Week 1:
Foundations of Calculus

MSIN00180 Quantitative Methods for Business

MSIN0180 Module Overview

Module Lead: Dr Andrew Whiter

- Associate Professor
- Programme Director, BSc/MSci Management Science
- PhD in Artificial Intelligence (Fuzzy Logic),
  BSc(Eng) in Engineering Mathematics
- Sloan Fellow London Business School
- Lots of real-world experience! HP, EY, software company founder ...

MSIN0180 Module Overview

Part I: Calculus for Business (Weeks 1-5)

- Why Calculus?
  - Optimisation, optimisation, optimisation
  - Modelling change over time (dynamics)
    e.g. modelling viral marketing campaigns

- Topics
  - Mathematical foundations, single and multi-variate differentiation, constrained and numerical optimisation, integration, ordinary differential equations and their numerical solution

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Part II: Linear Algebra for Business (Weeks 6-10)

- Why Linear Algebra?
  - Mathematics of large data
  - Underpins many modern algorithms, including machine learning
  - Also used to model interactions over time
    e.g. modelling of competitive and cooperative business interactions

- Topics
  - Matrices, Gauss-Jordan Elimination, Google Page Rank algorithm, least-squares data fitting, determinants, eigenvalues, discrete dynamic systems

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Focus on Software Tools

How maths is done in the real-world!

- *Mathematica*
    - Ensure you have *Mathematica* loaded and ready to use
    - Lecture and seminar slides are runnable *Mathematica* files
- A little Python as well
    - Some of the numerical methods will be illustrated using Python code

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Assessment

- 80% 2-hour in-person exam
  - A self-marked mock-exam paper will be released in Week 10
- 10% Individual Assignment 1 (Calculus)
  - Released in Week 4 - 2 weeks to complete (14th February)
  - *Mathematica* based
- 10% Individual Assignment 2 (Linear Algebra)
  - Released in Week 8 - 2 weeks to complete (20th March)
  - *Mathematica* based

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Seminars

The seminar each week will aim to cover at least:

- 1 worked “exam-style” question
- 1 *Mathematica*/python question/example

Every week please ensure you bring to the seminars both:

- Paper and pen for hand-worked “exam-style” questions
- Charged laptops with *Mathematica* ready to use

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Textbooks

- Required - None!
    - The weekly materials cover the topics required for this module.

- Optional
    - **Thomas' Calculus (12th Edition)**, G. Thomas, M. Weir, J. Hass and F. Giordano, Pearson Education (ISBN-10: 0321643631, ISBN-13: 978-032164363
    - **The Student's Introduction to Mathematica and the Wolfram Language (3rd Edition)**, B. Torrence and E. Torrence, Cambridge University Press (ISBN-10: 110840636X, ISBN-13: 978-1108406369)
    - **Linear Algebra with Applications (5th Edition)**, Otto Bretscher, Pearson (ISBN-10: 0-321-89058-2, ISBN-13: 978-0-321-89058-0)

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Support

- Dr Andrew Whiter Office Hour
  - Thursdays 2-3pm
  - Room 4.05 Engineering Front Building

- Joe Myers, Teaching Assistant
  - Joe will be marking your assignments and exam
  - While Joe is not currently providing support this may be possible for any students that need extra help.

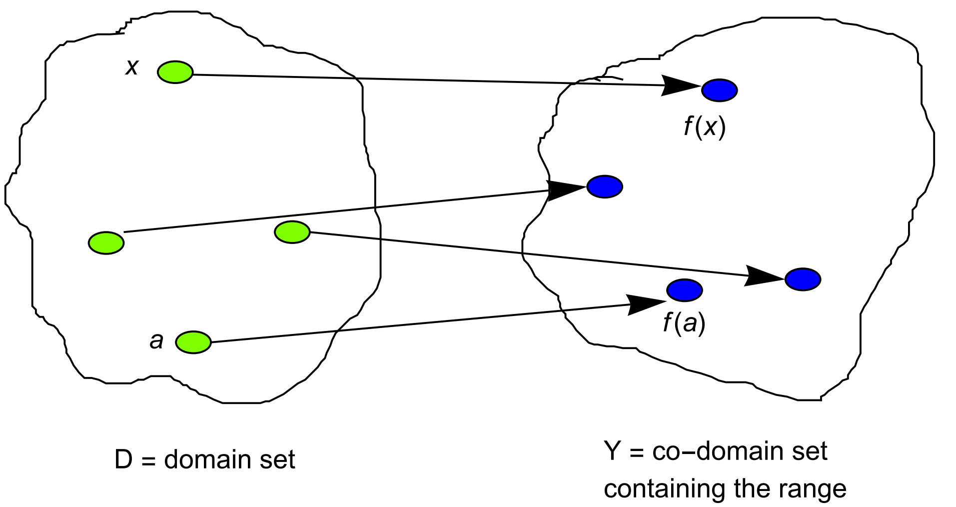
Learning Objectives for this Lecture: Foundations of Calculus

Understand of the foundations of calculus, including rigorous definitions for functions, limits and continuity.

- Functions
- Limits
- Continuous Functions
- Vectors

Functions, Domains, Co-domains & Ranges

A **function *f*** from a set D to a set Y is a rule that assigns a unique element $f(x) \in Y$ to each element $x \in D$



Co-domain is simply a set of values that includes the **range**, the definitive set of values that can be output by a function.
Usually, co-domains and ranges serve the same purpose.

Surjectivity

A function is **surjective** if

co-domain = range

Definition: Surjectivity

For every y in the co-domain there is a x in the domain for which $f(x) = y$

Intervals

We write the interval of numbers $x \geq -2$ using the notation $[-2, \infty)$

In general any interval can be written in this notation with square brackets, [, meaning the value is included and round brackets,), meaning it is not

$(0, \infty)$ = all **positive numbers**
$(-\infty, 0)$ = all **negative numbers**
$[0, \infty)$ = all **non-negative numbers**
$(-\infty, 0]$ = all **non-positive numbers**

Number Sets

We also have the following useful number sets:

$\mathbb{N}$ =natural numbers, 0, 1, 2, ...
$\mathbb{Z}$ =integers, ..., −2, −1, 0, 1, 2, ...
$\mathbb{R}$ =real numbers
$\mathbb{C}$ =complex numbers

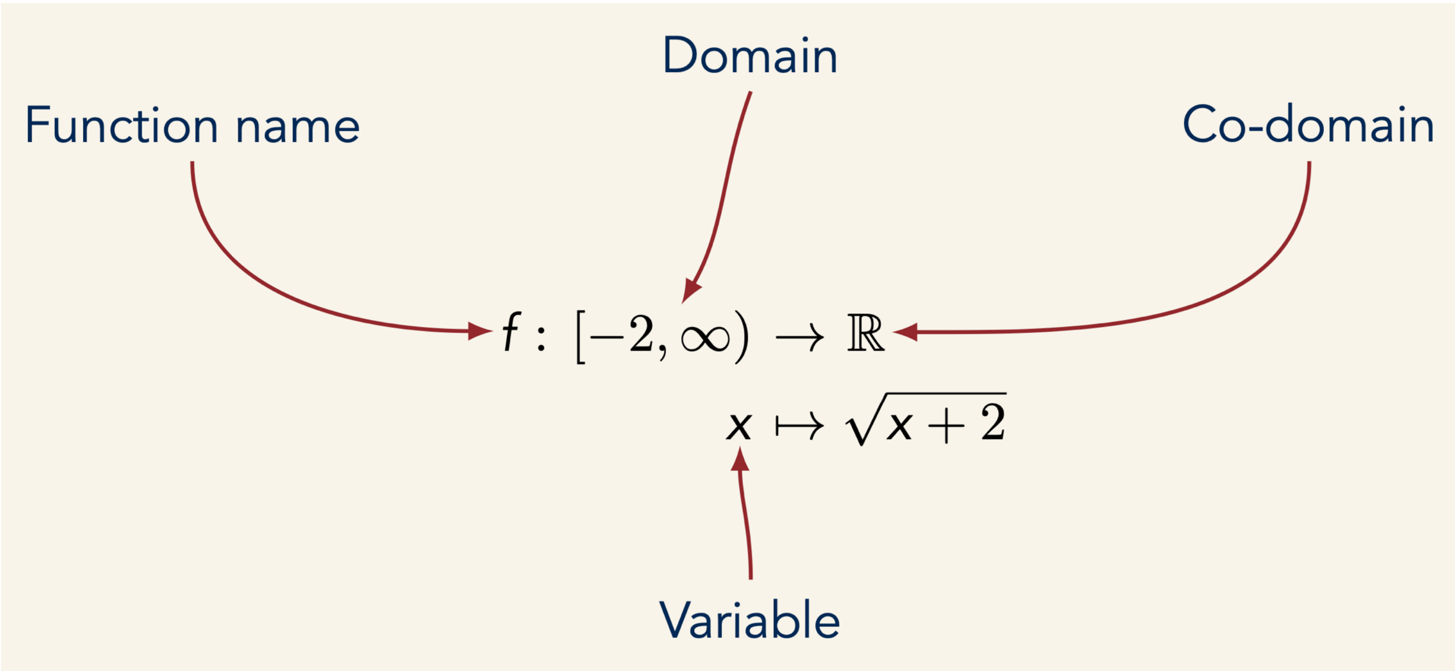
Natural Domains

The **natural domain** is the largest set of real *x*-values for which the formula gives real *y*-values.

Function	Natural Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$(-\infty, 0) \bigcup (0, \infty)$	$(-\infty, 0) \bigcup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Defining functions properly

$$\begin{aligned} f : [-2, \infty) &\to \mathbb{R} \\ x &\mapsto \sqrt{x+2} \end{aligned}$$



First Introduction to Mathematica

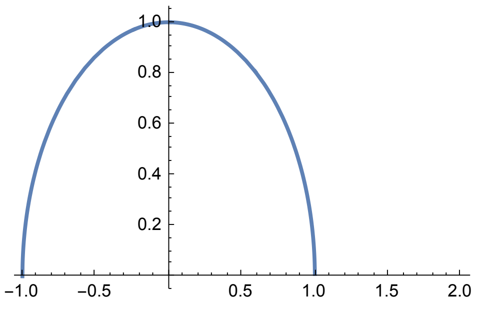
In[ ]:=
```
f[x_]:= √(1-x^2)
```

In[ ]:=
```
f[.8]
```

Out[ ]=
```
0.6
```

In[ ]:=
```
Plot[f[x],{x,-1,2}]
```

Out[ ]=



Set Notation Definition of Graphs of Functions

In set notation a graph of a function f with domain D consists of the points on the Cartesian plane

$$\{(x, f(x)) \mid x \in D\}$$

Representing Functions Numerically

Functions can also be represented as tables of values, which are often used to represent real data.

Strictly the domain of these functions consists only for the values of x for which values of the function are specified. They are not continuous. The nature of the system itself however may or may not be naturally continuous.

In[61]:= data =

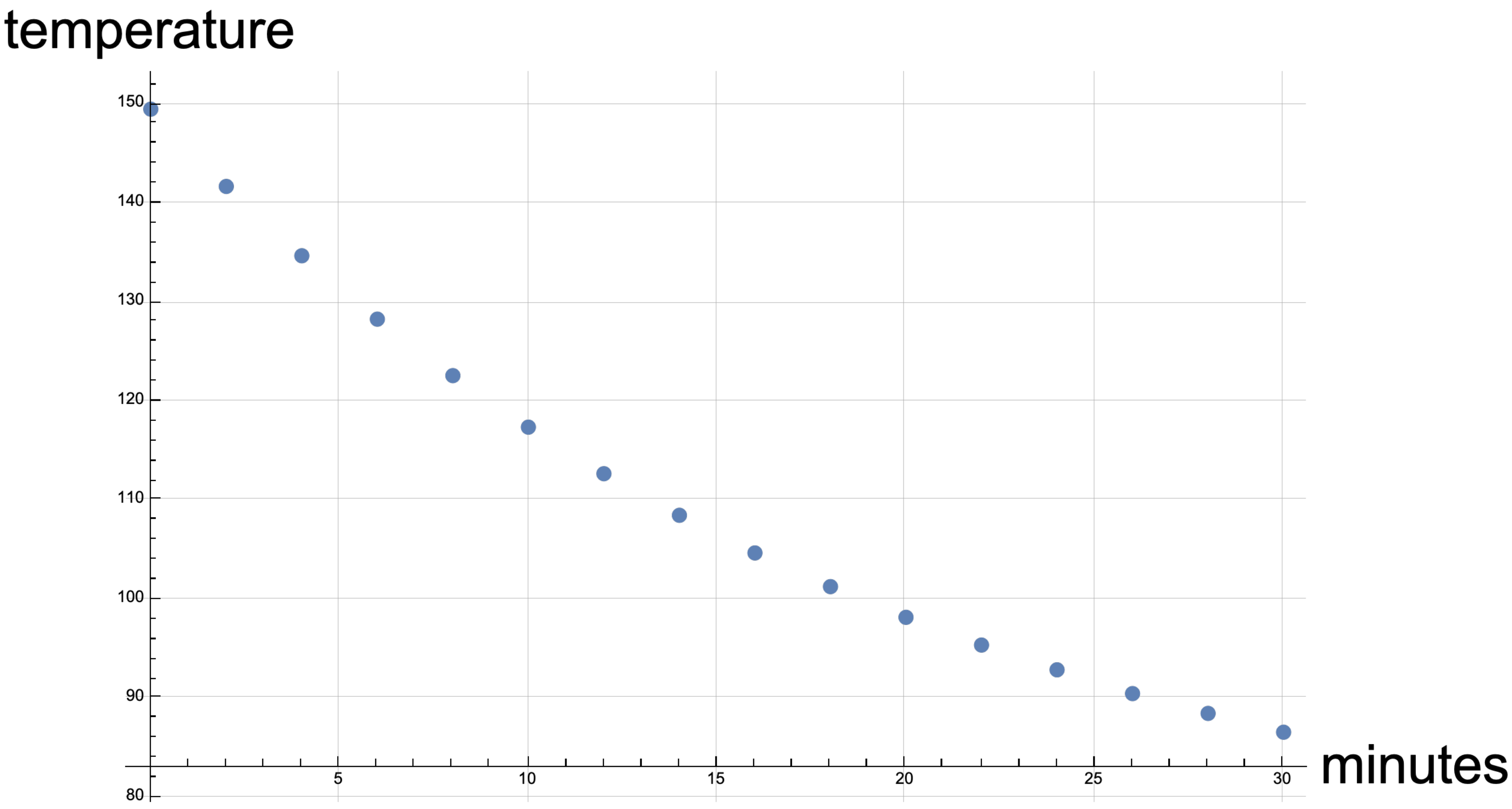
0	149.5
2	141.7
4	134.7
6	128.3
8	122.6
10	117.4
12	112.7
14	108.5
16	104.7
18	101.3
20	98.2
22	95.4
24	92.9
26	90.5
28	88.5
30	86.6

;

Representing Functions Numerically

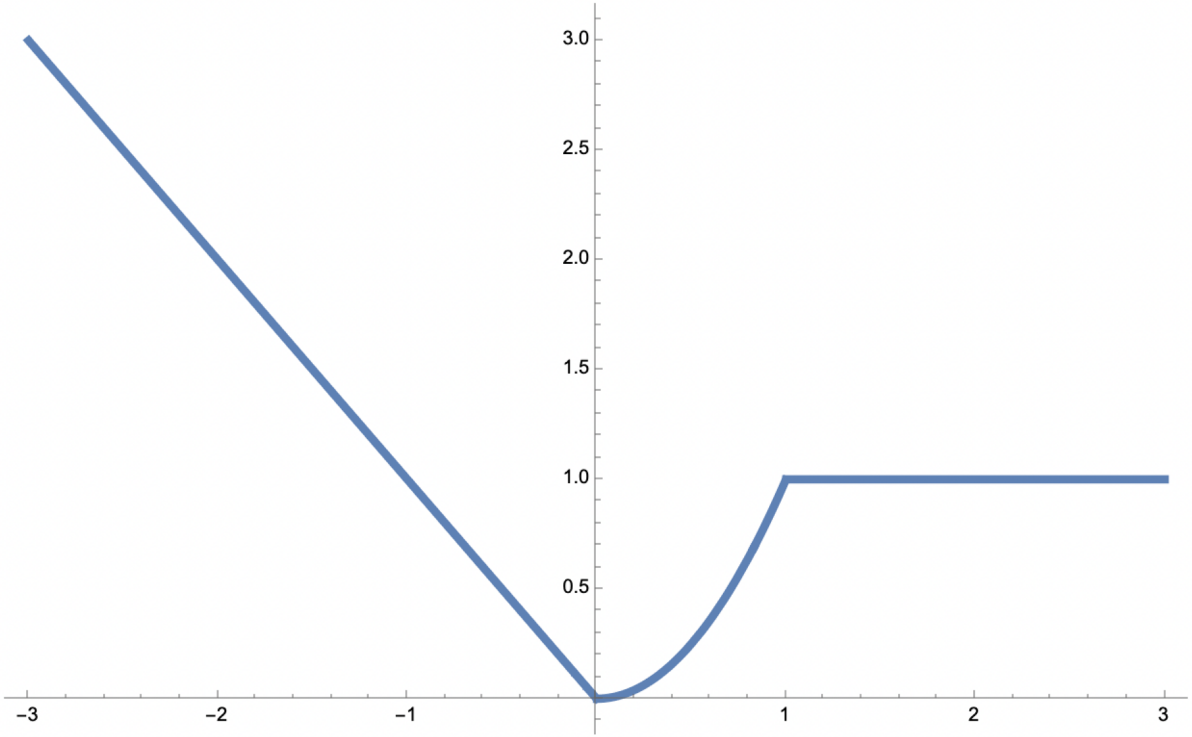
In[67]:=
```
ListPlot[data, AxesLabel→{Style["minutes",30],Style["temperature",30]},
         GridLines→Automatic]
```

Out[67]=



Piecewise-Defined Functions

$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

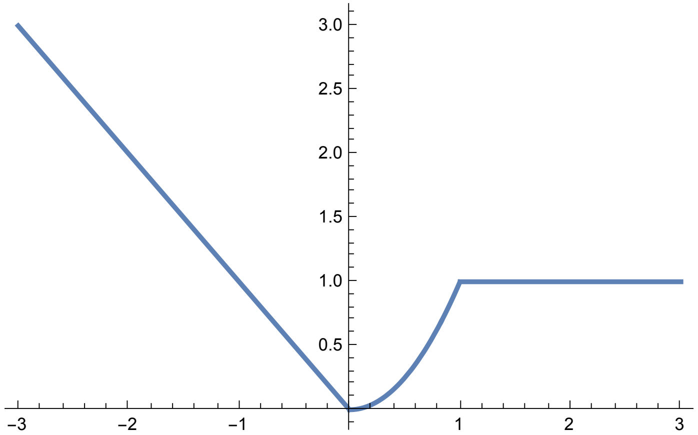


Piecewise Functions in *Mathematica*

In[69]:=
```
f[x_]:=Piecewise[{{-x,x<0},{x^2,0≤x≤1},{1,x>1}}]
```

In[78]:=
```
Plot[f[x],{x,-3,3},PlotStyle→Thickness[0.007]]
```

Out[78]=



Increasing and Decreasing Functions

Given x_1 and x_2 are any two points on the interval I.

If $f(x_2) \geq f(x_1)$ whenever $x_1 < x_2$ then f is said to be **increasing** on I

If $f(x_2) \leq f(x_1)$ whenever $x_1 < x_2$ then f is said to be **decreasing** on I

Strictly increasing/decreasing

If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$ then f is said to be **strictly increasing** on I

If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$ then f is said to be **strictly decreasing** on I

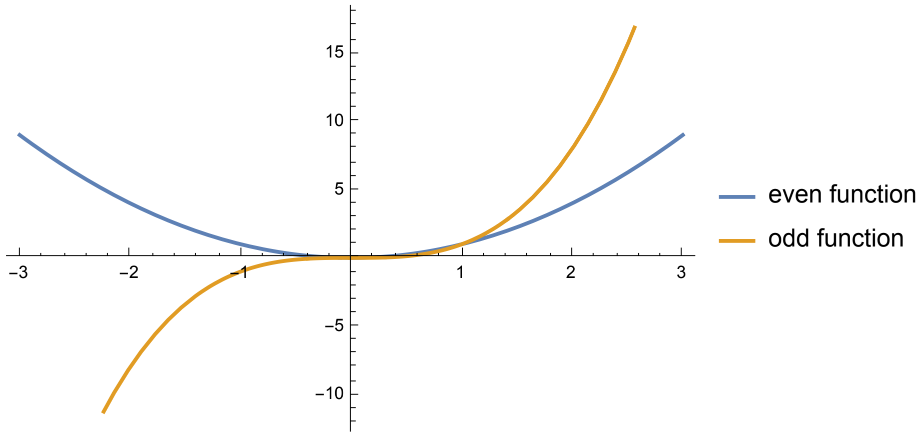
Even Functions and Odd Functions: Symmetry

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$

odd function of x if $f(-x) = -f(x)$

Out[ ]=



Composite Functions

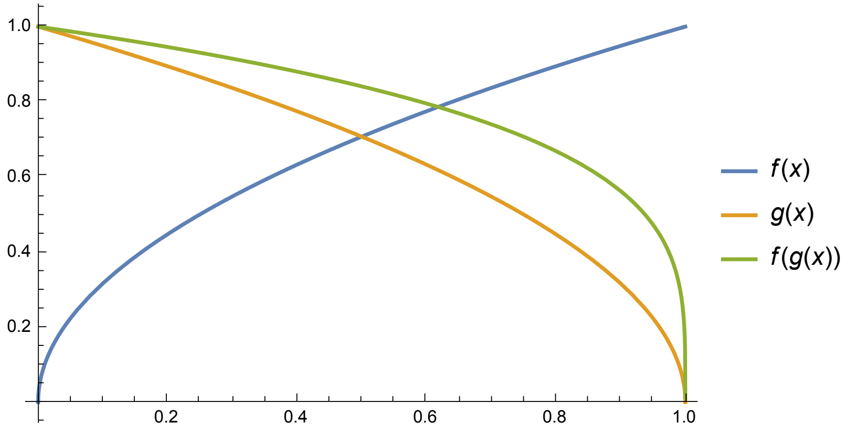
The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

$$(f \circ g)(x) = f(g(x))$$

In[∘]:=

```
f[x_]:=√x;
g[x_]:=√(1-x);
Plot[{f[x],g[x],f@g[x]},{x,0,1}, PlotLegends→"Expressions"]
```

Out[∘]=



One-to-one or injective functions

A function $f(x)$ is **one-to-one** on the domain D if

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2$$

A one-to-one function is also called an **injective** function.

Inverse Functions

Provided that f is a **one-to-one** function on the domain D with range R.
The **inverse function f^{-1}** is defined by

$$f^{-1}(b) = a \qquad \text{if } f(a) = b$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Note that: $\left(f^{-1} \circ f\right)(x) = x$ for all x∈D

The graph of $f^{-1}$is obtained by reflecting the graph of f about the line $y = x$

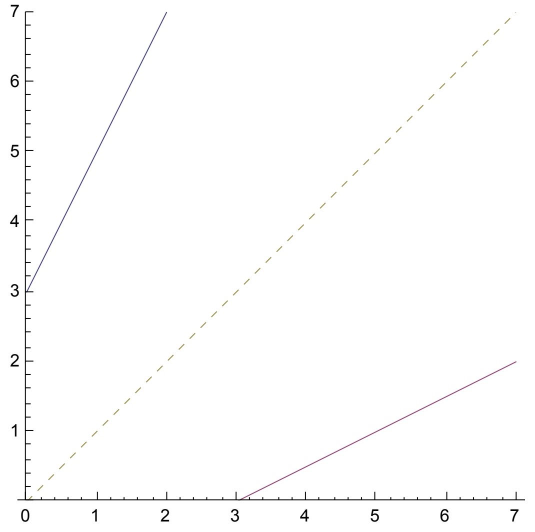
Finding inverses in *Mathematica*

In[ ]:=
```
f[x_]:=5x+4
```

In[ ]:=
```
InverseFunction[f][x]
```

Out[ ]=

$$\frac{1}{5}(-4+x)$$

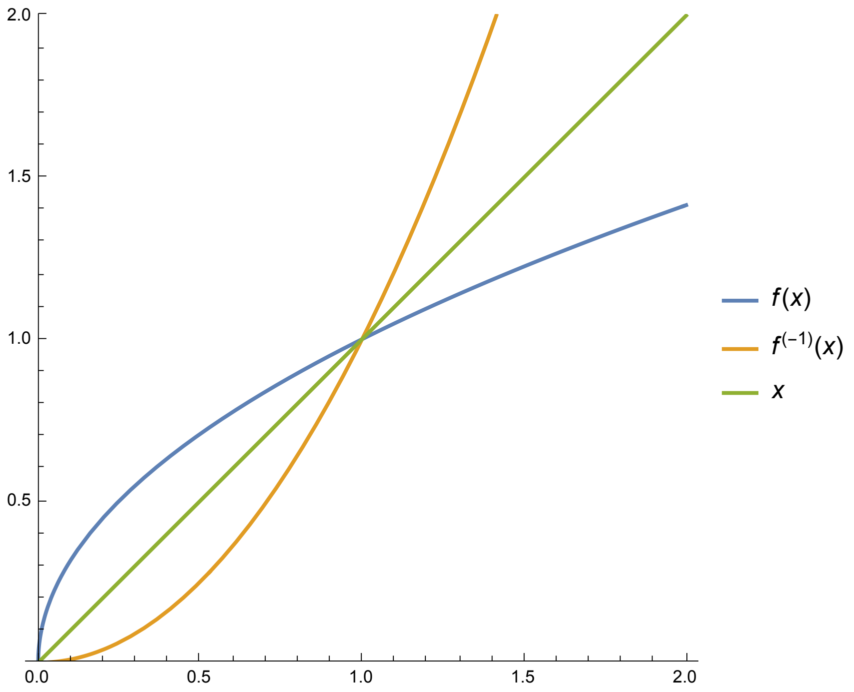


Inverse Functions: Example 2

In[ ]:=
```
f[x_]:= √x
```

In[ ]:=
```
Plot[{f[x],InverseFunction[f][x],x},{x,0,2}, PlotLegends→"Expressions",
    PlotRange→{0,2}, AspectRatio→ Automatic]
```

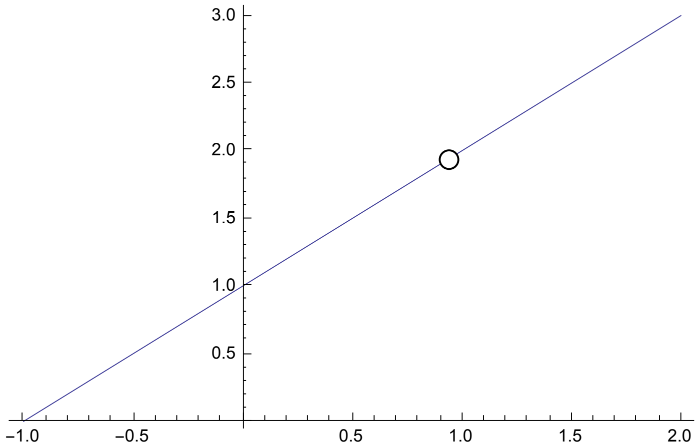
Out[ ]=



Limits and Continuity

Limits provide a precise way to describe the way in which functions behave. Limits underpin the definition of continuity.

Consider the behaviour of $f(x) = \frac{x^2-1}{x-1}$ near $x = 1$



$$f(x) = \frac{(x-1)(x+1)}{(x-1)} = (x+1) \quad \text{for } x \neq 1$$

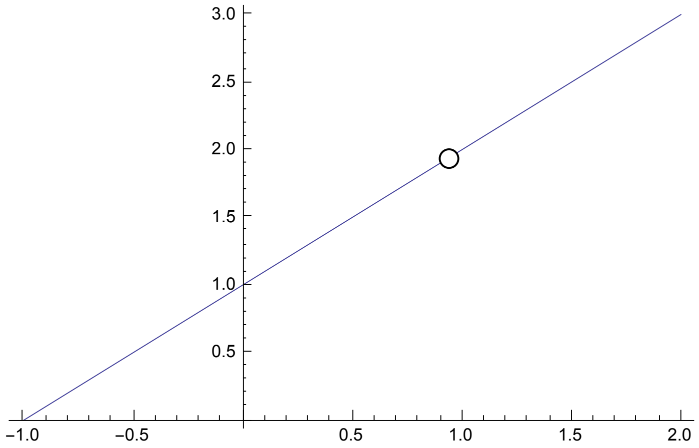
Limits: Informal Definition

If $f(x)$ is *arbitrarily close* to L for all x *sufficiently close* to x_0, we say that f approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L$$

So, for the previous example, we can informally observe that:

$$\lim_{x \to 1} f(x) = 2 \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$



Limit Laws

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use these rules:

Given: $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$

Sum Rule	$\lim_{x \to c} (f(x) + g(x)) = L + M$
Constant Multiplier Rule	$\lim_{x \to c} (k.f(x)) = k.L$
Product Rule	$\lim_{x \to c} (f(x).g(x)) = L.M$
Quotient Rule	$\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$, $M \neq 0$
Power Rule	$\lim_{x \to c} (f(x)^n) = L^n$, n a positive integer

Example

What is $\lim_{x \to c} \left(x^3 + 4\,x^2 - 3\right)$?

$= \lim_{x \to c} x^3 + \lim_{x \to c} 4\,x^2 - \lim_{x \to c}$

$= c^3 + 4\,c^2 - 3$

The Sandwich Theorem

Given $g(x) \leqslant f(x) \leqslant h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself.

Given also that:

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

Then

$$\lim_{x \to c} f(x) = L$$

The Sandwich Theorem: Example

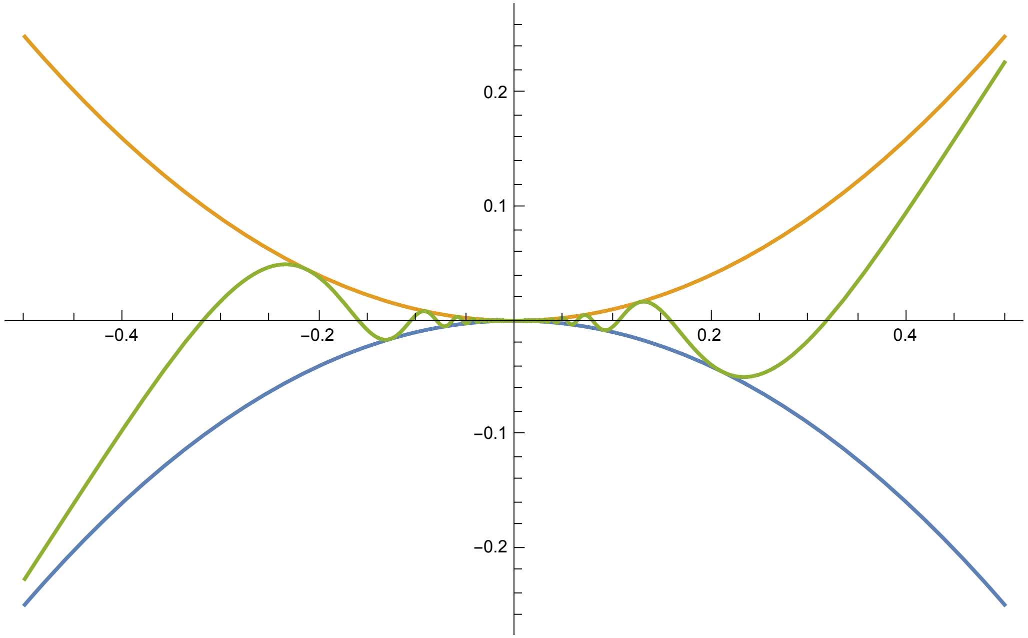
It is not obvious what the following limit is as $x \to 0$ ...

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

But given $-x^2 \leqslant x^2 \sin\left(\frac{1}{x}\right) \leqslant x^2$ for all $x \neq 0$

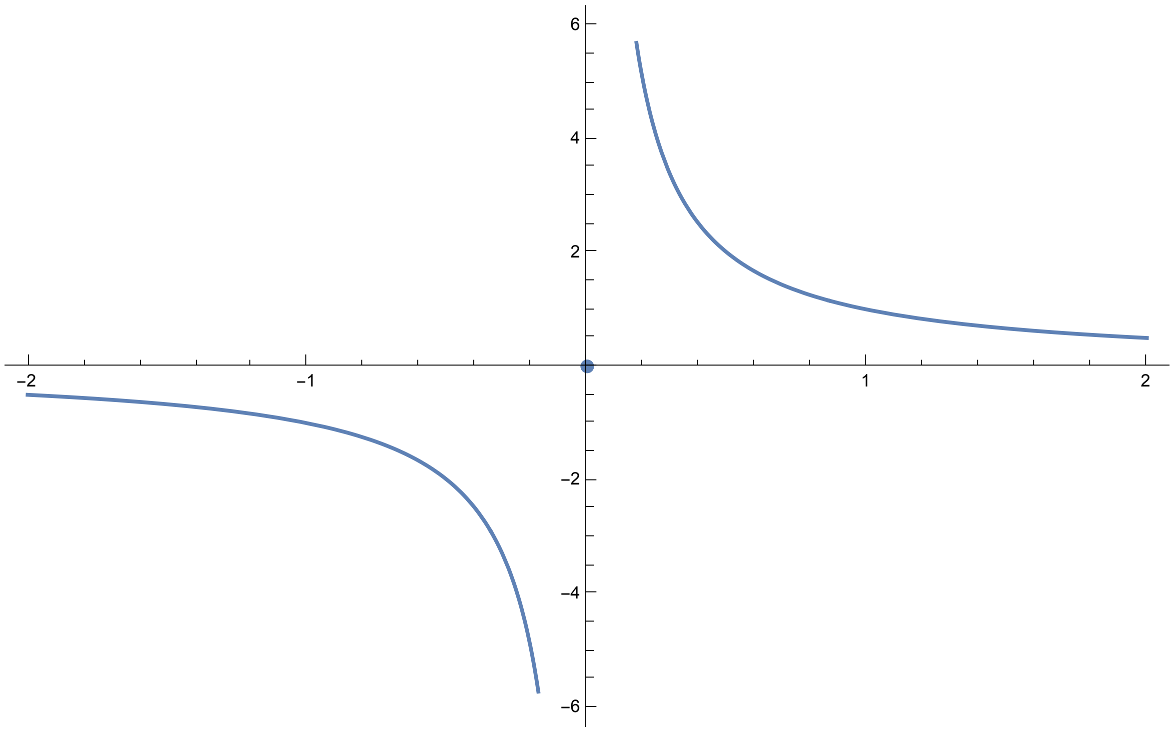
It follows that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

Out[ ]=



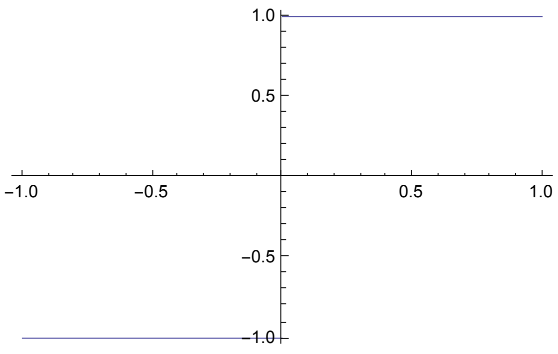
What is the $\text{limit}_{x \to 0}\, f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$?

Out[▫]=



One-sided Limits

Consider the function $f(x) = \frac{x}{|x|}$



Right-hand Limit

$$\lim_{x\to 0^+} f(x) = 1$$

Left-hand Limit

$$\lim_{x\to 0^-} f(x) = -1$$

Limit only exists if both its one-side limits have the same value

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L$$

Continuity

A function $f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x\to c} f(x) = f(c)$$

This implies that a function f(x) must meet 3 conditions to be continuous at an interior point c:

- $f(c)$ exists
- $\lim_{x\to c} f(x)$ exists
- $\lim_{x\to c} f(x) = f(c)$ exists

Discontinuity

If a function is not continuous at point c we say it is **discontinuous** at c and that the point c is a **point of discontinuity** of f.
Types of discontinuity include:

- Jump discontinuity
- Infinite discontinuity
- Oscillating discontinuity

Continuous Functions

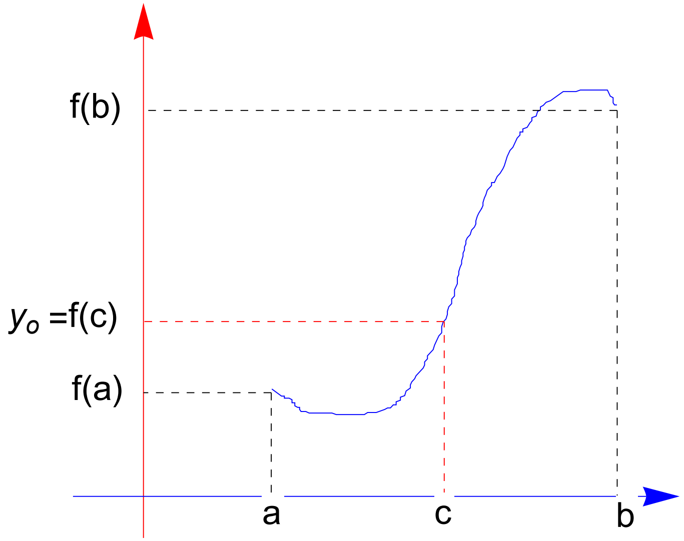
A function is **continuous on an interval** if and only if it is continuous at every point of the interval.

A **continuous function** is one that is continuous on every point of its domain.

Intermediate Value Theorem

The Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval [a,b] and if $y_0$is any value between $f(a)$ and $f(b)$ then $y_0 = f(c)$ for some c in [a,b].



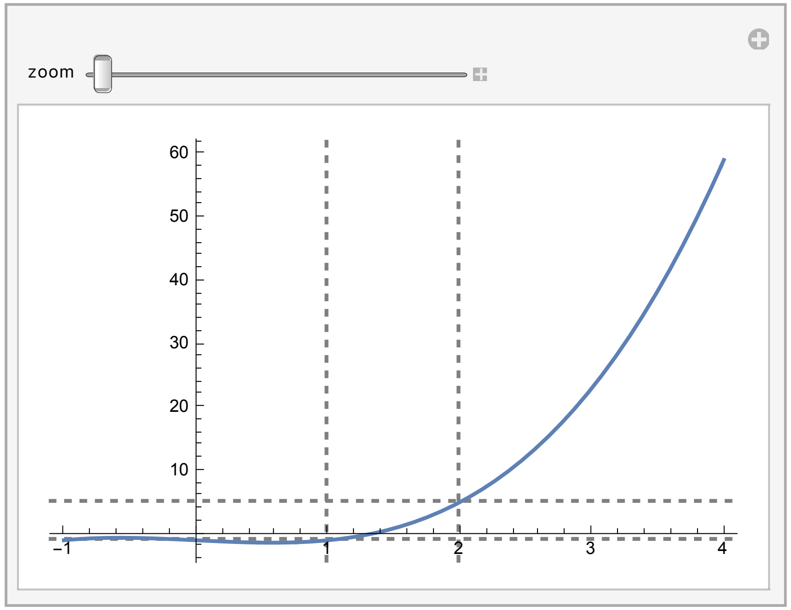
Geometrically this theorem says that any horizontal line $y = y_0$ crossing the y-axis between $f(a)$ and $f(b)$ crosses the curve $y = f(x)$ at least once over the interval [a,b].

Example

Use the Intermediate Value Theorem to show there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Let $f(x) = x^3 - x - 1$
Since $f(1) = -1 < 0$ and $f(2) = 5 > 0$ we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$.

Out[91]=



Asymptotes

A line $y = b$ is a **horizontal asymptote** of the graph of the function $y = f(x)$ if either:

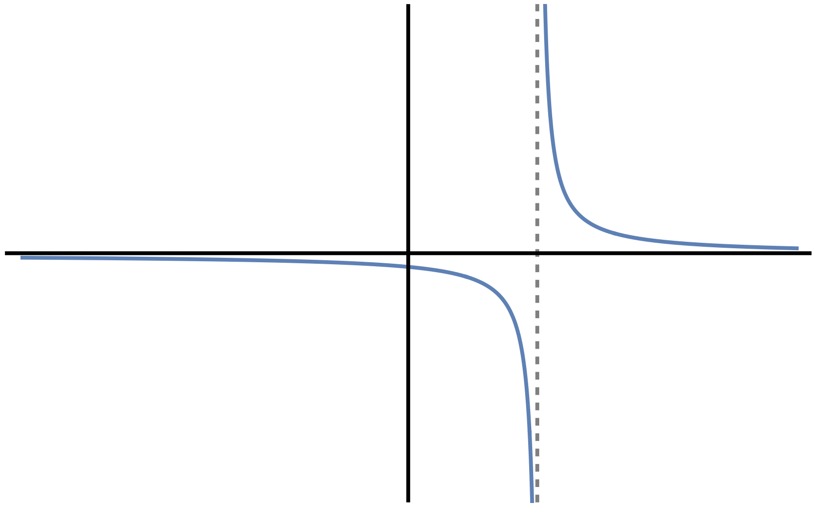
$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

A line $x = a$ is a **vertical asymptote** of the graph of the function $y = f(x)$ if either:

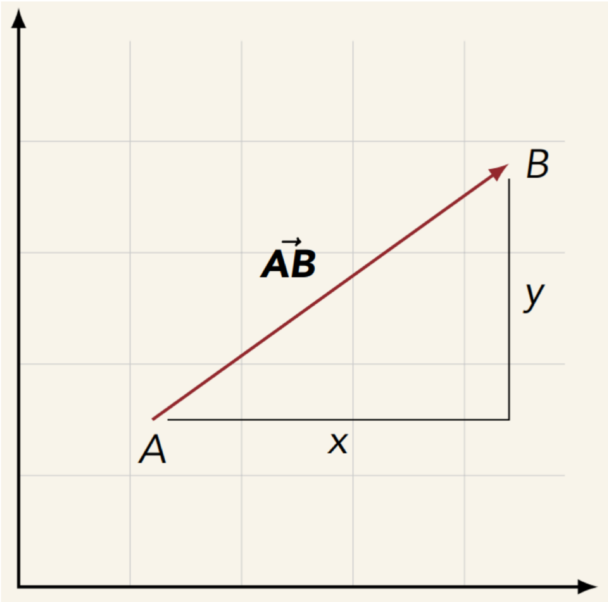
$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

Out[95]=

$$\frac{1}{(x-1)}$$

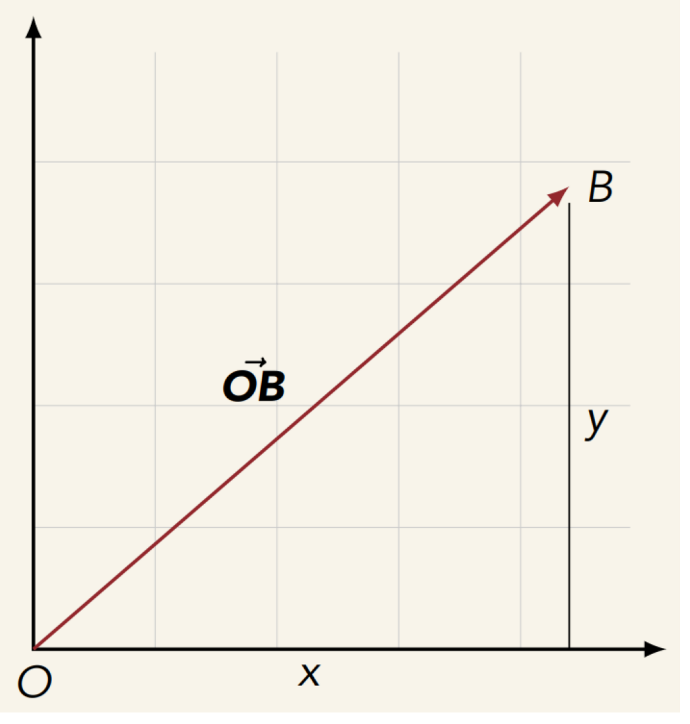


Vectors



- Vectors have magnitude and direction
- Notation options include using overbars or bold fonts
  e.g. $\overrightarrow{AB} = \vec{u} = \boldsymbol{u}$
- Can be expressed as row vectors $\begin{pmatrix} x & y \end{pmatrix}$ or as column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$

Position Vectors



- Same as vectors, except they start at the origin $O = (0, 0)$
- They represent a point
  $B = (x, y)$

Magnitude and Unit Vectors

- The magnitude or length of a vector $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$ is
  $|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$
- A vector is said to be a unit vector when its magnitude is 1
  e.g. when $|\overrightarrow{AB}| = 1$ $\overrightarrow{AB}$ is a unit vector
- Vectors be expressed using the base vectors $\boldsymbol{i}$, $\boldsymbol{j}$ which are unit vectors in the x, y directions respectively
  e.g. $\begin{pmatrix} x \\ y \end{pmatrix} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$

Dot products

- The **dot product** is a measure of how closely two vectors align, in terms of the directions they point.
- Given two vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and the angle between them θ their dot product is expressed and calculated as

  $\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}| \; |\boldsymbol{v}| \cos\theta$

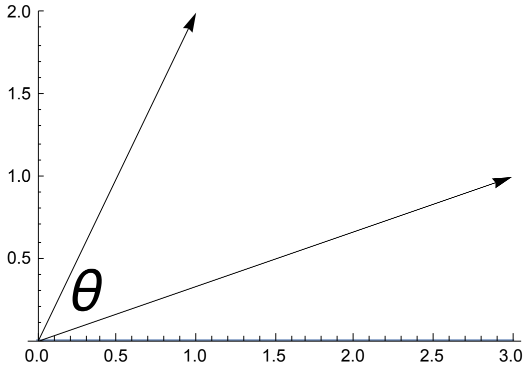
- Given two vectors $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ their dot product can also be calculated as

  $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2$

Dot Product: Example

Find the angle θ between $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Out[53]=



$$\cos\theta = \frac{u \cdot v}{|u|\ |v|} = \frac{(1 \times 3 + 2 \times 1)}{\sqrt{1^2+2^2}\ \sqrt{3^2+1^2}}$$

$$= \frac{5}{\left(\sqrt{5}\right)\left(\sqrt{10}\right)} = \frac{5}{\left(5\ \sqrt{2}\right)} = \frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$