# Research on Hexagonal Board Tiling Sequences and Its Recursive Formulas

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**Abstract**

This research investigates the tiling problem in hexagonal boards using the Fibonacci sequence and its relationship to recursive sequences such as Narayana's Cow sequence. While traditional tiling problems often focus on linear arrangements, this study explores tiling hexagonal boards using triangular and trapezoidal tiles. Through combinatoric casework, breakability analysis, and recursive subdivision, this research identifies patterns and relationships among tiling configurations as the board's dimensions increase.

The study begins by demonstrating how Fibonacci's recursive formula can be applied to linear tilings, where board configurations correspond to the sequence $F_n = F_{n-1} + F_{n-2}$. Building on this foundation, the research transitions to hexagonal boards, introducing new tiling possibilities involving trapezoidal tiles and layers of increasing complexity. Utilizing a breakability approach, the research categorizes tilings based on whether specific lines within the hexagonal structure allow for division into smaller sub-boards.

As board size increases, computational limitations of brute-force casework necessitate a shift toward recursive methods. The Narayana's Cow sequence is adapted to model tiling possibilities, yielding recursive formulas that predict tiling counts with greater efficiency. Further exploration develops novel recursive relationships for subdivided triangular boards, resulting in a new sequence that has been added to the OEIS database.

This study contributes to combinatorics and mathematical tiling by extending the understanding of recursive patterns in multidimensional structures, opening possibilities for applications in mathematical modeling and algorithmic design.

# 1 Introduction

## 1.1 Introduction to Fibonacci Tilings

The Fibonacci sequence is a well known sequence due to its wide applicability within various fields. Examples of this include its use when locating a specific term of an array in computer science or when predicting market volatility within investing. These representations, however, are dependent on the sequentially recursive definition of the Fibonacci, $F_n = F_{n-1} + F_{n-2}$ given that $F_0 = F_1 = 1$. In contrast, Art Benjamin and Jennifer Quinn's book *Proofs That Really Count* offers a unique approach, in which they break down theorems into smaller cases expressed as "tilings". Initially, the book establishes that when given a board of length $n$, or a strip made up of $n$ squares, and we tile it using square tiles and domino tiles:

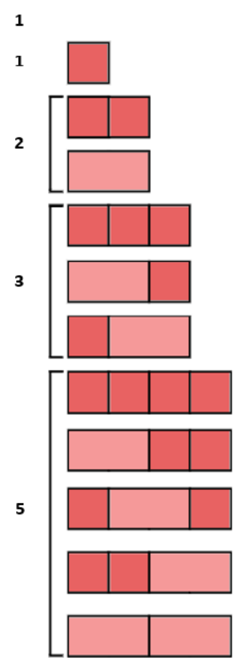


Figure 1: Linear tilings of boards of length $n = 0$ to $n = 4$

There is an observable pattern between the tiling problem and the Fibonacci. If $f_n$ represents the amount of ways to tile a board of length $n$ using squares and dominoes, the tiles in the diagram given above shows a relationship in which $f_n = F_n$ and can be observed to continue in following boards. And if we tile a board of length $n$ based on the condition of the last tile, either square or domino, we obtain two cases: $f_{n-1}$ or $f_{n-2}$, essentially tiling every cell of the board subtracted by the length of the final piece. Combining these variables together, we are able to obtain the recursive formula for the Fibonacci: $f_n = f_{n-1} + f_{n-2}$!

## 1.2 Introduction to Hexagonal Boards

The possible types of tilings for these boards seem limitless, yet most have already been explored plenty of times. However, a curious approach to this tiling problem is to reduce the number of sides each board cell contains by one, resulting in the use of triangular and trapezoidal tiles to tile boards in the shape of hexagons. Specifically, we can layer these hexagonal boards by observing the length of their horizontal diagonal and simply adding one to its length in order to obtain the next board. Notice that every consequent board increases by four equilateral triangles.

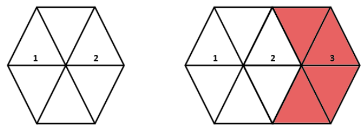


Figure 2: Hexagonal boards of horizontal diagonal of length $n = 2$ and $n = 3$

# 2 Combinatoric Approach Casework

Let us define $G_n$ to be the number of ways to tile a hexagonal board of horizontal diagonal length $n$ with triangles (of side length one) and trapezoids (three triangular tiles combined). Approaching this problem using simple casework, we are able to separate the tilings into several cases: using no trapezoidal tiles, using one trapezoidal tile, using two trapezoidal tiles, and so on scaling with the size of the board. For instance, we are able to categorize the different ways to arrange a board of horizontal diagonal length two through this form of casework, as shown:

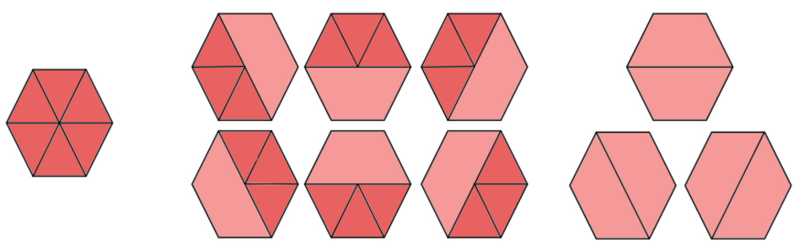

Figure 3: All tilings for a hexagonal board of horizontal diagonal length $n = 2$

Although organizing these tilings based on the number of trapezoidal tiles used provides us with $G_2 = 10$, the flaw with this "brute force-esque" method is that it becomes impractical as the horizontal diagonal length increases. For example, the amount of tilings for the next board of diagonal length $n = 3$ drastically increases due to the introduction of vertical trapezoidal tiles. Hence, we arrive at a more efficient approach to this method.

# 3 Breakability Approach

## 3.1 Introduction to Breakability

The condition of "Breakability" offers a new and further in-depth perspective towards the previous casework approach. We can observe this in the context of Fibonacci tilings as introduced in Section 1:

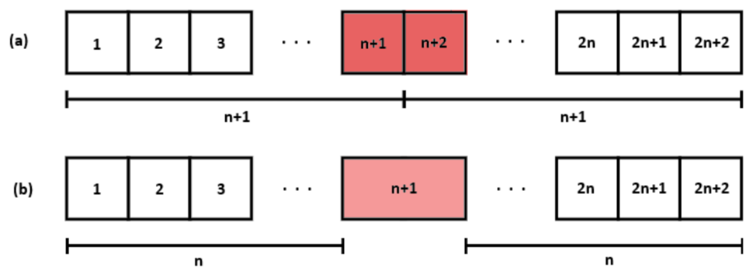


Figure 4: Two breakability cases for a board of length $2n+2$ based on cell $n+1$

As seen in part (a) of the diagram above, the board can be considered *breakable* at cell $n+1$ if there is no domino covering the cells $n+1$ and $n+2$, allowing the board to be broken up into two smaller sub-boards of equal length. Similarly, it may be considered unbreakable if a domino instead occupies those cells as seen in part (b), allowing us to break apart the tiling into two equal sub-boards of length $n$.

### 3.2 Breakability Casework

In the context of our hexagonal tilings, the line which determines breakability can be set on the side where the new layer is added on. As a result, all tiling possibilities can be similarly split into two cases: one where there is no trapezoidal tile overlapping the breakability line, and the other one where a trapezoidal tile does overlap or cross the line.

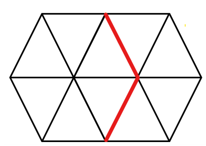

Figure 5: Breakability line of board $G_3$

Under the condition that no trapezoidal tile overlaps the line, three cases can be created in which the number of tilings that the previous board contains can be utilized:

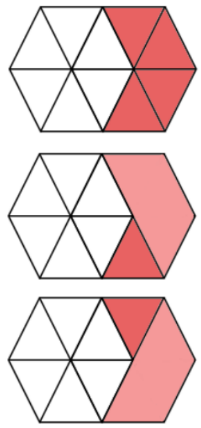

Figure 6: Three breakable cases of board $G_3$

Similar to the breakability diagram shown in Figure 4, these boards can be broken up into two smaller sub-boards of a hexagon and the layer of new triangles. Since we have previously calculated that $G_2 = 10$, we can conclude that the total number of tilings in the cases where no trapezoidal tile overlaps with the line in $G_3$ is equal to $10 \times 3 = 30$. Extending from this derivation, we learn that for the next boards, we can simply multiply the amount of tilings for the previous board by 3 to obtain the number of tilings where the board is *breakable* at the given line, and we simply have to calculate the number of tilings for the *unbreakable* cases. The cases of $G_3$ where the trapezoidal tile *does* overlap the line can be separated into six cases:

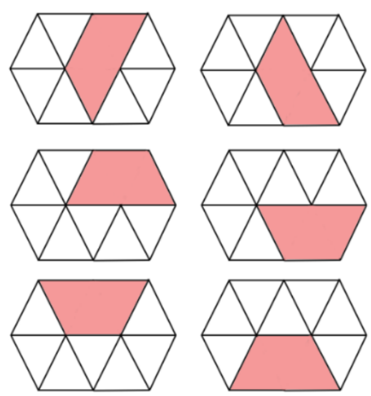

Figure 7: Six unbreakable cases of board $G_3$

Note that these tiling patterns seem to come in pairs of the same tile reflected vertically, so we can calculate the amount of possible tilings for one of each pair, double it, and then subtract the repeated cases to account for the other as well. In order to calculate the total amount of tilings for each of these six cases, rather than applying more casework, we are able to utilize an established sequence: The Narayana's Cow Sequence. In the tiling context, this establishes that if $a_n$ is to be defined as the number of ways to tile a linear board of length $n$, given that $a_0 = a_1 = a_2 = 1$, with squares of width one and trominoes of width three (paralleling the triangular and trapezoidal tiles made of one and three triangles), we can derive a recursive formula of $a_n = a_{n-1} + a_{n-3}$. In order to utilize this theorem, we can manipulate the shape of the untiled cells to a linear shape, such as this:

Figure 8: An unbreakable case of $G_3$ unfolded into a linear shape

And the cases where there are two separate smaller boards can also be manipulated into a linear shape:

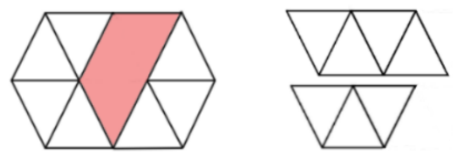

Figure 9: An unbreakable case of $G_3$ unfolded into two separate linear shape

Hence, without accounting for repeated cases, we are able to obtain a value of

$$\begin{aligned} G_3' &= 4(a_7) + 2(a_4 \times a_3) \\ &= 4(9) + 2(3 \times 2) \\ &= 36 + 12 \\ &= 48 \end{aligned}$$

for the unbreakable cases. For the repeated tiling cases, or the possibilities where two of the cases are present at the same time, we observe a total of four possibilities of two unbreakable cases appearing together:

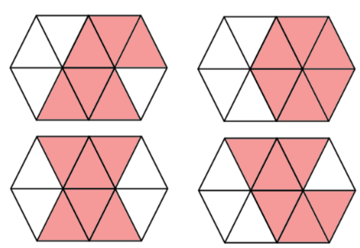

Figure 10: Duplicate tiling cases

Again utilizing Narayana's Cow Sequence on the leftover white tiles, we sum up all possibilities' tilings and obtain 8 duplicate tiling cases. Hence, subtracting 8 from the original 48 gives us a total of 40 possible tilings for the cases where the board is unbreakable at the line. Lastly, we can obtain the sum of the breakable and unbreakable cases, equating to $G_3 = 30 + 40 = 70$. However, as the horizontal diagonal length increases beyond 3, this method also appears to become impractical due to the Narayana's inability to account for the possibility of vertical tiles that may overlap opposite sides of a straightened sequence of cells, as seen below:

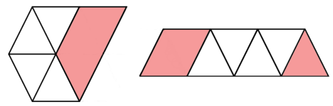

Figure 11: Vertical trapezoidal tiling case of a sub-board unaccounted for using Narayana's

# 4 Recursive Subdivision Approach

## 4.1 Triangular approach

Though casework is a valid approach towards solving tiling problems similar to those introduced in previous sections, both the simple combinatoric casework and the breakability casework are unable to offer us an easy pattern or recursive formula in the context of hexagonal boards and become impractical as the length of the board increases. In order to find the number of possible tilings for the larger boards, we can adopt a method that allows us to solve problems regarding number sequences without a clear common ratio or recursive formula. Such a method entails splitting up the hexagonal boards into smaller equilateral triangles, and a new series will be created which we will call $T_n$. Six boards of this new sequence $T_n$ are shown below:

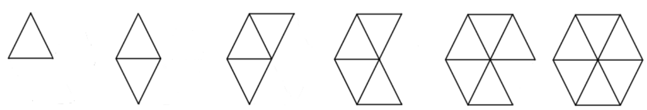

Figure 12: Boards of $T_1$ to $T_6$

We note that $T_2$ and $T_6$ have the same convex "ending" on the right-hand side, and so it is natural to consider the four "intermediate" cases as presented by $T_2$, $T_3$, $T_4$, and $T_5$. By then inspecting the ending tiles of each case, we deduct that each board can be represented as the tilings of certain previous boards depending on the combination of tilings in its last cells. We can visualize this concept through a board $T$ of length $n = 4k + 2$:

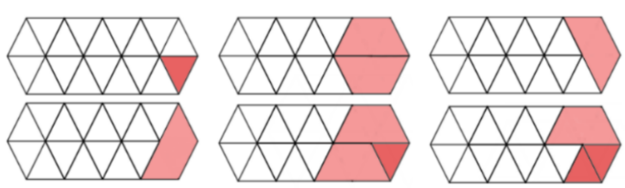

Figure 13: 6 possible ending-tiling combinations of a board $T_{4k+2}$

Because the cells left untiled can be expressed in terms of previous boards and the tilings above account for all possible ending combinations of a board $T$ of length $n = 4k+2$, the board can be expressed as the sum of all previous boards derived from ending tiling combinations, resulting in this equation for any board $T$ with length $n = 4k + 2$:

$$T_{4k+2} = T_{4k+1} + 2T_{4k-1} + T_{4k-3} + T_{4k-4} + T_{4k-5}$$

When this strategy is applied to other $T$ boards of length $n = 4k$, $4k + 1$, or $4k + 3$, we are able to derive a recursive formula for each:

**Theorem 4.1.** For $T_n$ defined as above, and for $n$ sufficiently large, we have the following formulas, based on whether $n = 4k, 4k + 1, 4k + 2$, or $4k + 3$.

$$\begin{aligned}
T_{4k} &= T_{4k-1} + T_{4k-4} + T_{4k-5} + T_{4k-6} + T_{4k-7} \\
T_{4k+1} &= T_{4k} + T_{4k-3} + T_{4k-5} \\
T_{4k+2} &= T_{4k+1} + 2T_{4k-1} + T_{4k-3} + T_{4k-4} + T_{4k-5} \\
T_{4k+3} &= T_{4k+2} + T_{4k} + T_{4k-1} + T_{4k-3}
\end{aligned}$$

Using these formulas, and using our initial values for $T_0, T_1, \ldots, T_5$, we are able to obtain all terms of the $T_n$ sequence, giving us the following sequence:

**The first few values for $T_n$**

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_n$ | 1 | 1 | 1 | 2 | 3 | 4 | 10 | 16 | 23 | 29 | 70 |

This is a new sequence, and we are delighted to have it included in the OEIS at A365352.

To then derive the terms of our original series $G_n$, we can define the $G_n$ boards as the number of equilateral triangles within them, or $G_1 = 6$, $G_2 = 10$, $G_3 = 14$, and so on, adding 4 each time with 6 as a starting point. We then find the matching number of equilateral triangles in the $T_n$ series, resulting in the following series:

**The first few values for $G_n$**

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $G_n$ | 10 | 70 | 505 | 3631 | 26105 | 187700 | 1349583 | 9703648 | 69770292 |

As seen from even just the first few values of $G_n$, attempting to list all cases and calculating the total would've been near impossible by brute force due to the exponential increase in the number of possible tilings for each new board.

## 4.2 Narayana's Cow Sequence Approach

Revisiting the approach utilizing Narayana's Cow Sequence as explored within our breakability casework, we are able to apply this concept towards the overall board rather than based on only the breakability line where the new layer is added on, allowing for us to reach another recursive relation of sequence $T_n$:

**Theorem 4.2.** For $T_n$ as defined in section 4.1, and for $n$ sufficiently large, we have the following formula so long as we agree that $T_{-1} = a_{-1} = 0$.

$$T_{4n+2} = a_{2n+1}^2 + 2\sum_{k=0}^{n} a_{2(n-k)}(T_{4k} \times a_{2(n-k)-1} + T_{4k-1} \times a_{2(n-k)})$$

We can prove this through proof by equivalence of the board $T_{4n+2}$.

**Question:** How many ways are there to tile a hexagonal board (made up of 4n+2 cells) using triangular and trapezoidal tiles?

**Answer 1:** By definition of the sequence $T$, there are $T_{4n+2}$ ways of tiling this board using triangular and trapezoidal tiles.

**Answer 2:** Conditioning on the presence of vertical trapezoids.

If no vertical trapezoids are present, we are able to break the board into its top and bottom half, each of length $2n+1$ as such:

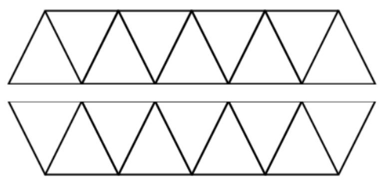

Figure 14: Board of $T_{4n+2}$ split into two equal halves

This divides the board into two cases in which the linear shape of each half with no vertical trapezoidal tiles allows us to apply Narayana's, or $a_{2n+1}$ if we again define $a_n$ to be the Narayana tiling number. Hence, the value of all possible tilings of the board $T_{4n+2}$ when there are no vertical trapezoidal tiles is $a_{2n+1}^2$.

When there are vertical trapezoids, however, we can condition the tilings based on the position of the last vertical trapezoidal tile, and therefore have to consider the four possible orientations of it:

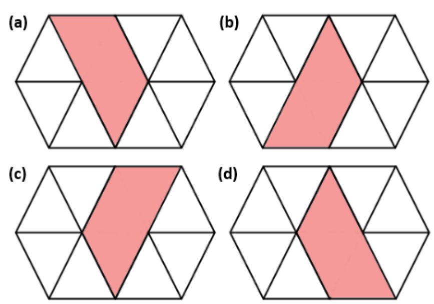


Figure 15: Four possible orientations of a vertical trapezoidal tile in a hexagonal board

If we observe orientations (a) and (b) in Figure 14, we can observe that the resulting sub-boards parallel each other, with three cells on the left and four on the right. This observation can be extended to the bottom two orientations as well, instead with four cells on the left and three on the right. Therefore, we would calculate the possible tilings for one of each pair and double it to account for all orientations.

If we observe orientation (a)'s sub-boards, we can conclude that the left sub-board would have $T_{4k-1}$ possible tilings due to the possibility of vertical trapezoidal tiles and its ending case. The right sub-board can be split into two halves like Figure 13 due the lack of vertical trapezoidal tiles, giving us $a_{2(n-k)}$ instead for each half. Therefore, when the orientation of the last vertical trapezoidal tile is that of orientation (a) in Figure 14, there are $T_{4k-1} \times a_{2(n-k)}^2$ possible tiling combinations. To then account for orientation (b) of the last vertical tile, we recognize that the equal sub-boards result in the expression $2(T_{4k-1} \times a_{2(n-k)}^2)$ being able to account for both orientations.

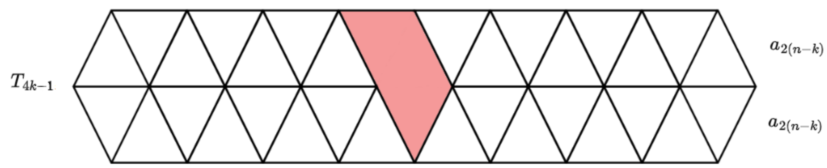


Figure 16: Possible tilings of each sub-board with the top left orientation given in Figure 14

If we instead observe orientation (c)'s sub-boards on each side, we can conclude with the same process that the left sub-board would have $T_{4k}$ possible tilings, while the right sub-board can instead be split into two unequal halves, giving us $a_{2(n-k)}$ possible tilings for the longer half and $a_{2(n-k)-1}$ for the other. Therefore, there are $T_{4k} \times a_{2(n-k)} \times a_{2(n-k)-1}$ possible tiling combinations. Similarly, to then account for orientation (d) of the last vertical tile, we recognize that the equal sub-boards result in the expression $2(T_{4k} \times a_{2(n-k)} \times a_{2(n-k)-1})$ being able to account for both orientations.

If we define the location of each vertical tile based on the horizontal diagonal it overlaps, starting at $k = 0$ and ending at $k = n$ as shown below:

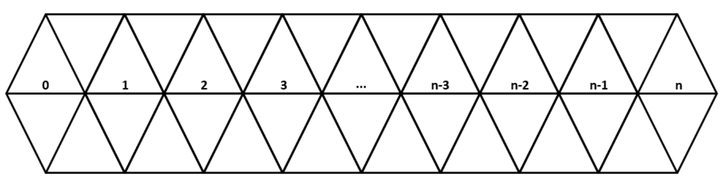


Figure 17: Four possible orientations of a vertical trapezoidal tile in a hexagonal board

We can account for all possible locations of the last vertical trapezoidal tile through utilizing the summation function starting at $k = 0$ and ending at $k = n$. Combining all possibilities, we can derive that all possible tilings of board $T_{4n+2}$ would be equal to the sum of the possible tilings with no vertical trapezoidal tile and possible tilings based on conditions of the last vertical trapezoidal tile, resulting in:

$$a_{2n+1}^2 + \sum_{k=0}^{n} (2(T_{4k} \times a_{2(n-k)} \times a_{2(n-k)-1}) \times 2(T_{4k-1} \times a_{2(n-k)}^2))$$

$$a_{2n+1}^2 + \sum_{k=0}^{n} 2a_{2(n-k)}((T_{4k} \times a_{2(n-k)-1}) \times (T_{4k-1} \times a_{2(n-k)}))$$

$$a_{2n+1}^2 + 2\sum_{k=0}^{n} a_{2(n-k)}((T_{4k} \times a_{2(n-k)-1}) \times (T_{4k-1} \times a_{2(n-k)}))$$

Due to the fact that any counting problem only contains one solution and therefore $A_1 = A_2$, we can arrive at the given theorem of:

$$T_{4n+2} = a_{2n+1}^2 + 2\sum_{k=0}^{n} a_{2(n-k)}(T_{4k} \times a_{2(n-k)-1} + T_{4k-1} \times a_{2(n-k)})$$

# 5 Bibliography

Benjamin, Arthur T., and Jennifer J. Quinn *Proofs That Really Count*, Mathematical Association of America, 2003.