Quantum Algorithms for Lattice-based Cryptography

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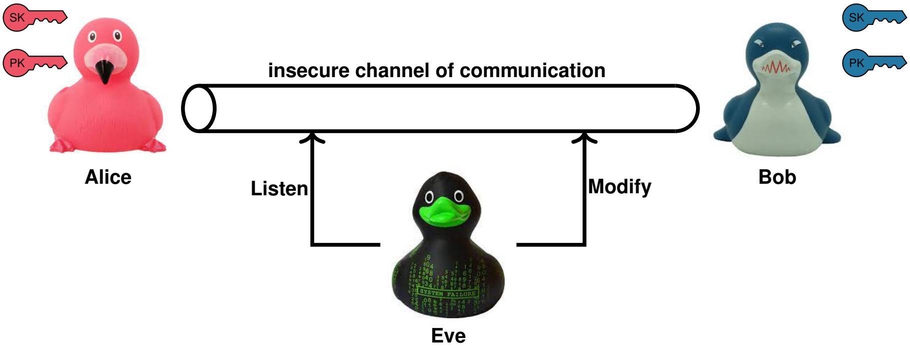
November 1, 2022

Hard Problems in Public Key Cryptography



factorisation, discrete logarithm

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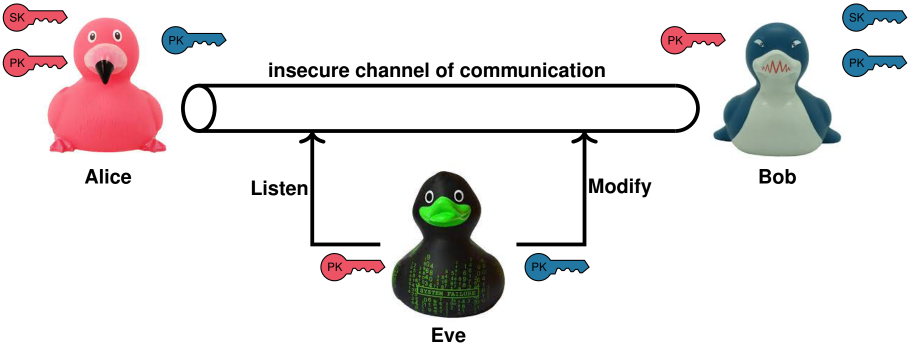

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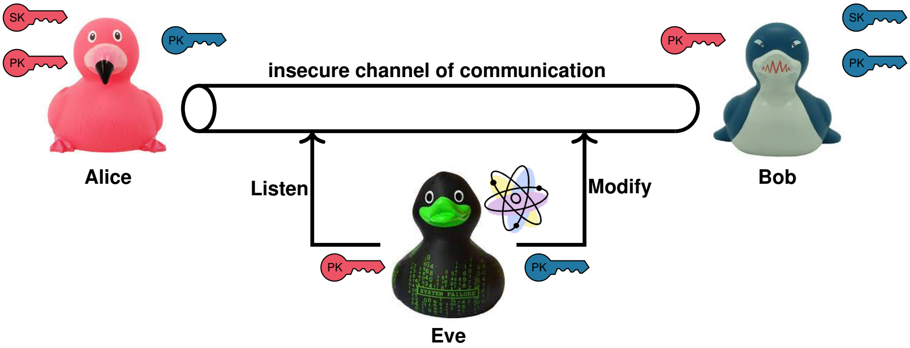

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lattices, codes, ...

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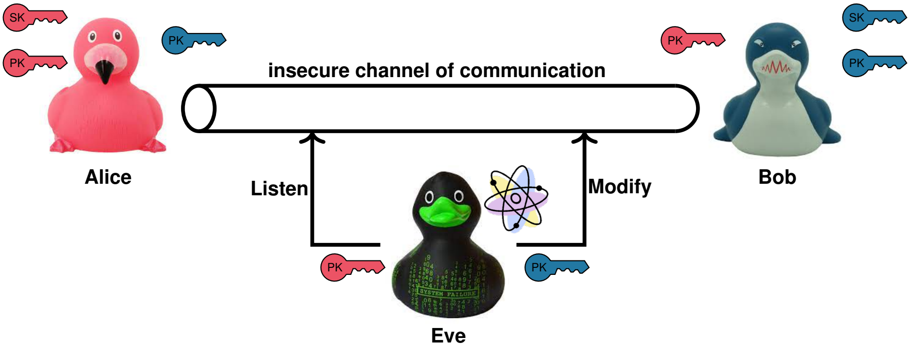

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lattices, codes, ...

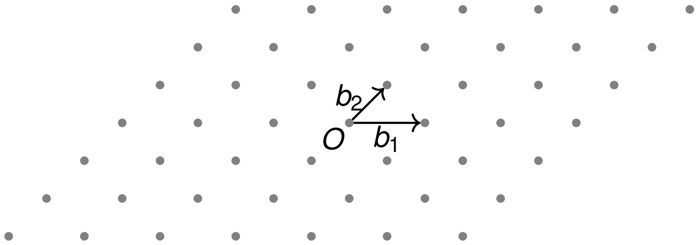
NIST selected algorithms:

- encryption: the only selected candidate is based on lattices
- signatures: 2 out of 3 based on lattices

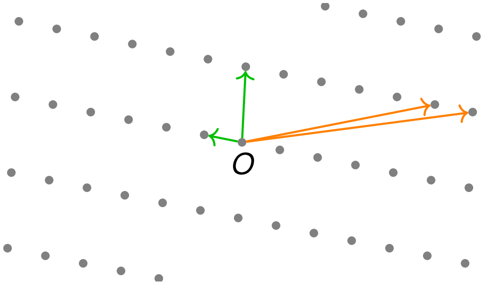
What is a (Euclidean) lattice?

Definition

$\mathcal{L}(\boldsymbol{b}_1, \dots, \boldsymbol{b}_n) = \left\{ \sum_{i=1}^{n} x_i \boldsymbol{b}_i : x_i \in \mathbb{Z} \right\}$ where $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ is a basis of $\mathbb{R}^n$.



Lattice-based cryptography: fundamental idea



- good basis: private information, makes problem easy
- bad basis: public information, makes problem hard

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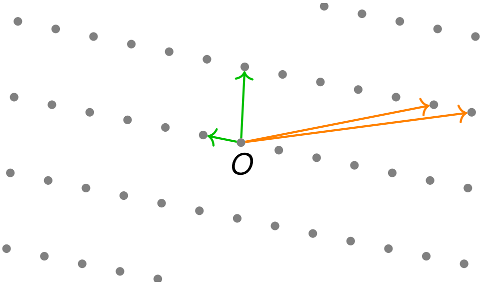

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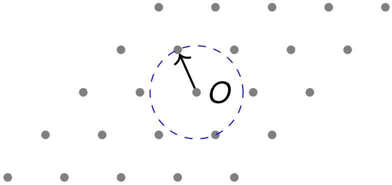
Basis reduction: transform a bad basis into a good one
Main tool: BKZ algorithm and its variants

Requires to solve the (approx-)SVP problem in smaller dimensions.

The Shortest Vector Problem



Shortest Vector Problem (SVP):
Given a basis for the lattice $\mathcal{L}$, find a shortest nonzero lattice vector.
$\lambda_1(\mathcal{L})$ = length of such a vector.

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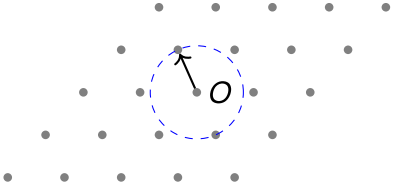

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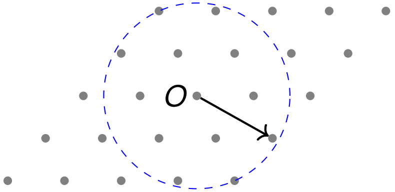

γ-approx-SVP ($\gamma > 1$):
Given a basis of $\mathcal{L}$, find a nonzero lattice vector of length at most $\gamma \cdot \lambda_1(\mathcal{L})$.
γ is approximation factor.

The Shortest Vector Problem

Depending on the dimension n:

- NP-Hardness (randomized reduction)
- NP $\cap$ co-NP
- Subexponential-time algorithms
- Poly-time algorithms

Approx factor:

- $O(1)$
- $\sqrt{n}$
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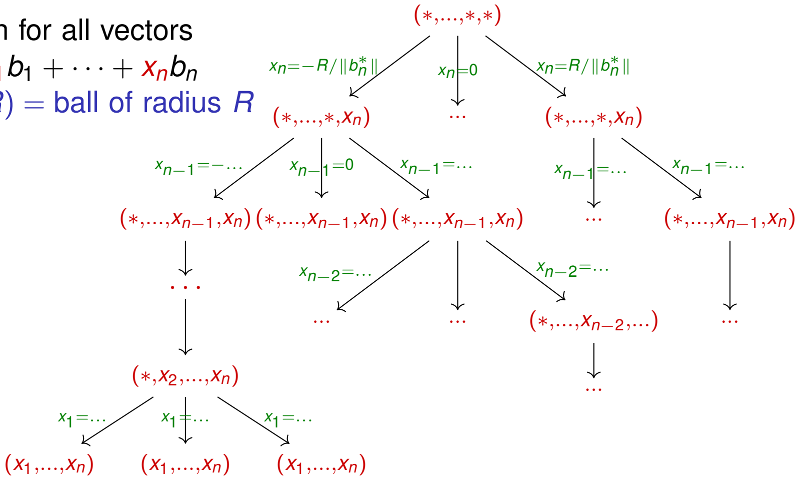
Main approaches for SVP:

- Enumeration: $2^{O(n \log(n))}$ time and $\text{poly}(n)$ space
- Sieving: $2^{O(n)}$ time and $2^{O(n)}$ space

Enumeration Algorithm

Search for all vectors
$X = x_1 b_1 + \cdots + x_n b_n$
in $B(R)$ = ball of radius R



Given $x_n, \ldots, x_{i+1}, \|x\| \leqslant R \Rightarrow$ the integer x_i belongs to an interval of small length

Quantum Speed-up for Enumeration

Quantum Backtracking [Montanaro15]

- blackbox access to a tree with marked nodes:
  - can only query the local structure of the tree
- tree of size T, depth n, constant max degree

$\Rightarrow$ $O^*(\sqrt{T})$ queries to find a marked node [ANS18]

Combine with clever ways of pruning the tree

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Variational Quantum Algorithm [APSW22]

- ignore the tree structure, encode SVP into a QUBO
- upper bound the x_is
- modify the classical loop of VQE to target directly the first excited state
- 1500 qubits suffice to solve SVP in dim 180
- time complexity? unknown...

Sieving Algorithms

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- Reduce basis
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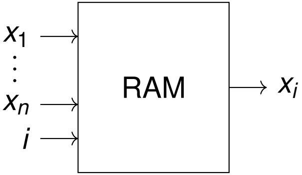
Many heuristic variants: local sensitive hash, tuple sieve, ...

Interlude: quantum memory models

classical access

quantum access

classical data



standard

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Assumption: $O(1)$ time cost

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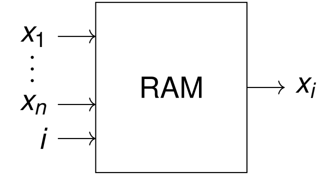

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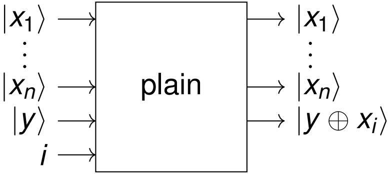

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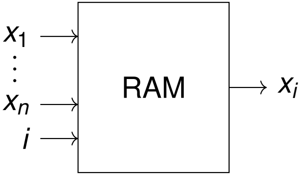

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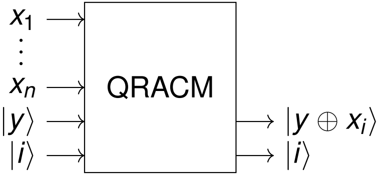
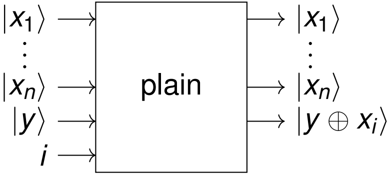


potentially strong assumption

quantum data



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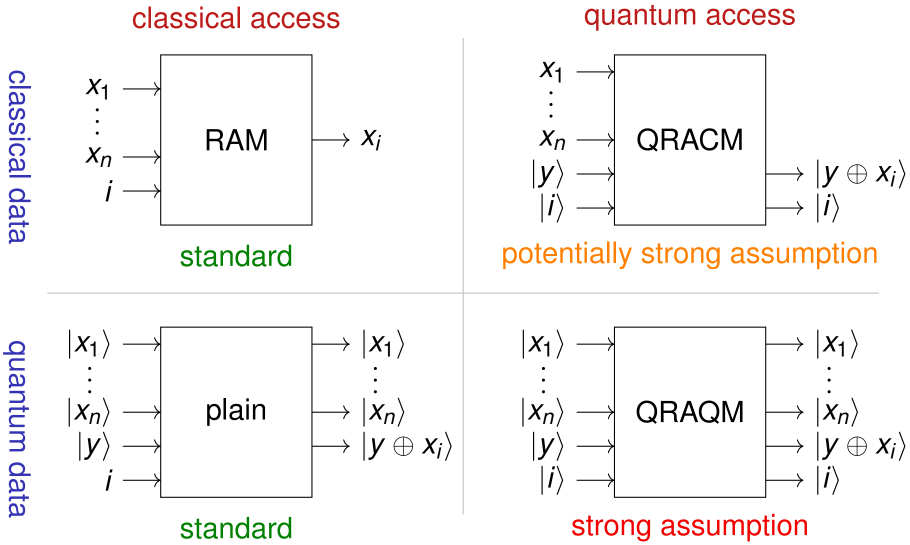

Assumption: $O(1)$ time cost

Results in the Classical and Quantum Setting

Time Complexity	PH	Space Complexity: Classical	Qubits	Model	Reference
$2^{n+o(n)}$	P	$2^{n+o(n)}$	N/A	N/A	[LMP15]
$2^{0.950n+o(n)}$	P	$2^{0.5n+o(n)}$	poly(n)	plain	[ACKS21]
$2^{0.835n+o(n)}$	P	$2^{0.5n+o(n)}$	poly(n)	QRACM	[ACKS21]
$2^{0.292n+o(n)}$	H	$2^{0.292n+o(n)}$	N/A	N/A	[BDGL16]
$2^{0.265n+o(n)}$	H	$2^{0.265n+o(n)}$	poly(n)	QRACM	[Laa15]
$2^{0.2563n+o(n)}$	H	$2^{0.2075n+o(n)}$	poly(n)	QRAQM	[BCSS22]

P=provable, H=heuristic

Tools: Quantum Amplitude Amplification/Estimation, Quantum Walks

Currently interested in: Heuristic algorithms for lattice sieving without QRACM. Such algorithm exists for collision finding: [CNS17].