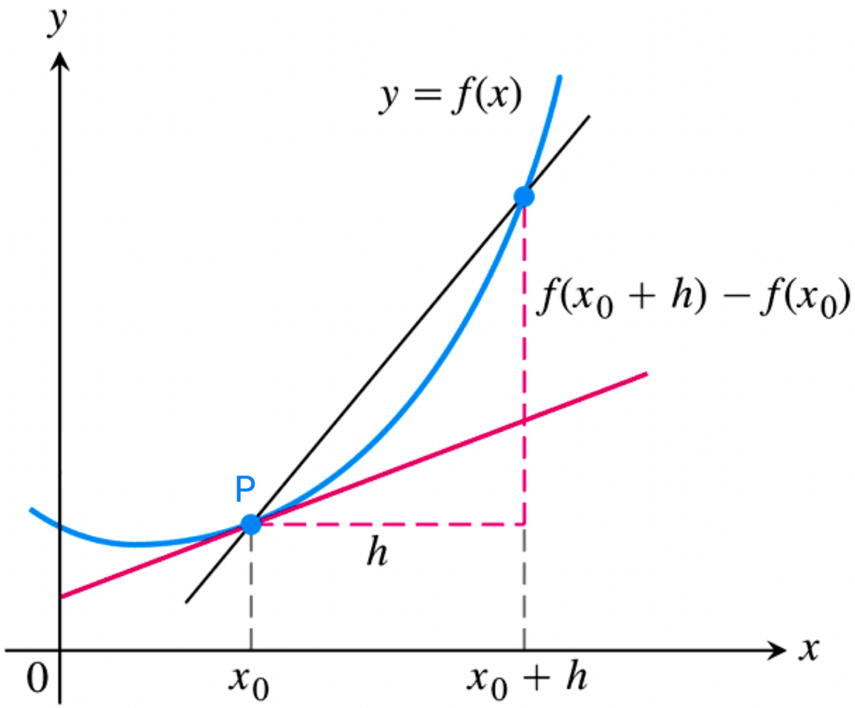
Week 2:
Differentiation & optimisation
of single variable functions

MSIN00180 Quantitative Methods for Business

Topics

- The Derivative
- Differentiation Rules
- The Chain Rule
- Implicit Differentiation
- Logarithmic Differentiation
- Extreme Values and Critical Points
- Optimisation

Slope of a Tangent to a Curve



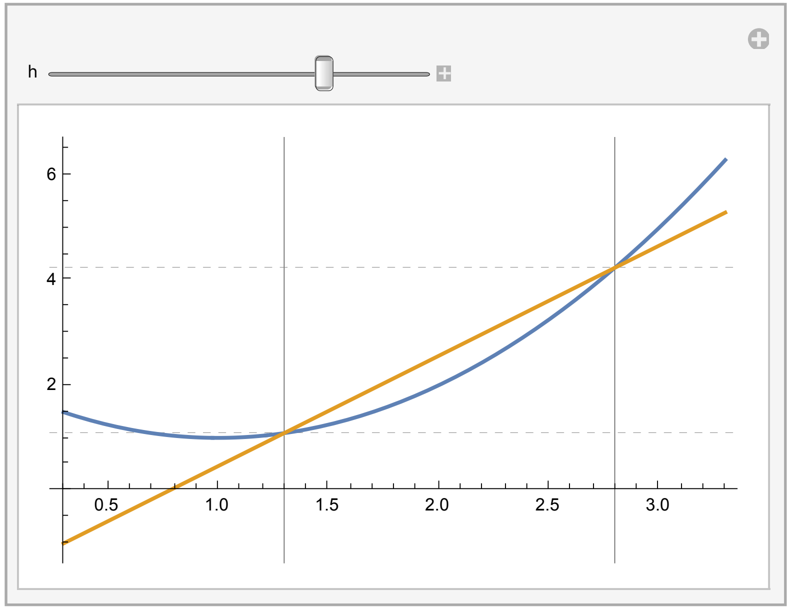
The slope of the tangent line at point P is given by

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

Tangent to a Curve

In[1]:=
```
f[x_]:=1+(x-1)^2;
tangent[f_,x_,a_,h_]:=((f[a+h]-f[a])/h)(x-a)+f[a];
a=1.3;
Manipulate[
        Plot[{f[x],tangent[f,x,a,h]},{x,a-1,a+2},
            GridLines → {{a,a+h}, {f[a],f[a+h]}}, GridLinesStyle→{Black,Dashed}],
        {{h,1.5},0.001,2}]
```

Out[4]=



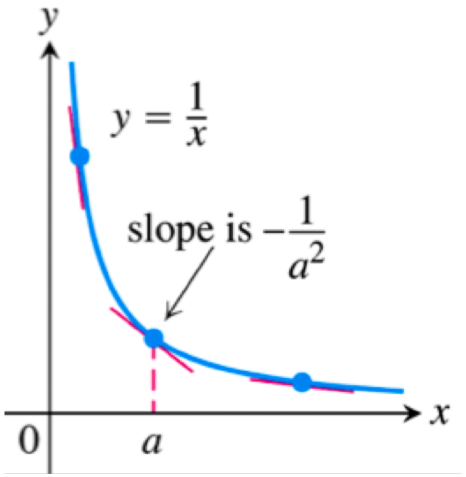
Derivative at a Point

The **derivative** of a function f at a point x_0, denoted $\boldsymbol{f'(x_0)}$, is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

provided this limit exists.

Example



When $f(x) = \frac{1}{x}$ the slope at $x = a$ is found thus:

$$\frac{f(a+h)-f(a)}{h} = \frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \frac{1}{h}\,\frac{a-(a+h)}{a(a+h)} = \frac{-h}{h\,a(a+h)} = \frac{-1}{a(a+h)}$$

$$\Longrightarrow \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = -\frac{1}{a^2}$$

Find the derivative of $2\,x^3$ from first principles in Mathematica

In[ ]:=
```
Clear[x]; f[x_]:= 1/x
slope[f_]:= (f[x+h]-f[x])/h
```

In[ ]:=
```
slope[f]
```

Out[ ]=

$$\frac{-\frac{1}{x}+\frac{1}{h+x}}{h}$$

In[ ]:=
```
Simplify[%]
```

Out[ ]=

$$-\frac{1}{h\,x+x^2}$$

In[ ]:=
```
Limit[%,h→0]
```

Out[ ]=

$$-\frac{1}{x^2}$$

Notation

There are many ways to denote the derivative of a function $y = f(x)$

$\boldsymbol{f'(x)}$: Lagrange notation

$\mathbf{\frac{dy}{dx}}$: Leibniz notation

Other notations include: $\frac{d}{dx} f(x)$, $D(f)(x)$, $D_x f(x)$

Another notation for a derivative due to Newton, if a function varies with time, i.e. $y = y(t)$ then a dot is used: $\dot{y}$

Notation: values at $x = a$

There are also many ways to denote the derivative of a function $y = f(x)$ at a point $x = a$

$f'(a)$

$\frac{dy}{dx}\Big|_{x=a}$

$\frac{d}{dx} f(x)\Big|_{x=a}$

Calculating the derivative in Mathematica

In[ ]:=
```
f[x_]=2x^3
```
Out[ ]=

$2\,x^3$

In[ ]:=
```
f'[x]
```
Out[ ]=

$6\,x^2$

alternatively

In[ ]:=
```
D[2x^3,x]
```
Out[ ]=

$6\,x^2$

or even

In[ ]:=
```
∂x f[x]
```
Out[ ]=

$6\,x^2$

Differentiable Functions on an open interval (a,b)

The function $f(x)$ is differentiable on an open interval (a, b) if it has a derivative at each point on the interval (a, b).

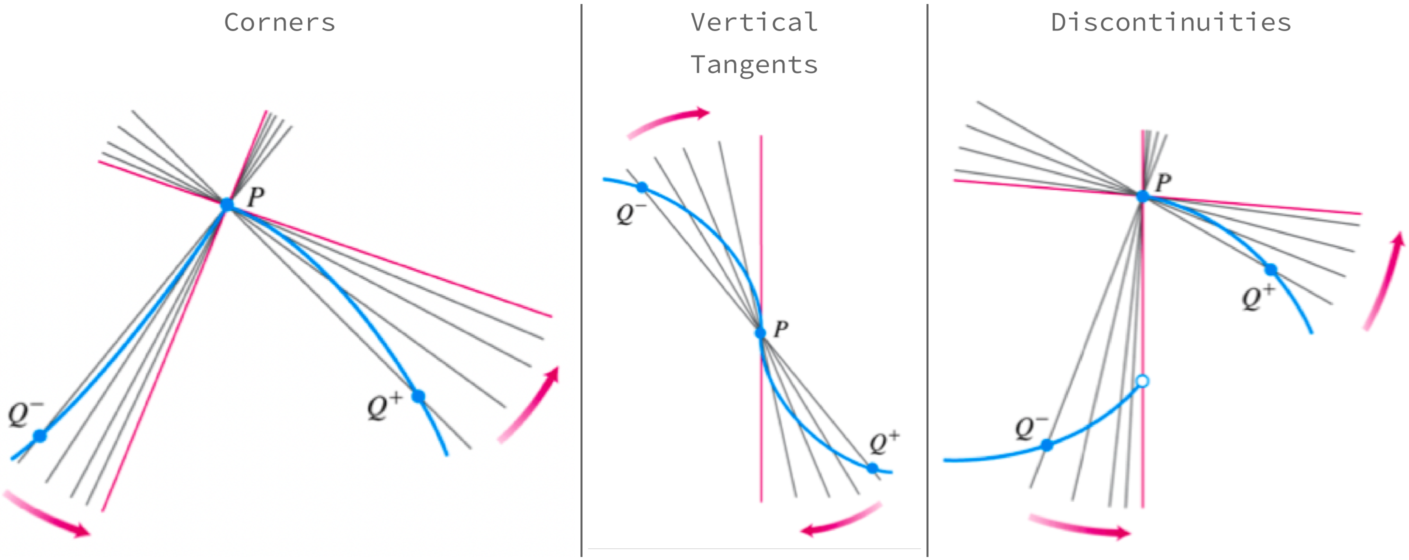
Differentiable functions on an closed interval [a,b]

The function $f(x)$ is differentiable on a closed interval $[a, b]$ if it has a derivative at each point on the interval and if the following limits exist at the endpoints of the interval:

$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ right-hand derivative at a

$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$ left-hand derivative at b

Points of a Function that do NOT have Derivatives



Differentiable functions are continuous functions

THEOREM: **Differentiability Implies Continuity**.

If f has a derivative at $x = c$, the f is continuous at $x = c$

Differentiation Rules

Each of these rules can be derived from the derivative definition.

Derivative of a Constant

If f has the constant value $f(x) = c$ then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$

Power Rule

If n is any real number then

$$\frac{d}{dx} x^n = n\, x^{n-1}$$

Differentiation Rules

Derivative Constant Multiple Rule

If u is a differentiable function of x and c is a constant then

$$\frac{d}{dx}(c\,u) = c\,\frac{du}{dx}$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable, and

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Example: $\frac{d}{dx}\left(2x^3 + \frac{3}{x^2} - 5\right)$

First use the sum/difference rule:

$\frac{d}{dx}\left(2x^3\right) + \frac{d}{dx}\left(\frac{3}{x^2}\right) - \frac{d}{dx}(5)$

Then use the constant multiple rule and derivative of a constant rule:

$2\frac{d}{dx}\left(x^3\right) + 3\frac{d}{dx}\left(\frac{1}{x^2}\right)$

Finally, use the power rule:

$2\left(3x^2\right) + 3\left(-2x^{-3}\right) \quad = \quad 6\left(x^2 - \frac{1}{x^3}\right)$

in Mathematica ...

In[ ]:=

```
D[2x^3 + 3/x^2 - 5, x]
```

Out[ ]=

$-\frac{6}{x^3} + 6x^2$

Differentiation Rules

Derivative Product Rules

If u and v are differentiable at x, then so is their product $u\,v$, and

$$\frac{d}{dx}(u\,v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

Derivative Quotient Rules

If u and v are differentiable at x and if $v(x) \neq 0$, then their quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$$

Example: $\frac{d}{dx}\left(\frac{1-x^2}{2x+3}\right)$

Use the quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$:

$$\frac{(2x+3)\frac{d}{dx}(1-x^2) - (1-x^2)\frac{d}{dx}(2x+3)}{(2x+3)^2}$$

Differentiate the remaining terms

$$\frac{(2x+3)(-2x) - (1-x^2)(2)}{(2x+3)^2}$$

$$\frac{(-2x^2-6x-2)}{(2x+3)^2} = \frac{-2(x^2+3x+1)}{(2x+3)^2}$$

in Mathematica ...

In[ ]:=

```
D[(1-x^2)/(2x+3), x] //Simplify
```

Out[ ]=

$$-\frac{2\left(1+3x+x^2\right)}{(3+2x)^2}$$

Example proof: derivative quotient rule

As the focus on this module is on the application of mathematics you do not need to learn theorem proofs for this module. This proof is provided for your interest only...

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0}\frac{\frac{u(x+h)}{v(x+h)}-\frac{u(x)}{v(x)}}{h} = \lim_{h\to 0}\frac{v(x)\,u(x+h)-u(x)\,v(x+h)}{h\,v(x+h)\,v(x)}$$

Add and subtract $v(x)\,u(x)$ in the numerator.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0}\frac{v(x)\,u(x+h)-v(x)\,u(x)+v(x)\,u(x)-u(x)\,v(x+h)}{h\,v(x+h)\,v(x)}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0}\frac{v(x)\,(u(x+h)-u(x))-u(x)\,(v(x+h)-v(x))}{h\,v(x+h)\,v(x)}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0}\frac{v(x)\,\frac{(u(x+h)-u(x))}{h}-u(x)\,\frac{(v(x+h)-v(x))}{h}}{v(x+h)\,v(x)}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(x)\lim_{h\to 0}\frac{(u(x+h)-u(x))}{h}-u(x)\lim_{h\to 0}\frac{(v(x+h)-v(x))}{h}}{\lim_{h\to 0}v(x+h)\,v(x)} \quad\Longrightarrow\quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(x)\,\frac{du(x)}{dx}-u(x)\,\frac{dv(x)}{dx}}{v(x)^2}$$

Differentiation Rules

Differentiation Rules for Exponential and Logarithmic Functions

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(a^x) = a^x \ln a$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Differentiation Rules

Differentiation Rules for the Basic Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

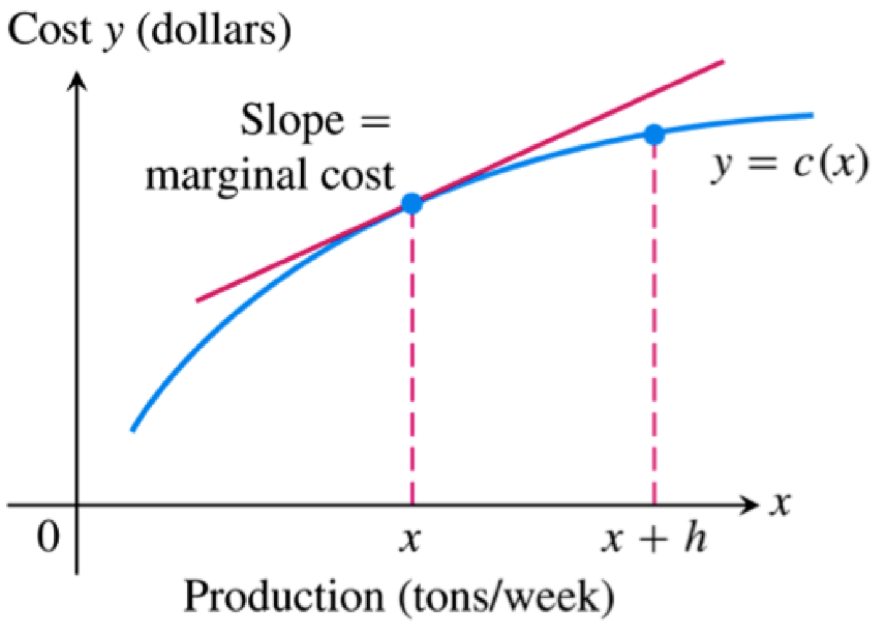
$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$$

In economics rates of changes and derivatives are referred to as **marginals**

Marginal Cost of Production

If cost of production $c(x)$ is a function of x, the number of units produced then marginal cost of production is given by $\frac{\mathrm{dc}}{\mathrm{dx}} = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h}$



Total Cost Function & Marginal Cost

Economists often represent a total cost function as the following cubic polynomial:

$c(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$

In which case the marginal cost will be given by:

$c'(x) = 3\,\alpha x^2 + 2\,\beta x + \gamma$

Example

Suppose that it costs $c(x) = x^3 - 6x^2 + 15x$ to produce x radiators.

The cost of producing one more radiator a day when 10 are produced is about $c'(10)$

$$c'(x) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195$$

The additional cost will be about £195.

The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)'(x) = f(g(x))$ is differentiable at x, and

$(f \circ g)'(x) = f'(g(x)).g'(x)$

Alternatively, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The Chain Rule: Examples

Find $\frac{dy}{dx}$ where $y = (3x^2 + 1)^2$

$y = (f \circ g)(x)$, where $f(u) = u^2$ and $u = g(x) = 3x^2 + 1$

$\frac{dy}{dx} = (f \circ g)'(x) = f'(g(x)).g'(x)$

$\frac{dy}{dx}(3x^2 + 1)^2 = \frac{d}{du}(u^2).\frac{d}{dx}(3x^2 + 1)$

$2u.6x = 2(3x^2 + 1).6x = 12x(3x^2 + 1)$

in Mathematica ...

In[ ]:=
```
D[(3x^2+1)^2, x] //Simplify
```

Out[ ]=

$12\,x\,\left(1+3\,x^{2}\right)$

Another Example

Find $\frac{dy}{dx}$ where $y = \frac{1}{(1-2x)^3}$

$y = (f \circ g)(x)$, where $f(u) = u^{-3}$ and $g(x) = 1 - 2x$

$$\frac{dy}{dx} = -3(1-2x)^{-4}(-2)$$
$$= \frac{6}{(1-2x)^4}$$

in Mathematica ...

In[ ]:=
```
D[1/(1-2x)^3, x]
```

Out[ ]=

$$\frac{6}{(1-2x)^4}$$

Implicit Differentiation

Implicit Differentiation is used to find $\frac{dy}{dx}$ when it is not possible (or convenient) to express y as a function of x, $y = f(x)$

This method uses the **Chain Rule** assuming y is a function of x, i.e. $y = f(x)$

Implicit Differentiation: Example

Find $\frac{dy}{dx}$ when $x^2 + y^2 = 25$

Differentiate both sides, using the **Chain Rule** to differentiate y^2 assuming $y = f(x)$, where $f(x)$, though unspecified is differentiable

$2x + 2(f(x))f'(x) = 0$

$2x + 2y\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{-x}{y}$

If answer required only in terms of *x* replace *y* with its equivalent *x* expression ...

$\frac{dy}{dx} = \frac{-x}{y} = \frac{-x}{\sqrt{25-x^2}}$

Implicit differentiation in Mathematica

In[ ]:=
```
ImplicitD[x^2+y^2 == 25, y, x]
```

Out[ ]=

$$-\frac{x}{y}$$

Alternatively, if the answer is needed only in terms of *x*

In[ ]:=
```
ImplicitD[x^2+y^2 == 25, y, x] /. Solve[x^2+y^2 == 25,y]
```

Out[ ]=

$$\left\{\frac{x}{\sqrt{25-x^2}}, -\frac{x}{\sqrt{25-x^2}}\right\}$$

Higher order derivatives

Repeated differentiation leads to higher order derivatives:

$$f''(x) = \frac{d^2 f}{d x^2} = \frac{d}{d x}\left(\frac{d f}{d x}\right) = D^2 f$$

$$f'''(x) = \frac{d^3 f}{d x^3} = \frac{d}{d x}\left(\frac{d^2 f}{d x^2}\right) = D^3 f$$

and so on.

Note

$$f^{(n)}(x) = \frac{d^n f}{d x^n} = D^n f$$

Higher order derivatives: Example

$f(x) = 4x^3$

$\longrightarrow f'(x) = 12x^2$

$\longrightarrow f''(x) = 24x$

$\longrightarrow f'''(x) = 24$

$\longrightarrow f^{(iv)}(x) = 0$

Finding second derivatives in Mathematica

In[ ]:=
```
f[x_]=2x^3;
```

In[ ]:=
```
f''[x]
```

Out[ ]=
```
12 x
```

alternatively ...

In[ ]:=
```
D[2x^3,{x,2}]
```

Implicit Differentiation: Second Order Example

Find $\frac{d^2 y}{dx^2}$ when $2 x^3 - 3 y^2 = 8$

differentiate both sides

$6 x^2 - 6 y \frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{x^2}{y}$

differentiate both sides again using the Quotient Rule

$\frac{d^2 y}{dx^2} = \frac{2 xy - x^2 \frac{dy}{dx}}{y^2} = \frac{2 x}{y} - \frac{x^2}{y^2} \frac{dy}{dx}$

substitute $\frac{dy}{dx} = \frac{x^2}{y}$

$\frac{d^2 y}{dx^2} = \frac{2 x}{y} - \frac{x^2}{y^2} \frac{x^2}{y} = \frac{2 x}{y} - \frac{x^4}{y^3}$

If answer required only in terms of x replace y with its equivalent x expression $y = \frac{1}{3} (2 x^3 - 8)^{\frac{1}{2}}$

$\frac{d^2 y}{dx^2} = \frac{2 x}{\frac{1}{3} (2 x^3 - 8)^{\frac{1}{2}}} - \frac{x^4}{\left(\frac{1}{3} (2 x^3 - 8)^{\frac{1}{2}}\right)^3}$

Logarithmic Differentiation

This useful technique is best illustrated with an example ...

Find $\frac{dy}{dx}$ when $y = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$, $x > 1$.

Log both sides and simplify using the properties of logarithms:

$$\ln y = \ln \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1} = \ln\left(x^2+1\right) + \ln(x+3)^{\frac{1}{2}} - \ln(x-1)$$

$$= \ln\left(x^2+1\right) + \frac{1}{2}\ln(x+3) - \ln(x-1)$$

Logarithmic Differentiation contd.

Take the derivative of both sides (i.e. using **implicit differentiation**):

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\ln\left(x^2+1\right) + \frac{1}{2}\ln(x+3) - \ln(x-1)\right)$$

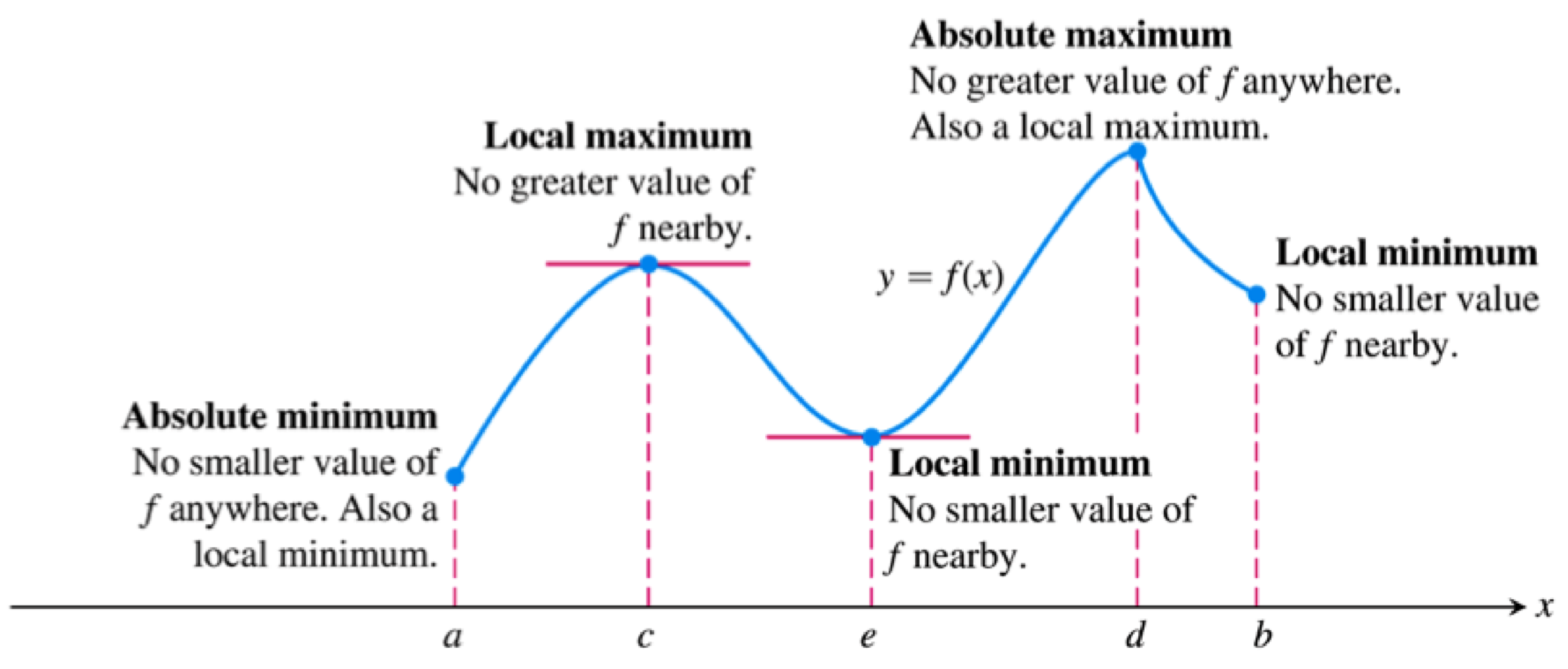
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{(x^2+1)}.2\,x + \frac{1}{2}\frac{1}{(x+3)} - \frac{1}{(x-1)}$$

$$\frac{dy}{dx} = y\left(\frac{2\,x}{(x^2+1)} + \frac{1}{(2\,x+6)} - \frac{1}{(x-1)}\right)$$

Substitute for y from the original equation:

$$\frac{dy}{dx} = \frac{(x^2+1)\,(x+3)^{\frac{1}{2}}}{x-1}\left(\frac{2\,x}{(x^2+1)} + \frac{1}{(2\,x+6)} - \frac{1}{(x-1)}\right)$$

Extreme Values of Functions



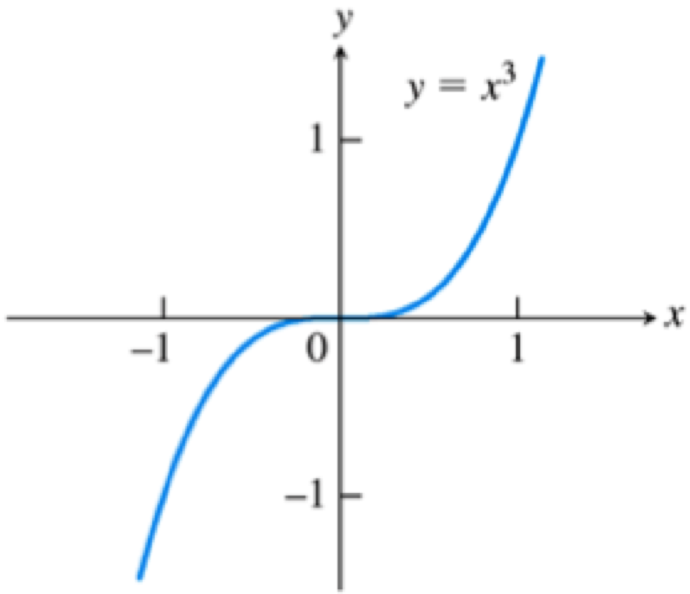
The First Derivative Theorem for Local Extreme Values

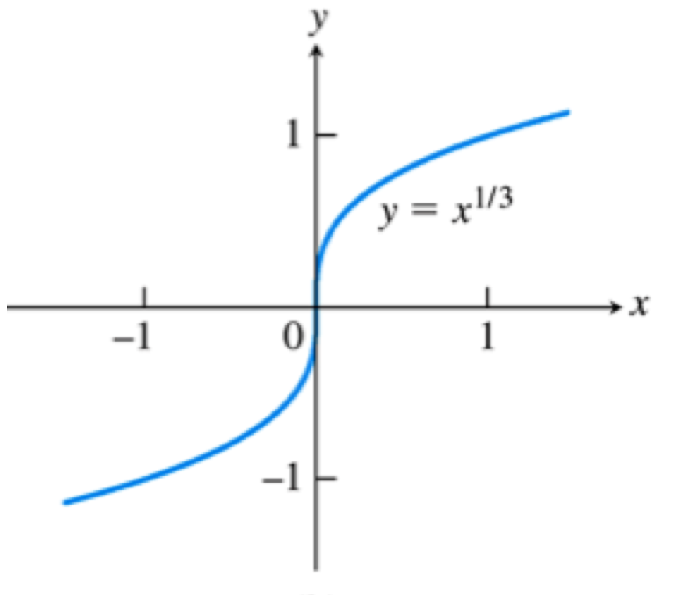
If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0$$

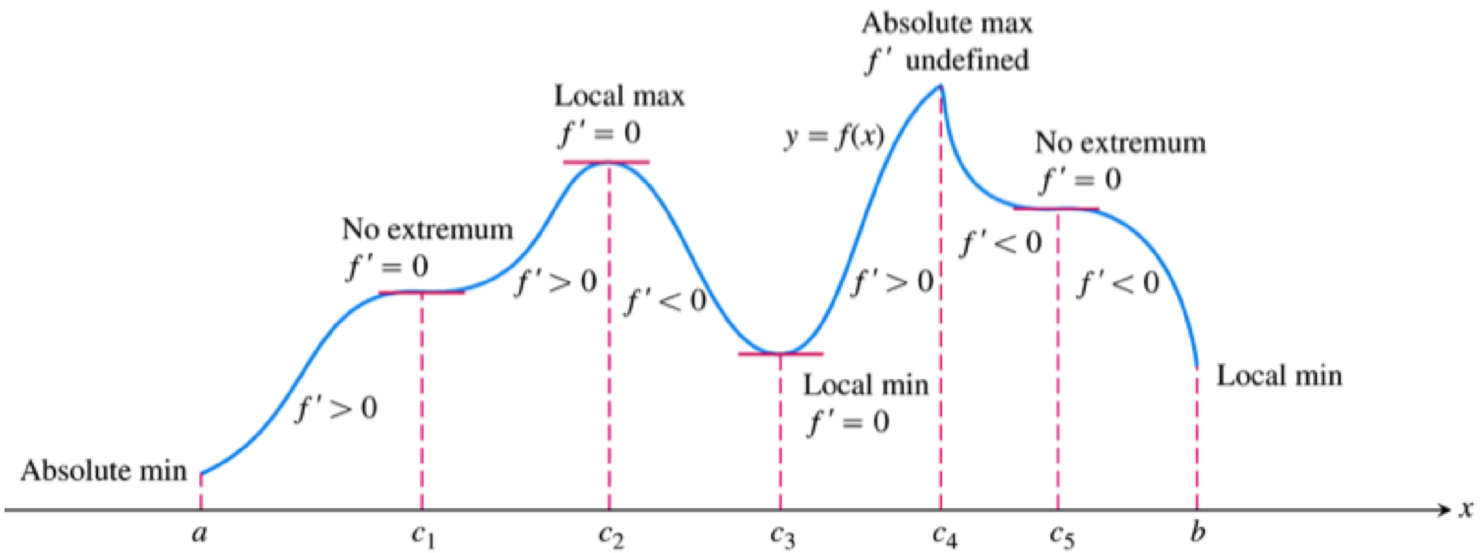
Critical Points

An interior point of the domain of a function f where **f' is zero or undefined** is a **critical point** of f.





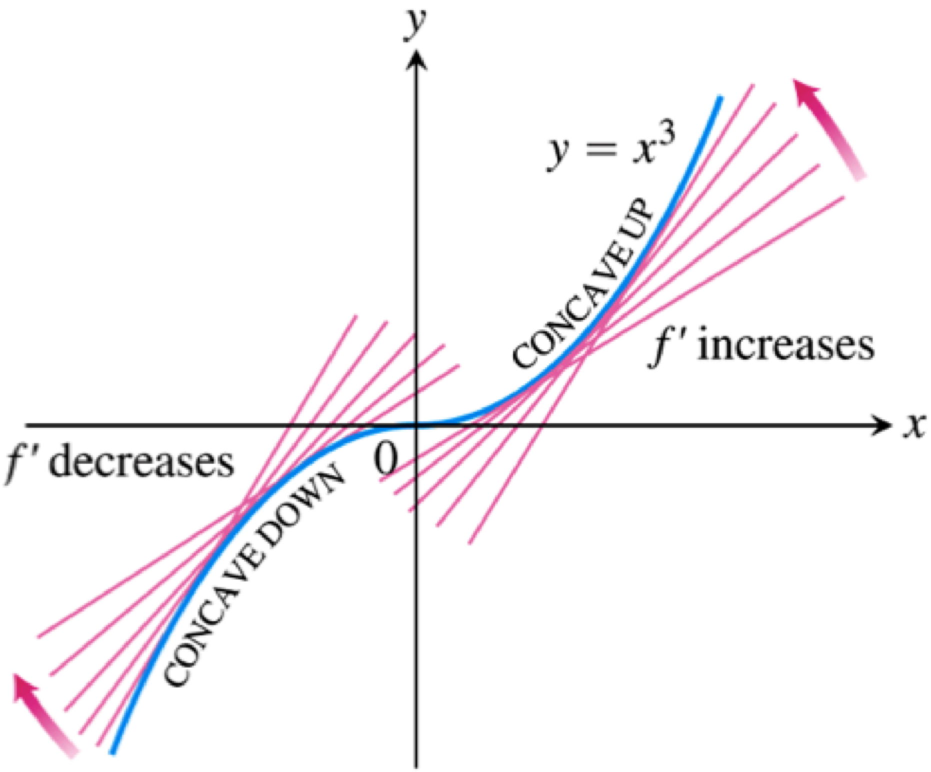
First Derivative Test for Local Extrema



Suppose that c is a **critical point** of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly c itself.

Moving across c from left to right,
f has a **local minimum** at c if $f\,'$ changes from negative to positive at c
f has a **local maximum** at c if $f\,'$ changes from positive to negative at c

Concavity



Let $y = f(x)$ be twice-differentiable on the interval I.

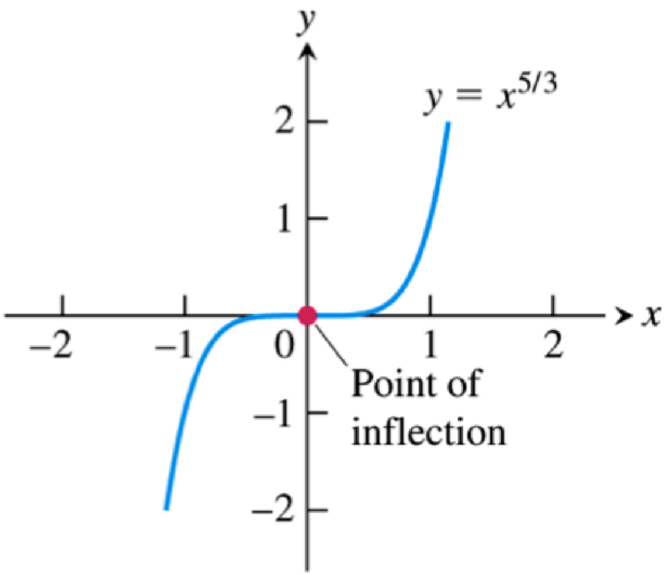
1. The graph of f over I is **concave up** if $f'' > 0$ on I.

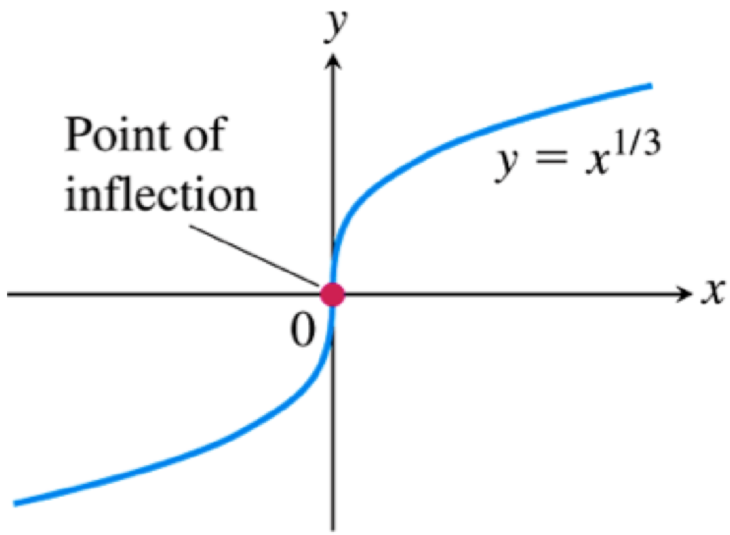
2. The graph of f over I is **concave down** if $f'' < 0$ on I.

Point of Inflection

A point where the graph of a function has a tangent line where the concavity changes is a **point of inflection**.

At a point of inflection $x = c$, either $f''(c) = 0$ or $f''(c)$ fails to exist.





Second Derivative Test for Local Extrema

The first derivative test for local extrema can alternatively be expressed in terms of the **second derivative**.

f has a **local minimum** at c if f' changes from negative to positive at c

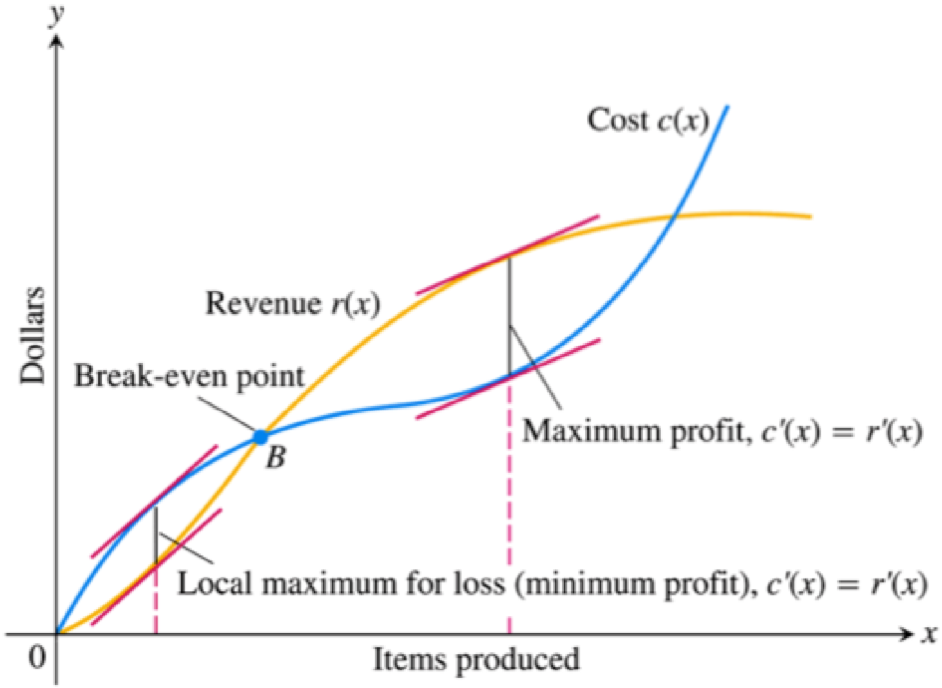
- This is equivalent to saying that at f has a **local minimum** at c when $\boldsymbol{f'(c) = 0}$ and the graph is **concave up** with $\boldsymbol{f''(c) > 0}$

f has a **local maximum** at c if f' changes from positive to negative at c

- This is equivalent to saying that at f has a **local maximum** at c when $\boldsymbol{f'(c) = 0}$ and the graph is **concave down** with $\boldsymbol{f''(c) < 0}$

Optimisation: Example from Business

A typical cost function graph starts concave down and later turns concave up.



To optimise profit $p(x) = r(x) - c(x)$ solve $p'(x) = r'(x) - c'(x) = 0$.
This is equivalent to solving $c'(x) = r'(x)$.

Optimisation: Example from Business

Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents millions of widgets produced. Is there a production level that maximises profit? If so, what is it?

Solution

To optimise profit $p(x) = r(x) - c(x)$ set $c'(x) = r'(x)$.

$3x^2 - 12x + 15 = 9$

$3x^2 - 12x + 6 = 0$

$x^2 - 4x + 2 = 0$

The two solutions are:

$x_1 = \frac{4-\sqrt{8}}{2} = 2 - \sqrt{2} \approx 0.586,\quad x_2 = 2 + \sqrt{2} \approx 3.414$

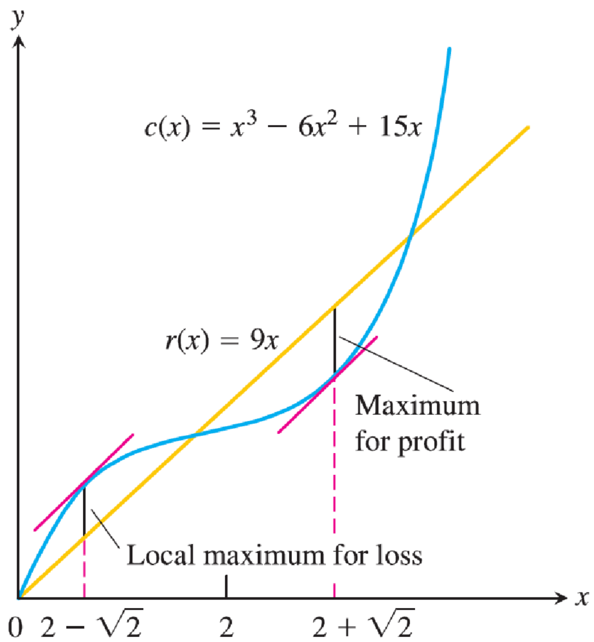
Optimisation: Example from Business contd.

Use the second derivative to identify nature of extremum.

$p''(x) = r''(x) - c''(x) = 0 - (6x - 12) = 6(2 - x)$

$p''(x_1) = 6\left(2 - \left(2 - \sqrt{2}\right)\right) > 0$ $\therefore$ profit minimum at x_1 (max loss)

$p''(x_2) = 6\left(2 - \left(2 + \sqrt{2}\right)\right) < 0$ $\therefore$ profit maximum at $x_2 \approx 3.414$



Optimisation: Example 2

Find the maximum and minimum point of the function: $y = \frac{x+3}{\sqrt{1+x^2}}$

$$\frac{dy}{dx} = \frac{-1}{2}\left(1+x^2\right)^{\frac{-3}{2}} 2x(x+3) + \left(1+x^2\right)^{\frac{-1}{2}}$$

$$= \left(1+x^2\right)^{\frac{-3}{2}}\left[-x(x+3) + \left(1+x^2\right)\right] \quad = \left(1+x^2\right)^{\frac{-3}{2}}(1-3x)$$

Solve for critical point $\frac{dy}{dx} = 0$

$$0 = \left(1+x^2\right)^{\frac{-3}{2}}(1-3x) \quad \Longrightarrow \quad x = \frac{1}{3},$$

$$\Longrightarrow \quad y = \frac{\frac{1}{3}+3}{\sqrt{1+\left(\frac{1}{3}\right)^2}} = \sqrt{10} \approx 3.16$$

Optimisation: Example 2 contd.

Find the second derivative:

$$\frac{d^2 y}{d x^2} = \frac{-3}{2} \left(1 + x^2\right)^{\frac{-5}{2}} 2 x (1 - 3 x) - 3 \left(1 + x^2\right)^{\frac{-3}{2}}$$

$$= \left(1 + x^2\right)^{\frac{-5}{2}} \left(-3 x(1 - 3 x) - 3 \left(1 + x^2\right)\right)$$

$$= \left(1 + x^2\right)^{\frac{-5}{2}} 3 \left(2 x^2 - x - 1\right)$$

$$\frac{d^2 y}{d x^2} \Big|_{x=\frac{1}{3}} = \left(1 + \left(\frac{1}{3}\right)^2\right)^{\frac{-5}{2}} 3 \left(2 \left(\frac{1}{3}\right)^2 - \frac{1}{3} - 1\right)$$

$$= \left(1 + \left(\frac{1}{3}\right)^2\right)^{\frac{-5}{2}} 3 \left(-\frac{10}{9}\right) \quad < 0 \quad \therefore \text{ maximum}$$

Solution in Mathematica

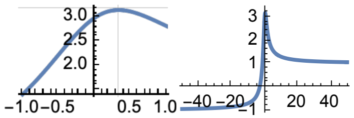
In[ ]:=

```
y[x_]:= (x+3)/Sqrt[1+x^2]
Maximize[y[x],x]
Row[{
Plot[y[x],{x,-1,1}, GridLines→{{1/3}, {y[1/3]}}],
Plot[y[x],{x,-50,50}]}]
```

Out[ ]=

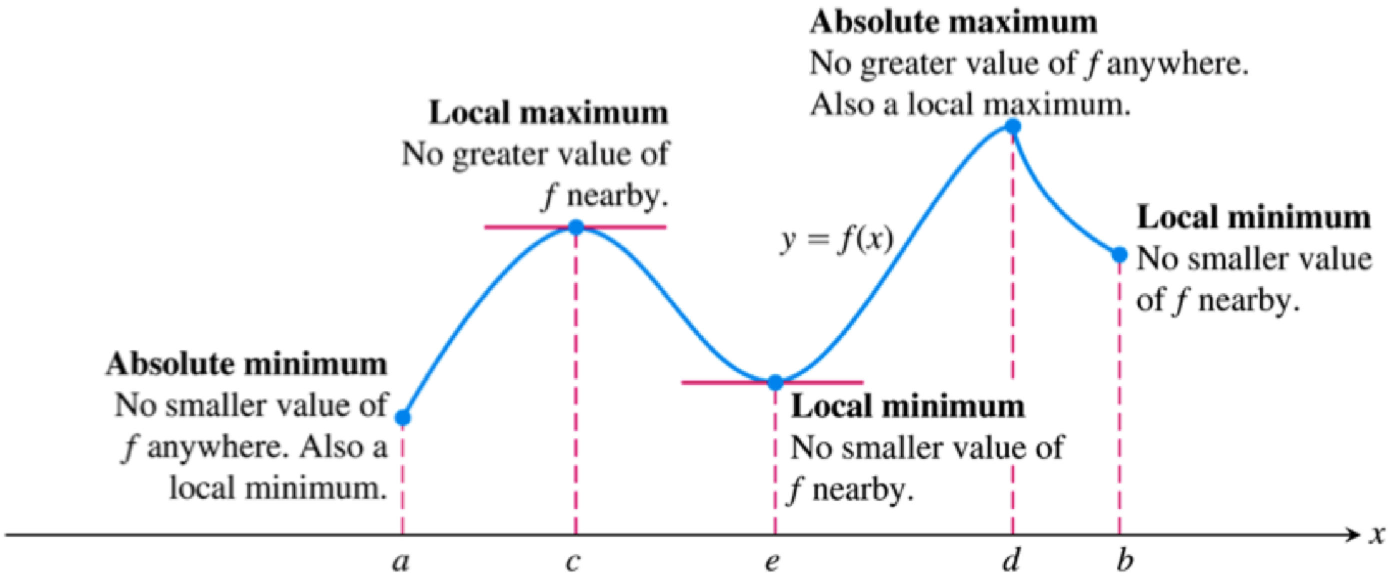
$$\left\{\sqrt{10}, \left\{x \to \frac{1}{3}\right\}\right\}$$

Out[ ]=



Finding ALL the Extreme Values of Functions

As you can see below extreme values do not only occur at points where $f'(x) = 0$.
They can also occur at boundary points and at critical points where $f'(x)$ is undefined.
We therefore need to consider all these points when optimising.



Example

Find and classify all the local and absolute extrema for the piecewise function

$$f(x) : [-5, 8] \to \mathbb{R}$$

$$x \mapsto \begin{cases} 3x + 15 & x \le 1 \\ (x-5)^2 + 2 & x > 1 \end{cases}$$

First consider critical points where $f'(x) = 0$

$$f'(x) = \begin{cases} 3 & x \le 1 \\ 2(x-5) & x > 1 \end{cases}$$

consider $x \le 1$ where $f'(x) = 3 \implies$ no critical points exist

consider $x > 1$ where $f'(x) = 2x - 10 = 0 \implies$ a critical point exists at $x = 5$
$x = 5$ is a valid point as $x > 1$ and $x \in [-5, 8]$, the domain of $f(x)$

test for minimum or maximum at $x = 5$:

$$f''(x) = \begin{cases} 0 & x \le 1 \\ 2 & x > 1 \end{cases}$$

at $x = 5$ $f''(x) = 2 > 0$ so this is a minimum with $f(5) = (5-5)^2 + 2 = 2$

Example

Next consider critical points where $f'(x)$ is undefined i.e. corners, discontinuities, and vertical tangents

For this piecewise function the only critical point of this kind is at the point the piecewise function changes i.e. at $x = 1$.

$f(x)$ is a corner at $x = 1$ rather than a discontinuity as the two different piecewise expressions of f have the same value at $x = 1$:

$f(1) = 3(1) + 15 = 18 \quad \& \, (1 - 5)^2 + 2 = 18$

As the gradient changes at this corner point from positive (3) to negative ($2(x-5)\,|_{x=1} = -8$) this is a maximum.

Example

Next consider the boundary points of $f(x)$

The domain boundary points for f were defined to be at $x = -5$, and at $x = 8$ ...
$f(x) : [-5, 8] \to \mathbb{R}$

$$f(-5) = 3x + 15 \;|_{x=-5} = 0$$

$$f(8) = (x-5)^2 + 2 \;|_{x=8} = 11$$

Finally compare the values of f at all the points we have considered and classify these point accordingly

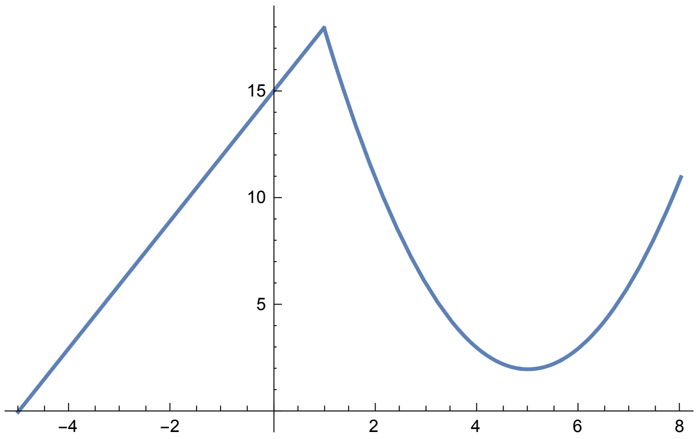
x	f(x)	extrema type
-5	0	absolute minimum
1	18	absolute maximum
5	2	local minimum
8	11	local maximum

Example: Mathematica solution

Extrema for a single variable function like $f(x)$ can be identified by plotting the function ...

In[ ]:=
```
f[x_]:=Piecewise[{{3x+15, x≤1},{(x-5)^2+2, x>1}}]
Plot[f[x],{x,-5,8}(*, AspectRatio→Full*)]
```

Out[ ]=



Maximise and Minimise return absolute extrema

In[ ]:=
```
Maximize[{f[x], -5≤x≤8}, x]
```

Out[ ]=

{18, {x → 1}}

In[ ]:=
```
Minimize[{f[x], -5≤x≤8}, x]
```

Out[ ]=

{0, {x → -5}}