Week 10:
Revision Lecture (Core Theories and Techniques)

MSIN00180 Quantitative Methods for Business

Calculus

Differentiation Rules

Given u and v are differentiable functions of x and c is a constant ...

Power Rule

$$\frac{d}{dx} x^n = n\, x^{n-1}$$

Constant Multiple Rule

$$\frac{d}{dx}(c\, u) = c \frac{du}{dx}$$

Sum Rule

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Product Rule

$$\frac{d}{dx}(u\, v) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Chain Rule

$$\frac{d}{dx} f(g(x)) = \frac{d\, f(u)}{du} . \frac{d\, g(x)}{dx}$$

alternatively ... $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$ where $y = f(u)$ and $u = g(x)$

Differential of an Exponential Function

$\frac{d}{dx} e^u = e^u \frac{du}{dx}$ where u is any differentiable function of x

Challenge 1

Find $\frac{d}{dx}\left[(2x-3)\left(x^2+1\right)^3\right]$ where $x = 0$

Challenge 1

Find $\frac{d}{dx}\left[(2x-3)\left(x^2+1\right)^3\right]$ where $x=0$

Solution

Product rule :

$(2x-3)\,\frac{d}{dx}\left[\left(x^2+1\right)^3\right] + \left(x^2+1\right)^3 \frac{d}{dx}[(2x-3)]$

$(2x-3)\,\frac{d}{dx}\left[\left(x^2+1\right)^3\right] + 2\left(x^2+1\right)^3$

Chain rule : $(2x-3)\,3\left(x^2+1\right)^2(2x) + 2\left(x^2+1\right)^3$

Factorise : $\left(x^2+1\right)^2\left[(2x-3)\,3\,(2x) + 2\left(x^2+1\right)\right]$

Simplify : $2\left(x^2+1\right)^2\left[7x^2-9x+1\right]$

When $x=0$: $2\left(0^2+1\right)^2\left[7\times0^2-9\times0+1\right] = 2$

Challenge 2

Find $\frac{d}{dx}\left[\frac{(2x-3)\left(x^2+1\right)^3}{(x-2)^2}\right]$ where $x = 0$

Challenge 2

Find $\frac{d}{dx}\left[\frac{(2x-3)(x^2+1)^3}{(x-2)^2}\right]$ where $x = 0$

Solution

Quotient rule:

$$\frac{d}{dx}\left[\frac{(2x-3)(x^2+1)^3}{(x-2)^2}\right] = \frac{(x-2)^2 \frac{d}{dx}[(2x-3)(x^2+1)^3]-(2x-3)(x^2+1)^3 \frac{d}{dx}[(x-2)^2]}{(x-2)^4}$$

Use previous challenge result : $\frac{(x-2)^2\, 2(x^2+1)^2[7x^2-9x+1]-(2x-3)(x^2+1)^3 \frac{d}{dx}[(x-2)^2]}{(x-2)^4}$

Chain rule : $\frac{(x-2)^2\, 2(x^2+1)^2[7x^2-9x+1]-(2x-3)(x^2+1)^3\, 2(x-2)^1 (1)}{(x-2)^4}$

Simplify – divide by $(x-2)$: $\frac{(x-2)\, 2(x^2+1)^2 (7x^2-9x+1)-(2x-3)(x^2+1)^3\, 2}{(x-2)^3}$

Factorise : $\frac{2(x^2+1)^2\, [(x-2)(7x^2-9x+1)-(2x-3)(x^2+1)]}{(x-2)^3}$

When $x = 0$: $\frac{2(0^2+1)^2\, [(0-2)(7\times 0^2-9\times 0+1)-(2\times 0-3)(0^2+1)]}{(0-2)^3} = -\frac{1}{4}$

Implicit Differentiation

Implicit differentiation is used when the dependent variable (typically y) is not the subject of a formula.

eg find $\frac{dy}{dx}$ where $x^2 + 3\,y\,x = (x + 2\,y)^2$

The key is to always treat the dependent variable as a function of the independent variable (typically x)

$$x^2 + 3\,y(x)\,x = (x + 2\,y(x))^2$$

Remember: implicit differentiation requires no new differentiation rules or techniques.

Challenge 3

Find $\frac{dy}{dx}$ where $x^2 + 3\,y\,x = (x + 2\,y)^2$

Challenge 3

Find $\frac{dy}{dx}$ where $x^2 + 3\,y\,x = (x + 2\,y)^2$

Solution

differentiate both sides by x : $\frac{d}{dx}\left(x^2 + 3\,y\,x\right) = \frac{d}{dx}\left((x + 2\,y)^2\right)$

use sum rule on LHS & chain rule on RHS : $2x + \frac{d}{dx}3\,y\,x = 2\,(x+2y)\,\frac{d}{dx}\,(x + 2\,y)$

use product rule on LHS & sum rule on RHS : $2x + 3\,(y\frac{dx}{dx} + x\frac{dy}{dx}) = 2\,(x+2y)\,\left(1 + 2\,\frac{dy}{dx}\right)$

expand out RHS : $2x + 3\,(y + x\frac{dy}{dx}) = 2\,(x+2y) + 2\,(x + 2\,y)\,2\,\frac{dy}{dx}$

collect $\frac{dy}{dx}$ terms on the RHS : $2x + 3y - 2\,(x+2y) = 2\,(x + 2\,y)\,2\,\frac{dy}{dx} - 3x\frac{dy}{dx}$

factorise out $\frac{dy}{dx}$: $2x + 3y - 2\,(x+2y) = \frac{dy}{dx}\,(4\,(x + 2\,y) - 3x)$

$$\frac{dy}{dx} = \frac{-y}{(x+8\,y)}$$

Partial Differentiation

Partial differentiation is used to differentiate functions of two or more independent variables.

eg find $\frac{\partial f}{\partial x}$ where $f(x, y) = (x + 2y)^2$

note this time both x and y are independent variables

The key is to treat all other variables (besides the variable being used differentiated with respect to) as constants

Challenge 4

find $\frac{\partial f}{\partial x}$ where $f(x, y) = (x + 2y)^2$

Challenge 4

find $\frac{\partial f}{\partial x}$ where $f(x, y) = (x + 2y)^2$

Solution

partially differentiate with respect to x:

chain rule:

$$\frac{\partial f}{\partial x} = 2(x + 2y)\frac{\partial}{\partial x}(x + 2y)$$

sum rule:

$$\frac{\partial f}{\partial x} = 2(x + 2y)\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial x}2y\right)$$

treat y as a constant when differentiating:

$$\frac{\partial f}{\partial x} = 2(x + 2y)(1 + 0)$$

$$\frac{\partial f}{\partial x} = 2(x + 2y)$$

Successive partial differentials

$$\frac{\partial f}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Order does not matter!

The same partial differentiation process applies - always treat the other variables as constants

$\frac{\partial f}{\partial x\,\partial y}$ is the same as $f_{x\,y}$

Challenge 5

find $f_{x\,y}$ where $f = e^{x\,y}$

Challenge 5

find $f_{x\,y}$ where $f = e^{x\,y}$

Solution

differentiate with respect to x

$f_x = e^{x\,y}(y)$

now differentiate again with respect to y using product rule :

$f_{x\,y} = e^{x\,y}(1) + y\,(e^{x\,y}\,x)$

$f_{x\,y} = e^{x\,y}(1 + x\,y)$

The Gradient Vector

The gradient vector of $f(x, y)$ is the vector

$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$ where i is the unit vector in the x direction and j is the unit vector in the y direction

- The gradient vector is the vector made up of the partial derivatives with respect to x, y, ...
- At a given point P_0 the gradient vector points in the direction of maximum upward slope

Challenge 6

Find the gradient vector of the function $f(x, y) = 2xy - 3y^2$ at the point $P_0 = (1, 1)$

Challenge 6

Find the gradient vector of the function $f(x, y) = 2\,x\,y - 3\,y^2$ at the point $P_0 = (1, 1)$

Solution

First find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial x} = 2\,y + 0$ so $\left(\frac{\partial f}{\partial x}\right)_{(1,1)} = 2 \times 1 = 2$

$\frac{\partial f}{\partial y} = 2\,x - 6\,y$ so $\left(\frac{\partial f}{\partial y}\right)_{(1,1)} = 2 \times 1 - 6 \times 1 = -4$

So

$\nabla f_{(1,1)} = 2\,i - 4\,j$

or representing vectors as lists we can write

$\nabla f_{(1,1)} = \{2, -4\}$

Directional Derivatives

The directional derivative of the function $f(x, y)$ at the point P_0 in the direction of the vector $u = u_1\, i + u_2\, j$ is

$$\left(\frac{df}{ds}\right)_{u,P_0} = (\nabla f)_{P_0} \cdot \frac{u}{|u|}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} \frac{u_1}{|u|} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{u_2}{|u|}$$

Challenge 7

Find the directional derivative of the function $f(x, y) = 2\,x\,y - 3\,y^2$ at the point $P_0(1, 1)$ in the direction $u = 3\,i + 4\,j$

Challenge 7

Find the directional derivative of the function $f(x, y) = 2\,xy - 3\,y^2$ at the point $P_0(1, 1)$ in the direction $u = 3\,i + 4\,j$

Solution

We have already found in challenge 6 that

$\nabla f_{(1,1)} = \{2, -4\}$

So the directional derivative is given by

$$\left(\frac{df}{ds}\right)_{u,(1,1)} = (\nabla f)_{(1,1)} \cdot \frac{u}{|u|}$$

$$\left(\frac{df}{ds}\right)_{u,(1,1)} = (2\,i - 4\,j) \cdot \frac{u}{|u|}$$

and $\quad |\,u\,| = |\,3\,i + 4\,j\,| = \sqrt{3^2 + 4^2} = 5$

so

$$\left(\frac{df}{ds}\right)_{u,(1,1)} = (2\,i - 4\,j) \cdot \frac{u}{5}$$
$$= (2\,i - 4\,j) \cdot \left(\frac{3}{5}\,i + \frac{4}{5}\,j\right)$$
$$= 2 \times \frac{3}{5} - 4 \times \frac{4}{5}$$
$$= -2$$

Extreme values of single variable functions

Critical point exists at an interior point $x = c$ when $f'(c) = 0$, or where $f'(c)$ does not exist.

- If $f''(c) < 0$ then a local maximum
- If $f''(c) > 0$ then a local minimum
- If $f''(c) = 0$ then an inflection point

For absolute minimum or maximum it is also necessary to test the values of f at the domain end points.

Extreme values of functions of 2 variables

Critical point exists at an interior point (a, b) when $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or where these values are do not exist.

- If $f_{xx} < 0$ and $f_{xx} f_{yy} - {f_{xy}}^2 > 0$ then a local maximum
- If $f_{xx} > 0$ and $f_{xx} f_{yy} - {f_{xy}}^2 > 0$ then a local minimum
- If $f_{xx} f_{yy} - {f_{xy}}^2 < 0$ then a saddle point
- If $f_{xx} f_{yy} - {f_{xy}}^2 = 0$ then test is inconclusive

Challenge 8

Find the local maximums or minimums of $f(x, y) = 2\,x\,y - x^2 - 2\,y^2 + 3\,x + 4$

Challenge 8

Find the local maximums or minimums of $f(x, y) = 2\,x\,y - x^2 - 2\,y^2 + 3\,x + 4$

Solution

$f_x = 2\,y - 2\,x + 3 = 0$ ....(*I*)
$f_y = 2\,x - 4\,y = 0$ ....(II)

From (II): $x = 2\,y$ ....(III)

Sub into (I): $2\,y - 2\,(2\,y) + 3 = 0$
$2\,y = 3 \longrightarrow y = \frac{3}{2}$

From (III): $x = 2 \times \frac{3}{2} = 3$

So there is an extreme point at $(3, \frac{3}{2})$.

Test for type of extreme point:

$f_{xx} = -2$ < 0 so could be a maximum but need to also test discriminant ...
$f_{yy} = -4$
$f_{xy} = 2$

discriminant $= f_{xx}\,f_{yy} - f_{xy}{}^2 = (-2)\,(-4) - (2)^2 = 4$ > 0 so $(3, \frac{3}{2})$ confirmed as local maximum

Value of f at $(3, \frac{3}{2})$: $f\left(3, \frac{3}{2}\right) = 2 \times 3 \times \frac{3}{2} - 3^2 - 2\left(\frac{3}{2}\right)^2 + 3 \times 3 + 4 = \frac{17}{2}$

Absolute Maxima and Minima on Closed Bounded Regions

1. First find and classify all the **critical points in the interior of the region** (which we have previously addressed)
2. Then consider any **end points of boundaries** as these may also be candidate points for absolute maxima or minima
3. Finally consider any **critical points that lie on the boundaries.** These need to be determined and classified individually using the equation of each boundary as a **constraint** on the function being analysed.
    - One way to do this is by using the **Lagrange Multiplier** method (next slide)

Lagrange Multiplier Method

To find the local maximum and minimum values of $f(x, y, y)$ subject to the constraint $g(x, y, z) = 0$ find the values of x, y, z and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and}$$

$$g(x, y, z) = 0$$

Challenge 9

Find the local extreme values of $f(x, y) = x^2 y$ on the line $x + y = 3$

Challenge 9

Find the local extreme values of $f(x, y) = x^2\, y$ on the line $x + y = 3$

Solution

$f_x = 2\, x\, y, \quad f_y = x^2$

$\longrightarrow \quad \nabla f = \{2\, x\, y,\ x^2\}$

The constraint is given by $g(x, y) = x + y - 3 = 0$

so $g_x = 1,\ g_y = 1 \ \longrightarrow\ \nabla g = \{1,\ 1\}$

Using the Lagrange equation we get

$\nabla f = \lambda \nabla g \quad \longrightarrow \quad \{2\, x\, y,\ x^2\} = \lambda \{1,\ 1\}$

Expressed as 2 simultaneous equations:

$2\, x\, y = \lambda$

$x^2 = \lambda$

$\longrightarrow \quad y = \frac{x}{2}$ or $x = 0$

Consider $x = 0$:

Substitute this into $g(x, y) = x + y - 3 = 0$ gives $y = 3$ when $x = 0$

At this extreme point $f(0, 3) = x^2\, y = 0$

Consider $y = \frac{x}{2}$:

Substitute this into $g(x, y) = x + y - 3 = 0$ gives $x + \frac{x}{2} - 3 = 0 \quad \Longrightarrow \quad x = 2$

$y = \frac{x}{2} = 1$ when $x = 2$

At this extreme point $f(2, 1) = x^2\, y = 4$

Basic integration formulae

$$\int k \, dx = k\,x + C$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \qquad a > 0,\ a \neq 1$$

Integration using the Substitution Rule

$\int f(g(x))\, g'(x)\, \mathrm{dx} = \int f(u)\, \mathrm{du}$ where $u = g(x)$

Challenge 10

Evaluate $\int \frac{2x}{\sqrt{x^2+1}}\, \mathrm{dx}$

Challenge 10

Evaluate $\int \frac{2x}{\sqrt{x^2+1}}\,dx$ using the substitution method

Solution

Try the substitution $u = x^2 + 1$:

$du = 2x\,dx$

so using the substitution rule:

$$\int \frac{2x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\,du$$

$$= 2u^{\frac{1}{2}} + C$$

$$= 2\sqrt{x^2+1} + C$$

Integration by Parts

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

alternatively this can be expressed as

$$\int u\, dv = u\, v - \int v\, du$$

- Use this method when the second integral is easier to evaluate than the first.
- Various choices may be available for u and dv
  - Choose u when its differential is simpler
  - Choose dv when its integral is simpler

Challenge 11

Evaluate $\int \ln x\, dx$.

Challenge 11

Evaluate $\int \ln x \, dx$.

Solution

Let $u = \ln x$ $\quad\Longrightarrow$(differentiate) $\quad du = \frac{1}{x}\, dx$

Let $dv = dx$ $\quad\Longrightarrow$(integrate) $\quad v = x$

Using the “by parts” formula:

$$\begin{aligned}\int u\, dv &= u\, v - \int v\, du \\ &= (\ln x)x - \int x \tfrac{1}{x}\, dx \\ &= x \ln x - x + C\end{aligned}$$

Solving Separable Differential Equations of the form $\frac{dy}{dx} = \frac{g(x)}{h(y)}$

In its differential form we can write

$$h(y)\,dy = g(x)\,dx$$

The solution can be then be found by integrating each side

$$\int h(y)\,dy = \int g(x)\,dx$$

Challenge 12

The unit price function p(x) decreases the price at the rate of £0.01 per unit order ie

$$\frac{dp}{dx} = -0.01\,x$$

Find $p(x)$ if $p(100) = 20$

Challenge 12

The unit price function p(x) decreases the price at the rate of £0.01 per unit order ie

$\frac{dp}{dx} = -0.01\,x$

Find $p(x)$ if $p(100) = 20$

Solution

$$\frac{dp}{dx} = -0.01\,x$$

separating variables and integrating both sides gives :

$\int dp = \int -0.01\ x\ dx$

$p = -0.01\,\frac{x^2}{2} + C$

Using p(100)=20

$20 = -0.01\,\frac{100^2}{2} + C$

$C = 20 + 50 = 70$

$p(x) = 70 - \frac{x^2}{200}$

Solving Linear Differential Equations of the form $\frac{dy}{dx} + P(x)\, y = Q(x)$

1. Find the function $v(x) = e^{\int P(x)\, dx}$

2. The solution is then given by $y = \frac{1}{v(x)} \int v(x)\, Q(x)\, dx$

Challenge 13

Solve the differential equation

$$e^x \frac{dy}{dx} + 2\, e^x\, y = 1$$

Challenge 13

Solve the differential equation

$$e^{x}\frac{dy}{dx}+2e^{x}y=1$$

Solution

Divide throughout by e^{x}:

$$\frac{dy}{dx}+2y=e^{-x} \qquad \Longrightarrow P(x)=2,\;\; Q(x)=e^{-x}$$

Find the function $v(x)=e^{\int P(x)\,dx}$

$$v(x) = e^{\int 2\,dx}=e^{2x}$$

The solution is then given by $y=\frac{1}{v(x)}\int v(x)\,Q(x)\,dx$

$$y=e^{-2x}\int e^{2x}e^{-x}\,dx$$

$$y=e^{-2x}\int e^{x}\,dx$$

$$y=e^{-2x}(e^{x}+C)$$

$$y=e^{-x}+C\,e^{-2x}$$

Linearization

The linearization of the function f at the point $x = a$ is given by

$L(x) = f(a) + f'(a)(x - a)$

- The linearization of a function is simply the equation of the tangent line at the point $x = a$
- The linearization is an approximation to f for values of x close to the point $x = a$

Challenge 14

Find the linearization of the function $f(x) = x^{\frac{1}{3}}$ around the point $x = -8$.

Use the linearization to estimate $(-6)^{\frac{1}{3}}$ to 2 decimal places.

Challenge 14

Find the linearization of the function $f(x) = x^{\frac{1}{3}}$ around the point $x = -8$.

Use the linearization to estimate $(-6)^{\frac{1}{3}}$ to 2 decimal places.

Solution

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= a^{\frac{1}{3}} + \frac{1}{3} a^{\frac{-2}{3}} (x - a) \\ &= (-8)^{\frac{1}{3}} + \frac{1}{3} (-8)^{\frac{-2}{3}} (x + 8) \\ &= -2 + \frac{1}{3} (-2)^{-2} (x + 8) \\ &= -2 + \frac{1}{3} \times \frac{1}{4} (x + 8) \\ &= \frac{x}{12} - \frac{4}{3} \end{aligned}$$

Use the linearization to estimate $(-6)^{\frac{1}{3}}$ to 2 decimal places:

$$L(-6) = \frac{-6}{12} - \frac{4}{3} = \frac{-11}{6} \approx -1.83$$

Taylor Series

The Taylor Series can be used to approximate any differentiable function $f(x)$ about a point $x = x_0$ thus

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \ldots$$

alternatively: $\boldsymbol{f(x) \approx f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n}$

Setting $x_0 = 0$ gives a special case of the Taylor Series called the Maclaurin Series

$$\boldsymbol{f(x) \approx f(0) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n}$$

Linear Algebra

Gauss-Jordan elimination method

1. Proceed from equation to equation, top to bottom, until the i_{th} equation is reached of the form $\boldsymbol{c\,x_j + \ldots = b}$ where c≠0.
   Divide this i_{th} equation by c: $m_{[[i]]} = \frac{1}{c}\, m_{[[i]]}$

2. Eliminate x_j from all the other equations, above and below the i_{th} equation by subtracting suitable multiples of the i_{th} equation.

3. Repeat from step (1) until all equations have been considered.

4. Finally convert back to normal form to solve each equation for its leading variable.

This may result in **no solutions** (due to an inconsistent set of equations), a **single solution** or an **infinite set of solutions**.

Rank governs the number of solutions

The **rank of a matrix** A is the **number of leading ones** in **rref**(A), denoted **rank**(A).

A system of equation is said to be **inconsistent** if there are no solutions. A linear system is **inconsistent** if (and only if) the RREF of its augmented matrix contains any row [0 0 ... 0 k] where k≠0.

If a linear system is **consistent**, then it has either:
• **infinitely many solutions** when there is at least one free variable
(**rank** < **number of variables**),
or
• **exactly one solution** when all the variables are leading (**rank**=**number of variables**)

Challenge 15

Solve the following simultaneous equations using Gaussian elimination

$4a + 7b = 1$

$5a + 8b = 2$

Challenge 15

Represent equations in **matrix form**

$$\begin{pmatrix} 4 & 7 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Gaussian elimination process:

Step 1: Express as an augmented matrix

$$\begin{pmatrix} 4 & 7 & 1 \\ 5 & 8 & 2 \end{pmatrix}$$

Step 2: Consider each row in turn

Step 2.1: Change the leading non-zero number to a 1 in row1:

row1 = row1 /4

$$\begin{pmatrix} 1 & \frac{7}{4} & \frac{1}{4} \\ 5 & 8 & 2 \end{pmatrix}$$

Step 2.2: Remove non-zero numbers above and below this new leading 1

row2 = row2 - 5×row1

$$\begin{pmatrix} 1 & \frac{7}{4} & \frac{1}{4} \\ 0 & -\frac{3}{4} & \frac{3}{4} \end{pmatrix}$$

Step 2: Consider next row

Step 2.1: Change the leading non-zero number to a 1 in row2:

row2 = row2 / $(-\frac{3}{4})$

$$\begin{pmatrix} 1 & \frac{7}{4} & \frac{1}{4} \\ 0 & 1 & -1 \end{pmatrix}$$

Step 2.2: Remove non-zero numbers above and below this new leading 1

row1 = row1 - $\frac{7}{4}$×row2

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

Step 3: Convert the final **reduced-row echelon form** matrix to a standard matrix equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Step 4: Determine whether there are many, one or no solutions

Step 4.1: **No solutions** (**inconsistent**) when zeros on one side and a non-zero number on the other

Step 4.2: **One solution** when the number of leading ones = number of variables (**rank = *m***)

- Each row gives a unique solution for each variable
- In this case: $a = 2$ and $b = -1$

Step 4.3: **Many solutions** when number of leading ones < number of variables (**rank < m**)

- For variables on all zero rows assign general variable values eg t, s, r ...
- Find the solutions for all other variables in terms of t, s, r ...

Find the inverse using Gaussian elimination

Consider an $n \times n$ matrix A. To find A^{-1}(if it exists):

1. Form the augmented matrix $[A \vdots I_n]$

2. Apply standard Gaussian elimination until you have $\left[I_n \vdots A^{-1} \right]$

Challenge 16

Find inverse of $\begin{pmatrix} 4 & 7 \\ 5 & 8 \end{pmatrix}$ using Gaussian elimination

Challenge 16

$$\begin{pmatrix} 4 & 7 & 1 & 0 \\ 5 & 8 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 5 & 8 & 0 & 1 \end{pmatrix} \text{row1} / 4$$

$$\begin{pmatrix} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{3}{4} & \frac{-5}{4} & 1 \end{pmatrix} \text{row2} - 5 \times \text{row1}$$

$$\begin{pmatrix} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-4}{3} \end{pmatrix} \text{row} \Big/ \left(\frac{-3}{4}\right)$$

$$\begin{pmatrix} 1 & 0 & -\frac{8}{3} & \frac{7}{3} \\ 0 & 1 & \frac{5}{3} & \frac{-4}{3} \end{pmatrix} \text{row 1} - \frac{7}{4} \times \text{row2}$$

The inverse is therefore $\begin{pmatrix} -\frac{8}{3} & \frac{7}{3} \\ \frac{5}{3} & \frac{-4}{3} \end{pmatrix}$

Least-squares solution

The normal equation

The least-squares solutions of the system $A\vec{x} = \vec{b}$ are the exact solutions of

$$A^T A \vec{x}^* = A^T \vec{b}$$

Challenge 17

Find the least-squares solution $\vec{x}^*$ of the system $A\,\vec{x} = \vec{b}$ where

$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

Challenge 17

$$A^T A \vec{x}^* = A^T \vec{b}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 15 \\ 15 & 45 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 5 \\ 15 \end{pmatrix}$$

Solve by Gaussian elimination:

$\begin{pmatrix} 5 & 15 & 5 \\ 15 & 45 & 15 \end{pmatrix}$ form augmented matrix

$\begin{pmatrix} 1 & 3 & 1 \\ 15 & 45 & 15 \end{pmatrix}$ row1 / 5

$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ row2 - 15×row1

Let $x_2 = t$

$$\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ t \end{pmatrix} = \begin{pmatrix} x_1 + 3\,t \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$x_1 = 1 - 3\,t$

so $\vec{x}^* = \begin{pmatrix} 1 - 3\,t \\ t \end{pmatrix}$ where t is an arbitrary constant

Fitting a straight line using a least-squares approach

x	y
1	0
2	4
3	6

We can represent each data point as a separate equation:

$c_0 + c_1 1 = 0$ for point (1,0)
$c_0 + c_1 2 = 4$ for point (2,4)
$c_0 + c_1 3 = 6$ for point (3,6)

We can represent these equations in matrix form thus:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix}$$

Now use the **normal equation** to find the least-squares solution for $\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$

Challenge 18

Fit a straight line to the following data points using least-squares

x	y
−1	0
0	−1
1	2

Challenge 18

Fit a straight line to the following data points using least-squares

x	y
-1	0
0	-1
1	2

Set up the matrix of equations for each data point for a straight line model

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

Form the normal equation where $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

$$A^T A \vec{x}^* = A^T \vec{b}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

so

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}$$

straight line equation therefore:

$$\frac{1}{3} + x = y$$

Determinants

Determinant of a 2×2 matrix

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\,d - b\,c)$$

Determinant of a 3×3 matrix: Sarrus’s Rule

$$\left(\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right)\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$$
$$+ \quad + \quad +$$

add the product of these diagonal elements

$$\left(\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right)\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$$
$$- \quad - \quad -$$

then

subtract the products of these opposite diagonal elements

Challenge 19

Find the determinant of $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

Challenge 19

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{matrix} 1 & 2 \\ 3 & 1 \\ 0 & 2 \end{matrix}$$

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = 1 \times 1 \times 1 + 2 \times 0 \times 0 + 3 \times 3 \times 2 - 0 \times 1 \times 3 - 2 \times 0 \times 1 - 1 \times 3 \times 2 \;\; = 13$$

Determinant of triangular and diagonal matrices

The determinant of an **upper or lower triangular matrix** or of a **diagonal matrix** is the product of the diagonal elements

eg $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \det\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11}\, a_{22}\, a_{33}$

Determinant of block matrix

If M is a **block matrix** such that $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ or $M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ where A and C are **square matrices** then

$$\det(M) = \det(A)\det(C)$$

Challenge 20

Find

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 3 & 5 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 2 \\ 3 & 0 & 2 & 3 & 4 \end{pmatrix}$$

Challenge 20

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 3 & 5 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 2 \\ 3 & 0 & 2 & 3 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 5 & 1 \end{pmatrix}\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1\times 2\times 1 \times (1\times 4 - 3\times 2) = -4$$

Eigenvalues and eigenvectors

A nonzero vector $\vec{v}$ is called an **eigenvector** of the square matrix A if

$$A\,\vec{v} = \lambda\,\vec{v}$$

for some scalar λ, called the associated **eigenvalue** of eigenvector $\vec{v}$.

Alternatively we can say:

When **diagonalising** a square matrix A to the form $A = S\,B\,S^{-1}$

- the **eigenvalues** of A are the scalar values λ_i that form the diagonal elements of B, and
- the **eigenvectors** of A are the corresponding column vectors $\vec{v}_i$ of S.

Finding Eigenvalues:
The Characteristic Equation

We can restate the defining eigenvalue equation $A\vec{v} = \lambda\vec{v}$ as:

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

By the definition of the **kernel**, since eigenvectors exist the matrix is **non-invertible**.

And if a matrix is non-invertible its **determinant is zero**, so

$$\boxed{\det(A - \lambda I_n) = 0}$$ ⇐ **Characteristic Equation**

Challenge 21

a) Find the eigenvalues of $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

b) Find the eigenvalues of $\begin{pmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{pmatrix}$

Challenge 21 (a)

a) Find the eigenvalues of $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

$$\begin{aligned} \det\begin{pmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{pmatrix} &= (5-\lambda)(-1-\lambda) + 2\times 4 \\ &= \lambda^2 - 4\lambda - 5 + 8 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda-3)(\lambda-1) \end{aligned}$$

The eigenvalues of $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$ are $\lambda_1 = 3,\ \lambda_2 = 1$

Challenge 21 (b)

b) Find the eigenvalues of $\begin{pmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{pmatrix}$

Use Sarrus's Rule:

$$\begin{pmatrix} -3-\lambda & 0 & 4 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{pmatrix} \begin{matrix} -3-\lambda & 0 \\ 0 & -1-\lambda \\ -2 & 7 \end{matrix}$$

$$\det\begin{pmatrix} -3-\lambda & 0 & 4 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{pmatrix} = 0 = (-3-\lambda)(-1-\lambda)(3-\lambda) + 0 + 0 - (-2)(-1-\lambda)4 - 0 - 0$$

$$= -\lambda^3 - \lambda^2 + 9\lambda + 9 - 8\lambda - 8$$

$$= -\lambda^3 - \lambda^2 + \lambda + 1$$

$$= (1-\lambda)(\lambda+1)^2$$

The eigenvalues of $\begin{pmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{pmatrix}$ are $\lambda_1 = 1$, $\lambda_2 = -1$ (algebraic multiplicity = 2)

- The $(1-\lambda)(\lambda+1)^2$ factorisation was possibly difficult to spot.
- However, it should be reasonably clear that 1 and -1 are roots of $-\lambda^3 - \lambda^2 + \lambda + 1 = 0$

Finding eigenvectors

We have already seen that

$$A \vec{v} = \lambda \vec{v} \qquad \Longrightarrow \qquad (A - \lambda I_n) \vec{v} = \vec{0}$$

To find eigenvectors solve the matrix equation: $$(A - \lambda I_n) \vec{v} = \vec{0}$$

Challenge 22

a) Find the eigenvectors of $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

b) Find the eigenvectors of $\begin{pmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{pmatrix}$

Challenge 22 (a)

a) Find the eigenvectors of $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

From Challenge 21 we found $\lambda_1 = 3,\ \lambda_2 = 1$

Consider $\lambda_1 = 3$

$$\begin{pmatrix} 5-3 & -4 \\ 2 & -1-3 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}$$

let $c_1,\ c_2$ be the column vectors of $\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}$

kernel basis vector $\rightarrow 2\,c_1 + 1\,c_2 = 0 \ \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

eigenvector $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Consider $\lambda_2 = 1$

$$\begin{pmatrix} 5-1 & -4 \\ 2 & -1-1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}$$

kernel basis vector $\rightarrow 1\,c_1 + 1\,c_2 = 0 \ \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

eigenvector $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Challenge 22 (b)

b) Find the eigenvectors of $\begin{pmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{pmatrix}$

From Challenge 21 we found $\lambda_1 = 1,\ \lambda_2 = -1$

Consider $\lambda_1 = 1$

$$\begin{pmatrix} -3-1 & 0 & 4 \\ 0 & -1-1 & 0 \\ -2 & 7 & 3-1 \end{pmatrix} = \begin{pmatrix} -4 & 0 & 4 \\ 0 & -2 & 0 \\ -2 & 7 & 2 \end{pmatrix}$$

by observation c_1, c_2 are independent but c_3 is redundant

kernel basis vector $\rightarrow\ 1\,c_1 + 0\,c_2 + 1\,c_3 = 0\ \rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Consider $\lambda_2 = -1$

$$\begin{pmatrix} -3+1 & 0 & 4 \\ 0 & -1+1 & 0 \\ -2 & 7 & 3+1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 4 \\ 0 & 0 & 0 \\ -2 & 7 & 4 \end{pmatrix}$$

by observation c_1, c_2 are independent but c_3 is redundant

kernel basis vector $\rightarrow\ 2\,c_1 + 0\,c_2 + 1\,c_3 = 0\ \rightarrow \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

eigenvector $v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

Discrete linear dynamical systems

General solution in terms of A^t

Consider the dynamical system $\vec{x}(t+1) = A\,\vec{x}(t)$ with $\vec{x}(0) = \vec{x}_0$

Then: $$\vec{x}(t) = A^t\,\vec{x}_0$$

Discrete linear dynamical systems

General solution in terms of eigenvectors and eigenvalues

1. Find the eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ for A, and the associated eigenvalues $\vec{\lambda}_1, \ldots, \vec{\lambda}_n$

2. Find the coefficients $c_1, \ldots, c_n$ that form a linear combination of the vector $\vec{x}_0$ with respect the eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$:

$$\vec{x}_0 = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n$$

3. Then: $$\vec{x}(t) = c_1 \lambda_1{}^t \vec{v}_1 + \ldots + c_n \lambda_n{}^t \vec{v}_n$$

The individual rows of this solution are referred to as the “Closed Formula” solutions.

Challenge 23

Find the closed formula solutions for $\vec{x}(t) = A^t \vec{x}_0$ where

$A = \begin{pmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{pmatrix} \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Challenge 23

Find eigenvalues:

$$\det\begin{pmatrix} 0.5-\lambda & 0.25 \\ 0.5 & 0.75-\lambda \end{pmatrix} = 0 = (0.5-\lambda)(0.75-\lambda) - 0.5 \times 0.25$$

$$= \lambda^2 - 1.25\,\lambda + 0.375 - 0.125$$

$$= \lambda^2 - 1.25\,\lambda + 0.25$$

$$= 4\,\lambda^2 - 5\,\lambda + 1$$

$$= (\lambda - 1)(4\,\lambda - 1)$$

$\lambda_1 = 1,\ \lambda_2 = \frac{1}{4}$

Solution contd.

Find eigenvectors:

Consider $\lambda_1 = 1$:

$$\begin{pmatrix} 0.5-1 & 0.25 \\ 0.5 & 0.75-1 \end{pmatrix} = \begin{pmatrix} -0.5 & 0.25 \\ 0.5 & -0.25 \end{pmatrix}$$

kernel basis vector $\to 1\,c_1 + 2\,c_2 = 0 \;\to \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

eigenvector $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Consider $\lambda_2 = \frac{1}{4}$:

$$\begin{pmatrix} 0.5-0.25 & 0.25 \\ 0.5 & 0.75-0.25 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}$$

kernel basis vector $\to -1\,c_1 + 1\,c_2 = 0 \;\to \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

eigenvector $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Solution contd.

Find the coefficients $c_1, \ldots, c_n$

$$\vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_n = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - c_2 \\ 2 c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Using standard simultaneous equations approach:

$c_1 - c_2 = 1$
$2 c_1 + c_2 = 0$

Add equations: $3 c_1 = 1 \quad \rightarrow \; c_1 = \frac{1}{3}$
From equation (1): $c_2 = c_1 - 1 \quad \rightarrow \; c_2 = -\frac{2}{3}$

Solution contd.

Closed formula solution:

$$\vec{x}(t) = c_1 \lambda_1{}^t \vec{v}_1 + c_2 \lambda_2{}^t \vec{v}_2 = \frac{1}{3} \times 1^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{2}{3} \left(\frac{1}{4}\right)^t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times \frac{1}{3} \times 1^t \\ 2 \times \frac{1}{3} \times 1^t \end{pmatrix} - \begin{pmatrix} -1 \times \frac{2}{3} \left(\frac{1}{4}\right)^t \\ 1 \times \frac{2}{3} \left(\frac{1}{4}\right)^t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \left(\frac{1}{4}\right)^t \\ -\frac{2}{3} \left(\frac{1}{4}\right)^t \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} + \frac{2}{3} \left(\frac{1}{4}\right)^t \\ \frac{2}{3} - \frac{2}{3} \left(\frac{1}{4}\right)^t \end{pmatrix}$$

Remember Other Topics May Arise ...

There obviously is not time to cover everything we have learned over this module in this revision lecture.

Other topics may arise in the exam besides those covered today!

Good luck!