x}) = A\,\vec{x}$ where $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

for simplicity we write $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ instead of $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$ i.e. $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

Without calculating the result what do you think $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ will be?

Inverse Matrices

The matrix that reverses the **transformation** A is called the **inverse** of A and is represented as $\mathbf{A^{-1}}$.

From the previous example:

$A = \begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{5}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

Inverse Matrices

Inverse matrices have the property:

$$A^{-1}(A\,\vec{x}) = \vec{x}$$

Inverse function in Mathematica

Mathematica provides a function to calculate the inverse of a matrix, if it exists.

In[ ]:=

```
Inverse[(2 5
         1 1)]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{3} & \frac{5}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

Example

Find the inverse of $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$ if it exists.

Solution

Find the inverse of $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$ if it exists.

$$\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

First set the augmented matrix with the identity matrix on the right hand side then perform a Gaussian elimination.

In[∘]:=
```
m=(2 3 y1
   6 9 y2);
```

In[∘]:=
```
m[[1]]=m[[1]]/2;
m[[2]]=m[[2]]-6 m[[1]];m
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{3}{2} & \frac{y1}{2} \\ 0 & 0 & -3\,y1 + y2 \end{pmatrix}$$

Though we have not finished this Gaussian elimination we can see from the bottom row that it is impossible to form an identity matrix so $\boxed{\text{this matrix is not invertible}}$.

Mathematica shows the following warning message when you try to find the inverse of a matrix that is not invertible

In[ ]:=
```
Inverse[(2 3
         6 9)]
```

Inverse: Matrix {{2, 3}, {6, 9}} is singular.

Out[ ]=
```
Inverse[{{2, 3}, {6, 9}}]
```

Example 2

Find the inverse of $A = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}$ if it exists.

Solution

Find the inverse of $A = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}$ if it exists.

In[ ]:=
```
m=(1 2 y1
   4 9 y2);
m[[2]]=m[[2]]-4m[[1]];  m
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & y1 \\ 0 & 1 & -4\,y1+y2 \end{pmatrix}$$

In[ ]:=
```
m[[1]]=m[[1]]-2m[[2]];  m
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & y1-2\,(-4\,y1+y2) \\ 0 & 1 & -4\,y1+y2 \end{pmatrix}$$

In[ ]:=
```
m //Simplify
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 9\,y1-2\,y2 \\ 0 & 1 & -4\,y1+y2 \end{pmatrix}$$

This is therefore invertible (we see an identity matrix) and the inverse is $\begin{pmatrix} 9 & -2 \\ -4 & 1 \end{pmatrix}$

Formula for the inverse of 2x2 matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

provided $ad \neq bc$

Derivation of 2x2 Inverse Formula

We can use Gauss-Jordan elimination to prove the 2x2 inverse formula ...

In[∘]:=
```
Clear[a,b,c,d,y1,y2];

m = (a b y1
     c d y2);

m[[1]] = 1/a m[[1]]; m
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{y1}{a} \\ c & d & y2 \end{pmatrix}$$

In[∘]:=
```
m[[2]]=m[[2]]-c m[[1]];  m
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{y1}{a} \\ 0 & -\frac{b\,c}{a}+d & -\frac{c\,y1}{a}+y2 \end{pmatrix}$$

In[∘]:=
```
m[[2]] = 1/(-b c/a + d) m[[2]]; m//Simplify
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{y1}{a} \\ 0 & 1 & \frac{c\,y1-a\,y2}{b\,c-a\,d} \end{pmatrix}$$

In[∘]:=
```
m[[1]]=m[[1]]-b/a m[[2]];   m//Simplify
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & \frac{d\,y1-b\,y2}{-b\,c+a\,d} \\ 0 & 1 & \frac{c\,y1-a\,y2}{b\,c-a\,d} \end{pmatrix}$$

So $\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} \frac{d\,y1-b\,y2}{-b\,c+a\,d} \\ \frac{c\,y1-a\,y2}{b\,c-a\,d} \end{pmatrix} = \frac{1}{\text{ad-bc}} \begin{pmatrix} d\,y1 - b\,y2 \\ -c\,y1 + a\,y2 \end{pmatrix} = \frac{1}{\text{ad-bc}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} y1 \\ y2 \end{pmatrix}$ Q.E.D.

Determinant of a Matrix

The **determinant** of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called det (A).

$$\det(A) = (a\,d - b\,c)$$

A is invertible provided det$(A) \neq 0$.

So when $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The Det function in Mathematica

In[ ]:=
```
Det[(a b
     c d)]
```
$$\mathrm{Det}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$$

Out[ ]=

$-b\,c + a\,d$

In[ ]:=

$$\mathrm{Det}\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right]$$

Out[ ]=

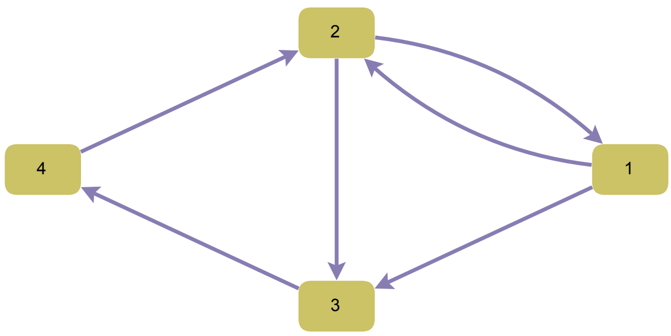
-2

Mini-Web Case Study

This case study will be used to introduce a number of new concepts and definitions, including **distribution vectors**, **transition matrices**, and **equilibrium distributions**.

Four web pages have links to each other as shown in this diagram.

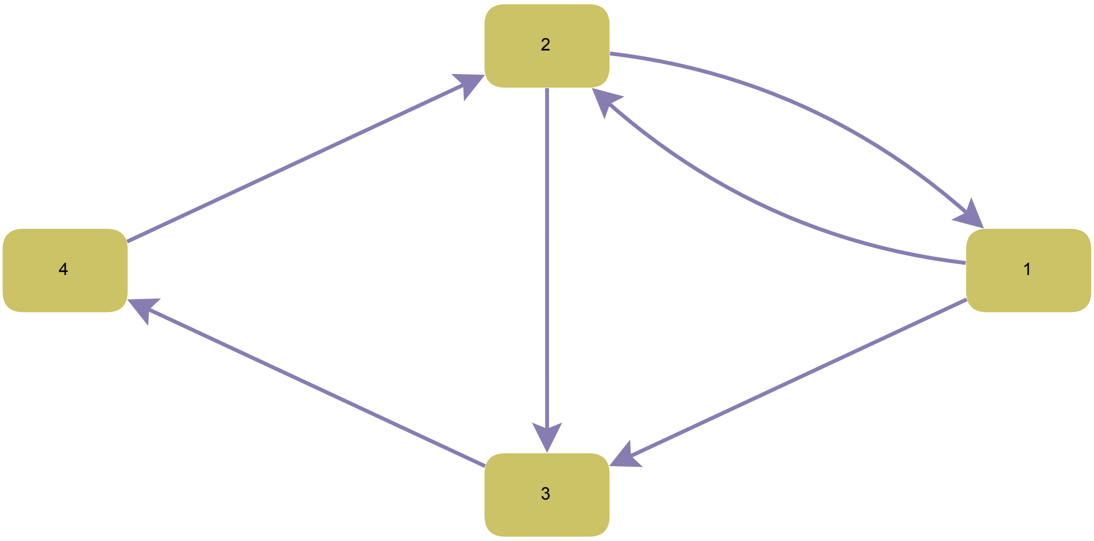
Out[ ]=



Let x_1, x_2, x_3, and x_4 be the initial proportions of the surfers at each site.

e.g. $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.1 \\ 0.3 \\ 0.2 \end{pmatrix}$ $\longleftarrow$ $x_1 = 0.4$ means 40% of the surfers are initially at site 1 etc.

Distribution Vectors



$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.1 \\ 0.3 \\ 0.2 \end{pmatrix}$$ $\longleftarrow$ Note that the components of $\vec{x}$ add to 1 (=100%).

$\vec{x}$ is an example of a **distribution vector.**

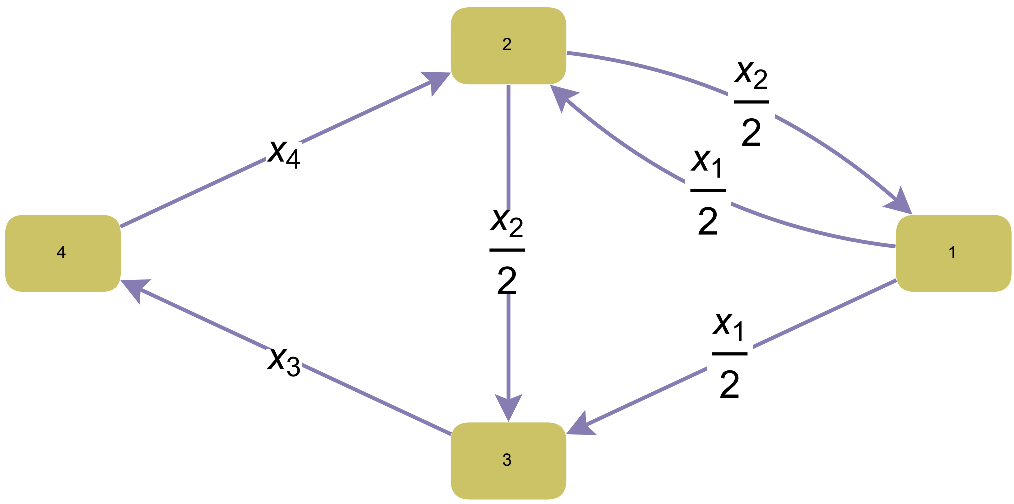
A vector $\vec{x}$ in $\mathbb{R}^n$ is said to be a **distribution vector** if its components add up to 1 and all the components are positive or zero.

Changing the distribution of web surfer

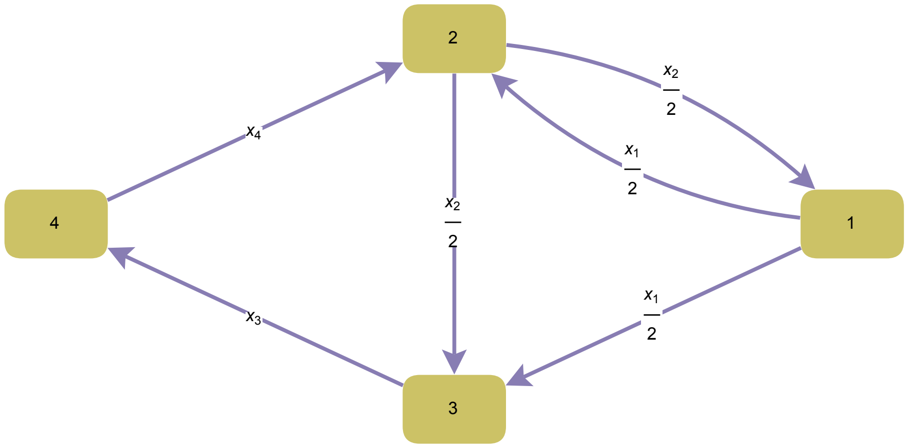
At a predetermined time each surfer randomly follows one of the links out of the page they are currently on.

e.g. The proportion of surfers following each of the 2 links out of page 1 (to pages 2 and 3) will be $\frac{x_1}{2}$.

Out[ ]=



Distribution after a change



Let the vector $\vec{y}$ represent the distribution after the first change.

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac{x_2}{2} \\ \frac{x_1}{2} + x_4 \\ \frac{x_1}{2} + \frac{x_2}{2} \\ x_3 \end{pmatrix}$$

Distribution after a change

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac{x_2}{2} \\ \frac{x_1}{2} + x_4 \\ \frac{x_1}{2} + \frac{x_2}{2} \\ x_3 \end{pmatrix}$$

$$\vec{y} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

this is the same as

$$\vec{y} = A\,\vec{x} \qquad \text{where } A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

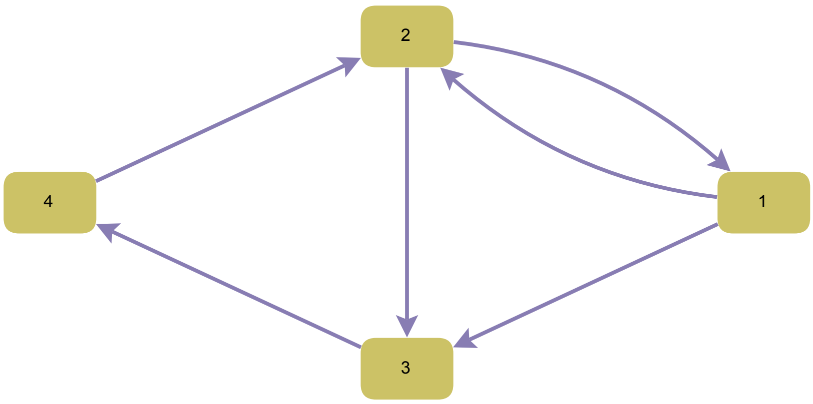
Transition Matrix

$\vec{y} = A\,\vec{x}$ where $A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

This is an example of a **linear transformation**.

Matrix A is referred to as the **transition matrix** of this linear transformation.

What do the column vectors of A indicate?



Transition Matrix

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The jth column of A indicate where the surfers go from page j.

Each column vector of A is a **distribution vector** as its components are all non-negative and add up to 1.

A square matrix A is said to be a **transition matrix** if all its column vectors are **distribution vectors**.

Equilibrium Distribution

An **equilibrium distribution** is any $\vec{x}$ such that $A\,\vec{x} = \vec{x}$.

In this case we are interested to see if there is a distribution of people over our mini-Web such that when everyone randomly follows a link to another page the overall distribution remains unchanged.

$$A\,\vec{x} = \vec{x}$$

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Equilibrium Distribution

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} \frac{x_2}{2} \\ \frac{x_1}{2} + x_4 \\ \frac{x_1}{2} + \frac{x_2}{2} \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Rearrange so each equation $= 0$.

$$\begin{pmatrix} -x_1 + \frac{x_2}{2} \\ \frac{x_1}{2} - x_2 + x_4 \\ \frac{x_1}{2} + \frac{x_2}{2} - x_3 \\ x_3 - x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} -1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

What do we do now?

Now solve this using Gauss-Jordan elimination ...

In[ ]:=

```
RowReduce[
```
$$\begin{pmatrix} -1 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$
```
]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

What does this result indicate ?

What are x_1, x_2, x_3 and x_4 ?

Equilibrium Distribution

In[ ]:=
$$\begin{pmatrix} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The above result indicates infinite solutions with x_4 being a free (non-leading) variable.

With $x_4 = t$ then:

$$\vec{x} = \begin{pmatrix} \frac{2}{3} t \\ \frac{4}{3} t \\ t \\ t \end{pmatrix}$$

Can we find a value for t?

Equilibrium Distribution

$$\vec{x} = \begin{pmatrix} \frac{2}{3}\,t \\ \frac{4}{3}\,t \\ t \\ t \end{pmatrix}$$

As $\vec{x}$ is a **distribution vector** then it also follows that $x_1 + x_2 + x_3 + x_4 = 1$

so $\quad \frac{2}{3}\,t + \frac{4}{3}\,t + t + t = 1 \qquad \longrightarrow \qquad t = \frac{1}{4}$

The **equilibrium distribution** is therefore

$$\vec{x} = \begin{pmatrix} \frac{2}{3}\,t \\ \frac{4}{3}\,t \\ t \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

Equilibrium Distributions = long-term distribution

Image we repeat the random movement of people along links over and over again.

We call each transition an **iteration**.

Perhaps surprisingly ....

The equilibrium distribution represents the distribution of surfers in the long run, whatever the initial distribution.

We'll come back to this important property later.

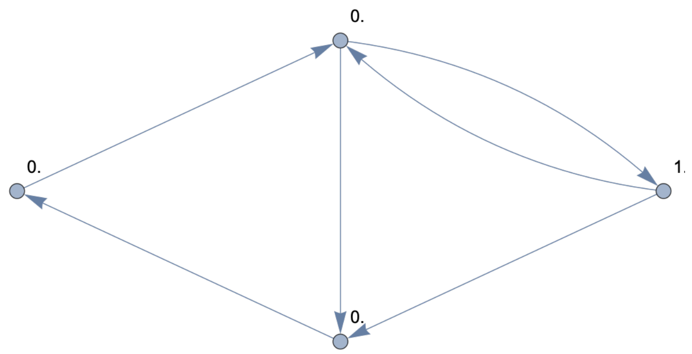
Mini-Web Iterations Demo: Mathematica

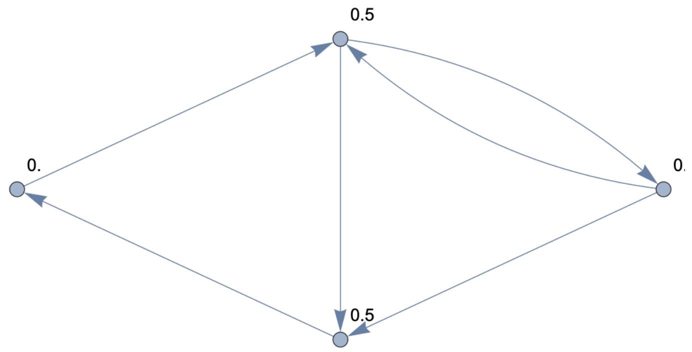
This code defines a Mathematica function called **IterateWeb**, that visually shows how the distribution changes step-by-step from some initial distribution. You do not need to understand this code.

In[ ]:=
```
IterateWeb[x0_,steps_]:=
Module[
{x=x0},
Print[Graph[{1→2,2→1,2→3,1→3,3→4,4→2},
VertexLabels→Table[i→Round[x[[i]],0.01],{i,4}]]];
For[i=0,i<steps,i++,
     x=( 0   0.5 0 0
         0.5 0   0 1
         0.5 0.5 0 0
         0   0   1 0 ).x;
     Print[Graph[{1→2,2→1,2→3,1→3,3→4,4→2},
         VertexLabels→Table[i→Round[x[[i]],0.01],{i,4}]]]
]];
```

Initially, we can just run this to show 1 step change in the distribution:

In[ ]:=
```
IterateWeb[x0={1,0,0,0},steps=1]
```

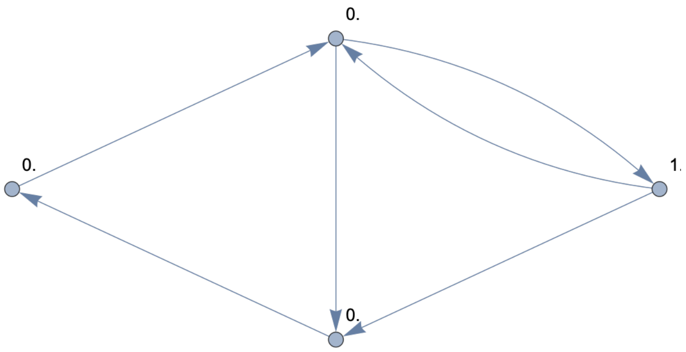


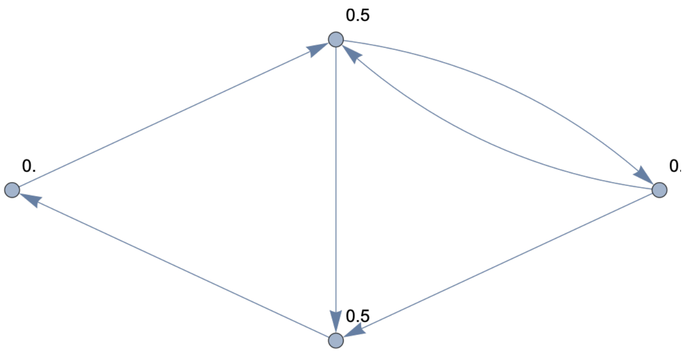


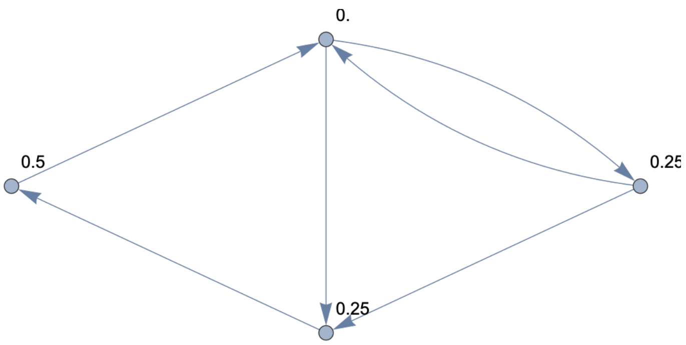
Let’s run this again over a larger number of steps and with different distributions ...

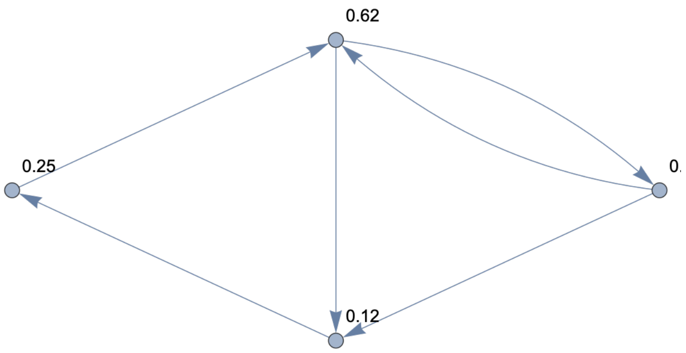
Try running these two examples.
What do you observe about the final outcomes in each case?

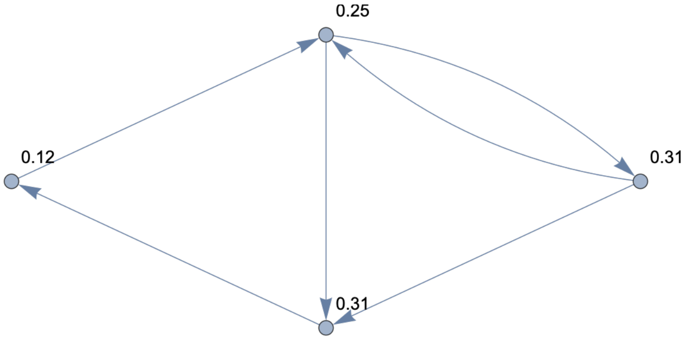
In[ ]:=
```
IterateWeb[x0={1,0,0,0}, steps=10]
```

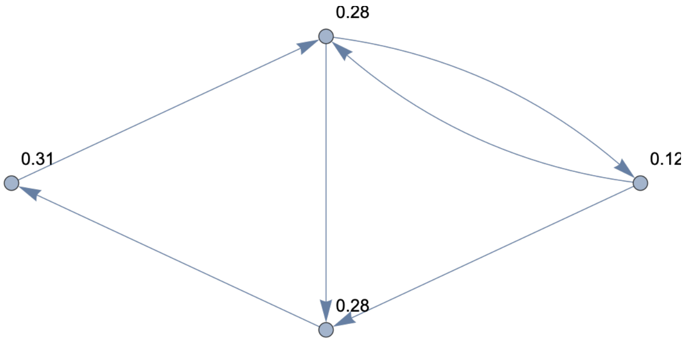


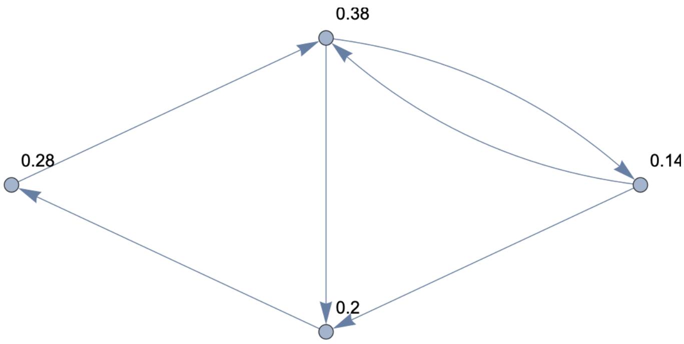


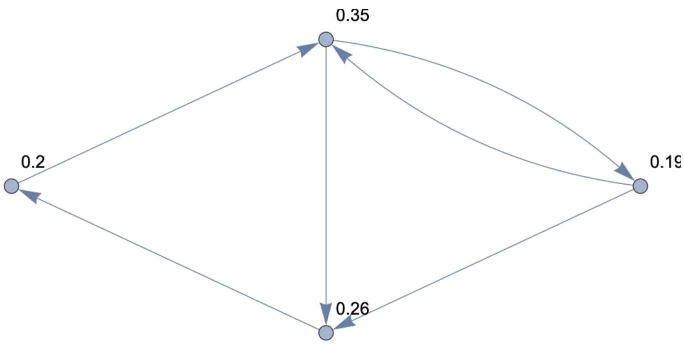


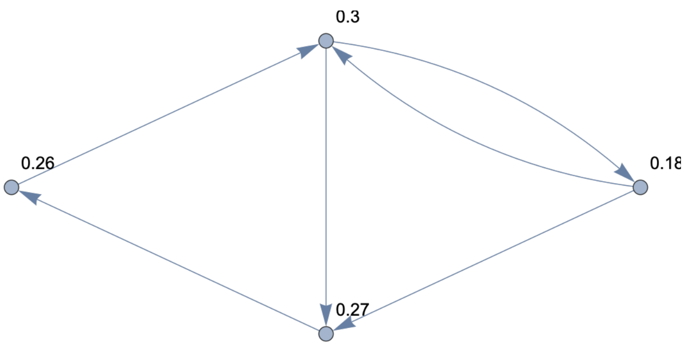


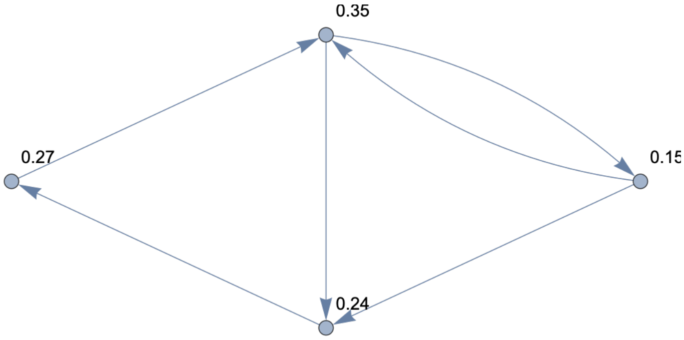

0.25
0.12
0.31
0.31

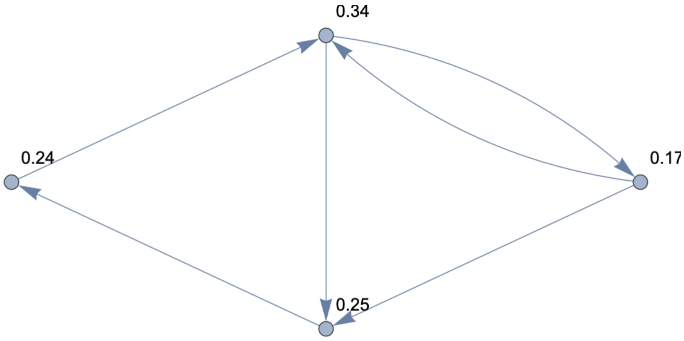

0.28
0.31
0.12
0.28


0.38
0.28
0.14
0.2

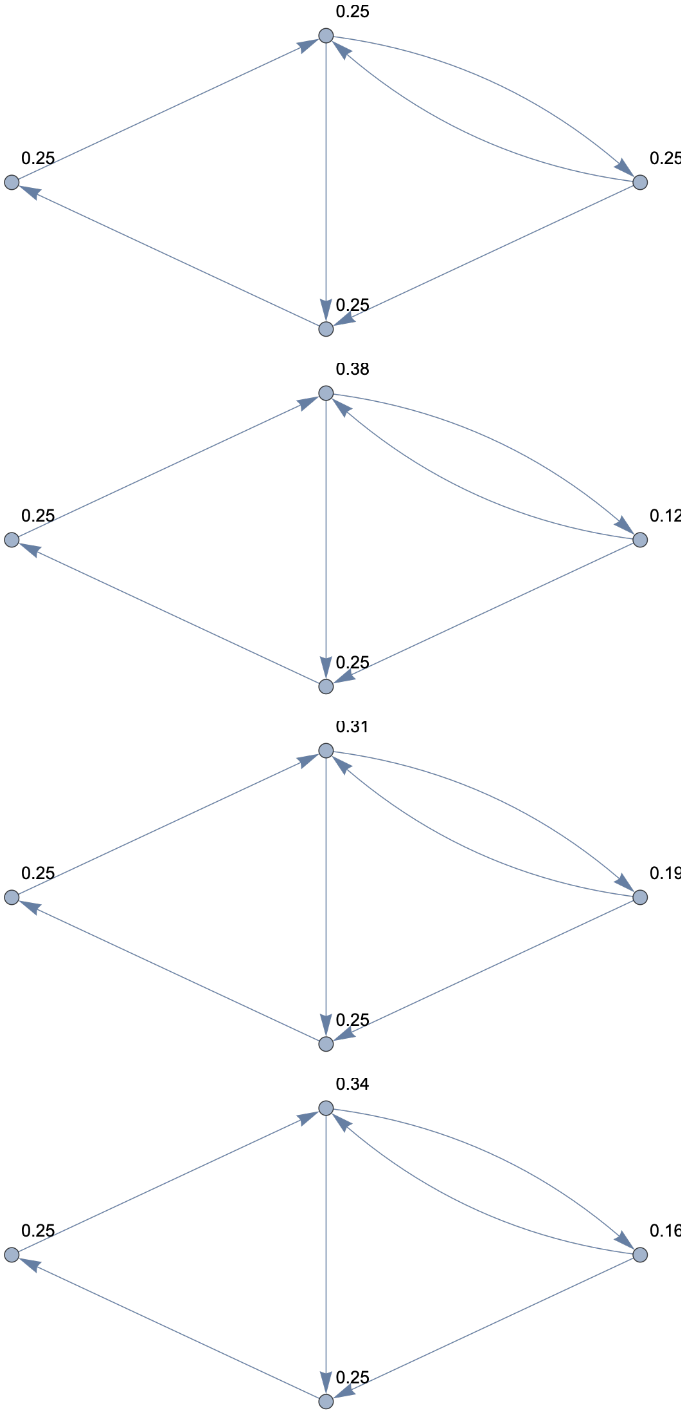

0.35
0.2
0.19
0.26

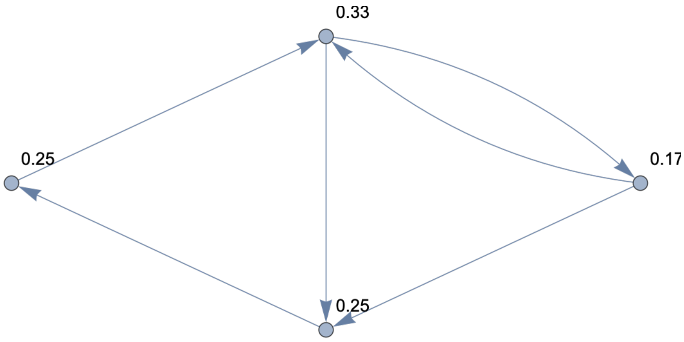

0.3
0.26
0.18
0.27

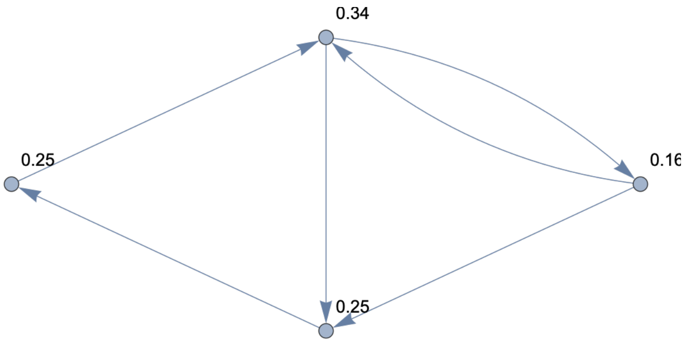

0.35
0.27
0.15
0.24


0.34
0.24
0.17
0.25

In[ ]:=
```
IterateWeb[x0={0.25,0.25,0.25,0.25}, steps=5]
```




0.33
0.25
0.17
0.25


0.34
0.25
0.16
0.25

Google, PageRank and Equilibrium Distributions

1998 Landmark Paper on Google Prototype

"*The Anatomy of a Large-Scale Hypertextual Search Engine*"
by Sergey Brin and Lawrence Page, Stanford University

PageRank

This paper introduced the **PageRank** measure which measures “link popularity” and is used to provide a quality ranking to search results.

Defines the popularity of a web page as the likelihood that random surfers will eventually find themselves at a given page.

PageRank is the same as a page’s component value in the **equilibrium distribution**!

$$\vec{x} = \begin{pmatrix} 0.167 \\ 0.333 \\ 0.25 \\ 0.25 \end{pmatrix}$$

This tells us that page 2 is the most popular page with a PageRank of 0.333.

Matrix Product

Let B be an $n \times p$ matrix and A a $p \times m$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$.

Then the product $B\,A$ is

$$B\,A = B\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{pmatrix} =$$

$$\begin{pmatrix} | & | & & | \\ B\,\vec{v}_1 & B\,\vec{v}_2 & \cdots & B\,\vec{v}_m \\ | & | & & | \end{pmatrix}$$

To find $B\,A$, we multiply B by the columns of A and combine the resulting vectors.

For a matrix product to be defined the number of columns of the first matrix must equal the number of rows of the second matrix.

Example

What is the product $B\,A = \begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$?

Example

$$B\,A = \begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

$$= \left(\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \;\; \begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} 6\times1+7\times3 \\ 8\times1+9\times3 \end{pmatrix} \;\; \begin{pmatrix} 6\times2+7\times5 \\ 8\times2+9\times5 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 27 & 47 \\ 35 & 61 \end{pmatrix}$$

Matrix Product in Mathematica

In Mathematica it is essential to **use a dot** between the matrices.

In[ ]:=
$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} . \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

Out[ ]//MatrixForm=
$$\begin{pmatrix} 27 & 47 \\ 35 & 61 \end{pmatrix}$$

In[ ]:=

What if we reverse the order of A and B?

What is the matrix product A B?

In[ ]:=

Matrix products are Noncommutative (generally)

$A B \neq B A$

In[ ]:=

```
(* B.A = *)
```

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} . \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 27 & 47 \\ 35 & 61 \end{pmatrix}$$

In[ ]:=

```
(* A.B = *)
```

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} . \begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 22 & 25 \\ 58 & 66 \end{pmatrix}$$

In general $A B \neq B A$.
However, at times it does happen that $A B = B A$; then we say the matrices **commute**.

Multiplying with the Identity Matrix

For an $n \times m$ matrix
A.

$$A\, I_m \;=\; I_n\, A \;=\; A$$

I_m is an $m \times m$ square matrix where all the diagonal elements are 1s and all other elements are 0s.

e.g. $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Matrix multiplication is Associative

$$(A\,B)\,C = A\,(B\,C)$$

We can therefore simply write $A\,B\,C$ for the product $(A\,B)\,C$ or $A\,(B\,C)$.

Distributive property of matrices

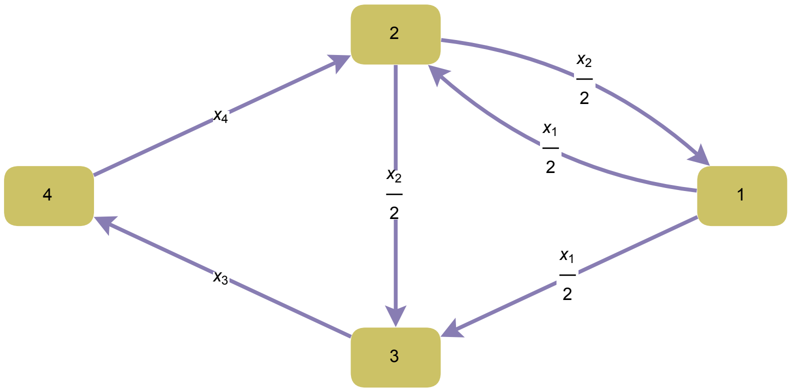
If A and B are $n \times p$ matrices, and C and D are $p \times m$ matrices,
then

$$A(C + D) = A\,C + A\,D,$$ and

$$(A + B)C = A\,C + B\,C$$

Powers of Transition Matrices

We return to our mini-Web case study.



Remember that the distribution after 1 iteration is given by

$$\vec{y} = A\,\vec{x} \qquad \text{where } A = \begin{pmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So after 2 iterations the distribution will be $A\left(A\,\vec{x}\right) = (A\,A)\,\vec{x} = A^2\,\vec{x}$

It follows that after m iterations the distribution vector will be $A^m\,\vec{x}$.

The MatrixPower function in Mathematica

In[ ]:=

$$A = \begin{pmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

```
MatrixPower[A,10]    (* = A^10 *)
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.172852 & 0.171875 & 0.171875 & 0.150391 \\ 0.34375 & 0.344727 & 0.300781 & 0.34375 \\ 0.24707 & 0.24707 & 0.269531 & 0.236328 \\ 0.236328 & 0.236328 & 0.257813 & 0.269531 \end{pmatrix}$$

In[ ]:=

```
MatrixPower[A,50]    (* = A^50 *)
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.166667 & 0.166667 & 0.166667 & 0.166667 \\ 0.333333 & 0.333333 & 0.333333 & 0.333333 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}$$

What do you observe about the columns of this matrix?

Transition matrices: how A^m behaves as m→∞

On the last slide we observed that when A is a **transition matrix** all the columns of A^m as m increases appear to tend towards the equilibrium vector $\vec{x}_{equ}$ for matrix A.

Equilibria for transition matrices

Let A be a **transition matrix** of size $n \times n$ and let $\vec{x}_{\text{equ}}$ be the **equilibrium distribution** of

If $\vec{x}$ is any distribution vector in $\mathbb{R}^n$ then

$$\lim_{m \to \infty} \left(A^m \vec{x}\right) = \vec{x}_{\text{equ}}$$

and

$$\lim_{m \to \infty} A^m = \begin{pmatrix} | & & | \\ \vec{x}_{\text{equ}} & \cdots & \vec{x}_{\text{equ}} \\ | & & | \end{pmatrix}$$

More on Inverse Matrices: Definitions

A square matrix A is said to be **invertible** if the linear transformation $\vec{y} = T(\vec{x}) = A\,\vec{x}$ is invertible.

In this case the matrix of T^{-1} is denoted by A^{-1}.

If the linear transformation $\vec{y} = T(\vec{x}) = A\,\vec{x}$ is invertible then its inverse is $\vec{x} = T^{-1}(\vec{y}) = A^{-1}\,\vec{x}$.

Invertibility

An $n \times n$ matrix A is **invertible** if (and only if)

$$\operatorname{rref}(A) = I_n$$

or, equivalently, if

$$\operatorname{rank}(A) = n$$

Multiplying with the inverse

$$A^{-1}A = A\,A^{-1} = I_n$$

Inverse of the product of matrices

$$(B\,A)^{-1} = A^{-1}\,B^{-1}$$

The order matters!

Finding the inverse of a matrix using Gauss-Jordan elimination

To find the inverse of an $n \times n$ matrix A, form the $n \times 2n$ matrix $[A \vdots I_n]$ and compute rref$[A \vdots I_n]$...

If

$$\text{rref}[A \vdots I_n] = [I_n \vdots B]$$

then A is invertible and $A^{-1} = B$.

Finding the inverse of a matrix using Gauss-Jordan elimination

Find the inverse of A= $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{pmatrix}$

First, form the augmented matrix $[A \vdots I_3] = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 1 & 0 & 0 & 1 \end{pmatrix}$

Now apply Gauss-Jordan elimination to reduce this to the RREF form:

In[ ]:=
```
m= (1 2 3 1 0 0
    4 5 6 0 1 0
    7 8 1 0 0 1);
m[[2]] -= 4 m[[1]];  m[[3]] -= 7 m[[1]];
m[[2]] /= -3;  m[[1]] -= 2 m[[2]];  m[[3]] -= -6 m[[2]];
m[[3]] /= -8;  m[[1]] -= -1 m[[3]];  m[[2]] -= 2 m[[3]];
m
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{43}{24} & \frac{11}{12} & -\frac{1}{8} \\ 0 & 1 & 0 & \frac{19}{12} & -\frac{5}{6} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \end{pmatrix}$$

As this RREF form is of the form $[I_3 \vdots B]$ we know that A is invertible where $A^{-1} = B = \begin{pmatrix} -\frac{43}{24} & \frac{11}{12} & -\frac{1}{8} \\ \frac{19}{12} & -\frac{5}{6} & \frac{1}{4} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \end{pmatrix}$