Math 116 — Practice for Exam 1

Generated September 30, 2020

Name: **SOLUTIONS**

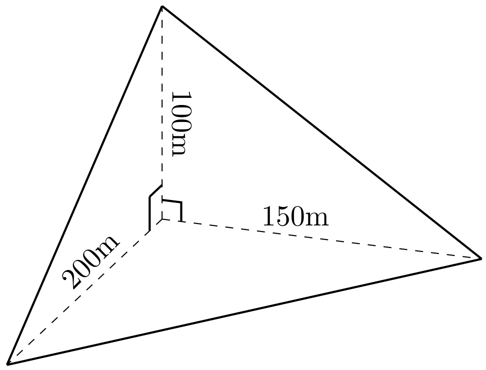
Instructor: ______________________ Section Number: ________

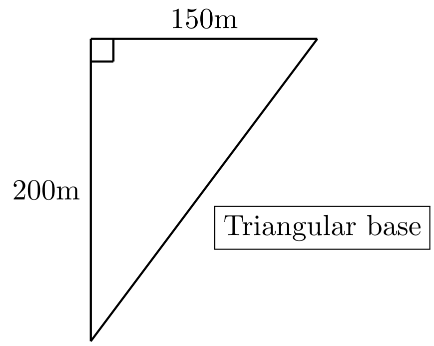
1. This exam has 12 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2017	1	9	nano pyramid	11	
Winter 2018	1	9	muck tank	9	
Fall 2017	1	4	alfalfa funnel	10	
Fall 2019	1	3	cave	9	
Winter 2018	1	7	pyramid	10	
Winter 2019	1	6	smoothies	12	
Winter 2020	1	5	heart rate	10	
Winter 2018	1	4		10	
Fall 2017	1	8		10	
Winter 2020	1	11		12	
Winter 2018	1	10		8	
Winter 2018	1	6		7	
Total				118	

Recommended time (based on points): 106 minutes

9. [11 points] Advanced beings from another planet recently realized they left a stockpile of nanotechnology here on Earth. These tiny devices are stacked in the shape of a pyramid with a triangular base that is flat on the ground. Its base is a right triangle with perpendicular sides of length 150m and 200m. Two of the other three sides are also right triangles, and all three right angles meet at one corner at the base of the pile. The fourth side is a triangle whose sides are the hypotenuses of the other three triangles. (See diagrams below.)





The density of the contents of the pile at a height of h meters above the ground is given by

$$\delta(h) = \frac{2}{\sqrt{1+h^2}} \text{ kg/m}^3.$$

For this problem, you may assume the acceleration due to gravity is $g = 9.8$ m/s^2.

a. [4 points] Write an expression in terms of h that approximates the volume (in cubic meters) of a horizontal slice of thickness Δh of the contents of the pile at a distance h meters above the ground.

Solution: The cross section of the pile h meters above the ground is a right triangle with legs of length $200-2h$ and $150-1.5h$ meters. (One can use e.g. linearity or similar triangles to find these lengths.) Therefore, the volume (in cubic meters) of the horizontal slice is approximately $\frac{1}{2}(200-2h)(150-1.5h)\Delta h$.

b. [3 points] Write, but do **not** to evaluate, an integral that gives the total mass of the pile of nanotechnology. Include units.

Solution: We mutiply the expression in (a) by the density to get an approximation for the mass of the slice and integrate that expression from 0 to 100 while replacing Δh by dh to find a total mass of

$$\int_0^{100} \frac{1}{2}(200-2h)(150-1.5h)\frac{2}{\sqrt{1+h^2}}\,dh \quad \text{kilograms.}$$

c. [4 points] The beings must return to Earth and collect the nanotech that they left behind. Suppose that the spaceship hovers 150 meters above the ground (directly above the pile) while recovering the nanotechnology. Write, but do **not** evaluate, an integral which gives the total work that must be done in order to lift all of the nanotech from the pile into the ship. Include units.

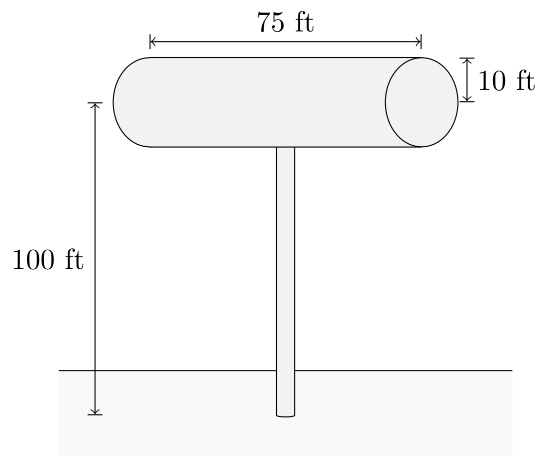
Solution: We multiply the expression in (a) by the density, acceleration due to gravity, and the distance required to lift it to a height of 150 m to estimate the work done to lift that slice up to the spaceship. Then we integrate this expression from 0 to 100 and replace Δh by dh to find that the total work done is

$$\int_0^{100} g\cdot(150-h)\cdot\frac{1}{2}(200-2h)(150-1.5h)\frac{2}{\sqrt{1+h^2}}\,dh \quad \text{Joules.}$$

9. [9 points]

De'von Baptiste is a shrewd industrialist. When energy costs are low, De'von pumps purified muck (which he gets *for free* from the city) into very tall tanks. In this way he stores cheap potential energy. Someday, when energy prices soar, Mr. Batiste will convert it all back into useful kinetic energy at a great profit.

His tanks are cylinders 75 ft long with radius 10 ft. The center of a tank is 100 ft above the ground. Purified muck has a density of 800 pounds/ft^3.



a. [3 points] What is the area, in square feet, of a cross-section parallel to the ground taken y feet above the **center** of the tank?

Solution: The cross-sections are rectangles with a length of 75 ft and a width $w(y)$ which depends on y. Using the Pythagorean Theorem we find that

$$10^2 = y^2 + (w(y)/2)^2 \longrightarrow w(y) = 2\sqrt{100-y^2} = \sqrt{400-4y^2}.$$

Hence the area of a cross-section is

$$\text{Area of cross-section} = 75w(y) = 75 \cdot 2\sqrt{100-y^2} = 150\sqrt{100-y^2}.$$

Answer: $150\sqrt{100-y^2}$

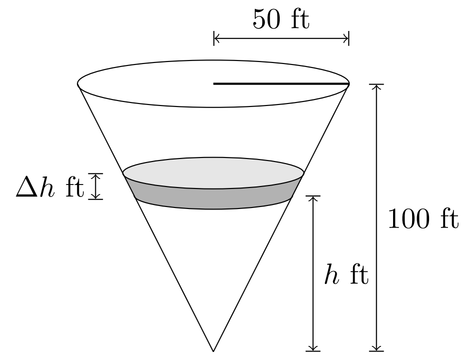
b. [6 points] Write an integral which represents the total work (in foot-pounds) required to fill one of De'von Batiste's tanks with purified muck. **Do not evaluate this integral.**

Solution: If we consider the tank after its filled, we can compute the work required to get each slice of muck y feet above the center of the tank from the ground to its height at $100+y$ ft above the ground. If $A(y) = 300\sqrt{100-y^2}$ is the area of a cross-section y feet above the center of the tank, then the total work is

$$\begin{aligned}
\text{Total work} &= \int_{-10}^{10} (\text{density})(\text{distance})(\text{slice volume}) \\
&= \int_{-10}^{10} 800(100+y)A(y)\,dy \\
&= \int_{-10}^{10} 800(100+y)150\sqrt{100-y^2}\,dy
\end{aligned}$$

4. [10 points]

A farming cooperative stores its alfalfa seed in a giant funnel. The funnel is in the shape of a right circular cone with height 100 feet and radius 50 feet at the top. A diagram of such a cone is shown in the figure on the right.



a. [4 points] Write an expression in terms of h that approximates the volume (in cubic feet) of a horizontal slice of the funnel of thickness Δh feet at a height of h feet above the bottom of the funnel. (Assume Δh is positive but very small.)

Solution: Using similar triangles, a horizontal cross section at height h will be a circle having radius r given by the proportion $\frac{50}{100} = \frac{r}{h}$. Hence $r = h/2$. The approximate volume of such a slice is then

$$V_{slice} \approx \pi r^2 \Delta h = \pi (h/2)^2 \Delta h \quad \text{ft}^3$$

b. [6 points] For parts i and ii below, assume that the funnel is full of alfalfa seed. The funnel is clogged, so the alfalfa seed must be removed from above in order to clear the clog. Assume that alfalfa seed weighs 48 pounds per cubic foot.

i. Using your answer to part (a), write an expression in terms of h that approximates the work, in foot-pounds, done in moving a horizontal slice of seed of thickness Δh that is h feet above the bottom of the funnel to the top of the funnel.

Solution: The weight of such a slice is

$$\text{Weight}_{slice} = 48 \cdot (V_{slice}) \approx 48(\pi(h/2)^2 \Delta h) \quad \text{lbs}$$

This slice needs to be moved up a distance of about $100 - h$ feet. Hence the work done on such a slice is

$$\text{Work}_{slice} \approx 48(100 - h)(\pi(h/2)^2 \Delta h) \quad \text{ft-lbs}$$

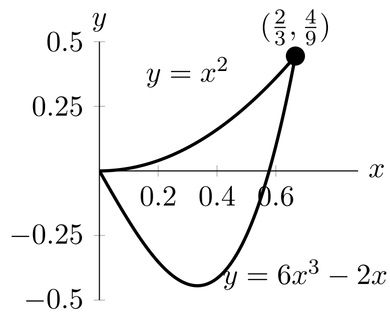
ii. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total work, in foot-pounds, that must be done to empty the tank of seed.

Solution:

$$\text{Work}_{\text{total}} = \int_0^{100} 48(100 - h)(\pi(h/2)^2)\, dh = \int_0^{100} 12\pi h^2 (100 - h)\, dh \quad \text{ft-lbs}$$

3. [9 points] Scientists are studying a cave. The inside of the cave can be modeled as a solid in the following way:

- the base is a region bounded by the curves $y = 6x^3 - 2x$ and $y = x^2$
- cross-sections perpendicular to the $x-$axis are squares
- the cave's entrance is at the origin
- x is measured in miles east of the entrance, and y is measured in miles north of the entrance



a. [6 points] The scientists want to know the volume of the cave.

(i) Write, but **do not evaluate**, an expression that gives the approximate volume, in cubic miles, of a vertical slice of the cave that is Δx miles thick and x miles east from the entrance of the cave.

Answer: Volume of slice ≈ $(x^2 - (6x^3 - 2x))^2 \Delta x$

(ii) Write, but do not evaluate, an expression involving one or more integrals that gives the total volume, in cubic miles, of the cave.

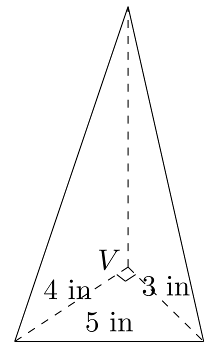
Answer: $\int_0^{2/3} (x^2 - (6x^3 - 2x))^2 dx$

b. [3 points] Salamanders live on the floor of the cave at a density that depends on the x distance from the entrance. Let $\delta(x)$ be their population density, in salamanders per square mile. Write, but do not evaluate, an expression involving one or more integrals that gives the total number of salamanders living in the cave.

Answer: $\int_0^{2/3} (x^2 - (6x^3 - 2x))\delta(x) dx$

7. [10 points]

Ms. Parth made a pyramid for her niece and nephew. The pyramid is 10 inches tall and the base has the shape of a right triangle. When the pyramid is sitting on the table it looks like the figure to the right. The three angles at vertex V are right angles. (Dashed lines are not visible from this point of view. The figure may not be drawn to scale.)



a. [6 points] Write an integral that represents the total volume of the pyramid in cubic inches and evaluate it.

Solution: Let h represent inches from the top of the pyramid. A cross-section h inches from the top of the pyramid and parallel to the table is a right triangle similar to the base. Using similar triangles we find that the cross-section has side lengths $\frac{3}{10}h$ and $\frac{4}{10}h$. Hence the integral representing its volume is

$$\begin{aligned}\text{Volume} &= \int_0^{10} \frac{1}{2}\left(\frac{3}{10}\right)\left(\frac{4}{10}\right) h^2\, dh \\ &= \frac{6}{10^2}\int_0^{10} h^2\, dh.\end{aligned}$$

Then the fundamental theorem of calculus gives

$$\text{Volume} = \frac{6}{10^2}\cdot \left.\frac{h^3}{3}\right|_0^{10} = 20\,\text{in}^3.$$

If y represents inches above the table, then the integral will be

$$\text{Volume} = \int_0^{10} \frac{1}{2}\left(\frac{3}{10}\right)\left(\frac{4}{10}\right)(10-y)^2\, dy.$$

b. [4 points] The children fail to share the pyramid, so Ms. Parth decides to cut it parallel to the table into two pieces of equal volume. How many inches H from the **top** of the pyramid should Ms. Parth cut? Round your answer to the nearest tenth of an inch.

Solution: Since the volume of the pyramid is 20 in^3, we want to find H satisfying

$$10 = \int_0^{H} \frac{1}{2}\left(\frac{3}{10}\right)\left(\frac{4}{10}\right) h^2\, dh.$$

Using the fundamental theorem of calculus this becomes

$$10 = \frac{2}{100}H^3 \longrightarrow H = \frac{10}{\sqrt[3]{2}} \approx 7.937.$$

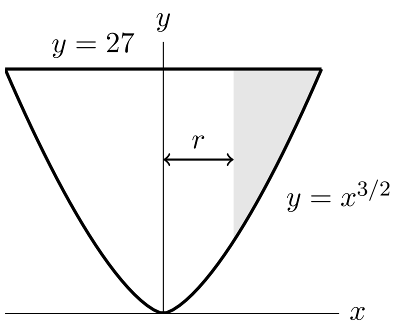
6. [12 points] Ryan Rabbitt is making a smoothie with his new electric drink mixer. Mathematically, the container of the mixer has a shape that can be modeled as the surface obtained by rotating the region in the first quadrant bounded by the curves $y = 27$ and $y = x^{3/2}$ about the y-axis, where all lengths are measured in centimeters.

a. [7 points] Write, but do not evaluate, two integrals representing the total volume, in cm^3, the mixer can hold: one with respect to x, and one with respect to y.

Answer (with respect to x): $\displaystyle\int_0^9 2\pi x \left(27 - x^{3/2}\right)\, dx$

Answer (with respect to y): $\displaystyle\int_0^{27} \pi \left(y^{2/3}\right)^2\, dy$

b. [5 points] Ryan adds 1600 cubic centimeters of liquid to his mixer. The container spins around the y-axis at a very high speed, causing the liquid to move away from the center of the container. The result is the solid made by rotating the shaded region around the y-axis in the diagram below. Note that this means that there is an empty space inside the liquid that has the shape of a cylinder.



Let r be the radius of this cylinder of empty space. Set up an equation involving one or more integrals that you would use to solve to find the value of r. **Do <u>not</u> solve for r.**

Solution:

$$\int_r^9 2\pi x \left(27 - x^{3/2}\right)\, dx = 1600,$$

or

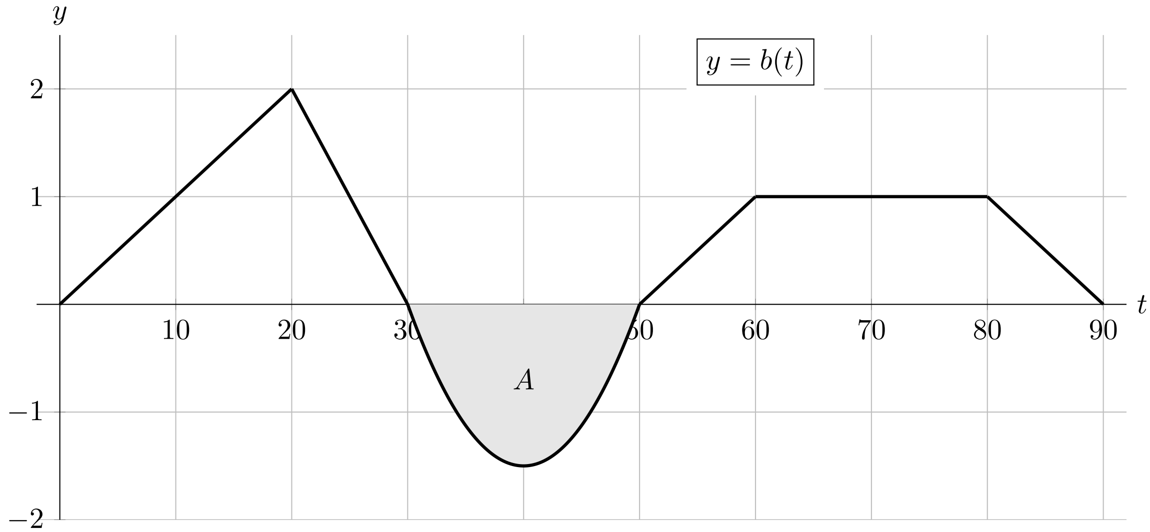
$$\int_{r^{3/2}}^{27} \pi \left(y^{2/3}\right)^2\, dy - \pi r^2 (27 - r^{3/2}) = 1600.$$

(There are other equations that would also work.)

Answer:

5. [10 points] After a long day filming a space walk scene for a movie, an actress attends a loud rock concert. Her fancy new watch monitors her stress levels during the concert. Among other data, it outputs the rate of change in her heart rate. Let $b(t)$ be the rate of change in her heart rate (in beats/minute^2) t minutes after the concert begins.

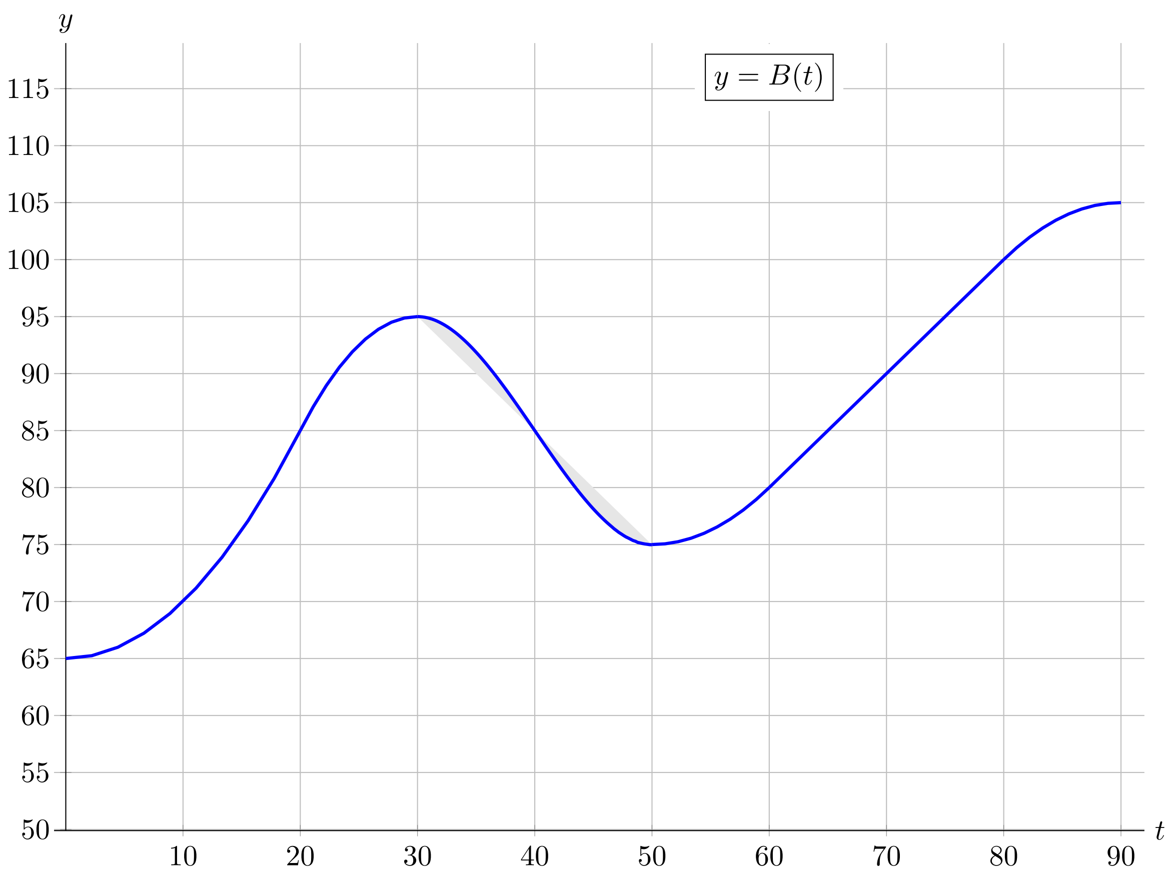
A graph of $b(t)$ is given below. The area of the shaded region A is 20.



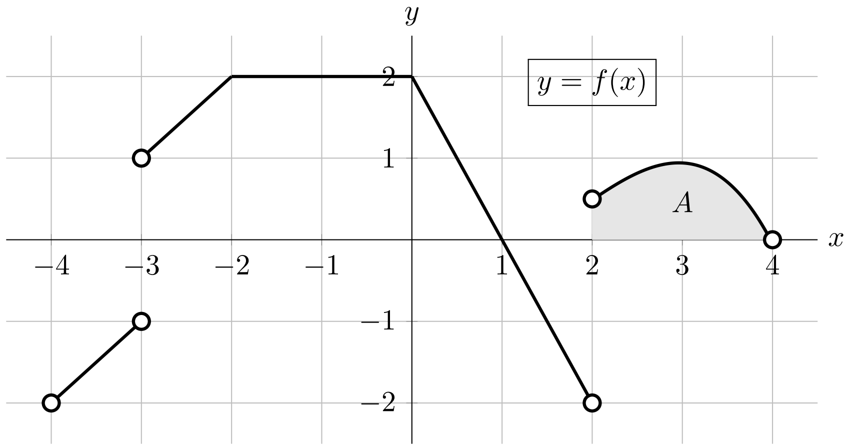
Let $B(t)$ be the actress's heart beat (in beats per minute) t minutes after the concert begins. Suppose that her heart rate 60 minutes into the concert is 80 beats/minute.

Sketch a detailed graph of $B(t)$ for $0 \le t \le 90$. Pay careful attention to where your graph is differentiable, increasing/decreasing, and concave up/concave down.

Solution:



4. [10 points] The entire graph of the function $f(x)$ is given below. Note that $f(x)$ is piecewise linear on $(-4, 2)$, and the area of the shaded region A is 1.5.



a. [2 points] Let $F(x)$ be the continuous antiderivative of $f(x)$ passing through $(2, 1)$. Circle all of the x-coordinates listed below at which $F(x)$ appears to have an inflection point.

$x = -3$

$x = 1$

$\boxed{x = 2}$

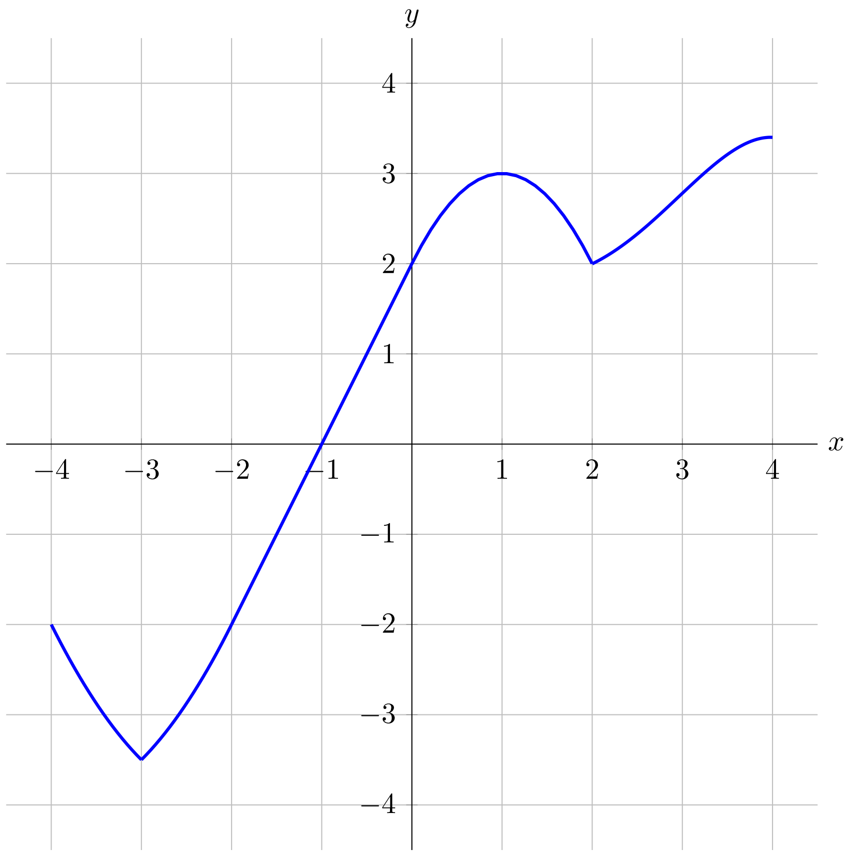
$\boxed{x = 3}$

NONE OF THESE

b. [8 points] On the axes to the right, sketch a graph of the function $G(x)$, a continuous antiderivative of $f(x)$ given on $(-3, 2)$ by

$$G(x) = \int_{-1}^{x} f(t)\, dt.$$

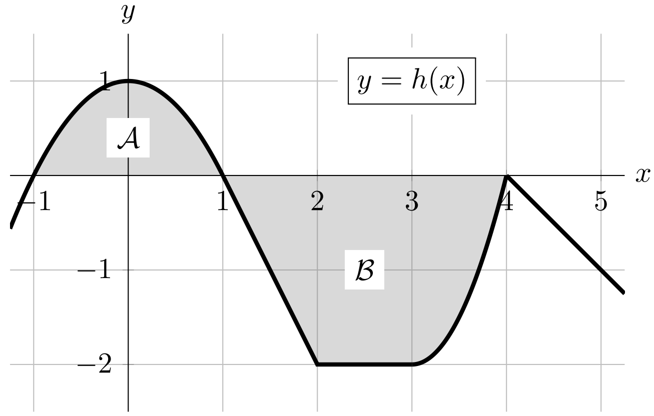
Make sure that local extrema and concavity are clear. If there are features that are difficult for you to draw, indicate these on your graph.



8. [10 points] A portion of the graph of a function h is shown below. The domain of $h(x)$ includes the interval $-1 \le x \le 5$.

Note the following:

- $h(x)$ is linear on each of the intervals $[1, 2]$, $[2, 3]$, and $[4, 5]$.
- The portion of the graph of $y = h(x)$ for $-1 < x < 1$ is symmetric across the y-axis.
- The area of shaded region $\mathcal{A}$ is $4/3$.
- The area of shaded region $\mathcal{B}$ is $13/3$.



Throughout this problem, the function H is the antiderivative of h satisfying $H(1) = 2$.

a. [2 points] For each of the following, compute the exact value. Show your work.

i. $H(-1)$

Solution:

$$H(-1) = H(1) + \int_1^{-1} h(x)\,dx = H(1) - \int_{-1}^{1} h(x)\,dx = 2 - \frac{4}{3} = \boxed{\frac{2}{3}}.$$

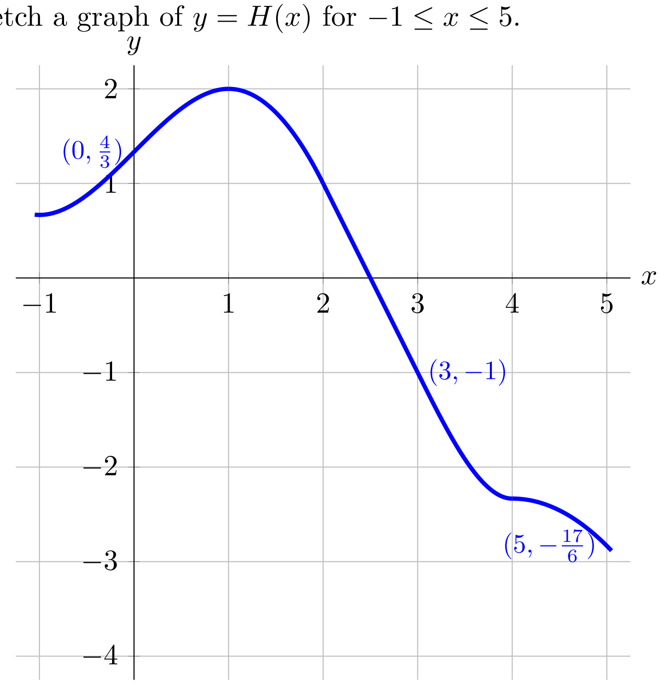
ii. $H(2)$

Solution:

$$H(2) = H(1) + \int_1^2 h(x)\,dx = 2 + (-1) = \boxed{1}.$$

b. [8 points] Use the axes below to carefully sketch a graph of $y = H(x)$ for $-1 \le x \le 5$.

- Clearly label the coordinates of the points on your graph at $x = 0$, 3, and 5.
- Be sure that local extrema and concavity are clear.
- If there are features of this function that are difficult for you to draw, indicate these on your graph.



11. [12 points] For each of the questions below, circle all of the available correct answers. Circle "NONE OF THESE" if none of the available choices are correct. No credit will be awarded for unclear markings. No justification is necessary.

a. [4 points] Suppose $f(x)$ is defined and continuous on $(-\infty, \infty)$. Which of the following MUST be true?

i. If a and b are constants with $a \neq b$, then $F(x) = \int_a^x f(t)\,dt$ and $G(x) = \int_b^x f(t)\,dt$ are different functions.

ii. $\boxed{\text{The function } F(x) = \int_a^x f(t)\,dt \text{ is an antiderivative of } f(x) \text{ with the property that } F(a) = 0}$.

iii. Every antiderivative of $f(x)$ is equal to $\int_c^x f(t)\,dt$, for some choice of constant c.

iv. $\boxed{\text{The function } J(x) = \int_{-x}^2 f(-t)\,dt \text{ is an antiderivative of } f(x)}$.

v. NONE OF THESE

b. [4 points] Suppose $g(t)$ has a positive second derivative for all values of t. Also suppose LEFT(10), RIGHT(10), TRAP(10), and MID(10) are all estimates of the integral $\int_2^5 g(t)\,dt$. Which of the following are POSSIBLE?

i. $\boxed{\int_2^5 g(t)\,dt < \text{RIGHT}(10)}$

ii. $\boxed{\int_2^5 g(t)\,dt < \text{TRAP}(10)}$

iii. $\int_2^5 g(t)\,dt < \text{MID}(10)$

iv. $\text{LEFT}(10) = \text{TRAP}(10) + 100$ and $\text{RIGHT}(10) = \text{TRAP}(10) + 50$

v. $\text{LEFT}(10) = \text{MID}(10) - 100$ and $\text{RIGHT}(10) = \text{MID}(10) - 50$

vi. $\text{LEFT}(10) = \text{MID}(10) - 100$ and $\text{RIGHT}(10) = \text{MID}(10) + 50$

vii. $\boxed{\text{LEFT}(10) = \text{MID}(10) + 100 \text{ and } \text{RIGHT}(10) = \text{MID}(10) - 50}$

viii. NONE OF THESE

c. [4 points] Which of the following are antiderivatives of $h(x) = e^x \cos x$?

i. $\boxed{J(x) = \int_1^{e^x} \cos(\ln t)\,dt}$

ii. $\boxed{K(x) = \frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x + 4}$

iii. $\boxed{L(x) = \int_0^x e^t \cos t\,dt}$

iv. $\boxed{M(x) = \int_0^{x+2\pi} e^{t-2\pi} \cos t\,dt}$

v. NONE OF THESE

10. [8 points] For each of the questions below, circle all of the available correct answers. Circle NONE OF THESE if none of the available choices are correct. You must circle at least one choice to receive any credit. No credit will be awarded for unclear markings. No justification is necessary.

a. [3 points] Which of the following integrals are equal to $\lim_{n\to\infty}\sum_{k=0}^{n-1}\left(3+\frac{2k}{n}\right)^4\cdot\frac{2}{n}$?

i. $\int (1+kx)^4 x\,dx$

ii. $\boxed{\int_3^5 x^4\,dx}$

iii. $\int_0^{n-1}\left(3+\frac{2x}{n}\right)^4\cdot\frac{2}{n}\,dx$

iv. $\int_2^3 x^4\,dx$

v. $\boxed{\int_0^2 (3+x)^4\,dx}$

vi. NONE OF THESE

b. [3 points] Which of the following expressions give the volume of the solid made by rotating around the y-axis the region bounded by $y=\sin(x)$, $y=0$, and $x=\frac{\pi}{2}$?

i. $\int_0^{\pi/2}\pi\left(\frac{\pi}{2}-\sin(x)\right)^2\,dx$

ii. $\int_0^{\pi/2}\pi\left(\left(\frac{\pi}{2}\right)^2-\sin^2(x)\right)\,dx$

iii. $\boxed{\int_0^{\pi/2}2\pi x\sin(x)\,dx}$

iv. $\int_0^{\pi/2}\pi\sin^2(x)\,dx$

v. $\int_0^1\pi\left(\frac{\pi}{2}-\arcsin(y)\right)^2\,dy$

vi. $\boxed{\int_0^1\pi\left(\left(\frac{\pi}{2}\right)^2-(\arcsin(y))^2\right)\,dy}$

vii. $\int_0^1 2\pi y\arcsin(y)\,dy$

viii. NONE OF THESE

c. [2 points] Let $f(x)$ be a function that is increasing on $(-3,3)$, concave up on $(0,3)$, and has a point of inflection at $x=0$. Consider the approximations for $\int_{-2}^2 f(x)\,dx$ given by LEFT(n) and TRAP(n). Which of the following statements **must** be true?

i. TRAP$(n) < \int_{-2}^2 f(x)\,dx$

ii. TRAP$(n) > \int_{-2}^2 f(x)\,dx$

iii. TRAP(n) is neither an overestimate nor an underestimate for $\int_{-2}^2 f(x)\,dx$.

iv. $\boxed{\text{LEFT}(n) < \int_{-2}^2 f(x)\,dx}$

v. LEFT$(n) > \int_{-2}^2 f(x)\,dx$

vi. LEFT$(n) = 0$

vii. NONE OF THESE

6. [7 points] For each of the questions below, circle **all** of the available correct answers. You must circle at least one choice to receive any credit. No credit will be awarded for unclear markings. No justification is necessary.

a. [3 points] Which $F(x)$ are antiderivatives of $f(x) = e^{x^2}$ with $F(3) = 5$ for $x > 0$?
Note: due to a typo in the original exam (corrected here), a student's answer to option IV did not impact their score.

I. $F(x) = \int_0^{x^2} e^u \, du + 5$ II. $F(x) = \int_3^x 5e^{u^2} \, du$

III. $F(x) = \frac{1}{x^2} e^{x^2} + 5$ IV. $\boxed{F(x) = \int_{x^2}^9 -\frac{1}{2\sqrt{u}} e^u \, du + 5}$

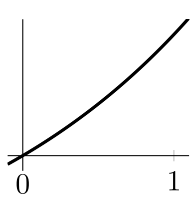
V. $\boxed{F(x) = \int_3^x e^{u^2} \, du + 5}$ VI. $F(x) = \frac{e^{x^2}}{2x} - \frac{e^9}{6} + 5$

b. [2 points] Suppose $f(x)$ is an odd function. Which values of b make the following equation true?

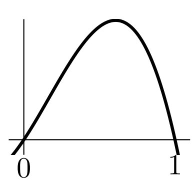
$$\int_{-\pi}^{b} \sin(f(x)) \, dx = 0$$

I. $\boxed{b = -\pi}$ II. $b = 0$ III. $\boxed{b = \pi}$ IV. $b = \frac{3\pi}{2}$ V. $b = 2\pi$

c. [2 points] Which of the following could be the graph of $f(x) = \int_x^{x^3} e^{\sqrt[3]{u}} \, du$?



I.



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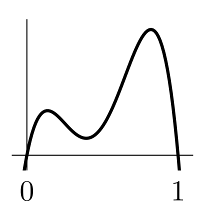

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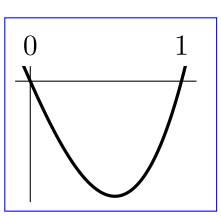

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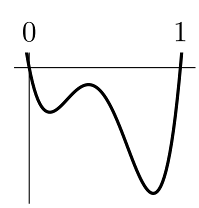

V.