Math 116 — Practice for Exam 1

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Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 3 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2017	1	1		12	
Fall 2018	1	1		12	
Fall 2016	1	7		6	
Total				30	

Recommended time (based on points): 27 minutes

1. [12 points] Suppose that f is a twice differentiable function with continuous second derivative. (That is, both f and f' are differentiable, and f'' is continuous.) The following table gives some values of f and f'.

x	0	1	2	3	4	5	6	e^3
$f(x)$	7	5	-1	0	11	-3	2	9
$f'(x)$	3	-4	-2	4	-5	0	-1	2

In parts (a) through (c) below, calculate the exact numerical value of the integral.
Write "NOT ENOUGH INFO" if there is not enough information to find the exact value.
Be sure to show your work clearly. No partial credit will be given for estimates.

a. [4 points] $\displaystyle\int_1^{e^3} \frac{f'(\ln x)}{x}\,dx$

Solution: The substitution $w = \ln(x)$ gives $dw = \frac{dx}{x}$ and

$$\int_1^{e^3} \frac{f'(\ln x)}{x}\,dx = \int_0^3 f'(w)\,dw = f(3) - f(0) = 0 - 7 = -7.$$

b. [4 points] $\displaystyle\int_0^4 x f''(x)\,dx$

Solution: Integration by parts with $u = x$ and $dv = f''(x)\,dx$ gives

$$\begin{aligned}\int_0^4 x f''(x)\,dx &= x f'(x)\Big|_0^4 - \int_0^4 f'(x)\,dx \\ &= \big(4 \cdot f'(4) - 0 \cdot f'(0)\big) - (f(4) - f(0)) \\ &= -20 - (11 - 7) = -24.\end{aligned}$$

c. [4 points] $\displaystyle\int_2^6 f'(x)\,[f(x)]^2\,dx$

Solution:
One Approach: substitution with $w = f(x)$ so $dw = f'(x)\,dx$

$$\int_2^6 f'(x)\,[f(x)]^2\,dx = \int_{f(2)}^{f(6)} w^2\,dw = \frac{w^3}{3}\Big|_{-1}^{2} = \frac{8}{3} - \frac{-1}{3} = 3.$$

Another Approach: integration by parts with $u = f(x)^2$ and $dv = f'(x)\,dx$

$$\int_2^6 f'(x)\,[f(x)]^2\,dx = [f(x)]^3\big|_2^6 - 2\int_2^6 f'(x)[f(x)]^2\,dx.$$

Moving the last term to the left hand side and dividing both sides of the resulting equation by 3 gives

$$\int_2^6 f'(x)\,[f(x)]^2\,dx = [f(x)]^3\big|_2^6 = \frac{8-(-1)}{3} = 3.$$

1. [12 points] The table below gives several values of a differentiable function f such that f' is also differentiable and f'' is continuous.

x	-3	-2	-1	0	1	2	3
$f(x)$	14	20	4	11	24	5	8
$f'(x)$	3	-4	-6	2	5	-3	4

For each of the following, calculate the exact numerical value of the integral. If it is not possible to do so based on the information provided, write "NOT POSSIBLE" and clearly indicate why it is not possible. Show your work.
Note that no variables or function names (such as f, f', or f'') should appear in your answers.

a. [3 points] $\int_0^1 f'(2x)\,dx$

let $w = 2x$
$dw = 2dx$
$x = 0 \Rightarrow w = 0$
$x = 1 \Rightarrow w = 2$

$= \int_0^2 f'(w) \cdot \frac{dw}{2}$

$= \frac{1}{2}\left[f(w)\right]_0^2 = \frac{1}{2}\left[f(2) - f(0)\right] = \frac{1}{2}\left[5 - 11\right]$

Answer: -3

b. [3 points] $\int_2^3 sf''(s)\,ds$ $= \int_2^3 uv' = uv\Big|_2^3 - \int_2^3 u'v = sf'(s)\Big|_2^3 - \int_2^3 (1)(f'(s))\,ds$

Parts:
Let $u = s$ $v' = f''(s)$
$u' = 1$ $v = f'(s)$

$= sf'(s) - f(s)\Big|_2^3 = \left[3f'(3) - f(3)\right] - \left[2f'(2) - f(2)\right]$

$= \left[3(4) - 8\right] - \left[2(-3) - 5\right] = \left[12 - 8\right] - \left[-6 - 5\right] = 4 + 11$

Answer: 15

c. [3 points] $\int_{-2}^{-1} q \cdot \left[\frac{d}{dq}\left(f'(q)e^{f(q)}\right)\right] dq$ $= \int_{-2}^{-1} uv' = \overset{①}{uv\Big|_{-2}^{-1}} - \overset{②}{\int_{-2}^{-1} u'v}$

Parts:
Let $u = q$ $v' = \frac{d}{dq}\left(f'(q)e^{f(q)}\right)$
$u' = 1$ $v = f'(q)e^{f(q)}$

① $uv\Big|_{-2}^{-1} = qf'(q)e^{f(q)}\Big|_{-2}^{-1} = (-1)f'(-1)e^{f(-1)} - (-2)f'(-2)e^{f(-2)}$
$= (-1)(-6)e^4 - (-2)(-4)e^{20} = 6e^4 - 8e^{20}$

② $\int_{-2}^{-1} u'v = \int_{-2}^{-1} (1)f'(q)e^{f(q)}\,dq$

let $w = f(q)$
$dw = f'(q)\,dq$
$q = -2 \Rightarrow w = f(-2) = 20$
$q = -1 \Rightarrow w = f(-1) = 4$

$= \int_{20}^{4} e^w\,dw = e^w\Big|_{20}^{4} = e^4 - e^{20}$

So we have ① − ② $= (6e^4 - 8e^{20}) - (e^4 - e^{20})$

Answer: $5e^4 - 7e^{20}$

d. [3 points] $\int_{-1}^{1} f'(y) \cdot f''(f(y))\,dy$

Let $w = f(y)$
$dw = f'(y)\,dy$
$y = -1 \Rightarrow w = f(-1) = 4$
$y = 1 \Rightarrow w = f(1) = 24$

$= \int_4^{24} f''(w)\,dw = f'(w)\Big|_4^{24} = f'(24) - f'(4).$

But we don't know $f'(24)$ and $f'(4)$, so we can't go any further.

Answer: NOT POSSIBLE

7. [6 points] Suppose that g is a continuous function, and define another function G by

$$G(x) = \int_0^x g(t)\,dt.$$

Given that $\int_0^7 g(x)\,dx = 5$, compute

$$\int_0^7 g(x)(G(x))^2\,dx.$$

Show each step of your computation.

Solution: Substitution gives

$$\int_0^7 g(x)(G(x))^2\,dx = \int_{G(0)}^{G(7)} u^2\,du = \left.\frac{u^3}{3}\right|_0^5 = \frac{125}{3}.$$

Alternatively, integrate by parts to obtain

$$\int_0^7 g(x)(G(x))^2\,dx = (G(x))^3\Big|_0^7 - 2\int_0^7 g(x)(G(x))^2\,dx,$$

which after rearranging gives

$$\int_0^7 g(x)(G(x))^2\,dx = \frac{1}{3}\left((G(x))^3\Big|_0^7\right) = \frac{125}{3}.$$