# Math 116 — Practice for Exam 1

Generated September 23, 2020

Name: **SOLUTIONS**

Instructor: ______________________________ Section Number: __________

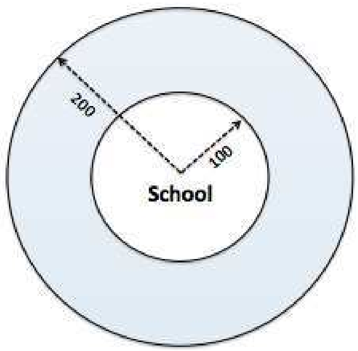
1. This exam has 6 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

| Semester | Exam | Problem | Name | Points | Score |
|---|---|---|---|---|---|
| Winter 2013 | 1 | 4 | property values | 8 | |
| Winter 2019 | 1 | 7 | carrot | 12 | |
| Fall 2018 | 3 | 6 | paperweight | 6 | |
| Winter 2016 | 1 | 7 | jewelry | 10 | |
| Fall 2019 | 2 | 9 | pond | 9 | |
| Fall 2019 | 2 | 6 | bucket | 8 | |
| Total | | | | 53 | |

**Recommended time (based on points): 50 minutes**

**4.** [8 points] In a small town, property values close to the school are determined primarily by how far the land is from the school. The function $\delta(r) = \dfrac{1}{ar^2+1}$ gives the value of the land (in thousands of dollars per $m^2$), where $r$ is the distance (in meters) from the school and $a$ is a positive contant.

**a.** [5 points] Find a formula containing a definite integral that computes the value of the land that lies in the annulus of inner radius 100 m and outer radius 200 m (figure shown below).




*Solution:* A thin annular slice has area $A_{slice} \approx 2\pi r\Delta r$, and so has an approximate value $V_{slice} \approx \frac{1}{ar^2+1}2\pi r\Delta r$. Summing these slice up and taking the limit as $\Delta r \to 0$, we get the integral

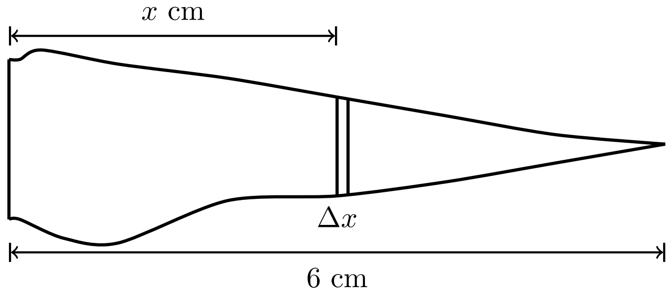
$$\text{Value of the land in the annulus} = \lim_{\Delta r\to 0}\sum \frac{1}{ar^2+1}2\pi r\Delta r = \int_{100}^{200} 2\pi r\frac{1}{ar^2+1}dr.$$

**b.** [3 points] Calculate the exact value of the land that lies in the annulus of inner radius 100 m and outer radius 200 m. Your answer should contain $a$. Show all your work.

*Solution:* Let $u = ar^2+1$. Then $du = 2ardr$. Our integral becomes

$$\int_{a100^2+1}^{a200^2+1} 2\pi r\frac{1}{u}\frac{du}{2ar} = \frac{\pi}{a}\int_{10{,}000a+1}^{40{,}000a+1}\frac{du}{u} = \frac{\pi}{a}\left(\ln(40,000a+1) - \ln(10,000a+1)\right).$$

**7**. [12 points] Hannah Haire has a carrot that is 6 cm long. Lying on its side, it looks like the diagram below, and cross-sections perpendicular to the $x$-axis are circles. The density of the carrot also varies with $x$.




Given a distance $x$ cm from the large end of the carrot, let $f(x)$ model the diameter, in cm, of the circular cross-section and $\delta(x)$ the density of the carrot, in g/cm$^3$.

**a**. [4 points] Write an expression that gives the approximate mass, in grams, of a slice of the carrot that is $\Delta x$ cm thick and $x$ cm from the large end of the carrot. (Assume here that $\Delta x$ is small but positive.) Your expression should not involve any integrals, but may include $f(x)$ and $\delta(x)$.

**Answer:** $\pi\left(\frac{f(x)}{2}\right)^2 \delta(x)\Delta x$

**b**. [3 points] Write an expression involving one or more integrals that gives the total mass of the carrot. Your answer may include $f(x)$ and $\delta(x)$.

**Answer:** $\pi\int_0^6 \left(\frac{f(x)}{2}\right)^2 \delta(x)\,dx$

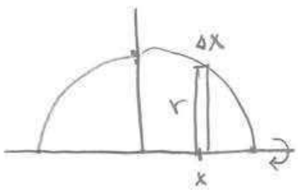
**c**. [5 points] Below is a table with some values of $f(x)$ and $\delta(x)$. Use MID(3) to estimate the mass, in grams, of the carrot. Write out every term in your sum.

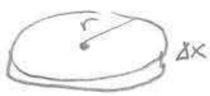
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 3.4 | 3.8 | 2.6 | 2.1 | 1.4 | 0.6 | 0 |
| $\delta(x)$ | 1.54 | 1.52 | 1.48 | 1.44 | 1.42 | 1.39 | 1.32 |

**Answer:** $2\pi((\frac{3.8}{2})^2\cdot 1.52 + (\frac{2.1}{2})^2 1.44 + (\frac{0.6}{2})^2 1.39) \approx 45.3$

**6.** [6 points] Consider the curve $y = \sqrt{1-x^2}$. Suppose a paperweight is formed by rotating this curve around the $x$-axis. This paperweight has a density given by $\rho(x) = 2 + \cos(x)$ g/cm$^3$. The units on both axes are centimeters (cm).

**a.** [3 points] Write an expression that gives the approximate mass, in grams, of a slice of the paperweight taken perpendicular to the $x$-axis at coordinate $x$ with thickness $\Delta x$. (Assume that $\Delta x$ is small but positive.) Your expression should not involve any integrals.




radius of slice $= \sqrt{1-x^2}$ cm

Volume of slice $= \pi r^2 \Delta x = \pi(1-x^2)\Delta x$ cm$^3$

mass of slice $= \rho(x) \cdot$ vol

**Answer:** Mass of slice $\approx$ $(2+\cos x)\,\pi(1-x^2)\,\Delta x$

**b.** [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the mass, in grams, of the paperweight.

**Answer:** Mass = $\int_{-1}^{1} (2+\cos x)\,\pi(1-x^2)\,dx$

**7.** [6 points] Determine whether the following series converges absolutely, converges conditionally, or diverges, and give a complete argument justifying your answer. In particular, be sure to show all work and include any convergence tests used.

Correction made at time of exam

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

*Circle one:* CONVERGES ABSOLUTELY $\boxed{\text{CONVERGES CONDITIONALLY}}$ DIVERGES

**Justification:**

- terms alternate in sign
- |terms| decreases
- |terms| $\to 0$.

So converges by the alternating series test.

But $\frac{\ln(n)}{n} \geq \frac{1}{n}$ eventually, and $\sum \frac{1}{n}$ diverges by the p-test ($p=1$). So $\sum \frac{\ln(n)}{n}$ diverges by comparison

**7**. [10 points] Maize and Blue Jewelry Company is trying to decide on a design for their signature aMaize-ing bracelet. There are two possible designs: type $W$ and type $J$. The company has done research and the two bracelet designs are equally pleasing to customers. The design for both rings starts with the function $C(x) = \cos\left(\frac{\pi}{2}x\right)$ where all units are in millimeters. Let $R$ be the region enclosed by the graph of $C(x)$ and the graph of $-C(x)$ for $-1 \leq x \leq 1$.

**a**. [5 points] The type $W$ bracelet is in the shape of the solid formed by rotating $R$ around the line $x = 50$. Write an integral that gives the volume of the type $W$ bracelet. Include **units**.

*Solution:* The volume of the type $W$ bracelet, in mm$^3$, using the shell method, is

$$\int_{-1}^{1} 2\pi(50 - x) \cdot 2C(x)\, dx.$$

**b**. [5 points] The type $J$ bracelet is in the shape of the solid formed by rotating $R$ around the line $y = -50$. Write an integral that gives the volume of the type $J$ bracelet. Include **units**.

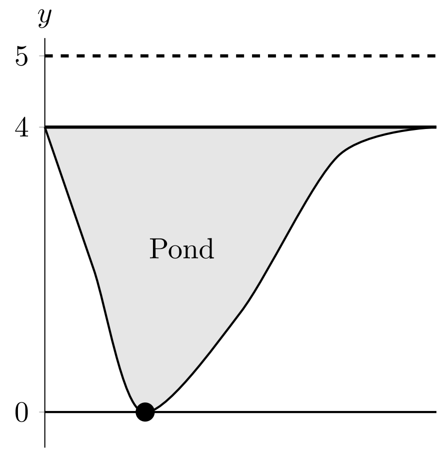
*Solution:* The volume of the type $J$ bracelet, in mm$^3$, using the washer method, is

$$\int_{-1}^{1} \pi(50 + C(x))^2 - \pi(50 - C(x))^2\, dx.$$

**9.** [9 points]

A small pond has murky water, and needs to be completely drained.

- A side view of the pond looks like the diagram at right.
- $y$ measures the distance, in meters, above the bottom of the pond.
- The surface of the pond is at $y = 4$.
- The water must be pumped to a height **1 meter above the surface.**
- The cross-sections perpendicular to the $y$-axis are **circles.**
- The **radius** of the circular cross-section $y$ meters above the bottom of the pond is $r(y)$ meters.




- The **density** of the murky water varies with $y$, and is given by $Q(y)$ kg/m$^3$.
- Note that the domain for both $r$ and $Q$ is $[0, 4]$.
- You may assume that acceleration due to gravity is $g = 9.8$ m/s$^2$.

Note that your answers below may include $r(y)$ and $Q(y)$.

**a.** [3 points] Write an expression that gives the approximate mass, in kilograms, of a slice of the murky water that is $\Delta y$ m thick and at a height of $y$ meters. Your expression should not involve any integrals.

**Answer:** Mass of slice ≈ $\pi r(y)^2 Q(y)\Delta y$ **kg**

**b.** [3 points] Write an expression in terms of $y$ that approximates the work, in joules, done in pumping a horizontal slice of murky water of thickness $\Delta y$ at a height of $y$ meters to 1 meter above the surface of the pond. Your expression should not involve any integrals.

**Answer:** Work ≈ $9.8(5-y)\pi r(y)^2 Q(y)\Delta y$ **Joules**

**c.** [3 points] Write an expression involving one or more integrals that gives the total work, in joules, to completely drain the pond by pumping all the water to 1 meter above the pond.

**Answer:** $\int_0^4 9.8(5-y)\pi r(y)^2 Q(y)\,dy$ **Joules**

**6.** [8 points] Derivative Girl lifts a bucket of water at a constant velocity from the ground up to a platform 50 meters above the ground. The bucket and water start at a total mass of 20 kg, but while it is being lifted, a total of 3 kg of water drips out at a steady rate through a hole in the bottom of the bucket.

For this problem, you may assume that acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.

**a.** [2 points] Give an expression giving the mass of the bucket and water when the bucket is $h$ meters above ground. Include units.

**Answer:** Mass of water = $20 - \frac{3}{50}h$ **kg**

**b.** [3 points] Suppose $\Delta h$ is small. Write an expression (not involving integrals) that approximates the work required to lift the bucket from a height of $h$ meters above the ground to a height of $h + \Delta h$ meters above the ground. Include units.

**Answer:** Work $\approx$ $9.8(20 - \frac{3}{50}h)\Delta h$ **joules**

**c.** [3 points] Write, but do not evaluate, an integral that gives the work required to lift the bucket from the ground to the platform. Include units.

**Answer:** $\int_0^{50} 9.8(20 - \frac{3}{50}h)\, dh$ **joules**