Improved (Provable) Algorithms for the Shortest Vector Problem via Bounded Distance Decoding

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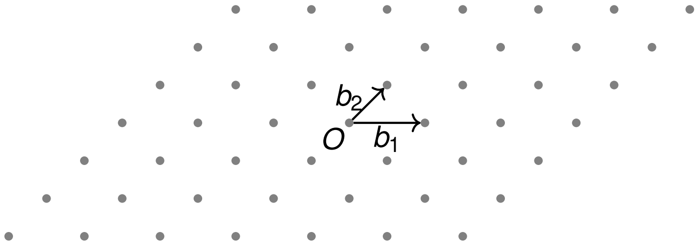
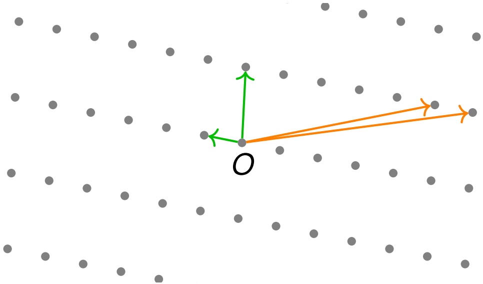
What is a (Euclidean) lattice?

Definition

$\mathcal{L}(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n) = \left\{\sum_{i=1}^{n} x_i \boldsymbol{b}_i : x_i \in \mathbb{Z}\right\}$ where $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ is a basis of $\mathbb{R}^n$.



Lattice-based cryptography: fundamental idea



- good basis: private information, makes problem easy
- bad basis: public information, makes problem hard

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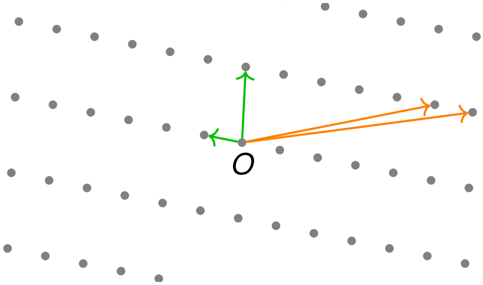

- good basis: private information, makes problem easy
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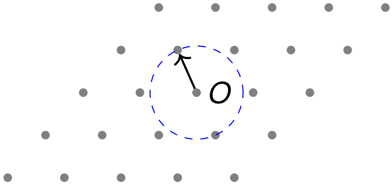
Basis reduction: transform a bad basis into a good one
Main tool: BKZ algorithm and its variants

Requires to solve the (approx-)SVP problem in smaller dimensions.

The Shortest Vector Problem



Shortest Vector Problem (SVP):
Given a basis for the lattice $\mathcal{L}$, find a shortest nonzero lattice vector.
$\lambda_1(\mathcal{L})$ = length of such a vector.

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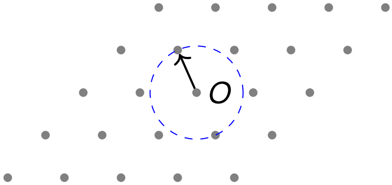

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Main approaches for SVP:

- Enumeration: $2^{O(n \log(n))}$ time and $\text{poly}(n)$ space
- Sieving: $2^{O(n)}$ time and $2^{O(n)}$ space

Sieving

- Heuristic algorithms: fastest in practice
- Provable algorithms: important for theory $\rightarrow$ our work

Results in the Classical Setting

Provable algorithms for SVP:

Time Complexity	Space Complexity	Reference
$n^{\frac{n}{2e}+o(n)}$	$\mathrm{poly}(n)$	[Kan87,HS07]
$2^{n+o(n)}$	$2^{n+o(n)}$	[ADRS15]
$2^{2.05n+o(n)}$	$2^{0.5n+o(n)}$	[CCL18]
$2^{1.7397n+o(n)}$	$2^{0.5n+o(n)}$	Our work

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Our work: first provable smooth time/space trade-off for SVP

time $q^{13n+o(n)}$ space $\mathrm{poly}(n) \cdot q^{\frac{16n}{q^2}}$ $q \in [4, \sqrt{n}]$

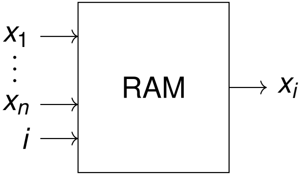
- $q = \sqrt{n}$: time $n^{O(n)}$ and space $\mathrm{poly}(n)$, not as good as [Kan87].
- $q = 4$: time $2^{O(n)}$ and space $2^{O(n)}$, not as good as [ADRS15].

Interlude: quantum memory models

classical access

quantum access

classical data



standard

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Assumption: $O(1)$ time cost

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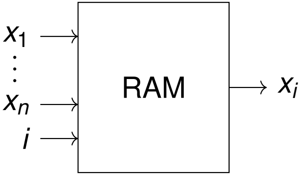

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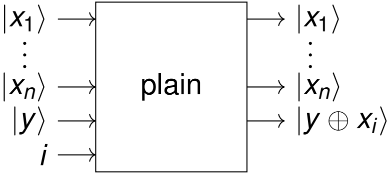

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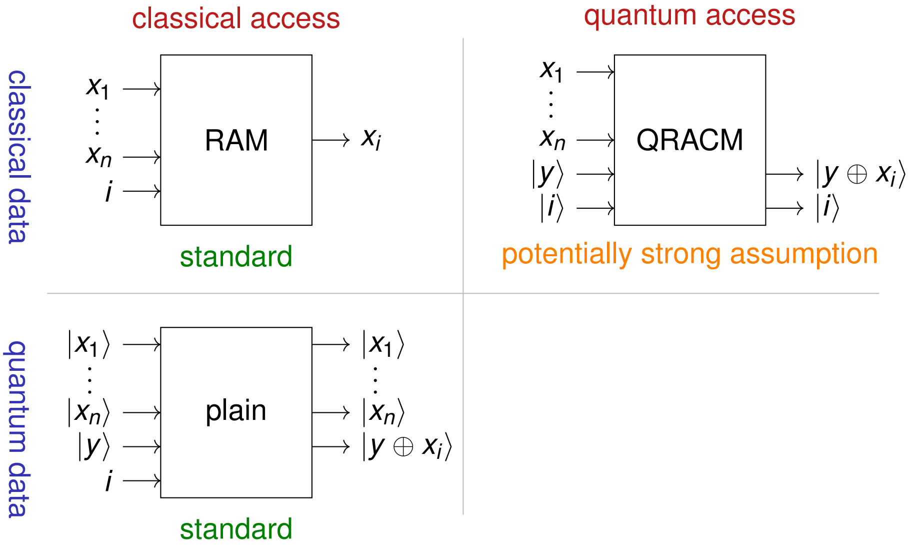

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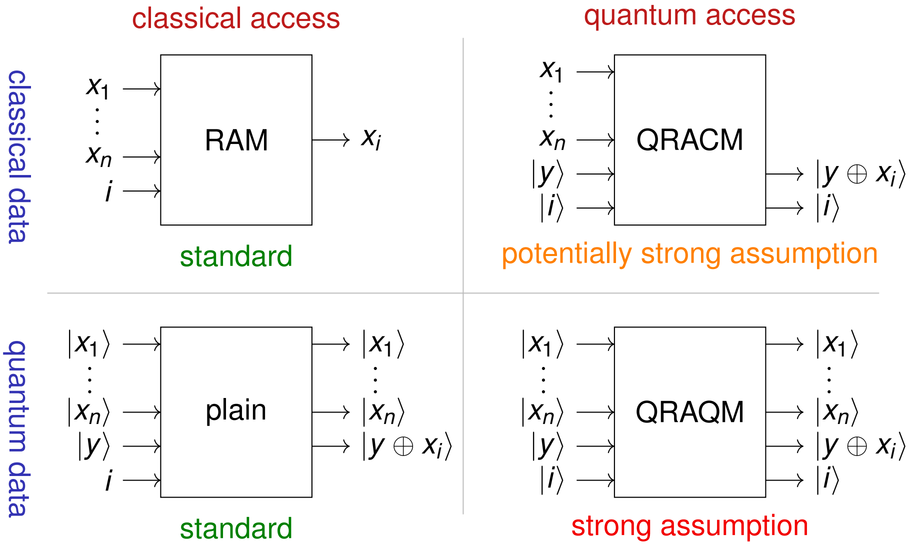

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Results in the Quantum Setting

Provable quantum algorithms for SVP:

Time Complexity	Space Complexity			Reference
	Classical	Quantum	Model	
$2^{1.799n+o(n)}$	$2^{1.286n+o(n)}$	$2^{1.286n+o(n)}$	QRACM	[LMP15]
$2^{1.2553n+o(n)}$	$2^{0.5n+o(n)}$	$\mathrm{poly}(n)$	plain	[CCL18]
$2^{0.9535n+o(n)}$	$2^{0.5n+o(n)}$	$\mathrm{poly}(n)$	plain	Our work
$2^{0.873n+o(n)}$	$2^{0.5n+o(n)}$	$2^{0.1604n+o(n)}$	QRACM	Our work

Remark on quantum heuristic algorithms:

- better complexity: $2^{0.265n+o(n)}$ [Laarhoven15]
- requires QRACM
- even better complexity: $2^{0.257n+o(n)}$ [CL21]
- requires QRAQM

Sieving Algorithms

Original idea [AKS01]:

- Reduce basis
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Many heuristic variants: local sensitive hash, tuple sieve, ...
All control the length of the vectors.

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All control the length of the vectors.

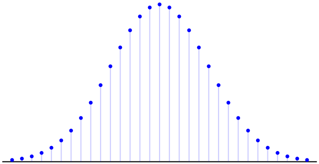
[ADRS15]'s new idea: control distribution instead of length of vectors

Discrete Gaussian Sampling

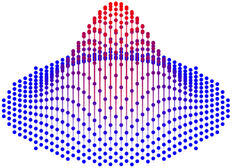
$$\rho_s(\boldsymbol{x}) = \exp\left(-\pi \frac{\|\boldsymbol{x}\|^2}{s^2}\right), \qquad D_{L,s}(\boldsymbol{x}) = \frac{\rho_s(\boldsymbol{x})}{\rho_s(L)}, \qquad \boldsymbol{x} \in \mathbb{R}^n, s > 0.$$

Definition (Discrete Gaussian Distribution)

On lattice L with parameter s: probability of $\boldsymbol{x} \in L$ is $D_{L,s}(\boldsymbol{x})$.



$L = \mathbb{Z}, s = 7$



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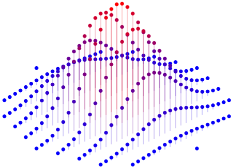

$L = \mathbb{Z} \times 4\mathbb{Z}, s = 7$

Discrete Gaussian Sampling (DGS)

- **input:** L and s
- **output:** random $\boldsymbol{x} \in L$ according to $D_{L,s}$.

DGS, BDD and SVP

$$\begin{tikzcd} \boxed{\text{SVP}} \arrow[r, bend left, "{[\text{ADRS15}]}"] & \boxed{\text{DGS}} \end{tikzcd}$$

DGS, BDD and SVP

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\text{SVP} \arrow[r, "\text{[CCL18]}"'] \arrow[rr, bend left, "\text{[ADRS15]}"] & \text{BDD} \arrow[r, "\text{[DSR14]}"'] & \text{DGS}
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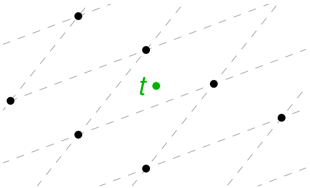

Bounded Distance Decoding ($\alpha-$BDD):
Given a lattice $\mathcal{L}$ and a target vector $t \in \mathbb{R}^n$

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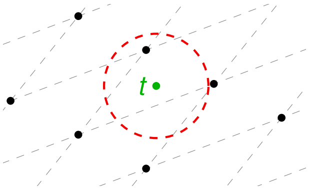

Bounded Distance Decoding (α−BDD):
Given a lattice $\mathcal{L}$ and a target vector $t \in \mathbb{R}^n$ with distance to lattice $\leq \alpha \cdot \lambda_1(\mathcal{L})$

The two reductions use completely different DGS parameter regimes!

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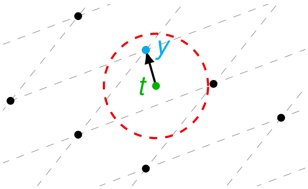

Bounded Distance Decoding ($\alpha-$BDD):
Given a lattice $\mathcal{L}$ and a target vector $t \in \mathbb{R}^n$ with distance to lattice $\leq \alpha \cdot \lambda_1(\mathcal{L})$, find the closest vector $y \in \mathcal{L}$.

- α is the decoding radius
- $\alpha < \frac{1}{2}$ for unique solution

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Hardness of Discrete Gaussian Sampling

Parameter s (width/standard deviation) of $D_{\mathcal{L},s}$:

- hard to sample
- SVP

- easy to sample

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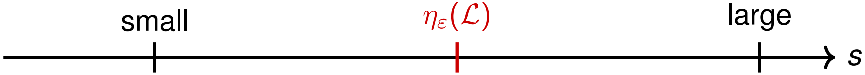

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small — $\eta_\varepsilon(\mathcal{L})$ — large → s

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- Open problem: $2^{O(n)}$ time, $2^{o(n)}$ space algorithm for $s = \eta_\varepsilon(\mathcal{L})$
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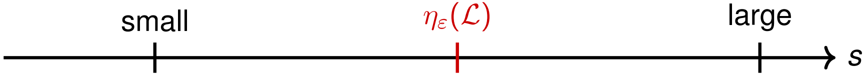

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Time-Space Tradeoff for SVP

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Smooth time-space tradeoff for BDD: create a $\frac{0.1}{q}$-BDD oracle in time q^{13n}, space q^{16n/q^2}, each call takes time q^{16n/q^2}.

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Theorem

Let $n \in \mathbb{N}, q \in [4, \sqrt{n}]$ be a positive integer. Let $\mathcal{L}$ be a lattice of rank n. There is a randomized algorithm that solves SVP in time $q^{13n+o(n)}$ and in space $\mathrm{poly}(n) \cdot q^{\frac{16n}{q^2}}$.

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SVP to BDD reduction [CCL18]

Lemma (CCL18, simplified)

Given a α-BDD oracle and p an integer, one can enumerate all lattice points in a ball of radius $p\alpha\lambda_1$ using p^n queries to the oracle.

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Solve SVP by using a α-BDD oracle:

- Set $p = \lceil \frac{1}{\alpha} \rceil$.
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The reduction is space efficient

But $\alpha < \frac{1}{2} \implies p \geq 3 \implies$ at least 3^n queries

Quantum SVP

Classical SVP to BDD: do 3^n queries to $1/3$-BDD and keep minimum

$$\boxed{\text{SVP}} \xrightarrow[\text{[CCL18]}]{} \boxed{1/3\text{-BDD}} \xrightarrow[\text{[DSR14]}]{} \boxed{\text{DGS}}$$

[ADRS15]
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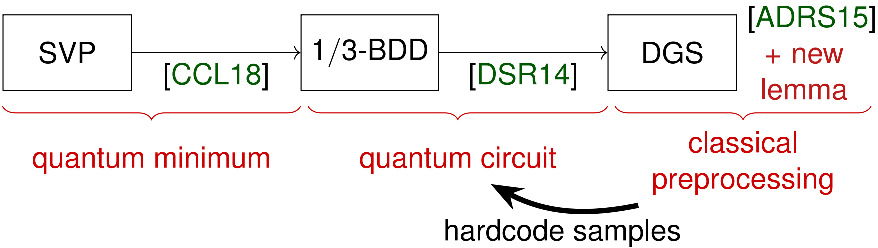

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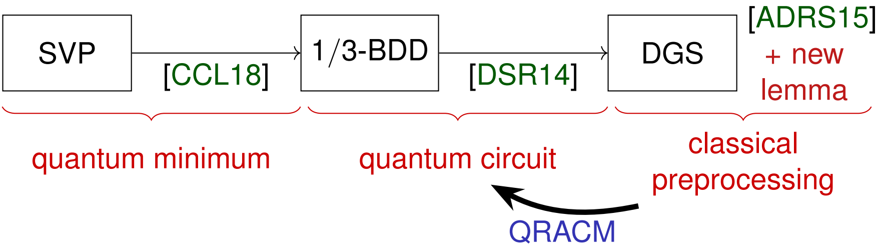

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DGS sampling: new lemma

- [ADRS15]: DGS of parameter $s \geqslant \sqrt{2}\eta_{1/2}(\mathcal{L})$ in time $2^{n/2}$
- BDD to DGS reduction requires $s = \eta_\varepsilon(\mathcal{L})$ for some $\varepsilon > 0$

Previous work [CCL18]: find ε such that $\eta_\varepsilon(\mathcal{L}) \geqslant \sqrt{2}\eta_{1/2}$
$\rightsquigarrow$ very small ε, larger than necessary BDD radius, too expensive BDD

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New idea:

- find a well-chosen lattice $\mathcal{L} \subset \mathcal{L}' \subset \mathcal{L}/2$ such that $\eta_{\varepsilon'}(\mathcal{L}') \leqslant \eta_\varepsilon(\mathcal{L})/\sqrt{2}$ for $\varepsilon' \approx \varepsilon$ [ADRS15]
- run DGS on $\mathcal{L}'$ at $s = \eta_{1/3}(\mathcal{L}) \geqslant \sqrt{2}\eta_{1/2}(\mathcal{L}')$ [ADRS15]
- only keep samples in $\mathcal{L}$ (rejection)

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Some details:

- $\mathcal{L}'$ is chosen randomly, works with high probability
- need that $|\mathcal{L}'/\mathcal{L}| \approx 2^{n/2}$ for $\varepsilon \approx \varepsilon'$
- rejection: $|\mathcal{L}'/\mathcal{L}| \approx 2^{n/2}$ slowdown, still better than previous work!
- allows to choose $\alpha = 1/3$ for BDD, improved from 0.391 [CCL18]

Reduction from BDD to DGS

Periodic Gaussian function $f(\boldsymbol{t}) := \frac{\rho(\boldsymbol{t}+\mathcal{L})}{\rho(\mathcal{L})} = \mathbb{E}_{\boldsymbol{w} \sim \mathcal{D}_{\mathcal{L}^*}}[\cos(2\pi\langle \boldsymbol{w}, \boldsymbol{t} \rangle)]$

- f achieves maximum on lattice points
- a constant number of gradient ascent steps solves BDD

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Approximate f by

$$f_W(t) = \frac{1}{N}\sum_{i=1}^{N}\cos(2\pi\langle w_i, t\rangle)$$

where $w_1, \ldots, w_n$ are i.i.d. DGS samples: small error if N is very large.

Theorem (Dadush, Regev, Stephens-Davidowitz (Informal))

There is an algorithm that solves α-BDD using N samples from $D_{\mathcal{L}^*,\eta_\varepsilon(\mathcal{L}^*)}$ in time $N \cdot \mathrm{poly}(n)$, where $N = O\left(n^{\frac{\log(1/\varepsilon)}{\sqrt{\varepsilon}}}\right)$ and $\alpha = \alpha(\varepsilon)$.

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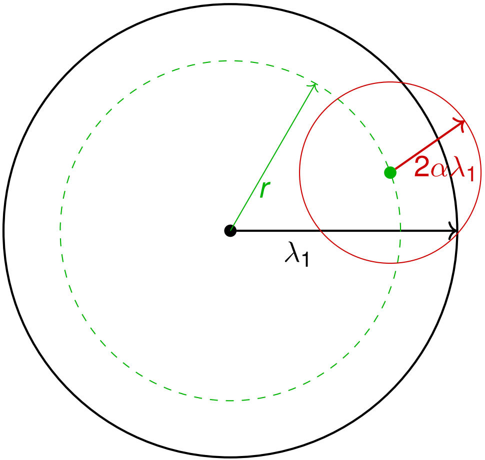
Our algorithm: approximate f_W quantumly in time $\sqrt{N}\cdot\mathrm{poly}(n)$
Use amplitude estimation and show that the error stays small

Theorem (Informal)

There is an quantum algorithm that solves α-BDD using N samples from $D_{\mathcal{L}^*,\eta_\varepsilon(\mathcal{L}^*)}$ in time $\sqrt{N}\cdot\mathrm{poly}(n)$, where $N = O\left(n^{\frac{\log(1/\varepsilon)}{\sqrt{\varepsilon}}}\right)$ and $\alpha = \alpha(\varepsilon)$. It requires a QRACM of size N and $O(N)$ preprocessing time.

Faster SVP to BDD reduction

Cover the sphere of radius $\lambda_1(\mathcal{L})$ by balls of radius $2\alpha\lambda_1(\mathcal{L})$:



Use 2^n $\alpha-$BDD queries to enumerate points in balls of radius $2\alpha\lambda_1$

Faster SVP to BDD reduction

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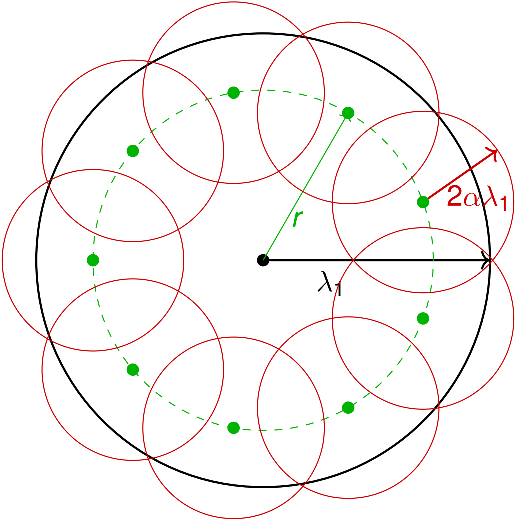

Use 2^n $\alpha-$BDD queries to enumerate points in balls of radius $2\alpha\lambda_1$

Each ball covers a spherical cap.

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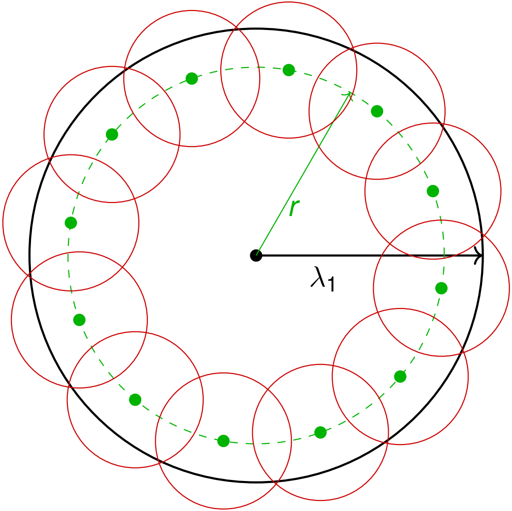

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Each ball covers a spherical cap.

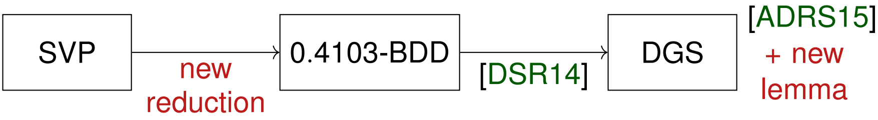
Smaller α:

- More balls
- Less expensive BDD

$\rightsquigarrow$ Trade-off

Improved classical SVP

Improved SVP to BDD: do 2^n queries to 0.4103-BDD



Theorem

There is a classical algorithm that solves SVP in time $2^{1.7397n+o(n)}$, *classical space* $2^{0.5n+o(n)}$.

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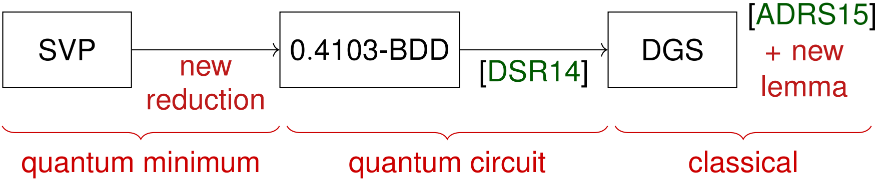

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Theorem

*There is a **quantum** algorithm that solves SVP in time $2^{1.051n+o(n)}$, classical space $2^{0.5n+o(n)}$ and $\mathrm{poly}(n)$ qubits.*

Not as good as our previous $2^{0.9529n+o(n)}$ algorithm but the story does not stop here...

SVP and Generalized Kissing Number

Number of lattice points in a ball of radius r is $\leqslant c^{n+o(n)} r^n$

$$\beta(\mathcal{L}) = \text{smallest } c \text{ that works for all } r$$

- Upper bound: $\beta(\mathcal{L}) \leqslant 2^{0.401}$ [KL78]
- Conjectured to be $\beta(\mathcal{L}) \approx 1$ for most lattices

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$$\boxed{\alpha\text{−BDD}} \xrightarrow[\text{reduce}]{} \boxed{\mathrm{DGS}_{\eta_\varepsilon}}$$

Best known relations between α and ε depends on $\beta(\mathcal{L})$:

small $\beta(\mathcal{L})$ $\rightsquigarrow$ bigger α for fixed ε $\rightsquigarrow$ less expensive BDD

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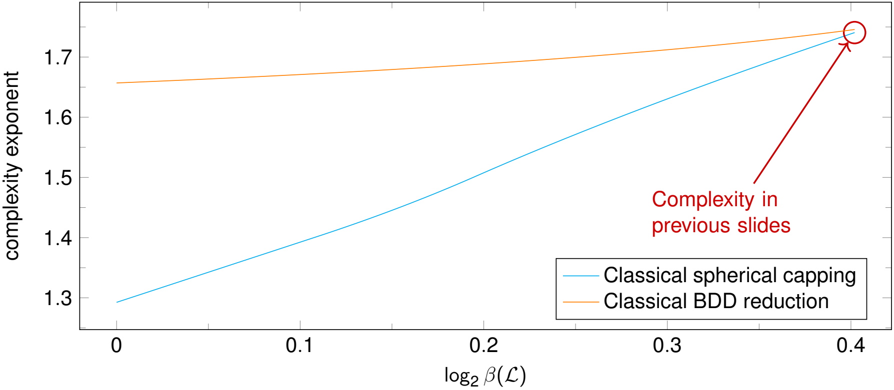

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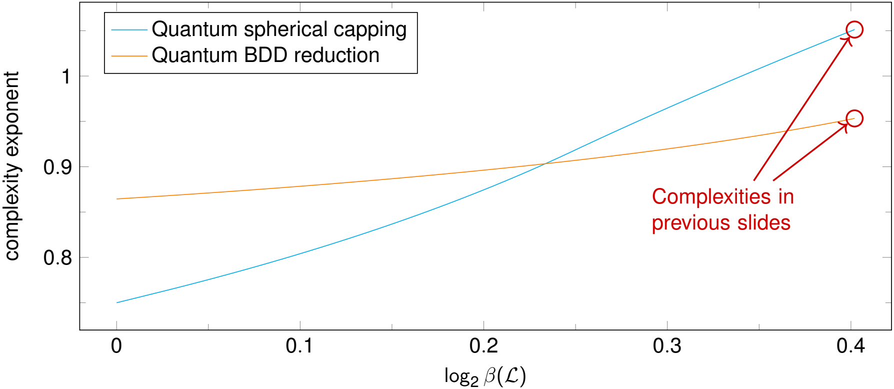

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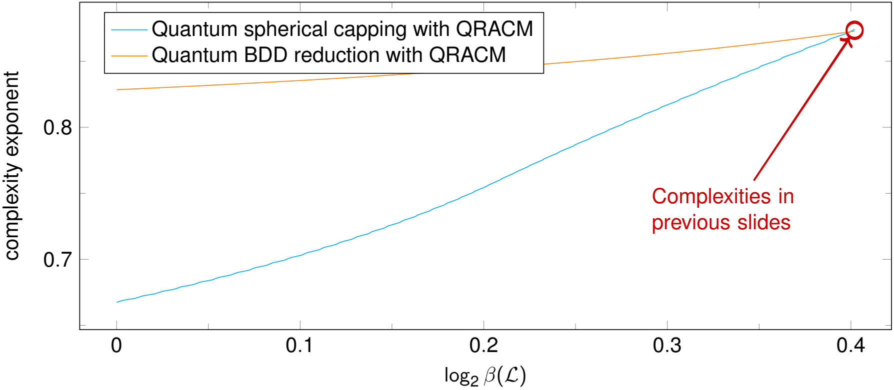

Conclusions and Future work

Provable SVP:

- classical: time $2^{1.7397n+o(n)}$, space $2^{0.5n+o(n)}$
- quantum: $2^{0.9529n+o(n)}$, classical space $2^{0.5n+o(n)}$ and $\mathrm{poly}(n)$ qubits
- quantum: $2^{0.873n+o(n)}$, classical space $2^{0.5n+o(n)}$ and QRACM $2^{0.1604n+o(n)}$
- first time/space tradeoff: time q^{13n}, space q^{16n/q^2} for $q \in [4, \sqrt{n}]$
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Open problems:

- Show that random lattices satisfy $\beta(\mathcal{L}) \approx 1$?
- Fill the gap between provable and heuristic algorithms for sieving?
- Exploit the subexponential space regime in our trade-off for SVP?
- $2^{O(n)}$ time, $2^{o(n)}$ space algorithm for DGS at smoothing parameter?