Math 116 — Practice for Exam 2

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Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

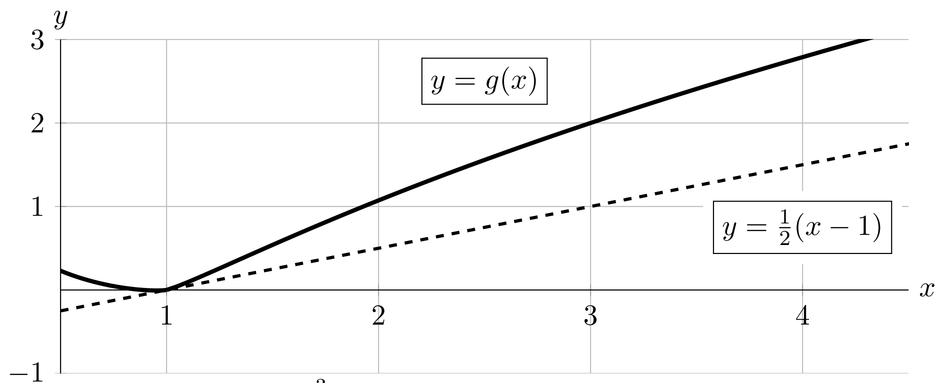
1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2020	1	10		7	
Fall 2018	2	9		9	
Winter 2016	2	9		12	
Fall 2015	3	2		8	
Total				36	

Recommended time (based on points): 35 minutes

10. [7 points] Consider functions f and g that satisfy all of the following:

- $f(x)$ is defined, positive, and continuous for all $x > 1$.
- $\lim_{x\to 1^+} f(x) = \infty$ (so $f(x)$ has a vertical asymptote at $x = 1$).
- $g(x)$ is defined and differentiable for all real numbers x, and $g'(x)$ is continuous.
- $\frac{d}{dx}\left(\frac{g(x)}{\ln x}\right) = f(x)$ for all $x > 1$.
- The tangent line to $g(x)$ at $x = 1$ is given by the equation $y = \frac{1}{2}(x-1)$. Graphs of $g(x)$ (solid) and this tangent line (dashed) are shown below.



Determine whether the integral $\int_1^3 f(x)\,dx$ converges or diverges.

- If the integral converges, circle "Converges", find its exact value, and write the exact value on the answer blank provided.
- If the integral diverges, circle "Diverges" and carefully justify your answer.

Show every step of your work carefully, and make sure that you use correct notation.

Solution: Since $f(x)$ has a vertical asymptote at $x = 1$, we write

$$\begin{aligned}
\int_1^3 f(x)\,dx &= \lim_{a\to 1^+}\int_a^3 f(x)\,dx \\
&= \lim_{a\to 1^+} \left.\frac{g(x)}{\ln x}\right|_a^3 \\
&= \lim_{a\to 1^+}\left(\frac{g(3)}{\ln 3} - \frac{g(a)}{\ln a}\right) \\
&= \frac{2}{\ln 3} - \lim_{a\to 1^+}\frac{g(a)}{\ln a} \\
&= \frac{2}{\ln 3} - \lim_{a\to 1^+}\frac{g'(a)}{1/a} \quad \text{where we applied l'Hopital's Rule} \\
&= \frac{2}{\ln 3} - \frac{1/2}{1}
\end{aligned}$$

So this improper integral converges.

Circle one:

$\boxed{\int_1^3 f(x)\,dx \textbf{ converges to } \underline{\quad \frac{2}{\ln 3} - \frac{1}{2} \quad}}$ or $\int_1^3 f(x)\,dx$ **diverges**

9. [9 points] Consider the function $f(x) = \begin{cases} \dfrac{\sin(x^2)}{x} & x \neq 0 \\ c & x = 0. \end{cases}$

a. [2 points] Find the value of c that makes the function $f(x)$ continuous at $x = 0$. Show your work carefully.

$$\lim_{x \to 0} \frac{\sin(x^2)}{x} \underset{\text{by L'Hôpital}}{=} \lim_{x \to 0} \frac{2x\cos(x^2)}{1} = 2 \cdot 0 \cdot 1 = 0$$

Answer: $c =$ 0

b. [2 points] Let b_n be the nth positive value of x such that $f(x) = 0$. Write a formula for b_n.

$$f(x) = 0 \Rightarrow \frac{\sin(x^2)}{x} = 0 \Rightarrow \sin(x^2) = 0 \Rightarrow x^2 = n\pi$$

for some integer n.

Answer: $b_n =$ $\sqrt{n\pi}$

For parts **c** and **d** below, let $a_n = \displaystyle\int_{b_{n-1}}^{b_n} f(x)\,dx$ for $n \geq 1$.

c. [2 points] Explain why a_n is an alternating sequence.

Because $\sin$ alternates from $+$ to $-$ between its zeroes:

$$a_n = \int_{\sqrt{(n-1)\pi}}^{\sqrt{n\pi}} \frac{\sin(x^2)}{x}\,dx \qquad \text{Let } w = x^2, \Rightarrow dw = 2x\,dx$$

$$= \int_{(n-1)\pi}^{n\pi} \frac{\sin(w)}{2w}\,dw$$

d. [3 points] Compute $\lim\limits_{n\to\infty} a_n$. Provide clear justification and show your work.

$$|a_n| = \left| \int_{(n-1)\pi}^{n\pi} \frac{\sin(w)}{2w}\,dw \right| \leq \int_{(n-1)\pi}^{n\pi} \left| \frac{\sin(w)}{2w} \right| dw \leq \int_{(n-1)\pi}^{n\pi} \frac{dw}{2w}$$

$$= \frac{1}{2} \ln w \Big|_{(n-1)\pi}^{n\pi} = \frac{1}{2}\left[\ln(n\pi) - \ln((n-1)\pi)\right] = \frac{1}{2}\ln\left(\frac{n}{n-1}\right).$$

as $n \to \infty$, $\dfrac{n}{n-1} \to 1$, so $|a_n| \to 0$, so $a_n \to 0$

Answer: $\lim\limits_{n\to\infty} a_n =$ 0

9. [12 points]

a. [6 points] Show that the following integral diverges. Give full evidence supporting your answer, showing all your work and indicating any theorems about improper integrals you use.

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\,dt$$

Solution: $t \geq 1 \Rightarrow \frac{1}{t} \leq 1 \Rightarrow \cos(\frac{1}{t}) \geq \cos(1)$ because the function $F(x) = \cos x$ is decreasing in the interval $[0, 1]$. Therefore,

$$\frac{\cos(\frac{1}{t})}{\sqrt{t}} \geq \frac{\cos(1)}{\sqrt{t}}$$

The improper integral

$$\int_1^\infty \frac{\cos(1)}{\sqrt{t}}\,dt = \cos(1)\int_1^\infty \frac{1}{\sqrt{t}}\,dt$$

diverges by the p-test since $p = \frac{1}{2} \leq 1$. So the integral

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\,dt$$

diverges by the comparison test (notice that $\cos(1) > 0$).

b. [6 points] Find the limit

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\,dt}{\sqrt{x}}$$

Solution: Notice that by (a), this is $\frac{\infty}{\infty}$. We use L'Hopital's rule along with the 2nd Fundamental Theorem in the numerator:

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\,dt}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{\cos(\frac{1}{x})}{\sqrt{x}}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} 2\cos\left(\frac{1}{x}\right) = 2\cos(0) = 2$$

2. [8 points] Let $f(x) = x^{2x}$. The first two derivatives of f are given below.

$$f'(x) = 2(1 + \ln x)x^{2x}$$
$$f''(x) = 2x^{2x-1} + 4(1 + \ln x)^2 x^{2x}$$

a. [4 points] Find the 2nd degree Taylor polynomial $P_2(x)$ of f centered at $x = 1$.

Solution: Using the formula for Taylor polynomials,

$$\begin{aligned} P_2(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 1 + 2(x-1) + 3(x-1)^2 \end{aligned}$$

$P_2(x) =$ $1 + 2(x-1) + 3(x-1)^2$

b. [4 points] Find

$$\lim_{x \to 1} \frac{x^{2x} - 1}{3x - 3}.$$

Clearly show your reasoning. Your answer from part **(a)** may be helpful.

Solution:

$$\begin{aligned} \lim_{x \to 1} \frac{x^{2x} - 1}{3x - 3} &= \lim_{x \to 1} \frac{1 + 2(x-1) + 3(x-1)^2 - 1}{3(x-1)} \\ &= \lim_{x \to 1} \frac{2 + 3(x-1)}{3} \\ &= \frac{2}{3} \end{aligned}$$