Week 6:
Introduction to Linear Algebra and Matrices

MSIN00180 Quantitative Methods for Business

What is linear algebra?

Linear algebra is particularly concerned with the **systematic solution of systems of linear equations** using vectors and matrices.

Linear algebra is very widely used to process, solve and optimise **large-scale data-rich problems** in business, economics, science, and engineering.

Linear algebra coupled with statistics provides the foundational mathematics for **data analytics**.

Computers and linear algebra

Using a computer is a practical necessity for anything other than the simplest/smallest of linear algebra problems.

Tools like Mathematica, Matlab, R, and a wide range of programming language libraries such as numpy (python) provide linear algebra capabilities.

We will be using **Mathematica** extensively.

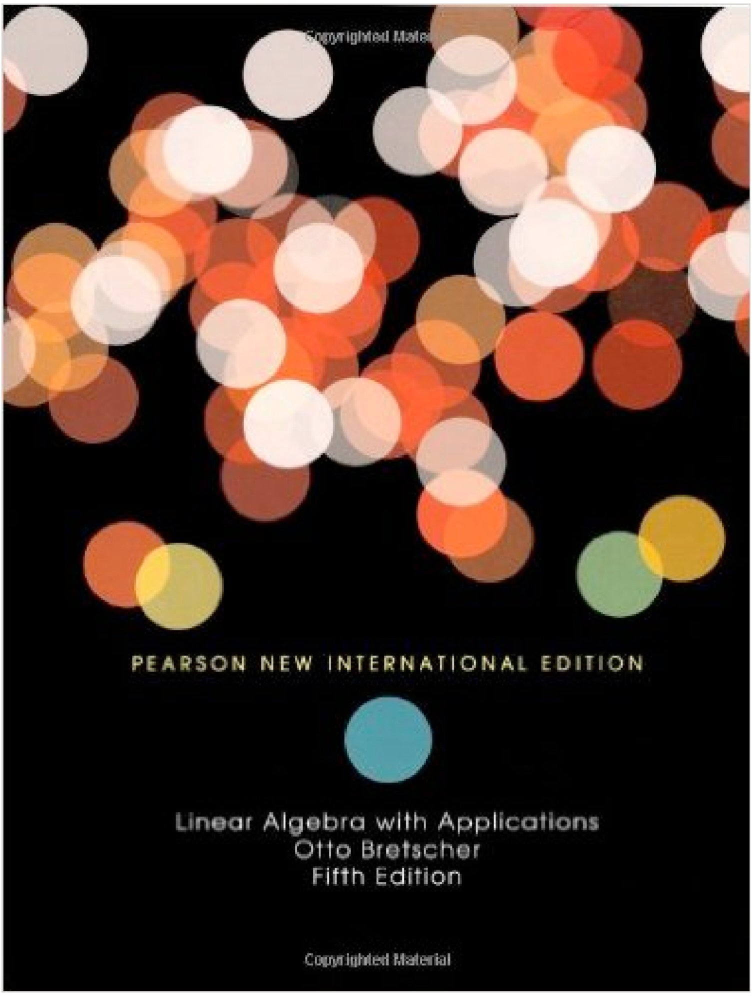
Week-by-Week Overview

- **Week 6:** Introduction to Linear Algebra
- **Week 7:** Matrix Algebra and State Transition Networks
- **Week 8:** Least Squares Data Fitting and Determinants
- **Week 9:** Eigenvalues and Discrete Dynamic Systems
- **Week 10:** Revision Lecture

Textbook

Linear Algebra with Applications (5th Edition)*, Otto Bretscher, Pearson (ISBN-10: 0-321-89058-2, ISBN-13: 978-0-321-89058-0)*

Unfortunately, as is common with maths textbooks, it is a little more expensive than we would ideally like.



Representing Linear Expressions as Dot Products

The dot product of a vector of coefficients and a vector of variables can be used to build an expression.

We generally layout such a **dot product** as a ‘row’ vector followed by a ‘column’ vector thus

$$\begin{pmatrix} 5 & 3 & 2 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5x + 3y + 2z \end{pmatrix}$$

Variable Names

In Linear Algebra variables names are typically expressed as $\boldsymbol{x_1, x_2, x_3, \ldots}$ rather than $x, y, z, \ldots$

$$\begin{pmatrix} 5 & 3 & 2 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Using $\boldsymbol{x_1, x_2, x_3}$ etc can be a little confusing to begin with so I will continue to use x, y, z for a while.

A matrix can be defined as an ordered set of row vectors or column vectors

$$\begin{pmatrix} 5 & 3 & 2 \\ 6 & 4 & 1 \\ 3 & 2 & 5 \end{pmatrix} \Longrightarrow \begin{pmatrix} \begin{pmatrix} 5 & 3 & 2 \end{pmatrix} \\ \begin{pmatrix} 6 & 4 & 1 \end{pmatrix} \\ \begin{pmatrix} 3 & 2 & 5 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 & 2 \\ 6 & 4 & 1 \\ 3 & 2 & 5 \end{pmatrix} \Longrightarrow \begin{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix} & \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \end{pmatrix}$$

Dot product of a matrix and a vector

The **dot product** of the matrix **m** and a (column) vector **v** is a vector where each element is the dot product of each row vector of **m** and the vector **v.**

$$\begin{pmatrix} 5 & 3 & 2 \\ 6 & 4 & 1 \\ 3 & 2 & 5 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 5 & 3 & 2 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} 6 & 4 & 1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} 3 & 2 & 5 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 5x + 3y + 2z \\ 6x + 4y + z \\ 3x + 2y + 5z \end{pmatrix}$$

A Set of Linear Expressions can be represented as a Dot Product of a Matrix and a Vector

$$\begin{pmatrix} 5 & 3 & 2 \\ 6 & 4 & 1 \\ 3 & 2 & 5 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5x + 3y + 2z \\ 6x + 4y + z \\ 3x + 2y + 5z \end{pmatrix}$$

Entering matrices into Mathematica: Method 1

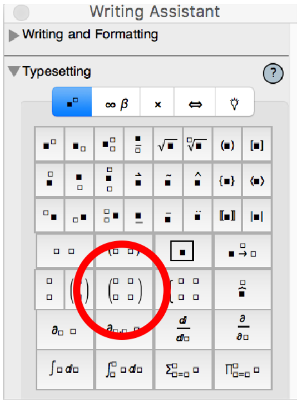
From the Mathematica menu bar click **Insert > Table/Matrix > New...**

In[ ]:=
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$ //MatrixForm

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Entering matrices into Mathematica: Method 2

Alternatively, use a Mathematica **pallet** to add an empty 2x2 matrix



Then to ...

- insert a **new row**: Ctrl-Enter
- insert a **new column**: Ctrl-Comma

In[128]:=

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Out[128]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Configuring Mathematica to output Matrices as expected

The default behaviour of **Mathematica** is to output matrices as a list of row lists:

In[ ]:=

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$ //StandardForm

Out[ ]//StandardForm=

```
{{1, 2}, {3, 4}}
```

To avoid this, you can add **//MatrixForm** after each matrix you want to output:

In[ ]:=

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$ //MatrixForm

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Configuring Mathematica to output Matrices as expected

Alternatively, you can run the following optional code **once** at the start of Mathematica notebooks to make Mathematica automatically output matrices as expected:

In[33]:=
```
$Post:=If[MatrixQ[#]||VectorQ[#],MatrixForm[#],#]&;
```

You will often see this code appear at the start of lecture and seminar Mathematica notebooks. Run this code cell before doing any work in these Mathematica notebooks.

Matrix calculations in Mathematica

This code finds the dot product of a matrix and a vector:

In[34]:=
```
m = {{5, 3, 2}, {6, 4, 1}, {3, 2, 5}};     v = {{x}, {y}, {z}};

m.v
```

$$m=\begin{pmatrix}5 & 3 & 2\\ 6 & 4 & 1\\ 3 & 2 & 5\end{pmatrix}; \qquad v=\begin{pmatrix}x\\ y\\ z\end{pmatrix};$$

Out[35]//MatrixForm=

$$\begin{pmatrix}5\,x+3\,y+2\,z\\ 6\,x+4\,y+z\\ 3\,x+2\,y+5\,z\end{pmatrix}$$

The “dot product” operator is an ordinary full stop in Mathematica.

Referencing matrix elements

$\mathbf{m_{ij}}$ refers to the element of matrix m on row **i** and column **j** :

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

Referencing elements of matrices in Mathematica

In[ ]:=
```
m=(5 3 2
   6 4 1
   3 2 5);
```

$$m=\begin{pmatrix}5 & 3 & 2\\6 & 4 & 1\\3 & 2 & 5\end{pmatrix};$$

To get the element m_{12} at row 1 and column 2 in Mathematica you write

In[ ]:=
```
m[[1,2]]
```

Out[ ]=

3

Referencing rows of matrices in Mathematica

In[ ]:=
```
m = (5 3 2
     6 4 1
     3 2 5);
```

$$m=\begin{pmatrix}5&3&2\\6&4&1\\3&2&5\end{pmatrix};$$

To get row 2

In[ ]:=
```
m[[2]] // StandardForm
```

Out[ ]//StandardForm=
```
{6, 4, 1}
```

Confusingly, Mathematica's **MatrixForm** output prints all vectors vertically but this can be ignored.

Referencing columns of matrices in Mathematica

In[∘]:=

$$m=\begin{pmatrix}5 & 3 & 2\\ 6 & 4 & 1\\ 3 & 2 & 5\end{pmatrix};$$

To get column 3

In[∘]:=

```
m⟦All,3⟧
```

Out[∘]//MatrixForm=

$$\begin{pmatrix}2\\ 1\\ 5\end{pmatrix}$$

Alternative Mathematica subscript notations

You can also use subscripted 〚 〛 symbols in **Mathematica** for references:

In[ ]:=
```
m〚1,2〛
```

Out[ ]=

3

In[ ]:=
```
m〚All,3〛
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$$

I will generally be using this more readable notation in the lecture slides but, in practice, it is quicker to type normal square brackets when using Mathematica, as per the previous slide.

Row operations in Mathematica

In[∘]:=

$$m=\begin{pmatrix}5 & 3 & 2\\ 6 & 4 & 1\\ 3 & 2 & 5\end{pmatrix};$$

For reasons we will get to later, it is often useful to multiply rows by a scalar.

eg to multiply row2 by 3

In[∘]:=

```
m[[2]] = 3m[[2]];
m
```

Out[∘]//MatrixForm=

$$\begin{pmatrix}5 & 3 & 2\\ 18 & 12 & 3\\ 3 & 2 & 5\end{pmatrix}$$

Row operations in Mathematica

In[129]:=

```
m = {{5, 3, 2}, {18, 12, 3}, {3, 2, 5}};
```

$$m=\begin{pmatrix} 5 & 3 & 2 \\ 18 & 12 & 3 \\ 3 & 2 & 5 \end{pmatrix};$$

Another row operation that is often used, is subtracting a multiple of one row from another row

eg to subtract $\frac{1}{6}\times$row2 from row3

In[130]:=

$$m_{[[3]]} = m_{[[3]]} - \frac{1}{6} m_{[[2]]};$$

```
m
```

Out[131]//MatrixForm=

$$\begin{pmatrix} 5 & 3 & 2 \\ 18 & 12 & 3 \\ 0 & 0 & \frac{9}{2} \end{pmatrix}$$

Vector Spaces

The entries of a vector are called its **components**

The **set of all vectors with n components** is denoted by $\mathbb{R}^n$

We refer to $\mathbb{R}^n$ as a **vector space**

$\begin{pmatrix} 1 & 2 & 9 & 1 \end{pmatrix}$ is a vector in $\mathbb{R}^4$

Dimensions of a Matrix

A matrix that has n **rows** and m **columns** is referred to as an

$n \times m$ **matrix**

$\boldsymbol{n \times m}$ are the **dimensions** of a matrix

Remember that the **number of rows is always stated first**

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

is a 2×3 matrix

with row vectors in $\mathbb{R}^3$

with column vectors in $\mathbb{R}^2$

Expressing Simultaneous Equations using Matrices

Consider the following set of linear simultaneous equations

$$5x + 3y + 2z = 24$$
$$6x + 4y + z = 23$$
$$3x + 2y + 5z = 34$$

These equations can be expressed in matrix form

$$\begin{pmatrix} 5 & 3 & 2 \\ 6 & 4 & 1 \\ 3 & 2 & 5 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24 \\ 23 \\ 34 \end{pmatrix}$$

↑ coefficient matrix

The Augmented Matrix

$$\begin{pmatrix} 5 & 3 & 2 \\ 6 & 4 & 1 \\ 3 & 2 & 5 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24 \\ 23 \\ 34 \end{pmatrix}$$

To solve a system of equations using matrices one method involves constructing a single matrix containing all the numerical information in the equations called the **augmented matrix**

$$\begin{pmatrix} 5 & 3 & 2 & 24 \\ 6 & 4 & 1 & 23 \\ 3 & 2 & 5 & 34 \end{pmatrix}$$

Solving Simultaneous Equations

Typically, solving simultaneous equations involves combinations of the following steps:

- Divide/multiply one equation by a scalar
- Add/subtract a scalar multiple of one equation to/from another equation

In linear algebra these equation-modifying steps are called row operations.

We will now consider a repeatable method that applies row operations to solve simultaneous equations ...

A method for solving systems of linear simultaneous equations using matrices

Consider our current example set of equations

$5x + 3y + 2z = 24$

$6x + 4y + z = 23$

$3x + 2y + 5z = 34$

First we define the **augmented matrix**

$$m = \begin{pmatrix} 5 & 3 & 2 & 24 \\ 6 & 4 & 1 & 23 \\ 3 & 2 & 5 & 34 \end{pmatrix}$$

Consider the first row...

Divide the first row by its **leading coefficient**, which is 5

$$m=\begin{pmatrix}5 & 3 & 2 & 24\\ 6 & 4 & 1 & 23\\ 3 & 2 & 5 & 34\end{pmatrix};$$

$$m_{[\![1]\!]}=\frac{1}{5}m_{[\![1]\!]};\quad m\ \text{//MatrixForm}$$

$$\begin{pmatrix}1 & \frac{3}{5} & \frac{2}{5} & \frac{24}{5}\\ 6 & 4 & 1 & 23\\ 3 & 2 & 5 & 34\end{pmatrix}$$

Now eliminate the first coefficient from the second row

In[ ]:= $m_{[\![2]\!]}=m_{[\![2]\!]}-6m_{[\![1]\!]};\quad m\text{//MatrixForm}$

$$\begin{pmatrix}1 & \frac{3}{5} & \frac{2}{5} & \frac{24}{5}\\ 0 & \frac{2}{5} & -\frac{7}{5} & -\frac{29}{5}\\ 3 & 2 & 5 & 34\end{pmatrix}$$

And also eliminate the first coefficient from the third row

In[ ]:= $m_{[\![3]\!]}=m_{[\![3]\!]}-3m_{[\![1]\!]};\quad m\text{//MatrixForm}$

$$\begin{pmatrix}1 & \frac{3}{5} & \frac{2}{5} & \frac{24}{5}\\ 0 & \frac{2}{5} & -\frac{7}{5} & -\frac{29}{5}\\ 0 & \frac{1}{5} & \frac{19}{5} & \frac{98}{5}\end{pmatrix}$$

Now consider the second row...

Divide by its leading (non-zero) coefficient, which is $\frac{2}{5}$

In[ ]:=
```
m[[2]] = 5/2 m[[2]];    m//MatrixForm
```

$$\begin{pmatrix} 1 & \frac{3}{5} & \frac{2}{5} & \frac{24}{5} \\ 0 & 1 & -\frac{7}{2} & -\frac{29}{2} \\ 0 & \frac{1}{5} & \frac{19}{5} & \frac{98}{5} \end{pmatrix}$$

Now eliminate the second coefficient from the other two rows

In[ ]:=
```
m[[1]] = m[[1]] - 3/5 m[[2]];
 m[[3]] = m[[3]] - 1/5 m[[2]];      m//MatrixForm
```

$$\begin{pmatrix} 1 & 0 & \frac{5}{2} & \frac{27}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{29}{2} \\ 0 & 0 & \frac{9}{2} & \frac{45}{2} \end{pmatrix}$$

Now consider the third row...

Divide its leading coefficient, which is $\frac{9}{2}$

In[ ]:=
```
m[[3]] = 2/9 m[[3]];    m//MatrixForm
```

$$\begin{pmatrix} 1 & 0 & \frac{5}{2} & \frac{27}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{29}{2} \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

Now eliminate the third variable coefficient from the other two rows

In[ ]:=
```
m[[1]] = m[[1]] - 5/2 m[[3]];
 m[[2]] = m[[2]] + 7/2 m[[3]];    m//MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

Convert back to normal form ...

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

We have reached a special form of the augmented matrix called the **reduced row-echelon form** of m.

Converting this augmented matrix back to the normal form shows we have now solved for x, y and z.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1x + 0y + 0z \\ 0x + 1y + 0z \\ 0x + 0y + 1z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \quad \longrightarrow x = 1, \; y = 3, \; z = 5$$

Identity matrix

A square matrix of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is called an **identity matrix**

The **dot product** of an **identity matrix** with a vector leaves the vector unchanged.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1\,a + 0\,b + 0\,c \\ 0\,a + 1\,b + 0\,c \\ 0\,a + 0\,b + 1\,c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Leading variables and leading coefficients

For an equation of the form $\boldsymbol{c\, x_j + \ldots = b}$, where **c is the first nonzero coefficient**, the variable is x_j is referred to as the **leading variable** and c as the **leading coefficient**.

e.g.

in the equation $0\, x_1 + \mathbf{3}\, \boldsymbol{x_2} - x_3 = 5$

$\boldsymbol{x_2}$ is the leading variable
3 is the leading coefficient

Gauss-Jordan elimination method

1. Proceed from equation to equation, top to bottom, until the i_{th} equation is reached of the form $\boldsymbol{c\, x_j + \ldots = b}$ where c≠0.
   Divide this i_{th} equation by c: $m_{[[i]]} = \frac{1}{c}\, m_{[[i]]}$

2. Eliminate x_j from all the other equations, above and below the i_{th} equation by subtracting suitable multiples of the i_{th} equation.

3. Repeat from step (1) until all equations have been considered.

4. Finally convert back to normal form to solve each equation for its leading variable.

This may result in **no solutions** (due to an inconsistent set of equations), a **single solution** or an **infinite set of solutions**.

Reduced Row-Echelon Form (RREF)

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

A matrix is in **reduced row-echelon** form:

1. If a row has nonzero entries the **first (leftmost) nonzero entry is 1**
   - called the "**leading one**" in this row

2. If a column contains a leading 1 **all the other entries in that column are 0**

3. If a row contains a leading 1 **each row above it contains a leading 1 further to the left**

Gauss-Jordan elimination is typically performed using a dedicated computer program

In **Mathematica** the function that performs Gauss-Jordan elimination to produce a reduced row-echelon matrix form is **RowReduce**

In[ ]:=
```
RowReduce[(5 3 2 24
           6 4 1 23
           3 2 5 34)]//MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

Inconsistent systems of linear equations

Consider the following system of equations

$$\begin{pmatrix} 1 & -3 & 0 & -5 \\ 3 & -12 & -2 & -27 \\ -2 & 10 & 2 & 24 \\ -1 & 6 & 1 & 14 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -7 \\ -33 \\ 29 \\ 17 \end{pmatrix}$$

Gauss-Jordan Elimination of the augmented matrix is

In[ ]:=
```
RowReduce[(1   -3   0   -5   -7
           3  -12  -2  -27  -33
          -2   10   2   24   29
          -1    6   1   14   17)]  //MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The final row is of the form ***zero=nonzero*** which shows that this system has no solutions

Systems of equations with infinite solutions

Consider the following system of equations

$$\begin{pmatrix} 2 & 4 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 & 0 \\ 3 & 6 & -2 & 1 & 9 \\ 5 & 10 & -4 & 5 & 9 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 9 \end{pmatrix}$$

The RREF for this augmented matrix is:

In[ ]:=
```
RowReduce[(2  4  -2 2 4 2
           1  2  -1 2 0 4
           3  6  -2 1 9 1
           5  10 -4 5 9 9)] //MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that the number of leading ones (3) is less than the number of variables (5). This indicates that there are infinite solutions.
But this is a constrained infinite solution and is found as follows .

Systems of equations with infinite solutions

Re-expressing this RREF augmented matrix in standard form we get

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix}$$

Expanding out the dot product gives

$$\begin{pmatrix} x_1 + 2 x_2 + 3 x_5 \\ x_3 - x_5 \\ x_4 - 2 x_5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix}$$

Systems of equations with infinite solutions

Rearranging to isolate the leading variables

$$\begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 - 2 x_2 - 3 x_5 \\ 4 + x_5 \\ 3 + 2 x_5 \end{pmatrix}$$

Use **parametric substitutions for the non-leading variables**

eg If we let $x_2 = t$ and $x_5 = r$ then there are infinitely many solutions of the form

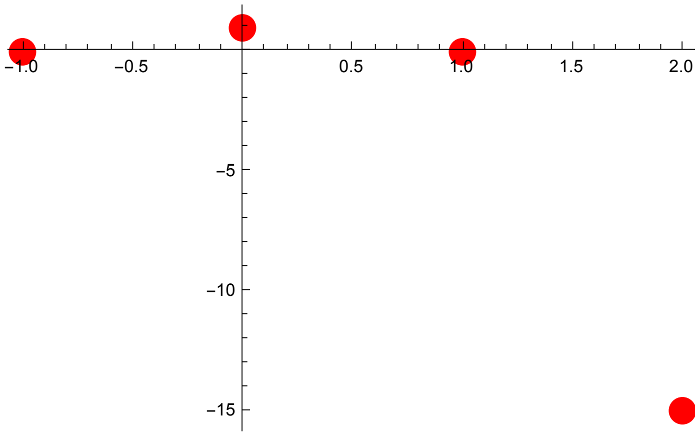
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 - 2 t - 3 r \\ t \\ 4 + r \\ 3 + 2 r \\ r \end{pmatrix}$$

Though infinite this solution is constrained by *t* and *r*.

Example: Fitting a cubic to 4 points

Find a polynomial of the form $f(t) = a + b\,t + c\,t^2 + d\,t^3$ whose graph goes through the points (0,1), (1,0), (-1,0) and (2,-15).

Out[56]=



How can we express this problem as a system of simultaneous linear equations?

Example: Fitting a cubic to 4 points

Consider the cubic equation

$$f(t) = a + b\,t + c\,t^2 + d\,t^3$$

Use each data point with this cubic to provide a linear equation in terms of a, b, c and d:

at point $(0, 1) \to \quad a + b\,0 + c\,0^2 + d\,0^3 = 1$

at point $(1, 0) \to \quad a + b\,1 + c\,1^2 + d\,1^3 = 0$

at point $(-1, 0) \to \quad a + b\,(-1) + c\,(-1)^2 + d\,(-1)^3 = 0$

at point $(2, -15) \to \quad a + b\,(2) + c\,(2)^2 + d\,(2)^3 = -15$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -15 \end{pmatrix}$$

Example: Fitting a cubic to 4 points

Express these equations as an augmented matrix:

In[ ]:=

$$m=\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & 4 & 8 & -15 \end{pmatrix};$$

What do we do now?

Example: Fitting a cubic to 4 points

In[◦]:=
```
m = (1  0  0  0    1
     1  1  1  1    0
     1 -1  1 -1    0
     1  2  4  8  -15);
```

$$m=\begin{pmatrix}1&0&0&0&1\\1&1&1&1&0\\1&-1&1&-1&0\\1&2&4&8&-15\end{pmatrix};$$

Make zeros in the first column under the leading one in row1

In[◦]:=
```
m[[2]]=m[[2]]-m[[1]];
m[[3]]=m[[3]]-m[[1]];
m[[4]]=m[[4]]-m[[1]];m//MatrixForm
```

Out[◦]//MatrixForm=

$$\begin{pmatrix}1&0&0&0&1\\0&1&1&1&-1\\0&-1&1&-1&-1\\0&2&4&8&-16\end{pmatrix}$$

Make zeros in the 2nd column above and below the leading one in row2

In[◦]:=
```
m[[3]]=m[[3]]+m[[2]];
m[[4]]=m[[4]]-2m[[2]];m//MatrixForm
```

Out[◦]//MatrixForm=

$$\begin{pmatrix}1&0&0&0&1\\0&1&1&1&-1\\0&0&2&0&-2\\0&0&2&6&-14\end{pmatrix}$$

Example: Fitting a cubic to 4 points

Make a leading one in row3

In[ ]:=
$$m_{[[3]]}=\frac{m_{[[3]]}}{2};m//\text{MatrixForm}$$

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 6 & -14 \end{pmatrix}$$

Make zeros in the 3rd column above and below the leading one in row3

In[ ]:=
$$m_{[[2]]}=m_{[[2]]}-m_{[[3]]};$$
$$m_{[[4]]}=m_{[[4]]}-2m_{[[3]]};m//\text{MatrixForm}$$

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 6 & -12 \end{pmatrix}$$

Example: Fitting a cubic to 4 points

Make a leading one in row4

In[ ]:=
```
m[[4]] = m[[4]]/6 ;m//MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Make zeros in the 4th column above the leading one in row4

In[ ]:=
```
m[[2]]=m[[2]]-m[[4]];m//MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Read off the solution from the last column

⟶ a=1, b=2, c=-1, d=-2

Example: Fitting a cubic to 4 points

Method 2: Gauss-Jordan elimination in one step in Mathematica

In[∘]:=

$$\text{RowReduce}\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & 4 & 8 & -15 \end{pmatrix}\right] \text{ //MatrixForm}$$

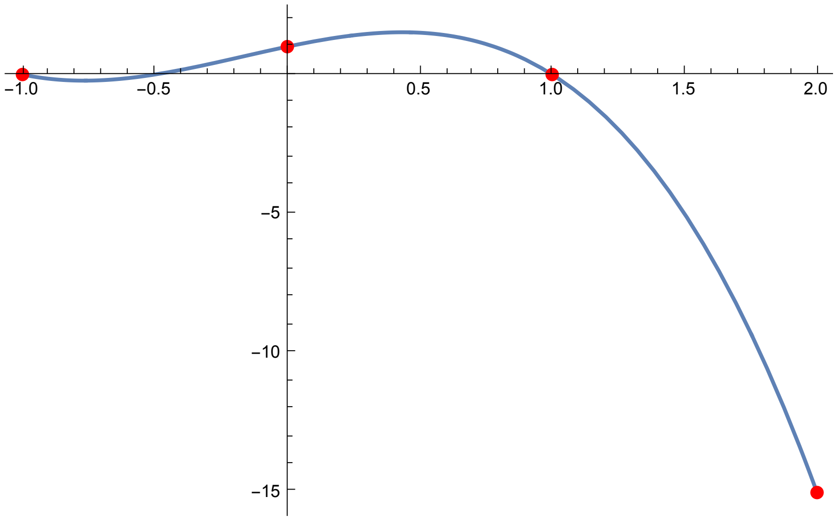
Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Sketch the graph of this cubic ...

```
Show[
    Plot[1+2t-t^2-2t^3,{t,-1,2}, PlotRange→All],
    ListPlot[{{0,1},{1,0},{-1,0},{2,-15}},
            PlotStyle→Directive[Red,PointSize[Large]]]]
```

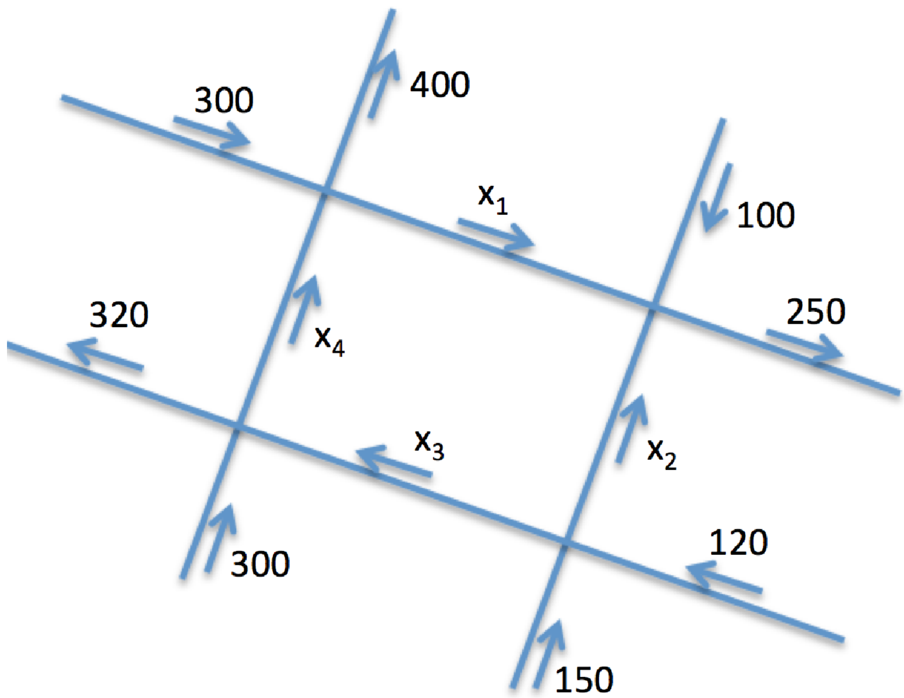
Out[48]=



Example: Traffic-Flow Analysis

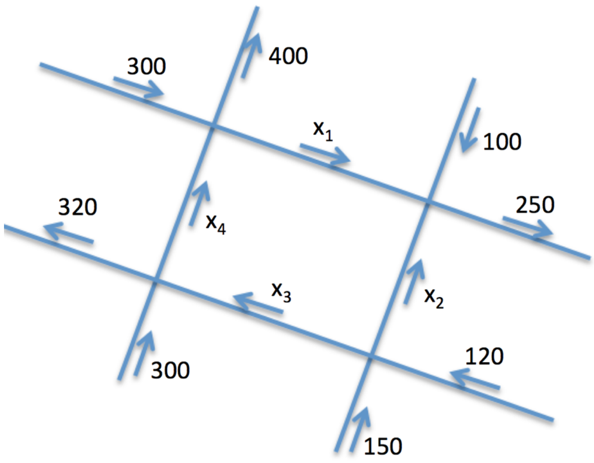
This diagram represents a maze of one-way streets in a city.

The traffic along certain streets during an hour has been measured.



What can you say about the traffic volumes x_1, x_2, x_3, x_4?
Describe a possible scenario.
For each of the 4 streets, find the highest and lowest possible traffic volume.

Example: Traffic-Flow Analysis



For each junction set up equations using: **traffic in = traffic out**

$x_4 + 300 = 400 + x_1$

$x_1 + x_2 + 100 = 250$

$120 + 150 = x_2 + x_3$

$x_3 + 300 = 320 + x_4$

Rearrange:

$-x_1 + x_4 = 100$

$x_1 + x_2 = 150$

$x_2 + x_3 = 270$

$x_3 - x_4 = 20$

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 150 \\ 270 \\ 20 \end{pmatrix}$$

Example: Traffic-Flow Analysis

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 150 \\ 270 \\ 20 \end{pmatrix}$$

Form the augmented matrix and determined the RREF:

In[ ]:=
```
m= (-1 0 0  1 100
     1 1 0  0 150
     0 1 1  0 270
     0 0 1 -1  20);

RowReduce[m]  //MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As the solution indicates 3 equations for 4 variables there must be infinite solutions. The next step is to determine the equations for these solutions.

Example: Traffic-Flow Analysis

Convert back to normal equations and rearrange for the three leading variables $x_1,\ x_2,\ x_3$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \implies \begin{aligned} x_1 &= x_4 - 100 \\ x_2 &= -x_4 + 250 \\ x_3 &= x_4 + 20 \end{aligned}$$

Let $x_4 = t$

$$x_1 = t - 100$$
$$x_2 = -t + 250$$
$$x_3 = t + 20$$

Combine these equations with the fact that $x_1, x_2, x_3 \geqslant 0$ to determine bounds on t

$$\text{as } x_1 \geqslant 0: \quad t - 100 \geqslant 0 \longrightarrow t \geqslant 100$$
$$\text{as } x_2 \geqslant 0: \quad -t + 250 \geqslant 0 \longrightarrow t - 250 \leqslant 0 \longrightarrow t \leqslant 250$$
$$\text{as } x_3 \geqslant 0: \quad t + 20 \geqslant 0 \longrightarrow t \geqslant -20$$
$$\text{as } x_4 \geqslant 0: \quad t \geqslant 0$$

Combining these inequalities gives $100 \leqslant t \leqslant 250$.

Example: Traffic-Flow Analysis

One possible scenario

The question asks for one possible scenario so consider when t = 100

$x_1 = t - 100 \quad = 0$

$x_2 = -t + 250 \quad = 150$

$x_3 = t + 20 \quad = 120$

$x_4 = t \quad = 100$

Minimums and maximums for each traffic flow

As $100 \leqslant t \leqslant 250$ we can infer the following minimums and maximums for each traffic flow:

$x_1 = t - 100 \quad \longrightarrow \quad 0 \leqslant x_1 \leqslant 150$

$x_2 = -t + 250 \quad \longrightarrow \quad 0 \leqslant x_2 \leqslant 150$

$x_3 = t + 20 \quad \longrightarrow \quad 120 \leqslant x_3 \leqslant 270$

$x_4 = t \quad \longrightarrow \quad 100 \leqslant x_4 \leqslant 250$

Rank of a matrix

The **rank of a matrix** A is the **number of leading ones** in **rref**(A), denoted **rank**(A).

Example

In[ ]:=
```
MatrixRank[(1 2 3
            4 5 6
            7 8 9)]
```

Out[ ]=

2

This is because the **rref** of this matrix has 2 leading ones:

In[ ]:=
```
RowReduce[(1 2 3
           4 5 6
           7 8 9)] //MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank governs the number of solutions

A system of equation is said to be **inconsistent** if there are no solutions. A linear system is **inconsistent** if (and only if) the RREF of its augmented matrix contains any row [0 0 ... 0 k] where k≠0.

If a linear system is **consistent**, then it has either:
• **infinitely many solutions** when there is at least one free variable
(**rank < number of variables**),
or
• **exactly one solution** when all the variables are leading (**rank=number of variables**)

Number of solutions of a linear system

a. $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

c. $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

How many solutions are implied for each of the above RREFs?

Answer: How many solutions are implied for each of the above RREFs?

a. $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

inconsistent: as contains row of the form $(0 \; \ldots \; 0 \; k)$, $k \neq 0$

b. $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

consistent with infinite solution: rank (2) < no. of variables (3)

c. $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

consistent with single solution: rank (3) = no. of variables (3)

Matrix Algebra: Linear Combination

The vector $\boldsymbol{u}$ given by $\boxed{\boldsymbol{u} = a\,\boldsymbol{v_1} + b\,\boldsymbol{v_2}}$ is an example of a **linear combination** of vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, where a, b are scalars and $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are vectors of the same dimension.

Linear combinations can be expressed in matrix form where $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are column vectors:

$$\begin{pmatrix} | & | \\ \boldsymbol{v_1} & \boldsymbol{v_2} \\ | & | \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} | \\ \boldsymbol{u} \\ | \end{pmatrix}$$

Example

$$2\begin{pmatrix}1\\2\\3\end{pmatrix}+3\begin{pmatrix}4\\5\\6\end{pmatrix} \Longrightarrow \begin{pmatrix}2\\4\\6\end{pmatrix}+\begin{pmatrix}12\\15\\18\end{pmatrix} \Longrightarrow \begin{pmatrix}14\\19\\24\end{pmatrix}$$

$\begin{pmatrix}14\\19\\24\end{pmatrix}$ is a **linear combination** of the vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\5\\6\end{pmatrix}$

In matrix form this can also be expressed thus:

$$\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}.\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}14\\19\\24\end{pmatrix}$$

Example Problem

Is the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ a linear combination of the vectors $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

How can this question be expressed as a matrix equation that can then solved using Gauss-Jordan elimination?

Example Problem

$$a\begin{pmatrix}4\\5\\6\end{pmatrix}+b\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

Which is same as the 3 simultaneous equations

$4a+7b=1$

$5a+8b=2$

$6a+9b=3$

Which can be expressed as a matrix equation

$$\begin{pmatrix}4&7\\5&8\\6&9\end{pmatrix}\cdot\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

We can now solve this using Gauss-Jordan Elimination ...

Example Problem

Now express this equation as an augmented matrix

In[60]:=
```
m= (4 7 1
    5 8 2
    6 9 3);
```

In[61]:=
```
m[[1]]=m[[1]]/4;
m[[2]]=m[[2]]-5m[[1]];
m[[3]]=m[[3]]-6m[[1]];m//MatrixForm
```

Out[63]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{7}{4} & \frac{1}{4} \\ 0 & -\frac{3}{4} & \frac{3}{4} \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{pmatrix}$$

In[64]:=
```
m[[2]]=m[[2]]/(-3/4);
m[[1]]=m[[1]]-7/4 m[[2]];
m[[3]]=m[[3]]+3/2 m[[2]];m//MatrixForm
```

Out[66]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence a solution exists ($a = 2$ and $b = -1$) so it is true that the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a linear combination of the vectors $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.

The solution can also be quickly found in Mathematica using Solve

In[58]:=

```
Clear[a,b]
Solve[a{4,5,6}+b{7,8,9}=={1,2,3},{a,b}]
```

Out[59]//MatrixForm=

$$\begin{pmatrix} a \to 2 & b \to -1 \end{pmatrix}$$