Week 3:
Partial differentiation & optimisation of multivariate functions

MSIN00180 Quantitative Methods for Business

Topics

- Functions of Several Variables (Multivariate Functions)
- Level Curves and Level Surfaces
- Limits in Higher Dimensions
- Partial Derivatives
- Implicit Differentiation using partial derivatives
- Higher-Order Partial Derivatives
- The Chain Rule for multivariate functions
- Optimisation of multivariate functions
- The Gradient Vector and Directional Derivatives

Functions of two variables

e.g. $z = f(x, y) = \sqrt{y - x^2}$

- z is the **dependent** variable
- x and y are the **independent** variables
- points in the domain are said to be **ordered pairs** (when there are 2 independent variables)

Functions of 2 variables in Mathematica

In[472]:=

```
f[x_,y_]:= √(y-x^2);
f[0,20] //N
```

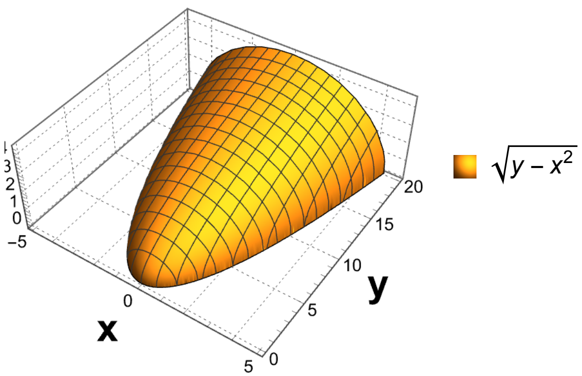
Out[473]=

4.47214

In[474]:=

```
Plot3D[ √(y-x^2),{x,-5,5},{y,0,20},
    AxesLabel→{Style["x", Bold, 20], Style["y", Bold, 20],Style["z", Bold, 20]},
    PlotTheme→"Detailed", MaxRecursion→5]
```

Out[474]=



Higher Dimension Names

For generalised numbers of dimensions it is common to use the labels

$x_1, x_2, x_3, \ \ldots, x_n$

instead of using names like x, y and z for each dimensions

Real-Valued Functions

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$.

A **real-valued function *f*** on D is a rule that assigns a unique real number

$w = f(x_1, x_2, x_3, \ldots, x_n)$

to each element in D.

- The set D is the function’s **domain**.
- The set of w-values taken on by f is the function’s **range**.
- The symbol w is the **dependent variable** of f.
- f is said to be a function of the n **independent variables** x_1 to x_n.

Natural Domains

As with singe variable functions, the natural domain of a multi-variate function is the largest set of values for which the defining rule generates real numbers. This implies exclusion of any inputs that lead to complex numbers and division by zero.

Function	**Domain**	**Range**
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

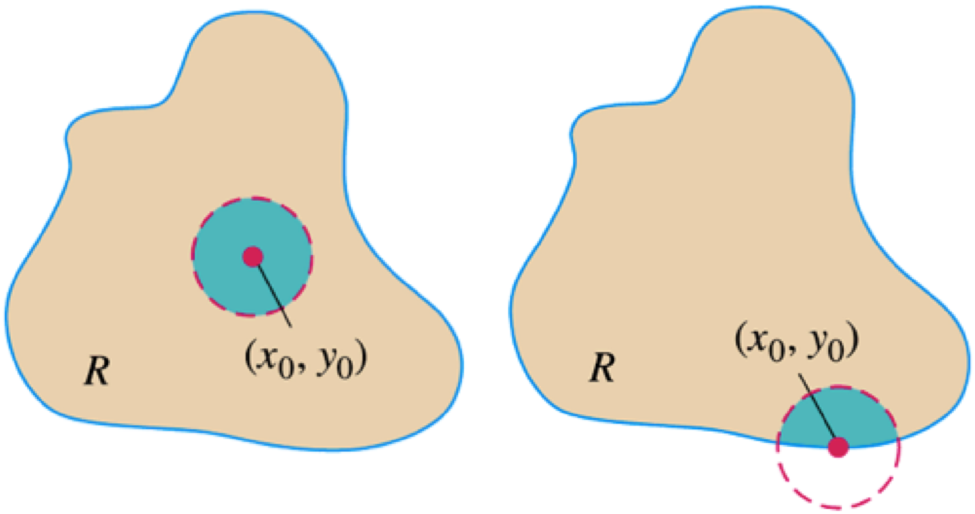
Regions in 2D Planes

The domain of functions with 2 variables are defined by regions rather than by intervals.

Regions in 2D planes can have **interior points** and **boundary points** just like intervals on the real line.

A region is **open** if it consists entirely of interior points.

A region is **closed** if it contains all its boundary points.



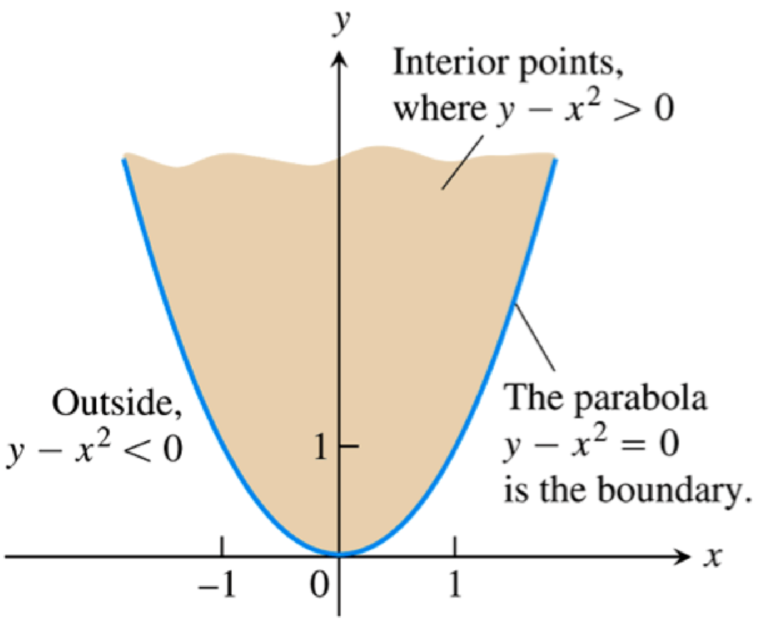
(a) Interior point (b) Boundary point

Bounded and Unbounded Regions in 2D Planes

A region in a plane is **bounded** if it lies inside a disk of fixed radius.

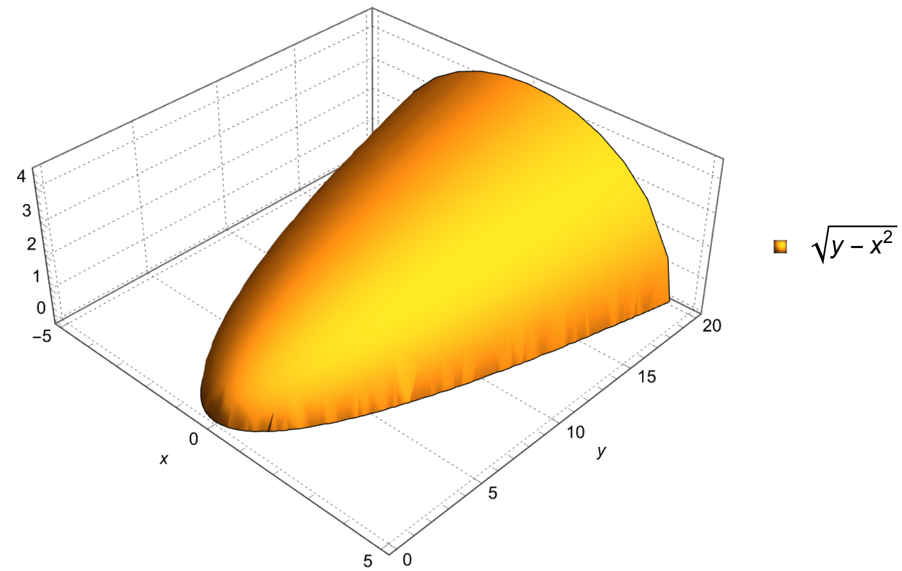
Example

The (natural) domain of the function $f(x, y) = \sqrt{y - x^2}$ is **closed** and **unbounded**.



Graphs and Surfaces

The **graph** of a function $f(x, y)$ is also called the **surface** $z = f(x, y)$

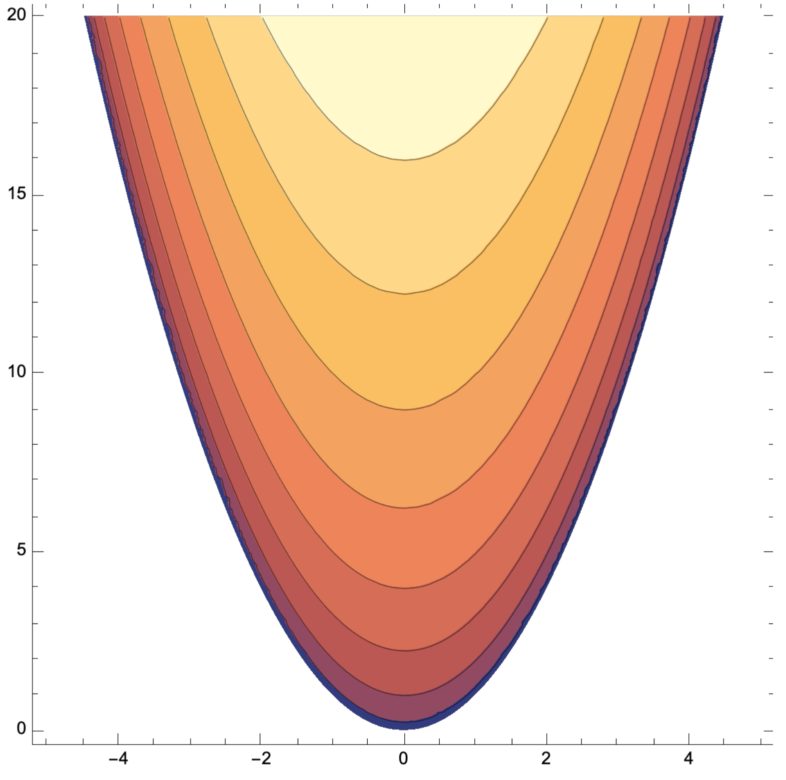


Level Curves

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f.

Level curves are also sometimes known as **contour curves**

Out[ ]=



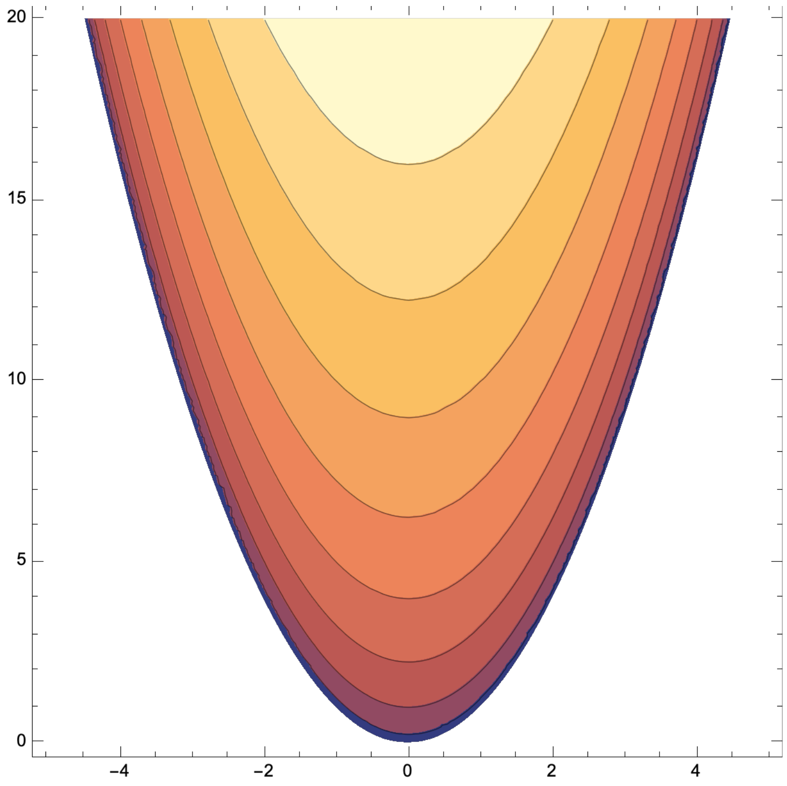
Level Curves in Mathematica

ContourPlot

In[475]:=

```
ContourPlot[ √(-x²+y) , {x,-5,5},{y,0,20}]
```

Out[475]=



Level Curves in Mathematica

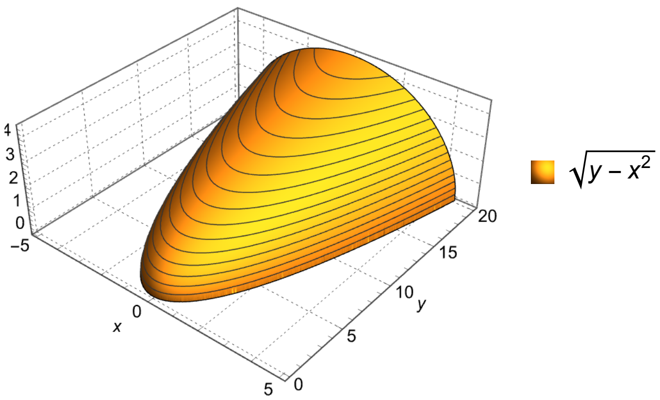
Plot3D with MeshFunctions

It is also possible to show **level curves** overlaid on a **surface ...**

In[476]:=

```
Plot3D[ √(y-x^2) ,{x,-5,5},{y,0,20},AxesLabel→{x,y,z},PlotTheme→"Detailed" ,
        PlotPoints→ 50,MaxRecursion→5,MeshFunctions→{#3&}]
```

Out[476]=



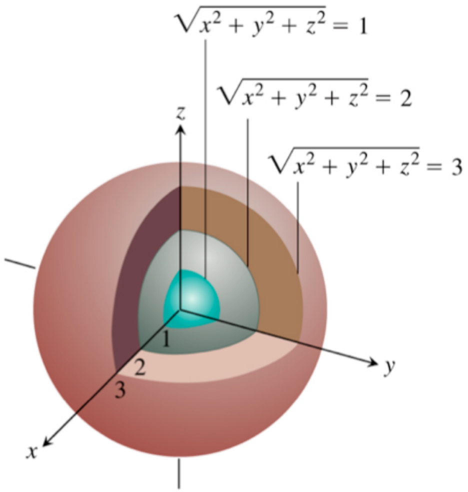
Level Surfaces for functions of 3 variables

Level surfaces for functions of 3 variables are the equivalent to level curves for functions of 2 variables

The set of points in the plane where a function f has a constant value $f(x, y, z) = c$ is called a level surface of f .

Example

Consider the level surfaces for the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

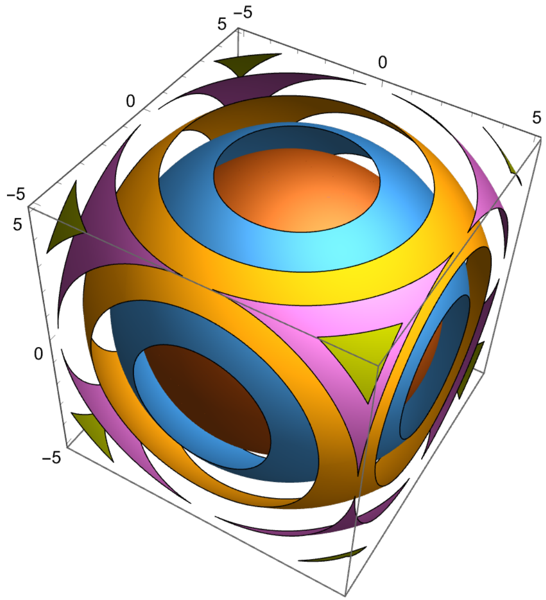


Level Surfaces in Mathematica

In[477]:=

```
ContourPlot3D[√(x²+y²+z²),{x,-5,5},{y,-5,5},{z,-5,5},Contours→10, Mesh→None]
```

Out[477]=



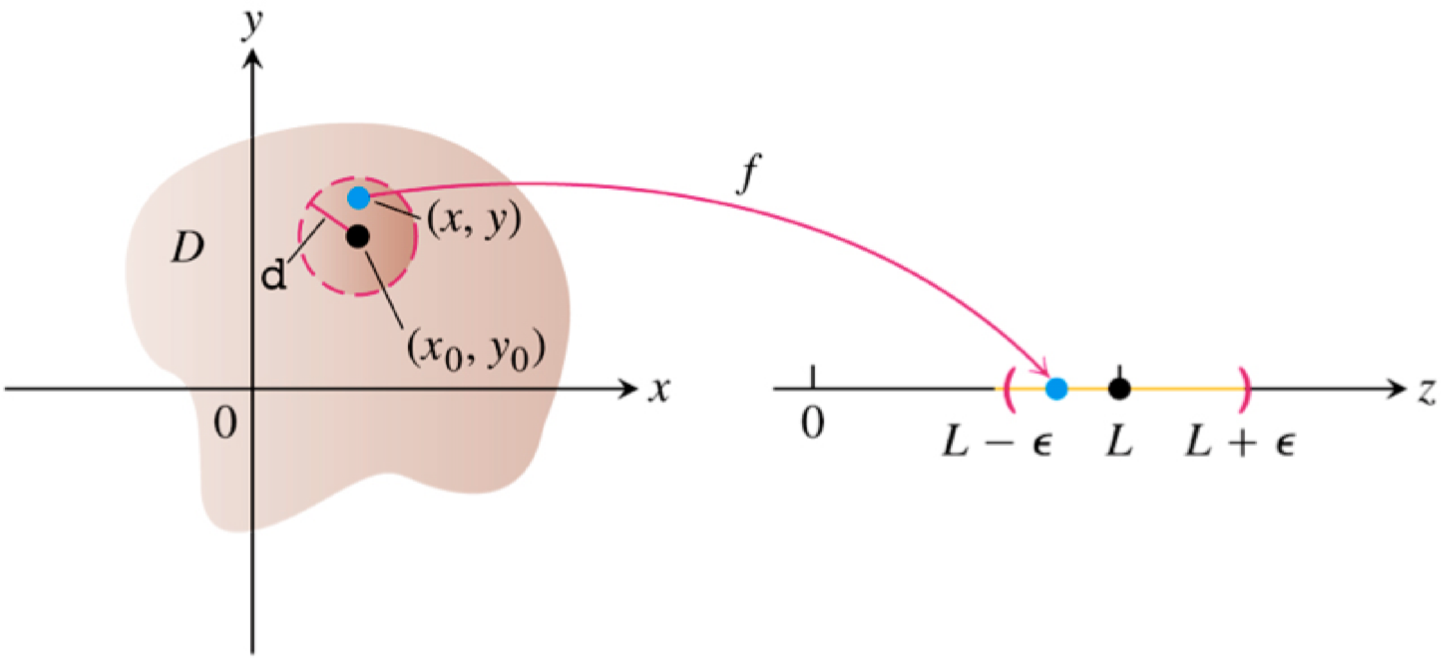
Limits for Functions of 2 Variables

We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$



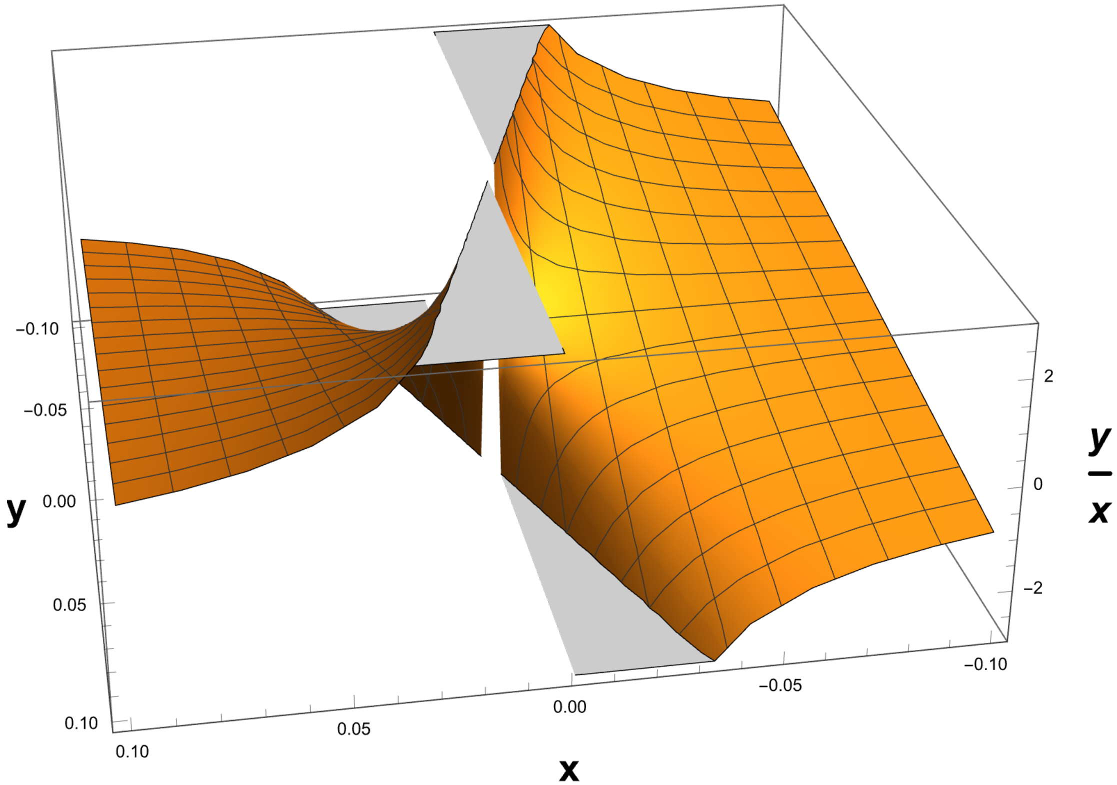
Does $\lim_{(x,y)\to(0,0)}\left(\frac{y}{x}\right)$ exist?

No it does not.

The value of $\frac{y}{x}$ depends on the path you take towards (0,0) and varies between 0 and ∞.

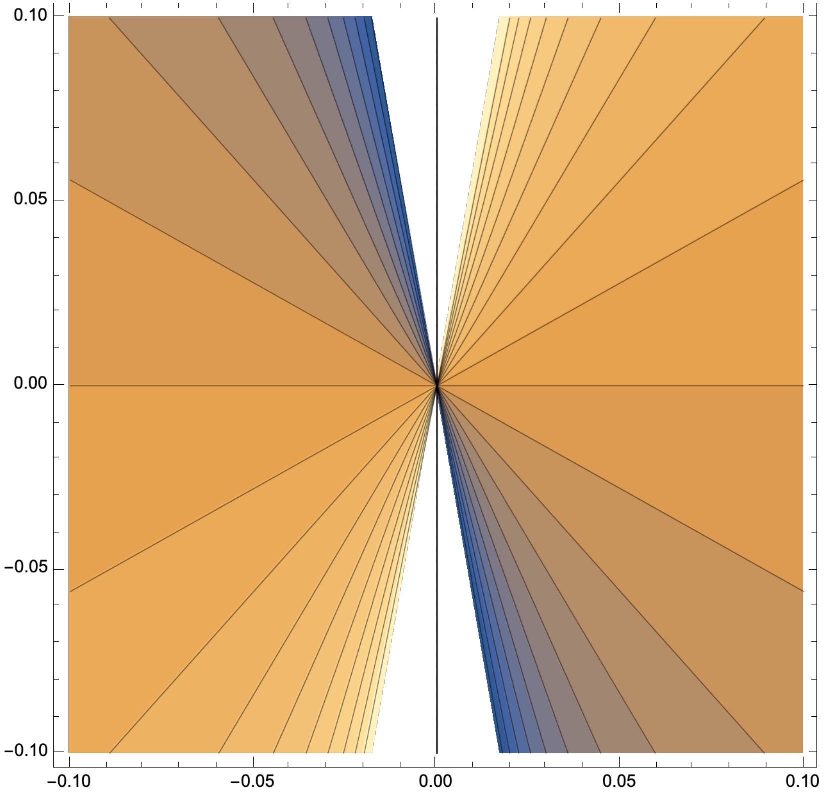
The limit definition requires that whatever path you take the limit is approached within an arbitrarily small error term, ϵ.

Out[ ]=



Contour Plots can help identify whether limits exist or not

Limits do not exist as points where Level Curves (Contour Curves) converge.



Properties of Limits of Functions of 2 Variables

The properties of limits of functions of 2 variables are similar to those of 1 variable.

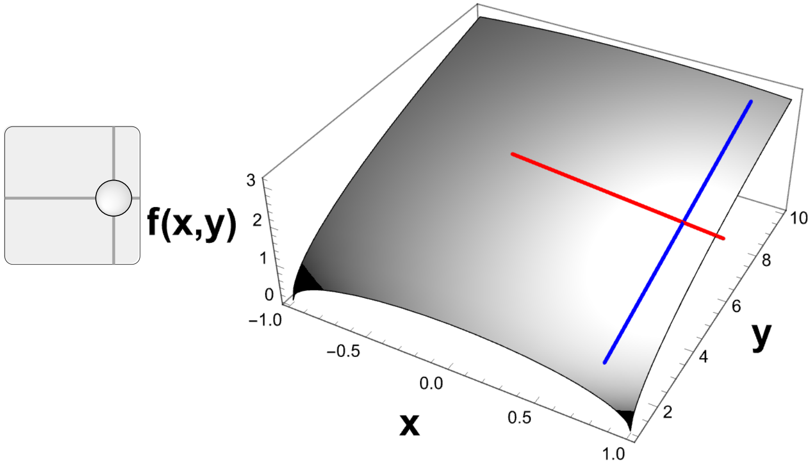
It can be shown that these properties extend to functions of 3 or more variables.

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, $M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,
   n a positive integer, and if n is even, we assume that $L > 0$.

Partial Derivatives

Out[2]=



The **partial derivative** of a function $f(x, y)$ **with respect to x** is the slope of the surface at a given point, holding y constant. It is the slope of the **red line** above.

The **partial derivative** of a function $f(x, y)$ **with respect to y** is the slope of the surface at a given point, holding x constant. It is the slope of the **blue line** above.

Partial derivative definitions

with respect to x

The partial derivative of $f(x, y)$ with respect to x at a point (x_0, y_0) is

$$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(\mathbf{x_0+h}, y_0)-f(x_0, y_0)}{h}$$ provided the limit exists.

Alternative notations for the partial derivative with respect to x include f_x and $\partial_x f$.

with respect to y

The partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is

$$\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, \mathbf{y_0+h})-f(x_0, y_0)}{h}$$ provided the limit exists.

Alternative notations for the partial derivative with respect to y include f_y and $\partial_y f$.

Calculating Partial Derivatives

To calculate the partial derivative with **respect to x**, differentiate in the normal way **treating y as a constant**

To calculate the partial derivative with **respect to y**, differentiate in the normal way **treating x as a constant**

Example

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x, y) = x^2 + 3\,x\,y + y - 1$

To find $\frac{\partial f}{\partial x}$ treat y as a constant:

$\frac{\partial f}{\partial x} = 2\,x + 3 \times 1\ \ y + 0 - 0 = 2\,x + 3\,y$

To find $\frac{\partial f}{\partial y}$ treat x as a constant:

$\frac{\partial f}{\partial y} = 0 + 3\,x\ \ 1 + 1 - 0 = 3\,x + 1$

Implicit Differentiation

Implicit differentiation works the same way for partial derivatives as it works for ordinary derivatives

Example

Find $\frac{\partial z}{\partial x}$ where

$$y\,z - \ln z = x + y$$

Differentiate both sides with respect to x, holding y constant and treating z as a function of x.

$$\frac{\partial}{\partial x}(y\,z) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\,\frac{\partial z}{\partial x} - \frac{1}{z}\,\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1 \qquad \Longrightarrow \qquad \frac{\partial z}{\partial x} = \frac{1}{\left(y - \frac{1}{z}\right)} = \frac{z}{y\,z - 1}$$

Partial Differentiation in Mathematica

In[478]:=

```
f[x_,y_]:= x^2 + 3 x y + y + 1
```

to find the partial derivative with respect to x ...

In[479]:=

```
D[f[x,y],x] (* or *)
∂x f[x,y]
```

Out[479]=

$2\,x + 3\,y$

Out[480]=

$2\,x + 3\,y$

to find the partial derivative with respect to y ...

In[481]:=

```
D[f[x,y],y] (* or *)
∂y f[x,y]
```

Out[481]=

$1 + 3\,x$

Out[482]=

$1 + 3\,x$

Partial derivatives of functions of more than 2 variables

To find the partial derivative of a function with more than 2 variables with respect to one of the variables, simply differentiate in the normal way holding all the other variables constant.

Example

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ when $f(x, y, z) = 1 + x\,y^2 - 2\,z^2$

Differentiate with respect to x holding y and z constant.

$\frac{\partial f}{\partial x} = 0 + 1 \cdot y^2 - 0 \quad = y^2$

Differentiate with respect to y holding x and z constant.

$\frac{\partial f}{\partial y} = 0 + x \cdot 2\,y - 0 \quad = 2\,x\,y$

Differentiate with respect to z holding x and y constant.

$\frac{\partial f}{\partial z} = 0 + 0 - 4\,z \quad = 4\,z$

Second-order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce **second-order derivatives**, usually denoted by

$\frac{\partial^2 f}{\partial x^2}$ or f_{xx}, $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}, $\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx}, and $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy}

Note the order in which the mixed partial derivatives are taken:

For $\frac{\partial^2 f}{\partial x \partial y}$ differentiate first with respect to y, then with respect to x.

$f_{yx} = (f_y)_x$

Example

Find $\frac{\partial^2 f}{\partial x\, \partial y}$ and $\frac{\partial^2 f}{\partial y\, \partial x}$ when $f(x, y, z) = 1 + x\, y^2 - 2\, z^2$

Differentiate with respect to x.

$\frac{\partial f}{\partial x} = 0 + 1\, y^2 - 0 \quad = y^2$

Now differentiate again with respect to y

$\frac{\partial^2 f}{\partial y\, \partial x} \; = 2\, y$

Differentiate with respect to y.

$\frac{\partial f}{\partial y} = 0 + x\, 2\, y - 0 \quad = 2\, x\, y$

Now differentiate again with respect to x

$\frac{\partial^2 f}{\partial x\, \partial y} = 2 \times 1\, y \quad = 2\, y$

The fact that $\frac{\partial^2 f}{\partial y\, \partial x}$ and $\frac{\partial^2 f}{\partial x\, \partial y}$ have the same result is not a coincidence.

The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivative f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (a, b) and are continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

This theorem is also known as **Clairaut's Theorem**.

Sometimes changing the order of differentiation helps get to the same answer more quickly.

The Chain Rule for functions of 2 variables

The Chain Rule for functions of 2 variables has a similar form to the Chain Rule for functions of 1 variable.

If $w = f(x, y)$ is differentiable and if $x(t)$ and $y(t)$ are differentiable functions of t, then

$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Example

Find $\frac{\mathrm{dw}}{\mathrm{dt}}$ when $w = f(x, y) = x + y^2$ and where $x(t) = t$ and $y(t) = \ln(t)$

First find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial x} = 1 + 0 = 1, \quad \frac{\partial f}{\partial y} = 0 + 2\,y = 2\,y$

Then find $\frac{\mathrm{dx}}{\mathrm{dt}}$ and $\frac{\mathrm{dy}}{\mathrm{dt}}$: $\quad \frac{\mathrm{dx}}{\mathrm{dt}} = 1, \quad \frac{\mathrm{dy}}{\mathrm{dt}} = \frac{1}{t}$

Using the Chain Rule:

$$\frac{\mathrm{dw}}{\mathrm{dt}} = \frac{\partial f}{\partial x}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\partial f}{\partial y}\frac{\mathrm{dy}}{\mathrm{dt}} = 1 \times 1 + 2\,y\,\frac{1}{t} = 1 + 2\,\frac{y}{t}$$

substituting $y(t) = \ln(t)$ gives: $\quad \frac{\mathrm{dw}}{\mathrm{dt}} = 1 + \frac{2\ln(t)}{t}$

Implicit Differentiation Formula

The following formula, which uses partial derivatives, can simplify the process of implicit differentiation.

Given the equation $F(x, y) = 0$ implicitly defines y as a function of x

Then $\frac{dy}{dx} = \frac{-F_x}{F_y}$

or, where $w = F(x, y)$

$$\frac{dy}{dx} = \frac{-\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial y}}$$

Implicit Differentiation Formula: Proof

$w = F(x, y) = 0 \quad \Longrightarrow \quad \frac{dw}{dx} = 0$ as w is a constant (0)

From the chain rule

$$0 = \frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx}$$

$$0 = F_x \cdot 1 + F_y \frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{-F_x}{F_y}$$

Example

Find $\frac{dy}{dx}$ when $F(x, y) = x^3 - 2y^2 + xy = 0$

The partial derivatives with respect to x, and with respect to y are

$F_x = 3x^2 - 0 + y \quad = 3x^2 + y$

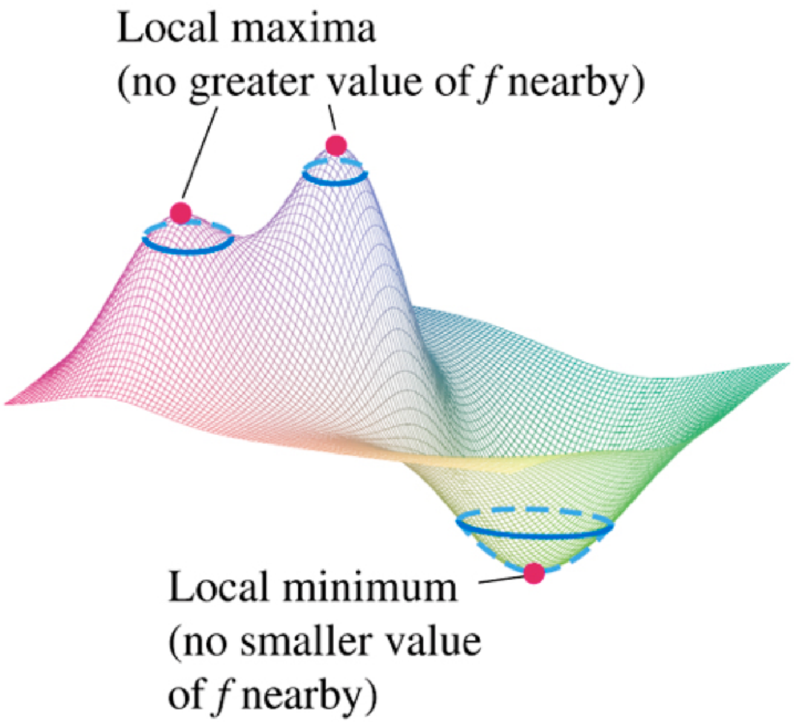
$F_y = \ 0 - 4y + x \quad = x - 4y$

Using the implicit differentiation formula

$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(3x^2+y)}{x-4y} = \frac{3x^2+y}{4y-x}$

Local Extreme Values

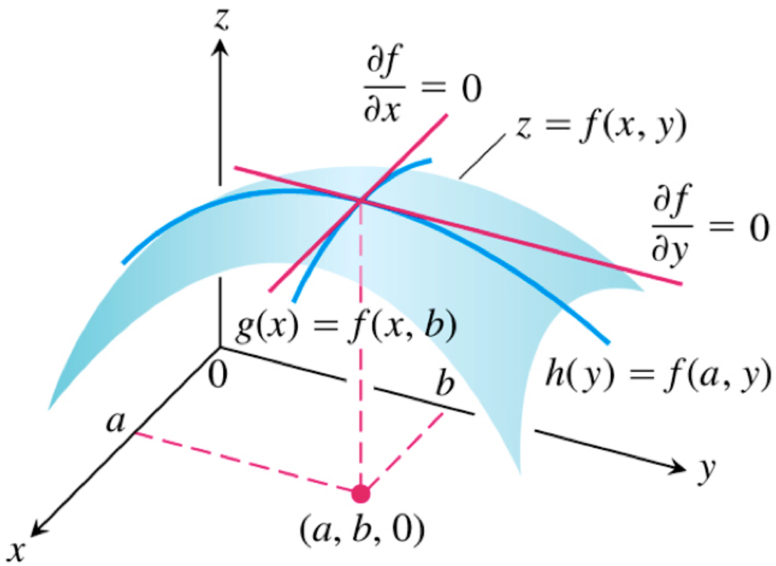
1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in the open disk centred at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in the open disk centred at (a, b).



First Derivative Test for Local Extreme Values

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then

$f_x(a, b) = 0$ and $f_y(a, b) = 0$



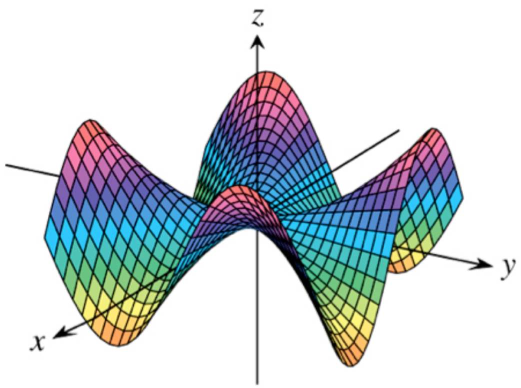
Critical Points

An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one of both of f_x and f_y do not exist is a critical point of f.

Saddle Points

A differentiable function $f(x, y)$ has a saddle point at a critical point (a, b) if in every open disk centred at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$.

The point $(a, b, f(a, b))$ on the surface $z = f(x, y, z)$ is called a saddle point of the surface.



$$z = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

Second Derivative Test for Local Extreme Values

Suppose $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centred at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

1. f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b).
2. f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b).
3. f has a **saddle point** at (a, b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a, b).
4. The test is **inconclusive** at (a, b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a, b).
   In this case, we must find some other way to determine the behaviour of f at (a, b).

The expression $\boldsymbol{f_{xx} f_{yy} - f_{xy}^2}$ is called the **discriminant** or **Hessian** of f.

The proof for the use of the **discriminant** in these tests depends on the use of **Taylor's Formula**, which we have not covered yet.

Example

Find the local extreme values of the function $f(x, y) = x\,y - x^2 - y^2 - 2\,x - 2\,y + 4$

$f_x = y - 2\,x - 2 = 0, \qquad f_y = x - 2\,y - 2 = 0$

$\Longrightarrow \; x = y = -2$

$f_{x\,x} = -2, \qquad f_{y\,y} = -2, \qquad f_{x\,y} = 1$

$\Longrightarrow \; f_{x\,x}\,f_{y\,y} - {f_{x\,y}}^2 = (-2)\,(-2) - (1)^2 = 3$

as $f_{x\,x} < 0$ and $f_{x\,x}\,f_{y\,y} - {f_{x\,y}}^2 > 0$

f has a local maximum at $(-2, -2)$

with $f(-2, -2) = 8$

Using *Mathematica* to find extreme values

In[483]:=

```
Maximize[x y-x^2-y^2-2x-2y+4,{x,y}]
```

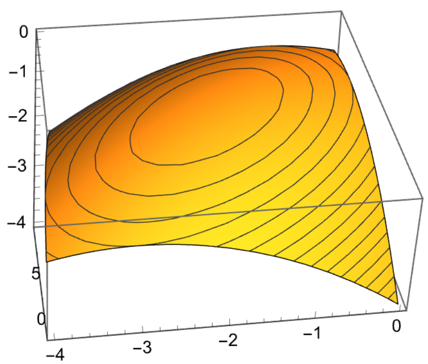
Out[483]=

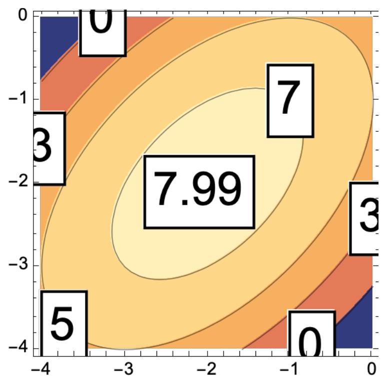
```
{8, {x → -2, y → -2}}
```

In[484]:=

```
Row[{Plot3D[x y-x^2-y^2-2x-2y+4,{x,-4,0},{y,-4,0},MeshFunctions→{#3&}],
ContourPlot[x y-x^2-y^2-2x-2y+4,{x,-4,0},{y,-4,0},Contours→{7.99,7,5,3,0},
ContourLabels→Function[{x,y,z},Text[Framed[Style[z,Large],Background→White],{x,y}]]]}]
```

Out[484]=





Absolute Maxima and Minima on Closed Bounded Regions

1. First find and classify all the **critical points in the interior of the region** (previously addressed)

2. Then consider any **end points of boundaries** as these may also be candidate points for absolute maxima or minima

3. Finally consider any **critical points that lie on the boundaries.** These need to be determined and classified individually using the equation of each boundary as a **constraint** on the function being analysed.

Solving extreme value problems with algebraic constraints usually requires the method of **Lagrange Multipliers** (see Week 4).

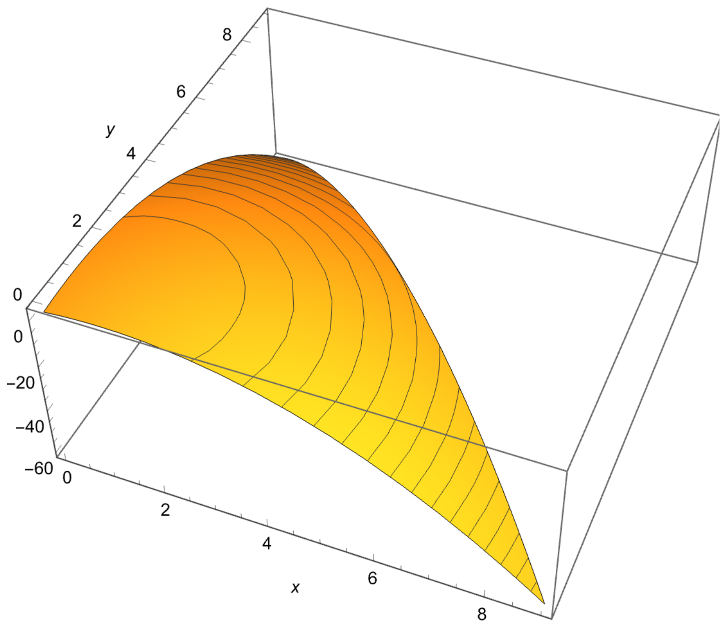
Example Question

Find the absolute maximum and minimum values of

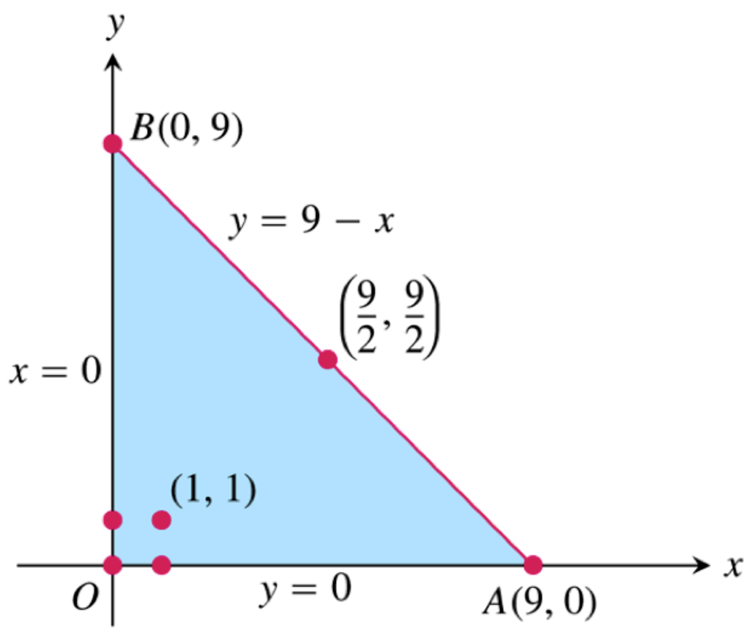
$f(x, y) = 2 + 2x + 2y - x^2 - y^2$

on the triangular region bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

Out[ ]=



Solution (steps 1): Find Interior Critical Points



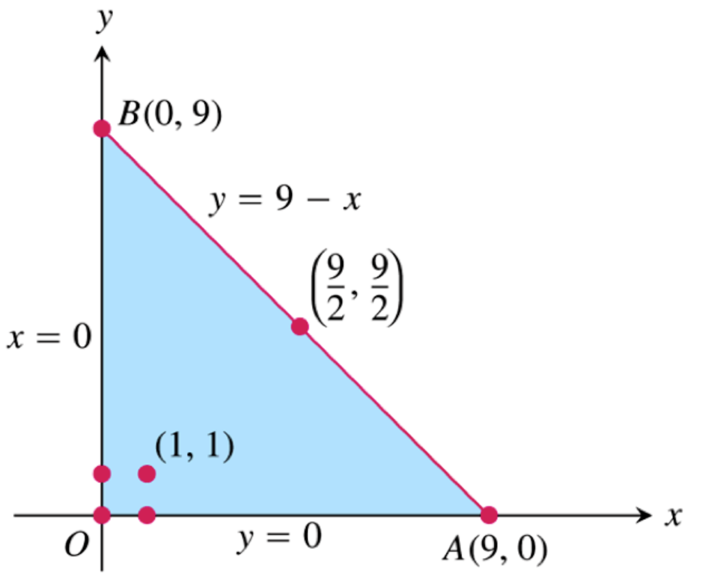
For the interior critical we solve for $f_x = 0$ and $f_y = 0$

$f_x = 2 - 2\,x = 0 \qquad \longrightarrow x = 1$

$f_y = 2 - 2\,y = 0 \qquad \longrightarrow y = 1$

$f(1, 1) = 2 + 2\,(1) + 2\,(1) - (1)^2 - (1)^2 = 4$

Solution (steps 2): Consider Boundary End Points



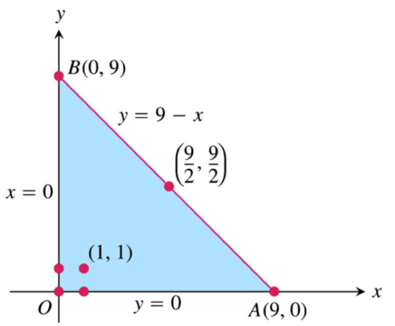
There are 3 boundary end points at O, A and B

O: $f(0, 0) = 2 + 2(0) + 2(0) - (0)^2 - (0)^2 = 2$

A: $f(9, 0) = 2 + 2(9) + 2(0) - (9)^2 - (0)^2 = -61$

B: $f(0, 9) = 2 + 2(0) + 2(9) - (0)^2 - (9)^2 = -61$

Solution (step 3): Consider Boundary Critical Points



For the **boundary OA**, $y = 0$
$f(x, y) = f(x, 0) = 2 + 2x - x^2$
Along this boundary critical points satisfy $\frac{d}{dx}\left(2 + 2x - x^2\right) = 0$

$2 - 2x = 0 \longrightarrow x = 1 \quad \longrightarrow f(1, 0) = 3$

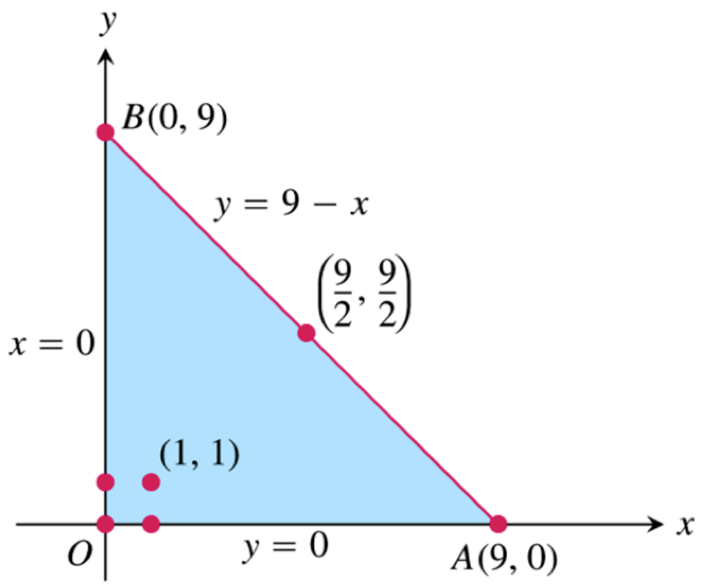
Similarly, for the **boundary OB**, $x = 0$
$f(x, y) = f(0, y) = 2 + 2y - y^2$
Along this boundary critical points satisfy $\frac{d}{dy}\left(2 + 2y - y^2\right) = 0$

$2 - 2y = 0 \longrightarrow y = 1 \quad \longrightarrow f(0, 1) = 3$

For the **boundary AB**, $y = 9 - x$
$f(x, y) = f(x, 9 - x) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2$
$= -61 + 18x - 2x^2$
Along this boundary critical points satisfy $\frac{d}{dx}\left(-61 + 18x - 2x^2\right) = 0$

$18 - 4x = 0 \quad \longrightarrow x = \frac{9}{2} \quad \longrightarrow y = 9 - x = \frac{9}{2} \quad \longrightarrow f\left(\frac{9}{2}, \frac{9}{2}\right) = -20.5$

Solution (final step):
Review all Critical Points for Absolute Maxima and Minima



The absolute maximum is at the point (1, 1) where $f = 4$

The absolute minima are the points (0, 9) and (9, 0) where $f = -61$

Solution in *Mathematica*

In[485]:=

```
Maximize[{2+2x+2y-x^2-y^2,y≤9-x,x≥0,y≥0},{x,y}]
Minimize[{2+2x+2y-x^2-y^2,y≤9-x,x≥0,y≥0},{x,y}]
```

Out[485]=

{4, {x → 1, y → 1}}

Out[486]=

{-61, {x → 0, y → 9}}

The *Mathematica* function Minimize only returns one point, even others exist with the same value. To find other absolute minima re-evaluate excluding any previously found absolute minima thus ...

In[487]:=

```
Minimize[{2+2x+2y-x^2-y^2,y≤9-x,x≥0,y≥0,x≠0,y≠9},{x,y}]
```

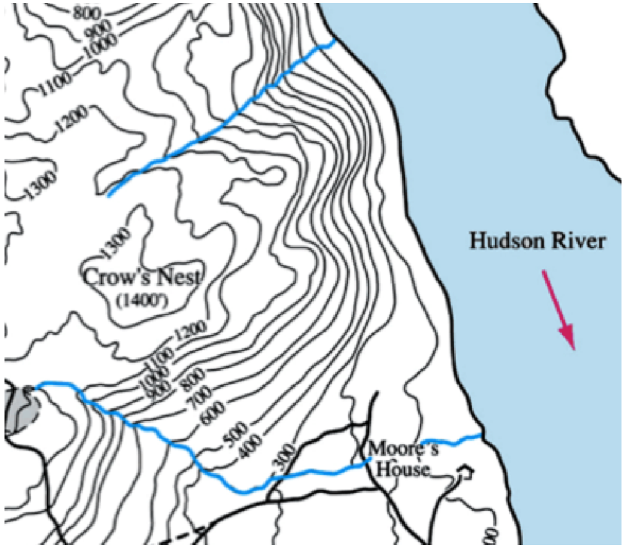
Out[487]=

{-61, {x → 9, y → 0}}

Directional Derivatives

For functions of 2 variables, we have so far considered slopes (partial derivatives) in just two directions.

Now we consider derivatives in any arbitrary direction. These are called **directional derivatives**.



We will also consider finding the **direction with the greatest slope**.

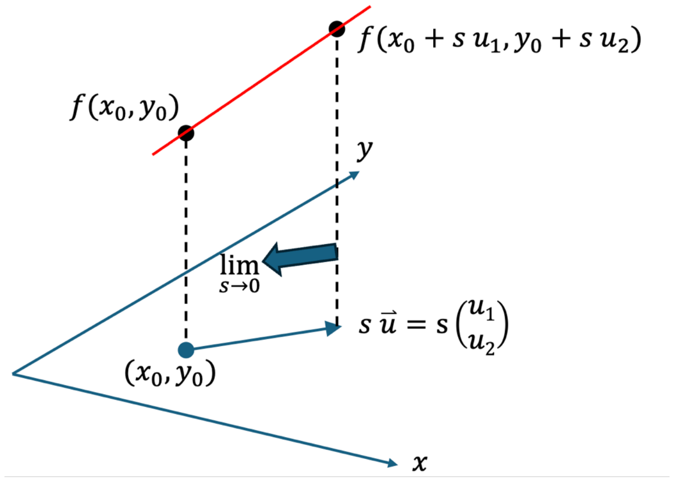
Directional Derivative

Let the vector $u = u_1\, i + u_2\, j$ be the unit vector in the direction of the derivative we wish to find.

The derivative of f at the point $P_0 = (x_0, y_0)$ in the direction of the vector $u = u_1\, i + u_2\, j$
is

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s\to 0} \frac{f(x_0+s\,u_1, y_0+s\,u_2) - f(x_0, y_0)}{s}$$

provided the limit exists.



The Gradient Vector

The **Gradient Vector** $\nabla \boldsymbol{f}$ at a point P_0 has the property that it is **normal to the level curve** of function f at the point P_0.

The notation $\nabla \boldsymbol{f}$ is read as "**grad** $\boldsymbol{f}$" as well as "gradient of f" and "del f".

Definition

The **gradient vector of** $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

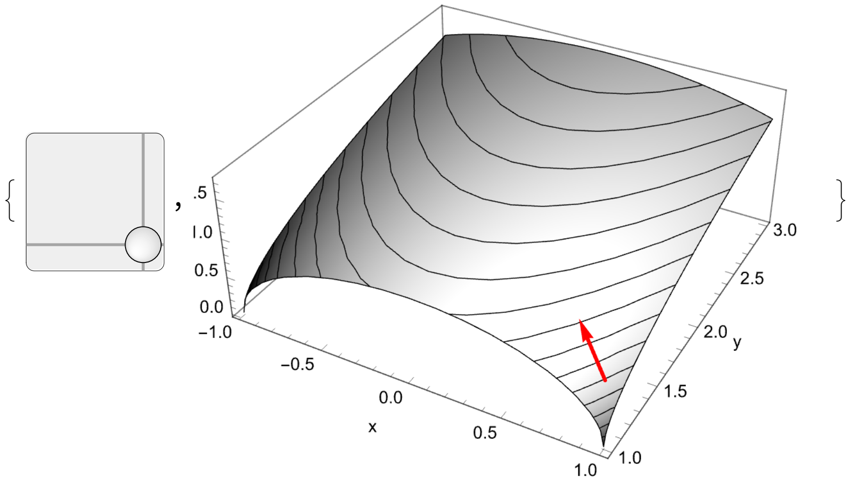
$$\nabla f = \frac{\partial f}{\partial x}\, \boldsymbol{i} + \frac{\partial f}{\partial y}\, \boldsymbol{j}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ are the unit vectors in the directions of $\boldsymbol{x}$, $\boldsymbol{y}$ respectively.

Gradient Vector Visualisation

The plot below show a vector in the same direction as ∇f.

Out[489]=



Calculating Directional Derivatives

Directional derivatives can be efficiently calculated by a formula that uses the **gradient vector** ∇f.

Theorem

The directional derivative of the function $f(x, y)$ at the point $P_0(x_0, y_0)$ in the direction of the unit vector $\boldsymbol{u} = u_1\, \boldsymbol{i} + u_2\, \boldsymbol{j}$ is the following dot product.

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

Note: the vector **u** is a unit vector if its magnitude $|\mathbf{u}| = 1$.

Directional Derivative: Alternative Notation

$D_u f$ is an alternative notation for $\left(\frac{df}{ds}\right)_u$

with an alternative form of the directional derivative formula

$D_u f = \nabla f . u$

Directional Derivative when Direction Vector is Not a Unit Vector

If a question asks you to find a directional derivative in the direction of a vector **u** that is NOT a unit vector you generally need to first convert it to a unit vector:

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \frac{\mathbf{u}}{|\mathbf{u}|}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} \frac{u_1}{|\mathbf{u}|} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{u_2}{|\mathbf{u}|}$$

The exception to this rule is if you are using directional directives to estimate change in given direction and magnitude (which we will look at in Week 4).

Example

Find the directional derivative of the function $f(x, y) = 2\,x\,y - 3\,y^2$ at the point $P_0(5, 5)$ in the direction $\boldsymbol{v} = 4\,\boldsymbol{i} + 3\,\boldsymbol{j}$.

First find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial x} = 2\,y + 0$ so $\left(\frac{\partial f}{\partial x}\right)_{(5,5)} = 2 \cdot 5 = 10$

$\frac{\partial f}{\partial y} = 2\,x - 6\,y$ so $\left(\frac{\partial f}{\partial y}\right)_{(5,5)} = 2 \cdot 5 - 6 \cdot 5 = -20$

gradient vector $\nabla f = 10\,\boldsymbol{i} - 20\,\boldsymbol{j}$

Example contd.

Next convert the direction vector v to a unit vector

$$u = \frac{v}{|v|} = \frac{4i+3j}{\sqrt{4^2+3^2}} = \frac{4}{5}i + \frac{3}{5}j$$

unit vector $u = \frac{4}{5}i + \frac{3}{5}j$

Now using the directional directive formula, where $u_1 = \frac{4}{5}$ and $u_2 = \frac{3}{5}$

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} = \nabla f \cdot \boldsymbol{u} = \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} = 10 \times \frac{4}{5} - 20 \times \frac{3}{5} = -4$$

Example *Mathematica Solution*

The previous directive derivative calculation can be performed in a single line in *Mathematica*

In[490]:=

```
Grad[2x y-3y^2,{x,y}].Normalize[{4,3}]   /.{x→5,y→5}
```

Out[490]=

-4

Note:

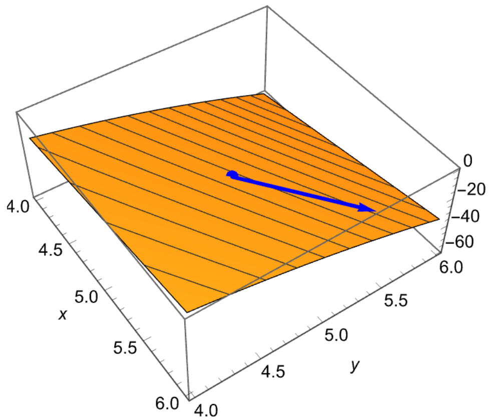
Grad returns a vector of the partial differentials with respect to *x* and *y*.

Normalize returns the unit vector in the same direction.

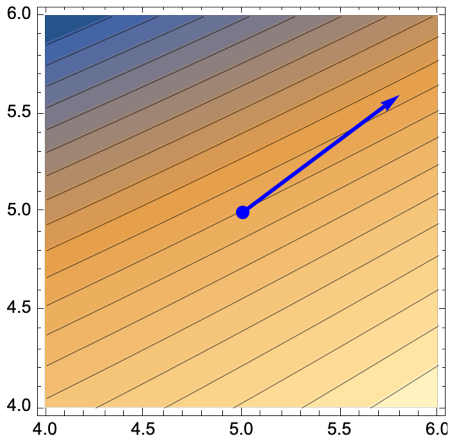
Visualising this solution in *Mathematica*

First using Plot3D.

Out[448]=



Then using a ContourPlot.



Directional Derivatives can also be expressed using Cos θ

Remember that any vector dot product can be written as

$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}|\ |\boldsymbol{v}| \cos\theta$,   where θ is the angle between vectors **u** and **v**

so, we can alternatively state the directional derivatives thus

$D_u f = \nabla f \cdot u$

$= |\nabla f|\ |u| \cos\theta$

$= |\nabla f| \cos\theta$

Note: $|\boldsymbol{u}| = 1$ as $\boldsymbol{u}$ is a unit vector

Direction of greatest increase/decrease

The direction of greatest increase in f is in the same direction as ∇f with a value of $D_u f = |\nabla f|$

$D_u f = |\nabla f| \cos\theta$ implies the direction of greatest increase in f is when $\cos\theta = 1$ which is when $\theta = 0$.

The direction of greatest decrease in f is in the direction $-\nabla f$ with a value of $D_u f = -|\nabla f|$

$D_u f = |\nabla f| \cos\theta$ implies the direction of greatest decrease in f is when $\cos\theta = -1$ which is when $\theta = \pi$.

Direction of Zero Change in *f*

$D_u f = |\nabla f| \cos\theta$ implies the direction of zero change in f is when $\cos\theta=0$ which is when $\theta = \frac{\pi}{2}$.

This confirms that the gradient vector ∇f (at a give point P) is normal to the level curves of f (passing through P), which are defined to lie in the direction of zero change in f.