Week 5:
Integration
First-order differential equations

MSIN00180 Quantitative Methods for Business

Sigma Notation

The index k ends at $k = n$.

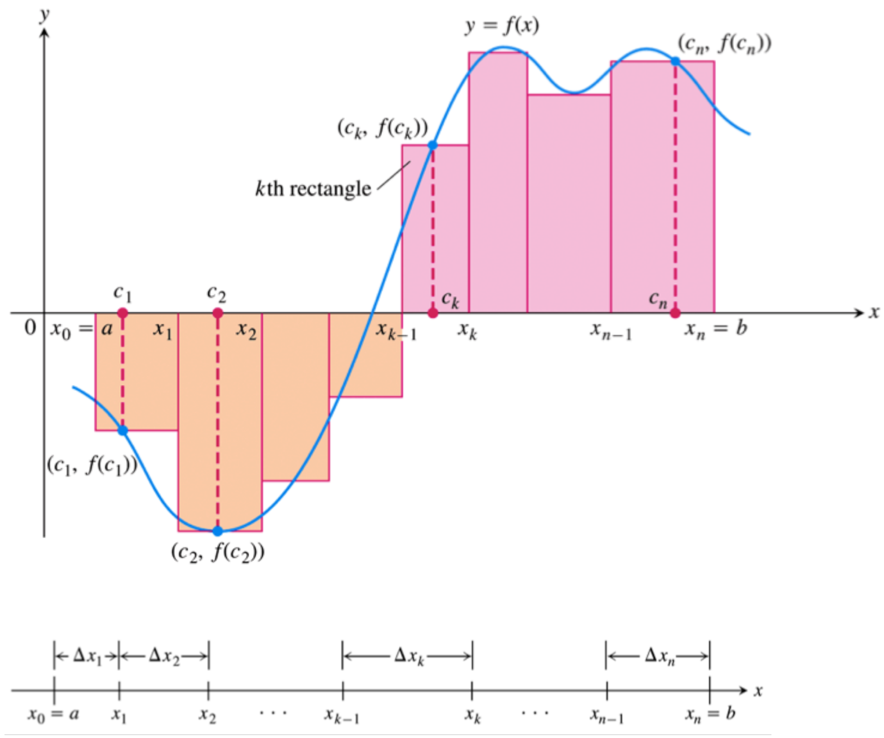
The summation symbol (Greek letter sigma) $\displaystyle\sum_{k=1}^{n} a_k$ — a_k is a formula for the kth term.

The index k starts at $k = 1$.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

Riemann Sum for *f* on the interval [*a*, *b*]

$$S_p = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

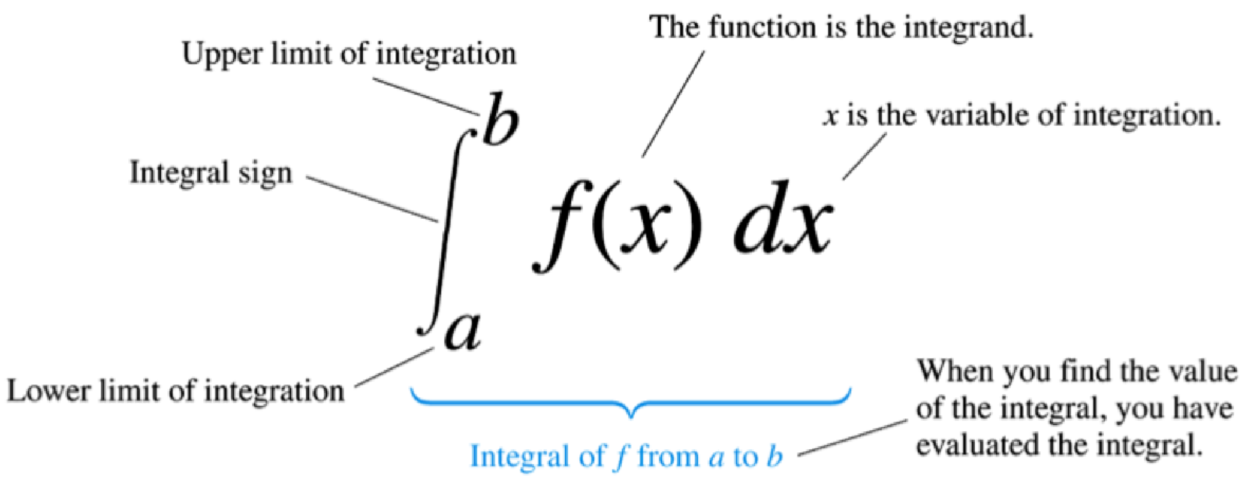


Integral defined in terms of Riemann Sums

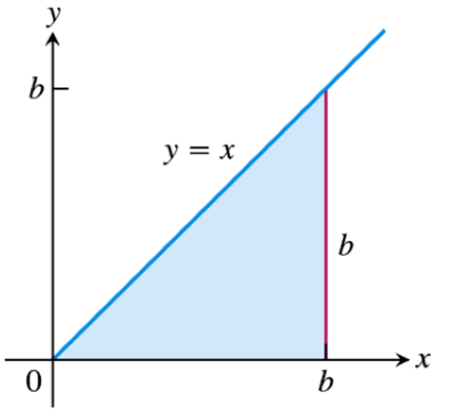
When the interval $[a, b]$ is partitioned into n equal subintervals, each of width $\Delta x = \frac{(b-a)}{n}$, we can write

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x = \int_a^b f(x)\,dx$$

Leibniz Notation



Calculating a Definite Integral from First Principles



$\sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} \frac{k\,b}{n}\cdot\frac{b}{n}$

$= \sum_{k=1}^{n} \frac{k\,b^2}{n^2} = \frac{b^2}{n^2}\sum_{k=1}^{n} k$

$= \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2}$ (sum of first n integers)

$= \frac{b^2}{2}\left(1+\frac{1}{n}\right)$

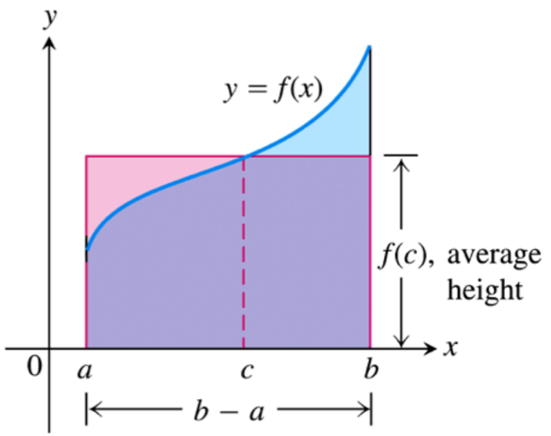
As $n \to \infty$ this expression has the limit $\frac{b^2}{2}$. Therefore,

$$\int_0^b x\,dx = \frac{b^2}{2}$$

Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$ the mean value of $f(x)$ over $[a, b]$ is given by

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$



We will now use this to informally prove the Fundamental Theorem of Calculus ...

The Fundamental Theorem of Calculus

Fundamentally connects integration and differentiation

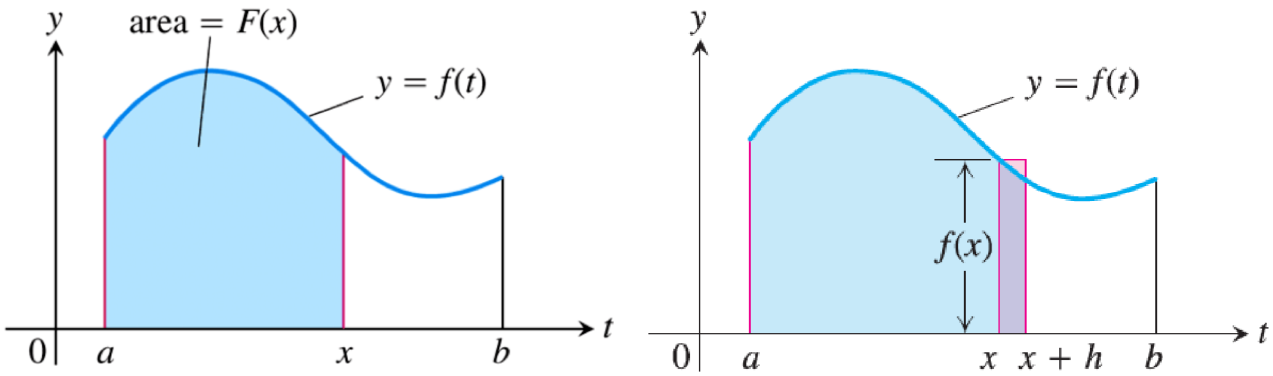
Provides a way of calculating integrals using antiderivatives rather than by using limits of Riemann sums

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Informal Proof



As we can see illustrated above

$$F(x+h) - F(x) \approx h.f(x)$$

Dividing both sides by h and taking the limit as $h \to 0$, it is reasonable to expect that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$$

The Fundamental Theorem of Calculus, Part 2 (The Evaluation Theorem)

If f is continuous on $[a, b]$ and F is any antiderivative* of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

This theorem is important as it says that to calculate the definite integral of f over the interval $[a, b]$ we need do only two things:

1) Find the antiderivative of F of f, and
2) Calculate $F(b) - F(a)$.

* A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

General Antiderivatives

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$F(x) + C$

where C is an arbitrary constant.

Antiderivatives and Differential Equations

Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x)$$

Example

Function f(x)	General antiderivative y(x)
x^n	$\frac{1}{n+1} x^{n+1} + C, \quad n \neq 1$

Common Antiderivatives / Integration Rules

1. $\int k \, dx = k x + C$

2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq 1)$

3. $\int \frac{1}{x} \, dx = \ln |x| + C$

4. $\int e^x \, dx = e^x + C$

5. $\int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a > 0, \; a \neq 1)$

We must distinguish carefully between definite and indefinite integrals.
A **definite integral** $\int_a^b f(x) \, dx$ is a *number*.
An **indefinite integral** $\int f(x) \, dx$ is a *function* plus an arbitrary constant C.

Integration in Mathematica

Indefinite Integrals

In[404]:=

```
Integrate[x^2, x]          (*or*)
∫x^2 dx
```

Out[404]=

$$\frac{x^3}{3}$$

Out[405]=

$$\frac{x^3}{3}$$

Definite Integrals

In[406]:=

```
Integrate[x^2,{x,0,2}]          (*or*)
∫_0^2 x^2 dx
```

Out[406]=

$$\frac{8}{3}$$

Out[407]=

$$\frac{8}{3}$$

Integration by the Substitution Method (Running the Chain Rule Backwards)

If $u = g(x)$ is a differentiable function over the interval I, and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

This method requires that you recognise that the expressions being integrated contains a subexpression $g(x)$ and its derivative $g'(x)$.

Example

Find the integral $\int (x^3 + x)^5 (3x^2 + 1) \, dx$

To realise that you can use the substitution method to solve this you need to have recognised that this contains a subexpression $(x^3 + x)$ and its derivative $(3x^2 + 1)$ i.e.

$\int (x^3 + x)^5 (3x^2 + 1) \, dx \iff \int f(g(x)) \, g'(x) \, dx$,where $f(x) = x^5$ and $g(x) = x^3 + x$

Given this we can use the substitution $u = g(x)$

u=$x^3 + x$

and solve

$\int u^5 \, du = \frac{u^6}{6} + C$

$= \frac{(x^3+x)^6}{6} + C$

Solution in Mathematica

Find the integral $\int (x^3 + x)^5 (3x^2 + 1) \, dx$

In[62]:=
```
∫(x^3+x)^5 (3x^2+1) dx //Simplify
```

Out[62]=

$$\frac{1}{6} \left(x + x^3\right)^6$$

Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)\, g(x)\, dx$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty.

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

This is more typically expressed as

$$\int u\, dv = u\, v - \int v\, du$$

Example

Find $\int \ln(x)\, x^2\, dx$

$u = \ln(x) \implies$(differentiate) $\frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \implies$(integrate) $v = \frac{x^3}{3}$

$$\int u\, dv = u\, v - \int v\, du$$
$$= \ln(x)\, \frac{x^3}{3} - \int \frac{x^3}{3}\, \frac{1}{x}\, dx$$
$$= \ln(x)\, \frac{x^3}{3} - \frac{x^3}{9} + C$$
$$= \frac{x^3}{9} \left(\frac{9}{3} \ln(x) - 1\right) + C$$
$$= \frac{1}{9}\, x^3 (3 \ln(x) - 1) + C$$

Solution in Mathematica

Find $\int \ln(x)\, x^2 \, dx$

In[63]:=
```
∫ln(x) x^2 dx //Simplify
```

Out[63]=

$$\frac{1}{9} x^3 (-1 + 3 \text{Log}[x])$$

Integration of Rational Functions by Partial Fractions

This method is best illustrated by an example ...

Find the integral of the rational function $\frac{5x-3}{x^2-2x-3}$.

Factorise the denominator and express as a sum of **partial fractions**

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

Solve for A and B:

$5x-3 = A(x-3) + B(x+1) = (A+B)\,x - 3A + B$

Equate the like powers of x:

$A + B = 5, \quad -3A + B = -3 \qquad \Longrightarrow \qquad$ A=2 and B=3

so $\quad \int \frac{5x-3}{x^2-2x-3}\,\mathrm{dx} = \int \frac{2}{x+1}\,\mathrm{dx} + \int \frac{3}{x-3}\,\mathrm{dx}$

These integrals are finally found using the Substitution Method

$$\int \frac{5x-3}{x^2-2x-3}\,\mathrm{dx} = 2\ln|x+1| + 3\ln|x-3| + C$$

Solution in Mathematica

Find the integral of the rational function $\frac{5x-3}{x^2-2x-3}$

In[64]:=

$$\int \frac{5x-3}{x^2-2x-3} dx \text{ //Simplify}$$

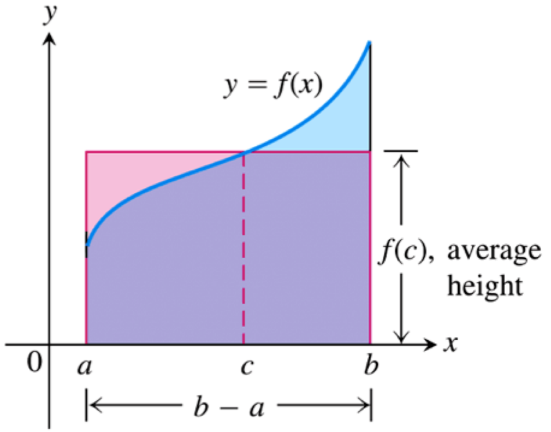
Out[64]=

```
3 Log[3 - x] + 2 Log[1 + x]
```

Using the Mean Value Theorem for Definite Integrals to find Averages

Remember:
If f is continuous on $[a, b]$, then at some point c in $[a, b]$ the mean value of $f(x)$ over $[a, b]$ is given by

mean $= f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$



We can directly use this to find the average value of a function over an interval as per the following example ...

Example Problem

The annual seasonal pattern of sales s of a company is projected to be

$s(t) = 50 + 20\sin\left(\frac{2\pi}{365}(t-100)\right) + \frac{t}{20}$, where t is the day number from the start of next year.

Find the average daily sales projected for next year and for the year after .

Using the mean value theorem for definite integrals we get:

$$\text{mean}_{\text{year 1}} = \frac{1}{365-0}\int_0^{365}\left(50 + 20\sin\left(\frac{2\pi}{365}(t-100)\right) + \frac{t}{20}\right)dt$$

$$\text{mean}_{\text{year 2}} = \frac{1}{730-365}\int_{365}^{730}\left(50 + 20\sin\left(\frac{2\pi}{365}(t-100)\right) + \frac{t}{20}\right)dt$$

Example Problem contd.

In Mathematica these integrals can be evaluation thus:

$$s[t\_]:=50+20\mathrm{Sin}\left[\frac{2\pi}{365}(t-100)\right]+\frac{t}{20}$$

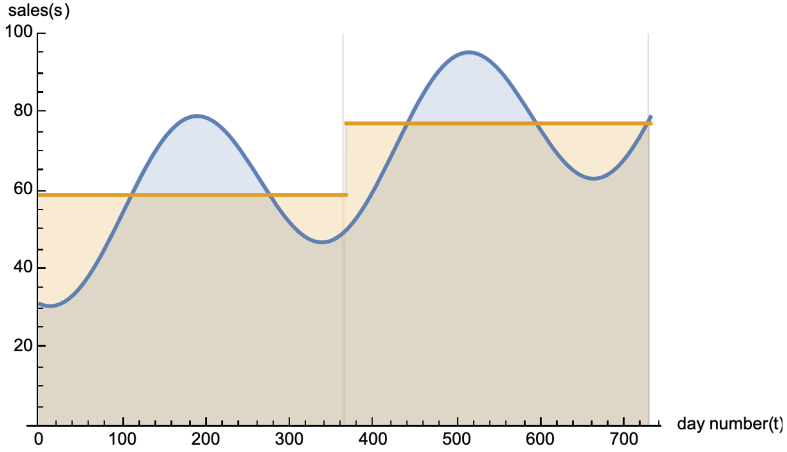
$$\mathrm{Echo}\left[\frac{1}{365-0}\int_0^{365} s[t]\,dt\ //N,\ \text{"mean}_{\text{year 1}}\text{ = "}\right];$$

$$\mathrm{Echo}\left[\frac{1}{730-365}\int_{365}^{730} s[t]\,dt\ //N,\ \text{"mean}_{\text{year 2}}\text{ = "}\right];$$

» $\text{mean}_{\text{year 1}}$ = 59.125

» $\text{mean}_{\text{year 2}}$ = 77.375

Out[392]=



Have a go at evaluating these integrals by hand before the seminar this week.

First-Order Differential Equations

A first-order differential equation is an an equation of the form

$$\frac{dy}{dx} = f(x, y)$$

It is first-order because there are no second-order derivatives or higher.

A solution is a differentiable function $y(x)$.

The general solution to a first-order differential equation is a solution that contains all possible values.

The general solution always contains an arbitrary constant.

Solving Separable Differential Equations

If the first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed in the form

$$\frac{dy}{dx} = g(x)\, H(y) \implies \frac{dy}{dx} = \frac{g(x)}{h(y)}, \text{ where } H(y) = \frac{1}{h(y)}$$

It is said to be a separable differential equation.

In its differential form we can write

$$h(y)\, dy = g(x)\, dx$$

The solution can be then be found by integrating each side

$$\int h(y)\, dy = \int g(x)\, dx$$

Example

Solve $\frac{dy}{dx} = (1 + y)\, e^x, \quad y > -1$

Solution

$\frac{dy}{(1+y)} = e^x\, dx$

$\int \frac{dy}{(1+y)} = \int e^x\, dx$

$\ln(1 + y) = e^x + C$

$1 + y = e^{(e^x + C)}$

$y = e^C\, e^{e^x} - 1$

$\boldsymbol{y = C\, e^{e^x} - 1}$ (replace the constant e^C with new constant C)

Solution in Mathematica

In[ ]:=
```
DSolve[y'[x]==(1+y[x])E^x,y[x],x]
```

Out[ ]=

$\{\{y[x] \to -1 + e^{e^x} c_1\}\}$

Modelling a viral marketing campaign as the spread of an infectious disease

A simple model for modelling the spread of a viral infection

Consider the spread of an infection, like flu, in a school with 1000 students.

What factors determine how quickly the infection spreads?

A simple model for modelling the spread of a viral infection

Assume

1. The speed of the spread of the flu depends on the strain of the flu.
2. The flu spreads more quickly when more students are infected.
3. The spread of the flu slows down when most students are infected.

Can you express these factors in a simple differential equation that models the spread of this infection?

A simple model for modelling the spread of a disease

Assume

1. The speed of the spread of the flu as depends on the strain of the flu.
2. The flu spreads more quickly when more students are infected.
3. The spread of the flu slows down when most students are infected.

These 3 assumptions can be expressed as a differential equation:

$$\frac{dx}{dt} = k\, x\, (1000 - x)$$

where x = number of infected students

k = some factor that represents how quickly the disease spreads

t = time

Logistic Growth Model and Carrying Capacity

$$\frac{dx}{dt} = k\,x\,(1000 - x)$$

The above model is known as the **logistic growth model**, which mimics that as a quantity is growing other factors will influence the growth and slow it down until a certain maximum size is being approached.

The value of 1000 above is the **carrying capacity** in this logistic growth model. In business, carry capacity could, for example, represent a maximum market size, with *x* being sales or the spread of a meme in that market.

Solving this differential equation

Assume $k = 1/250$

$$\frac{dx}{dt} = \frac{1}{250} x (1000 - x)$$

separate the variables and integrate each side:

$$\int \frac{1}{x(1000-x)}\, dx = \int \frac{1}{250}\, dt$$

express the LHS as the sum of partial fractions and solve the RHS

$$\int \left(\frac{A}{x} + \frac{B}{(1000-x)}\right) dx = \frac{t}{250} + C$$

using the method of partial fractions it can be seen that

$A = B = \frac{1}{1000}$ so

$$\frac{1}{1000} \int \left(\frac{1}{x} + \frac{1}{(1000-x)}\right) dx = \frac{t}{250} + C$$

Solving this differential equation (contd.)

$$\frac{1}{1000}\int\left(\frac{1}{x}+\frac{1}{(1000-x)}\right)dx=\frac{t}{250}+C$$

$$\ln|x|+\ln|1000-x|=4t+C$$

make both sides powers of e gives

$$e^{\ln|x|+\ln|1000-x|}=e^{4t+C}$$

$$\frac{x}{1000-x}=C\,e^{4t}$$

rearranging this gives: $\boxed{x(t)=\frac{1000}{1+C\,e^{-4t}}}$

Assume that the spread of flu starts with 2 infected students.

$2=\frac{1000}{1+C\,e^{0}}\longrightarrow$ C=499

$$\boxed{x(t)=\frac{1000}{1+499\,e^{-4t}}\quad\text{or}\quad x(t)=\frac{1000\,e^{4t}}{e^{4t}+499}}$$

Mathematica solution

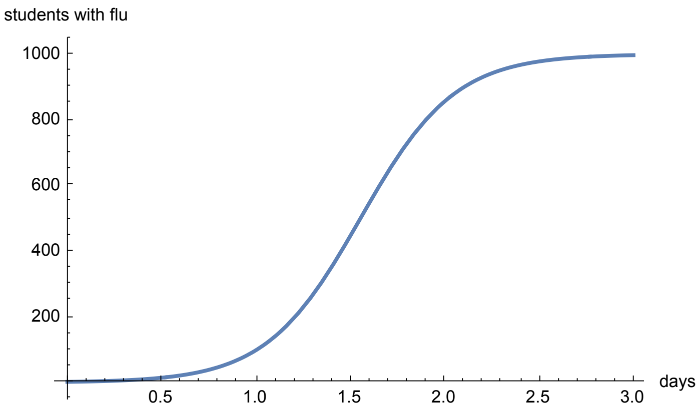
In[ ]:=
```
Clear[x, t, x0, k]; Off[Solve::ifun];
```

In[ ]:=
$$\text{DSolve}\left[\left\{x'[t]==\frac{1}{250}x[t]\,(1000-x[t]),\ x[0]==2\right\},\ x[t],\ t\right]$$

We can then plot this result in *Mathematica*

In[ ]:=
$$\text{Plot}\left[\frac{1000\ e^{4\,t}}{499+e^{4\,t}},\{t,0,3\},\ \text{AxesLabel}\to\{\text{"days"},\text{"students with flu"}\}\right]$$

Out[ ]=



Mathematica general solution

In *Mathematica* we can solve this equation generally for any value of k and x_0, the initial number of students infected.

In[ ]:=
```
DSolve[{x'[t] == k x[t] (1000-x[t]), x[0] == x0}, x[t], t]
```

Out[ ]=

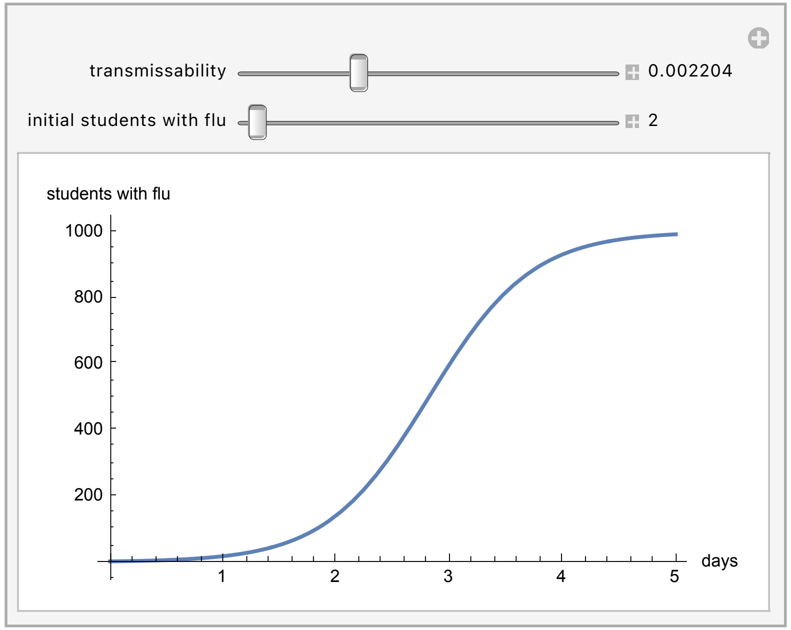
$$\left\{\left\{x[t] \to \frac{1000\, e^{1000\, k\, t}\, x0}{1000 - x0 + e^{1000\, k\, t}\, x0}\right\}\right\}$$

General solution dynamic plot ...

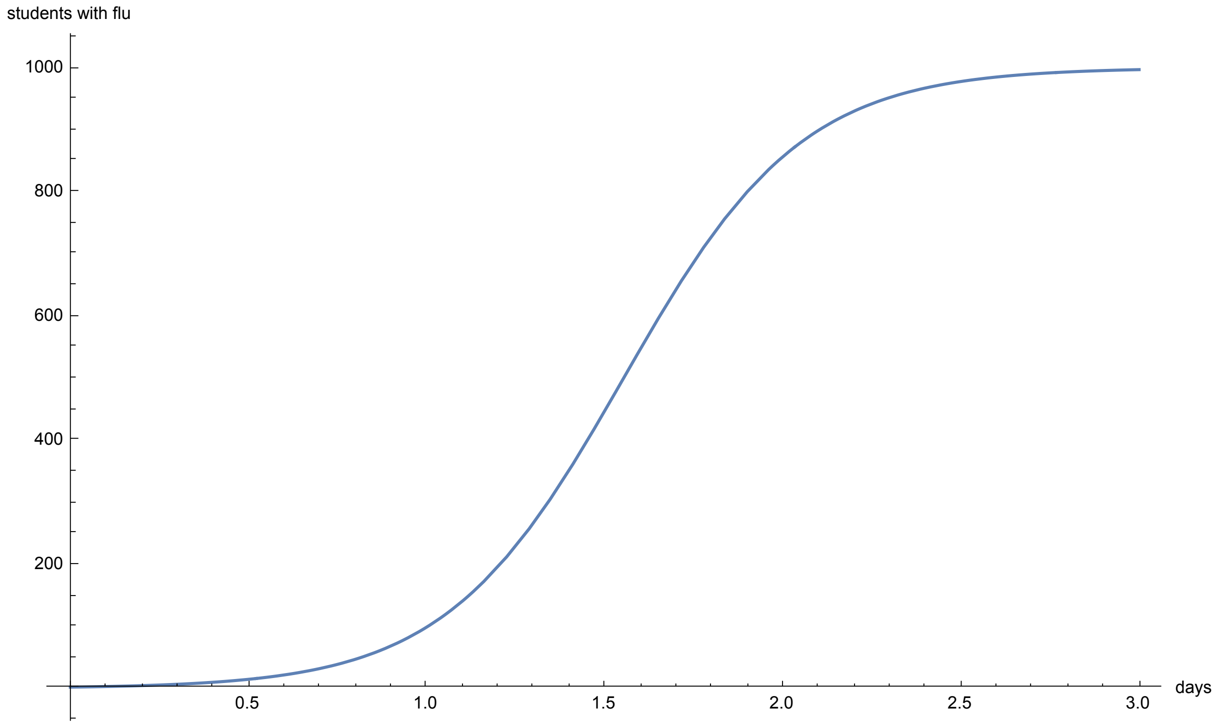
In[ ]:=

```
Manipulate[
Plot[(1000 E^(1000 k t) x0)/(1000-x0+E^(1000 k t) x0),{t,0,5},AxesLabel→{"days","students with flu"}],
{{k, 1/250,"transmissability"}, 1/1000, 1/200,Appearance → "Labeled"},
{{x0, 2,"initial students with flu"},1,100,1,Appearance → "Labeled"}]
```

Out[ ]=



What’s wrong with this model?



Modelling immunity and getting better

How could we change our model to include immunity?

- If 50% of students are immune then

$$\frac{dx}{dt} = k\,x\,(500 - x)$$

or if $\boxed{p = \text{probability of being immune}}$ we could say

$$\frac{dx}{dt} = k\,x\,((1-p)\,1000 - x)$$

What about getting better and no longer being infectious?

- For this we need a **system of differential equations**, sometimes called a **compartment model**.

SIR model of an infectious diseases

$$\boxed{S} \longrightarrow \boxed{I} \longrightarrow \boxed{R}$$

Imagine people moving over time between three “compartments” S, I and R where:

S = number susceptible
I = number infected
R = number recovered

SIR model of an infectious diseases

$$\boxed{S} \longrightarrow \boxed{I} \longrightarrow \boxed{R}$$

The system of differential equations that models the spread of an infectious disease in this way is given by

$\frac{dS}{dt} = -b\,S\,I$ where b = transmissability

$\frac{dI}{dt} = b\,S\,I - c\,I$ where c = recovery rate

$\frac{dR}{dt} = c\,I$

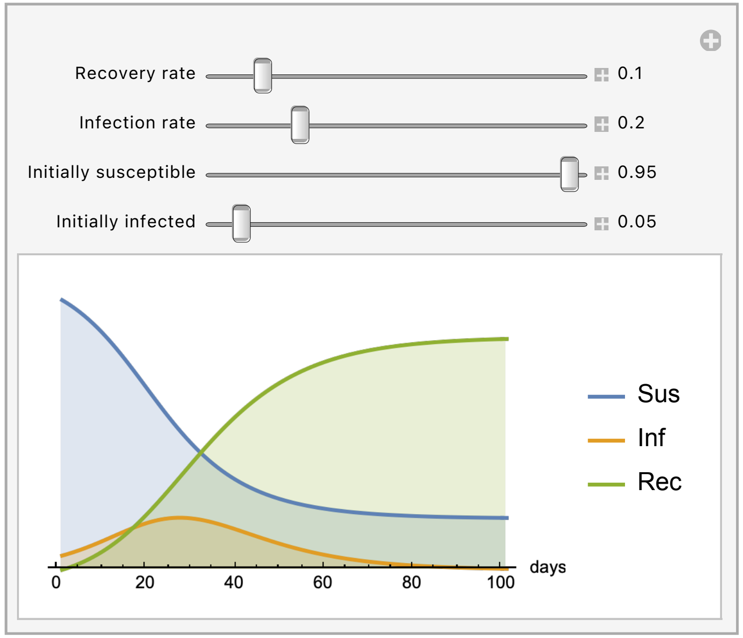
More complex models like this are typically solved numerically rather than algebraically.

This code uses Euler's Method which we will come to shortly ...

In[408]:=

```
Manipulate[
If[x0>1-y0,x0=1-y0];
ListLinePlot[Transpose[NestList[
{#[[1]]*(1-b #[[2]]),(1-c)#[[2]]+b*#[[2]]×#[[1]],#[[3]]+c #[[2]]}&,
{x0,y0,1-x0-y0},100]],PlotLegends→{"Sus","Inf","Rec"},Filling→0,Axes→{True,False},AxesLabel→{"days"}],
{{c,.1,"Recovery rate"},.0,.9,Appearance → "Labeled"},
{{b,.2,"Infection rate"},.0,.9,Appearance → "Labeled"},
{{x0,1-y0,"Initially susceptible"},.0,1-y0,Appearance → "Labeled"},
{{y0,.05,"Initially infected"},0,1,Appearance → "Labeled"}]
```

Out[408]=



Viral marketing and infectious diseases

$$\boxed{S} \longrightarrow \boxed{E} \longrightarrow \boxed{D}$$

Imagine now people moving over time between “compartments” S, E and D where:

S = number susceptible

E = number engaged

D = number disengaged

We can use the same equations as are in the SIR model and solve it using similar methods and tools:

$\frac{dS}{dt} = -b\,S\,E$ where $b =$ factor for campaign appeal

$\frac{dE}{dt} = b\,S\,E - c\,E$ where $c =$ factor for sustained interest

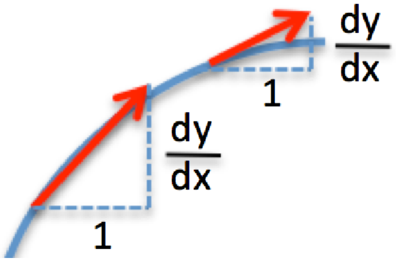
$\frac{dD}{dt} = c\,E$

Plotting Slope Fields in *Mathematica* Using VectorPlot

A slope field can help understand the general nature of solutions to differential equations, across all initial values.

A slope field is composed of many small vector arrows, each of which representing a tangent line to the solution at that point.

One easy way to represent these vectors is with the vector $\left(1 \ \frac{dy}{dx}\right)$ using the value of $\frac{dy}{dx}$ at each point.



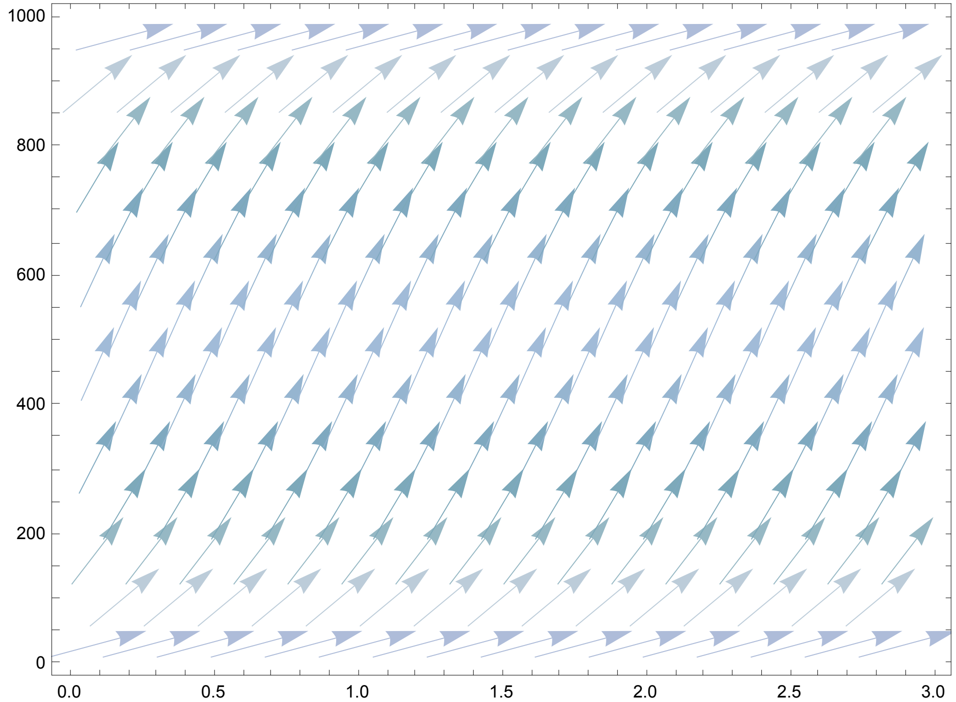
VectorPlot for the school infection model

Using this in Mathematica we can plot the slope field for our original school infection model $\frac{dx}{dt} = \frac{1}{250} x(1000 - x)$ using VectorPlot thus

In[403]:=

```
VectorPlot[{1,x(1000-x)/250},{t,0,3},{x,0,1000},
          VectorScale→{0.0004,0.5},AspectRatio→0.75, VectorColorFunction→"Aquamarine"]
```

Out[403]=



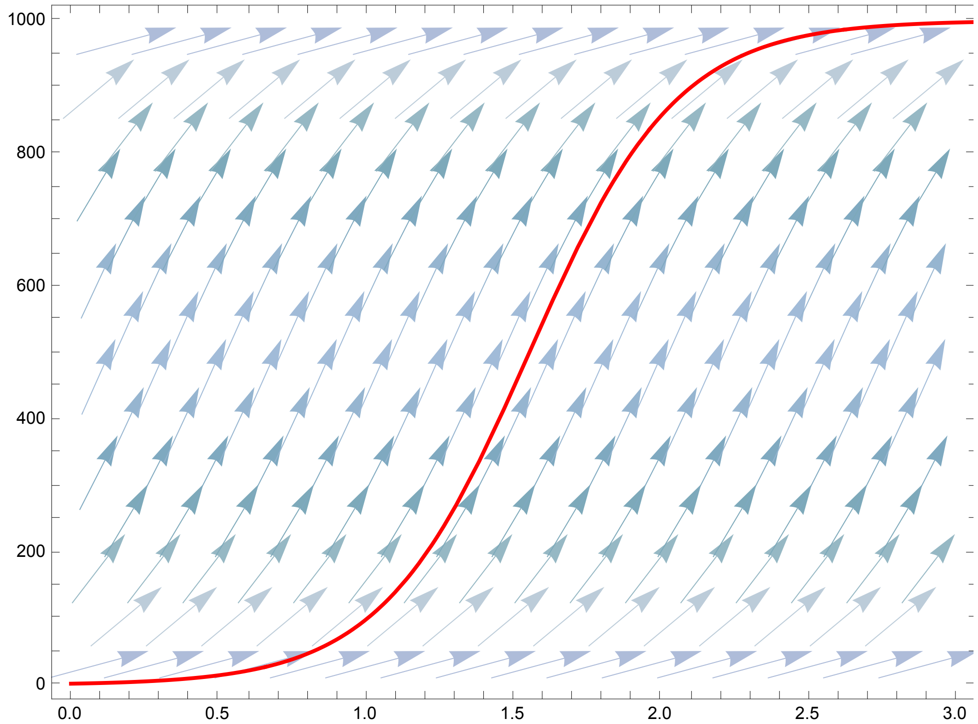
Combining SlopeFields and Solution Curves in *Mathematica*

We now can additionally show the solution we found earlier when the initial infections (*i.e.* x_0) = 2

In[400]:=

```
Show[
VectorPlot[{1,x(1000-x)/250},{t,0,3},{x,0,1000}, VectorScale→{0.0004,0.5},AspectRatio→0.75,
VectorColorFunction→"Aquamarine"],
Plot[(1000 ⅇ^(4 t))/(499+ⅇ^(4 t)),{t,0,4},PlotStyle→Red]]
```

Out[400]=

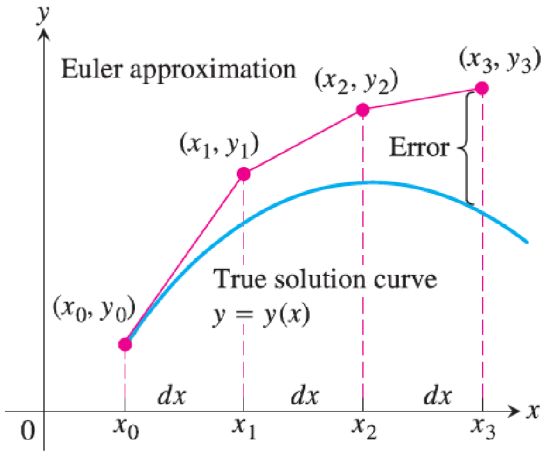


Euler’s Method

Given $\frac{dy}{dx} = f(x, y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution $y = y(x)$ over a short interval by its linearization

$$L(x) = y_0 + f(x, y)(x - x_0)$$

Euler’s method patches together a string of linearizations to approximate the solution over longer intervals.



If the interval dx is small then

$y_1 = y_0 + f(x_0, y_0)\,\mathrm{dx} \longrightarrow\ y_2 = y_1 + f(x_1, y_1)\,\mathrm{dx} \longrightarrow\ y_3 = y_2 + f(x_2, y_2)\,\mathrm{dx}$ etc.

This method builds an approximation by following the **slope field** of the differential equation.

Example

Find the first three approximations y_1, y_2, y_3 using Euler's method for the problem

$y' = 1 + y, \qquad y(0) = 1$

starting at $x_0 = 0$ with $\mathrm{dx} = 0.1$.

Solution

First:

$y_1 = y_0 + f(x_0, y_0)\,\mathrm{dx}$

$= y_0 + (1 + y_0)\,\mathrm{dx}$

$= 1 + (1 + 1)\,0.1 = 1.2$

Second:

$y_2 = y_1 + f(x_1, y_1)\,\mathrm{dx}$

$= 1.2 + (1 + 1.2)\,0.1 = 1.42$

Third:

$y_3 = y_2 + f(x_2, y_2)\,\mathrm{dx}$

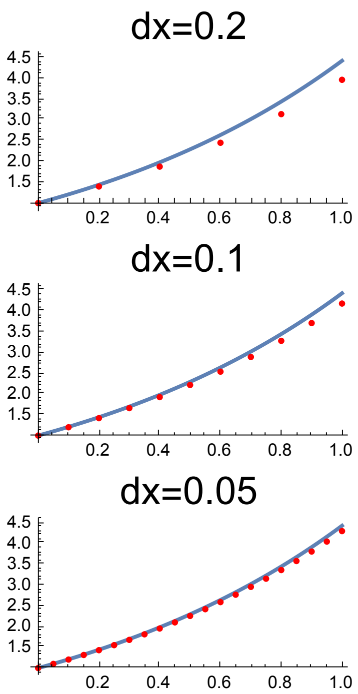
$= 1.42 + (1 + 1.42)\,0.1 = 1.662$

Euler's Method: *Mathematica* Example

In[564]:=

```
Column[Table[y0=1;  y={y0};
        For[i=1,i*dx≤1,i++, y=Append[y,y⟦i⟧+(1+y⟦i⟧)dx]];
        points=Table[{(n-1)dx,y⟦n⟧},{n,1,Length[y]}];
        Show[Plot[2E^x-1,{x,0,1}, PlotLabel→Style["dx="<>ToString[dx], 20], AspectRatio→1/2],
            ListPlot[points, PlotStyle→Directive[Red, PointSize[0.02]]]]
    , {dx,{0.2,0.1,0.05}}]]
```

Out[564]=



Euler's Method:
School Infection Problem Revisited

The system of differential equations that models the spread of an infectious disease was given by

$\frac{dS}{dt} = -b\,S\,I$ where b = transmissability

$\frac{dI}{dt} = b\,S\,I - c\,I$ where c = recovery rate

$\frac{dR}{dt} = c\,I$

We can turn these three differential equations into three Euler Method linearizations:

$S_{i+1} = S_i - b\,S_i\,I_i\,\text{dt}$
$I_{i+1} = I_i + (b\,S_i\,I_i - c\,I_i)\,\text{dt}$
$R_{i+1} = R_i + c\,I_i\,\text{dt}$

Which we can now solve using Mathematica code ...

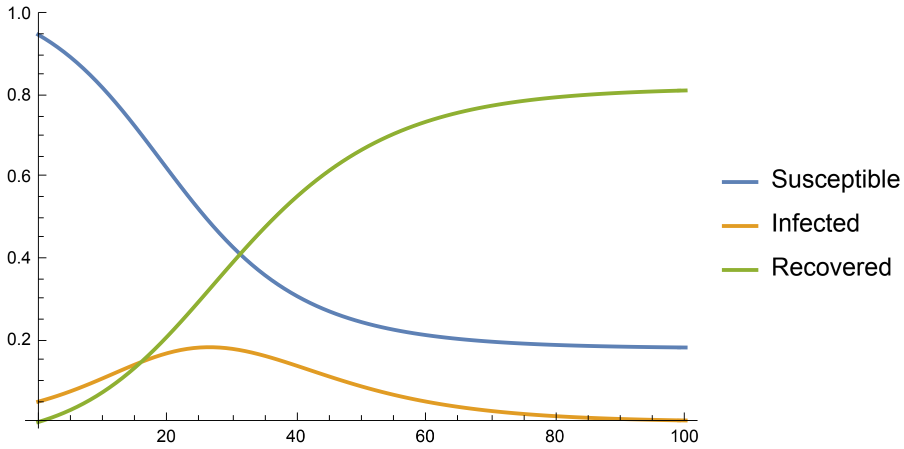
School Infection Problem Revisited

Using the same set of initial values as used previously $b = 0.2$, $c = 0.1$, $S_0 = 0.95$, $I_0 = 0.05$, $R_0 = 0$

In[ ]:=

```
dt=1;   S0=0.95; S={S0};  Inf0=0.05; Inf={Inf0};  R0=0; R={R0};
b=0.2;  c=0.1;
For[i=1,i≤100,i++,
    S=Append[S,S[[i]] - b S[[i]]×Inf[[i]]dt];
    Inf=Append[Inf,Inf[[i]] +(b S[[i]]×Inf[[i]] - c Inf[[i]]) dt];
    R=Append[R,R[[i]] + c Inf[[i]]dt]
    ]
Spoints=Table[{(n-1)dt,S[[n]]},{n,1,101}];
Infpoints=Table[{(n-1)dt,Inf[[n]]},{n,1,101}];
Rpoints=Table[{(n-1)dt,R[[n]]},{n,1,101}];
ListLinePlot[{Spoints, Infpoints, Rpoints}, PlotLegends→{"Susceptible","Infected","Recovered"}]
```

Out[ ]=



First-Order Linear Differentiation Equations

A first-order linear differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)\,y = Q(x) \qquad \text{, where P(x) and Q(x) are continuous functions}$$

This is the standard form of a linear differential equation.

The equation is linear because $\frac{dy}{dx}$ and y only occur to the first power, they are not multiplied together, nor do they appear as arguments of any functions.

Solving Linear Differential Equations

To solve $\frac{dy}{dx} + P(x)\, y = Q(x)$

1. Find the function $v(x) = e^{\int P(x)\, dx}$.

This function has the special property that $v(x)\left(\frac{dy}{dx} + P(x)\, y\right) = \frac{d}{dx}(v(x).y)$

2. Multiply both sides of the linear equation by v(x). Because of the above property this gives

$$\frac{d}{dx}(v(x).y) = v(x)\, Q(x)$$

3. Integrate both sides

$$v(x).y = \int v(x)\, Q(x)\, dx$$

giving the solution $y = \frac{1}{v(x)} \int v(x)\, Q(x)\, dx$

Example

Solve $x \frac{dy}{dx} = x^2 + 3y, \quad x > 0.$

Put the equation in standard form: $\frac{dy}{dx} - \frac{3}{x} y = x$

so $P(x) = -\frac{3}{x}$

$v(x) = e^{\int P(x)\, dx} = e^{\int \left(-\frac{3}{x}\right) dx} = e^{-3 \ln |x|} = e^{\ln x^{-3}} = \frac{1}{x^3}$

multiply both sides of the standard form by $v(x)$ and integrate:

$\frac{1}{x^3}\left(\frac{dy}{dx} - \frac{3}{x} y\right) = \frac{1}{x^3} x \quad \longrightarrow \quad \frac{1}{x^3} \frac{dy}{dx} - \frac{3}{x^4} y = \frac{1}{x^2}$

recognise that left hand side is $\frac{d}{dx}(v\, y)$ so $\frac{d}{dx}\left(\frac{1}{x^3} y\right) = \frac{1}{x^2}$

$\frac{1}{x^3} y = \int \frac{1}{x^2} dx = -\frac{1}{x} + C \quad \longrightarrow \quad y = -x^2 + C x^3, \quad x > 0$

Mathematica solution

In[2]:=
```
DSolve[x y'[x]==x^2+3y[x], y[x], x]
```

Out[2]= $\{\{y[x] \to -x^2 + x^3 \, c_1\}\}$