计算方法编程作业 5 实验报告

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1 实验目的

对于给定的 21 个点，实现三次样条插值算法，利用大 M 法和自然边界条件计算样条插值函数。

2 问题描述与算法

2.1 求解结点处的二阶导数

由书中公式，可知在自然边界条件下，应求解如下方程：

$$\begin{bmatrix} 2 & \lambda_1 & & & \\ \mu_2 & 2 & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda_{n-2} & 2 & \lambda_{n-2} \\ & & & \mu_{n-1} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - \mu_1 M_0 \\ d_2 \\ \vdots \\ d_{n-2} \\ d_{n-1} - \lambda_{n-1} M_n \end{bmatrix}$$

图 1: 待解方程组

此处 $\lambda_i = \mu_i = 0.5,\ h = 1, d_i = \frac{6}{h_i + h_{i-1}}(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}) = 3(y_{i+1} - 2y_i + y_{i-1})$

在本次实验中，选择复用 lab2 中代码，使用 Gauss-Seidel 迭代算法来求解上述线性方程组。

2.2 通过 M 求解多项式 S

由教材 P42 公式，有

$$
\begin{aligned}
S(x) &= \frac{(x_{i+1}-x)^3 M_i + (x-x_i)^3 M_{i+1}}{6h_i} + \frac{(x_{i+1}-x)y_i - (x-x_i)y_{i+1}}{h_i} - \frac{h_i}{6}[(x_{i+1}-x)M_i + (x-x_i)M_{i+1}] \\
&= \frac{M_{i+1}-M_i}{6h}x^3 + \frac{M_i x_{i+1} - M_{i+1}x_i}{2h}x^2 + [\frac{M_{i+1}x_i^2 - M_i x_{i+1}^2 + 2(y_{i+1}-y_i)}{2h} - \frac{h}{6}(M_{i+1}-M_i)]x \\
&\quad + \frac{M_i x_{i+1}^3 - M_{i+1}x_i^3}{6h} + \frac{x_{i+1}y_i - x_i y_{i+1}}{h} - \frac{h}{6}(M_i x_{i+1} - M_{i+1}x_i)
\end{aligned}
$$

代入计算即可得到多项式系数

3 实验结果与可视化

对初始数据进行三次样条插值，结果如下：

```
S(x)=-0.254241x^3+-6.8645x^2+-60.7398x+-175.849,  x in [-9, -8].
S(x)=0.203803x^3+4.12855x^2+27.2045x+58.6696,  x in [-8, -7].
S(x)=0.25253x^3+5.15181x^2+34.3673x+75.3829,  x in [-7, -6].
S(x)=-0.244121x^3+-3.7879x^2+-19.2709x+-31.8937,  x in [-6, -5].
S(x)=0.0954545x^3+1.30573x^2+6.19724x+10.5533,  x in [-5, -4].
S(x)=-0.0828972x^3+-0.83449x^2+-2.36364x+-0.86122,  x in [-4, -3].
S(x)=-0.309266x^3+-2.87181x^2+-8.47559x+-6.97317,  x in [-3, -2].
S(x)=0.90846x^3+4.43455x^2+6.13712x+2.76864,  x in [-2, -1].
S(x)=-0.889575x^3+-0.959557x^2+0.743018x+0.9706,  x in [-1, 0].
S(x)=0.203139x^3+-0.959557x^2+0.743018x+0.9706,  x in [0, 1].
S(x)=0.445017x^3+-1.68519x^2+1.46865x+0.728722,  x in [1, 2].
S(x)=-0.738108x^3+5.41356x^2+-12.7289x+10.1937,  x in [2, 3].
S(x)=0.747415x^3+-7.95615x^2+27.3803x+-29.9154,  x in [3, 4].
S(x)=-0.339953x^3+5.09227x^2+-24.8134x+39.6762,  x in [4, 5].
S(x)=-0.111902x^3+1.67151x^2+-7.7096x+11.1698,  x in [5, 6].
S(x)=0.0796627x^3+-1.77666x^2+12.9794x+-30.2082,  x in [6, 7].
S(x)=0.163651x^3+-3.54043x^2+25.3258x+-59.0164,  x in [7, 8].
S(x)=-0.319969x^3+8.06646x^2+-67.5293x+188.597,  x in [8, 9].
S(x)=0.354323x^3+-10.1394x^2+96.3237x+-302.962,  x in [9, 10].
S(x)=-0.163425x^3+5.39301x^2+-59.0007x+214.786,  x in [10, 11].
```

图 2: 初始数据结果

更改第十个点的坐标为 (0,10), 再次进行插值得

```
更改第十个点后，结果为:
S(x)=-0.254626x^3+-6.87491x^2+-60.8332x+-176.126,  x in [-9, -8].
S(x)=0.205731x^3+4.17367x^2+27.5555x+59.5768,  x in [-8, -7].
S(x)=0.245201x^3+5.00253x^2+33.3575x+73.1148,  x in [-7, -6].
S(x)=-0.216734x^3+-3.3123x^2+-16.5315x+-26.6632,  x in [-6, -5].
S(x)=-0.0067636x^3+-0.162739x^2+-0.783674x+-0.416839,  x in [-5, -4].
S(x)=0.298589x^3+3.50149x^2+13.8732x+19.1257,  x in [-4, -3].
S(x)=-1.73299x^3+-14.7827x^2+-40.9794x+-35.7269,  x in [-3, -2].
S(x)=6.22188x^3+32.9465x^2+54.479x+27.912,  x in [-2, -1].
S(x)=-11.6901x^3+-20.7895x^2+0.743018x+10,  x in [-1, 0].
S(x)=11.0037x^3+-20.7895x^2+0.743018x+10,  x in [0, 1].
S(x)=-4.8684x^3+26.8267x^2+-46.8732x+25.8721,  x in [1, 2].
S(x)=0.685617x^3+-6.49736x^2+19.775x+-18.5601,  x in [2, 3].
S(x)=0.365929x^3+-3.62017x^2+11.1434x+-9.92847,  x in [3, 4].
S(x)=-0.237734x^3+3.62379x^2+-17.8325x+28.706,  x in [4, 5].
S(x)=-0.139292x^3+2.14716x^2+-10.4493x+16.4007,  x in [5, 6].
S(x)=0.0870017x^3+-1.92613x^2+13.9904x+-32.4787,  x in [6, 7].
S(x)=0.161685x^3+-3.49448x^2+24.9689x+-58.0952,  x in [7, 8].
S(x)=-0.319443x^3+8.05258x^2+-67.4076x+188.242,  x in [8, 9].
S(x)=0.354185x^3+-10.1354x^2+96.2838x+-302.832,  x in [9, 10].
S(x)=-0.163397x^3+5.3921x^2+-58.9907x+214.75,  x in [10, 11].
```

图 3: 更改点后插值结果

分别对前后两次插值结果作比较，计算其系数各自的差，得：

```
更改前后，多项式的系数变化为：
0.000385697, 0.0104138, 0.0933387, 0.277702
-0.00192857, -0.0451287, -0.351001, -0.907204
0.00732878, 0.149276, 1.00983, 2.26807
-0.0273867, -0.475603, -2.73944, -5.23047
0.102218, 1.46847, 6.98092, 10.9701
-0.381486, -4.33598, -16.2369, -19.9869
1.42373, 11.9109, 32.5038, 28.7538
-5.31342, -28.5119, -48.3419, -25.1434
10.8005, 19.8299, -8.28078e-09, -9.0294
-10.8005, 19.8299, 1.78495e-08, -9.0294
5.31342, -28.5119, 48.3419, -25.1434
-1.42373, 11.9109, -32.5038, 28.7538
0.381486, -4.33598, 16.2369, -19.9869
-0.102219, 1.46848, -6.98096, 10.9702
0.0273895, -0.475646, 2.73967, -5.23086
-0.00733893, 0.149465, -1.011, 2.27047
0.00196628, -0.0459445, 0.356866, -0.921216
-0.000526188, 0.0138747, -0.121688, 0.354927
0.00013847, -0.00407103, 0.0398241, -0.129608
-2.76941e-05, 0.000913905, -0.0100253, 0.0365562
```

图 4: 前后结果比较

使用 Python+matplotlib 进行数据可视化，结果如下：

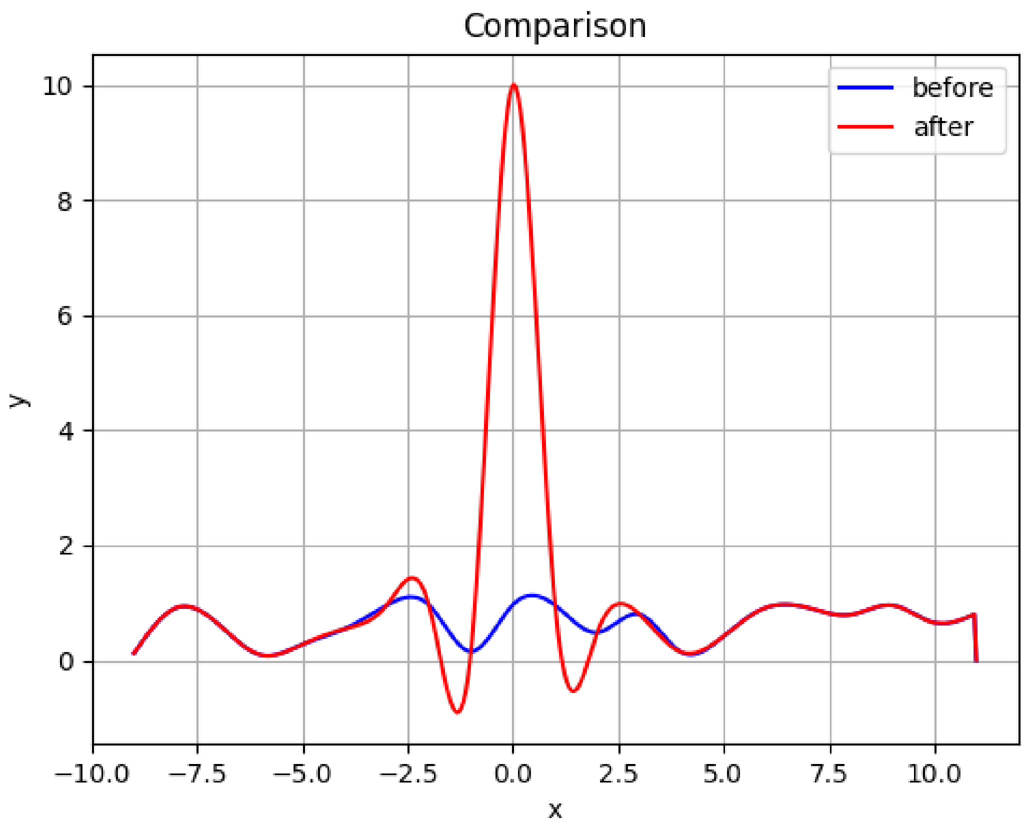

图 5: 可视化结果

可见，函数仅在更改坐标的点附近变化较大，在其他区间变动几乎可以忽略不计。

4 实验总结

综合以上实验结果可以看出，三次样条插值具有良好的连续性和数值稳定性，可以有效缓解 Runge 现象，局部数据点的变化不会对较远处的插值函数产生显著影响，是一种强大而灵活的插值方法。