Statistical Inference Assignment

i. Location Family

- **Definition**: Let $f(.\,; \theta, \theta \in R)$ be a family of densities indexed by a parameter θ. The parameter θ is defined to be a **location parameter** if and only $f(x; \theta)$ can be written as a function of $x - \theta$. $f(x; \theta) = h(x - \theta)$ for some function $h(.\,)$ Equivalently, θ is a location parameter for the density $f(x; \theta)$ of a random variable X if and only if the distribution of $X - \theta$ does not depend on θ
- **Importance**: To determine the location where the distrbution is centered (population mean) along horizontal axis. If θ is increased, the graph of the probability function shifts to the right on the horizontal axis and vice versa.
- **Example**: Let $X_1, X_2, \ldots, X_n$ are iid (identical and independent distributed) random sample from logistic distribution with location parameter $\theta \in R$ and scale parameter $b = 1$.
    - The probability density function (pdf): $f(x; \theta)$ is given by:

$$f(x; a) = \frac{e^{(x-\theta)}}{1 + e^{x-\theta}}, x \in R$$

    - By letting $y = x - \theta$, we have the following expression in which the distribution of $y = x - \theta$ does not depend on θ.

$$f(y; a) = \frac{e^y}{1 + e^y}, y \in R$$

ii. Scale Family

- **Definition**: Let $f(.\,; \theta, \theta > 0)$ be a family of densities indexed by a parameter θ. The parameter θ is defined to be a **location parameter** if and only $f(x; \theta)$ can be written as a function of $\frac{1}{\theta} h(\frac{x}{\theta})$. $f(x; \theta) = h(x - \theta)$ for some function $h(.\,)$. Equivalently, θ is a location parameter for the density $f(x; \theta)$ of a random variable X if and only if the distribution of $\frac{X}{\theta}$ does not depend on θ.
- **Importance**: To scale the shape of a distribution, which usually steches, squeeze or even change the general shape of distribution along horizontal axis.
- **Example**: Let $X_1, X_2, \ldots, X_n$ are iid random sample from exponential distribution with mean of θ.
    - The probability density function (pdf): $f(x; \theta)$ is given by:

$$f(x; \theta) = \frac{1}{\theta} e^{(-\frac{x}{\theta})},\ for\ x \geq 0$$

    - By letting $y = \frac{x}{\theta}$, we have the following expression in which the distribution of $y = \frac{x}{\theta}$ does not depend on θ.

$$f(y; \theta) = \frac{1}{\theta} \cdot e^y,\ for\ x \geq 0$$

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iii. Sufficient statistic

- **Definition**: Let $X_1, X_2, \ldots, X_n$ be random sample from density $f(\cdot, \theta)$ where θ may be a vector. A statistic S = $s(X_1, X_2, \ldots, X_n)$ is defined to be sufficient statistics iff the conditional distribution of $X_1, X_2, \ldots, X_n$ given S =s does not depend on θ.

- **Importance**:
  - To act as a form of data reduction.
  - To condense all the inportant information of sample in a statistics in such as way that no information of parameter θ is lost.
- **Factorization Theorem (single sufficient statistic)**: Let $X_1, X_2, \ldots, X_n$ be random sample from density $f(\cdot, \theta)$ where θ may be a vector. A statistics S = $s(X_1, X_2, \ldots, X_n)$ is sufficient iff the joint density of $X_1, X_2, \ldots, X_n$ can be expressed as:

$$\prod_{i=1}^{n} f(x_i; \theta) = g(s(x_1, \ldots, x_n); \theta) h(x_1, \ldots, x_n) = g(s; \theta) h(x_1, \ldots, x_n)$$

- **Example**: Let $X_1, X_2, \ldots, X_n$ are iid random sample from a Poisson distribution with mean of $\theta > 0$.
  - The probability mass function (pmf): $f(x; \theta)$ is given by:

$$f(x; \theta) = \frac{\theta^x e^{-\theta}}{x!}$$

  - Using a statistic of $s = \sum_{i=1}^{n} x_i$, its likelihood is expressed by:

$$\begin{aligned} L(\lambda; x_i) &= \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!} && (1) \\ &= \frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} (x_i)!} && (2) \\ &= \underbrace{\theta^s e^{-n\theta}}_{g(s;\theta)} \cdot \underbrace{\frac{1}{\prod_{i=1}^{n} (x_i!)}}_{h(x_1, \ldots, x_n)} && (3) \end{aligned}$$

  - Hence by factorization theorem, we can say that a statistic of $s = \sum_{i=1}^{n} x_i$ is **sufficient** to condense all important information of sample drawn from a Poisson distribution, without losing the information of θ.

iv. Minimal sufficient statistic

- **Definition**: A set of jointly sufficient statistics is defined to be **minimal sufficient** iff it is a **function** of every other set of sufficient statistics. Given two sample points of x and y, $S(X)$ is a minimal sufficient iff the ratio of likelihood, $\frac{L(\theta;x)}{L(\theta;y)}$ is a constant as a function of θ iff $S(X) = S(Y)$.

- **Importance**: A statistic captures all possible information of θ most efficiently.

- **Example**: Let $X_1, X_2, \ldots, X_n$ are iid random sample from a **Bernoulli distribution**, to show a test statistic $s = \sum X_i$ is minimal sufficient statistic:

  - The ratio of likelihood of 2 samples, X and Y is expressed as below:

$$\begin{aligned} \frac{L(\theta; x)}{L(\theta; y)} &= \frac{\prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i}}{\prod_{i=1}^{n} p^{y_i} (1-p)^{1-y_i}} && (4) \\ &= \frac{p^{\sum_{i=1}^{n} x_i} (1-p)^{\sum_{i=1}^{n} (1-x_i)}}{p^{\sum_{i=1}^{n} y_i} (1-p)^{\sum_{i=1}^{n} (1-y_i)}} && (5) \end{aligned}$$

- By algebra, we know that $\sum_{i=1}^n x_i = n\bar{X}$ and $\sum_{i=1}^n y_i = n\bar{Y}$

$$\frac{L(\theta;x)}{L(\theta;y)} = \frac{p^{n\bar{X}}(1-p)^{n(1-\bar{X})}}{p^{n\bar{Y}}(1-p)^{n(1-\bar{Y})}} \tag{6}$$

$$= p^{n(\bar{X}-\bar{Y})}(1-p)^{n(\bar{Y}-\bar{X})} \tag{7}$$

- The ratio of likelihood is a constant iff $\bar{X} = \bar{Y}$, hence, it implies that the sample mean $\bar{X}$ is minimum sufficient statistic. Any one-to-one function of $\bar{X}$ such as $\sum_{i=1}^n X_i$ is also a sufficient statistic.

v. Complete sufficient statistic

- **Definition**: Let $X_1, X_2, \ldots, X_n$ be random sample from density $f(\cdot, \theta)$ with parameter space Ω, and let T = $t(X_1, X_2, \ldots, X_n)$ be a statistic. The family of densitiies of T is defined to be complete iff $E_\theta[(z(T)] \equiv 0$ for all $\theta \in \Omega$ implied that $P_\theta[(z(T) = 0] \equiv 1$ for all $\theta \in \Omega$, where $z(T)$ is a setatistic. The statistic T is said to be complete iff its family of densities is complete.
- **Importance**:
  - It ensures that the distributions corresponding to different values of the parameters are distinct.
  - **Lehmann–Scheffé theorem**, it states that leting T = $t(X_1, X_2, \ldots, X_n)$ be a complete sufficient statistic for a parameter θ if there is a function of T that is an unbiased estimator of θ, then this function of T is the unique **minimum variance unbiased estimator (MVUE)** of θ.
  - If T is a complete sufficient statistics, then it is an unbiased estimator for θ that has **lower variance** than any other unbiased estimator for all possible values of the parameter.
- **Example**: Let $X_1, X_2, \ldots, X_n$ are iid random sample from a **Exponential distribution** with mean of $\frac{1}{\theta}$ and $\theta > 0$, to show a test statistic $T = \sum X_i$ is a complete sufficient statistic:
  - The pdf of Exponential distribution is:

$$f(x;\lambda) = \theta e^{-\theta x_i} \tag{8}$$

  - We need to show that $E_\theta[(z(T)] \equiv 0$

$$E_\theta[(z(T)] = \int z(t) P_\theta(t)\, dt \tag{9}$$

$$= \int_0^\theta z(t)\theta e^{-\theta t}\, dt \tag{10}$$

$$= \underbrace{\theta}_{>0} \int_0^\theta z(t) e^{-\theta t}\, dt \tag{11}$$

  - Looking at the fundamental theorem of calculus:

$$\frac{d}{d\theta} \int_0^{k(\theta)} f(u)\, du = f(k(\theta)) \cdot \frac{dk(\theta)}{d\theta} \tag{12}$$

  - Let $k(\theta) = \theta$ which implies $\frac{dk(\theta)}{d\theta} = 1$, and $f(u) = z(t)e^{-\theta t}$:

$$\frac{d}{d\theta} \int_0^\theta z(t) e^{-\theta t}\, dt = f(k(\theta)) \cdot 1 = 0 \tag{13}$$

  - This also implies that

$$z(t) \underbrace{e^{-\theta t}}_{e^x > 0\ for x \in R} = 0 \tag{14}$$

$$z(t) = 0 \tag{15}$$

- This also implies that $z(t) = 0$ for all $t \in R$, hence, $T = \sum X_i$ is a complete sufficient statistic.

vi. Ancillary statistic

- **Definition**: A statistic S = $s(X_1, X_2, \ldots, X_n)$ whose distribution **does not depend** on the parameters.
- **Importance**: The value of statistic does not change as the parameter of the distribution changes.
- **Example**: Show sample variance S^2 from a Normal distribution with unknown mean of μ, and **known variance** and σ^2 is an **ancillary statistic**.
  - Let $X_1, X_2, \ldots, X_n$ are iid random sample from a Normal distribution with unknown mean of μ, and known variance of σ^2.
  - Sample variance is known as:

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$$

  - There is a standard distribution of sample variance, given population variance is known that :

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

  - This has shown that sample variance can be distributed from a χ^2_{n-1} without explicitly having population mean to be known.
  - The sample variance S^2 does not depend on population mean of μ, hence it is an ancillary statistic.

2. Exponential Family of Distributions

i. Definition

- A one-parameter family (θ is undimensional of densities $f(\cdot;\theta)$ that can be expressed as the following:

$$f(x;\theta) = a(\theta)b(x)\exp[c(\theta)d(x)]$$

is defined to belong to the **exponential family** for $-\infty < x < \infty$, for all $\theta \in R$.

- A k-parameter family (θ is undimensional of densities $f(\cdot;\theta_1, \ldots, \theta_k)$ that can be expressed as the following:

$$f(x;\theta_1, \ldots, \theta_k) = a(\theta_1, \ldots, \theta_k)b(x)\exp\sum_{j=1}^{k}[c_j(\theta_1, \ldots, \theta_k)d_j(x)]$$

is defined to belong to the **exponential family** for $-\infty < x < \infty$, for all $\theta \in R$.

ii. Advantages

a. Mathematical convenience.
b. Demonstration of **complete** minimal sufficient statistics

a. Mathematical convenience.

- The form is designed for **mathematical convenience** to express useful algebraic properties.
- It is much clearer in determining likelihood expression; direct applying **multiplication and summation** at $a(\theta)b(x)$ and $\exp[c(\theta)d(x)]$, respectively.
- For example, given $X_1, X_2, ..., X_n$ are iid. (identical and independent distributed) observation from an exponential distribution with mean of $\frac{1}{\lambda}$.
  - The pdf is in the form of:

$$f(x;\lambda) = \lambda e^{(-\lambda x)} \; for \; x \geq 0$$

  - The pdf can also be expressed in the form of exponential family as such:

$$f(x;\lambda) = \lambda \cdot 1 \cdot exp(-\lambda \cdot x)$$

  - Hence, for likelihood, it can be easily expressed in the form of:

$$L(\lambda; x_i) = \prod_{i=1}^{n} \lambda \cdot 1 \cdot exp(-\lambda \cdot x) \tag{16}$$

$$= \lambda^n exp(-\lambda \cdot \sum x_i) \tag{17}$$

b. Demonstration of complete minimal sufficient statistics

- **Theorem**: given $X_1, X_2, ..., X_n$ are iid random sample from an exponential family distribution, then $\sum d(X_i)$ is a complete minimal sufficient statistics.
- For example, given $X_1, X_2, ..., X_n$ are iid from an exponential distribution with mean of $\frac{1}{\lambda}$.
  - The pdf expression in the form of exponential family is:

$$f(x;\lambda) = \lambda \cdot 1 \cdot exp(-\lambda \cdot x)$$

  - with:
    - $a(\lambda) = \lambda$
    - $b(x) = 1$
    - $c(\lambda) = -\lambda$
    - $d(x) = x$
  - We have $d(x) = x$, hence by convenience of theorem without explicitly proving, $\sum d(X_i) = \sum(X_i)$ is a complete minimal sufficient statistic.

3 Basu's Theorem

- **Theorem**: If T(X) is a complete and minimal sufficient statistics, then T(X) is independent of every ancillary statistics.

- **Example**: Given $X_1, X_2, ..., X_n$ are iid from a normal distribution with unknown mean and known variance of μ and $\sigma^2 = 1$, respectively. We want to show that the sample mean as a complete and minimal sufficient statistic: $T(X) = \frac{\sum_{i=1}^{n} X_i}{n}$ is independent of an ancillary statistic such as sample variance, S^2.
  - Probability density function (pdf) of $N(\mu, 1)$ is defined as :

$$f(x;\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} \tag{18}$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-\mu^2}{2}} \cdot 1 \cdot exp[-\frac{x^2}{2} + \mu x] \tag{19}$$

- The pdf is in the form of k-parameter exponential family in the form of $a(\theta_1, \ldots, \theta_k)b(x)\exp\sum_{j=1}^{k}[c_j(\theta_1, \ldots, \theta_k)d_j(x)]$, with k = 2, by having:
  - $a(\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-\mu^2}{2\sigma^2}}$
  - $b(x) = 1$
  - $c_1(\mu, \sigma) = -\frac{1}{2}$
  - $c_2(\mu, \sigma) = \mu$
  - $d_1(x) = x^2$
  - $d_2(x) = x$
- By theorem, $\sum d_1(X_i) = \sum_i^n x_i^2$ and $\sum d_2(X_i) = \sum_i^n x_i$ are complete minimal statistics.
- Only referring to $\sum d_2(X_i) = \sum_i^n x_i$, this implies that its one-to-one function, the sample mean, $\bar{X}$ **is also a complete minimal statistic**.
- **Sample variance** $S^2\sigma^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$, is one of the **ancilliary statistics** as it does not depend on the parameters of μ.
- According to Basu's Theorem, $T(X) = \bar{X} = \sum_i^n x_i$ is **independent** of sample variance S^2.

Citation

1. Casella, G., & Berger, R. L. (2002). Statistical inference. Belmont, CA: Duxbury.