Assignment 2 Writeup

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December 3, 2020

1 Fishing and extinction

(a)

$$
\begin{aligned}
&H = py_t - c(y_t) + \mu_t(f(x_t) - y_t) \\
&\quad FOC_{y_t}: \quad p - c'(y_t) \leq \mu_t, \quad y_t \geq 0, \quad c.s. \\
&\quad FOC_{x_t}: \quad \dot{\mu}_t = r\mu_t - \mu_t f'(x_t) \\
&\quad TVC: \quad \lim_{t\to\infty} x_t \geq 0, \quad \lim_{t\to\infty} \mu e^{-rt} \geq 0, \quad \lim_{t\to\infty} \mu_t x_t e^{-rt} \geq 0
\end{aligned}
$$

(b) If in the steady state, $x_t = 0$, then we know from FOC_{x_t} that $f'(0) = r$. Thus in order for extinction to not be optimal, it must be that $f'(0) > r$.

(c) T find the optimal steady state stock, note that $\dot{y}_t = 0$ implies $\dot{\mu}_t = 0$ from the FOC on y_t. Combining this with the FOC on x_t, we can solve for x_{ss}:

$$f'(x_{ss}) = r$$

Dynamics

$\dot{x}_t = 0$: From the LOM, we have that $y_t = f(x_t)$

- If $y_t > f(x_t) \Rightarrow \dot{x}_t < 0$
- If $y_t < f(x_t) \Rightarrow \dot{x}_t > 0$

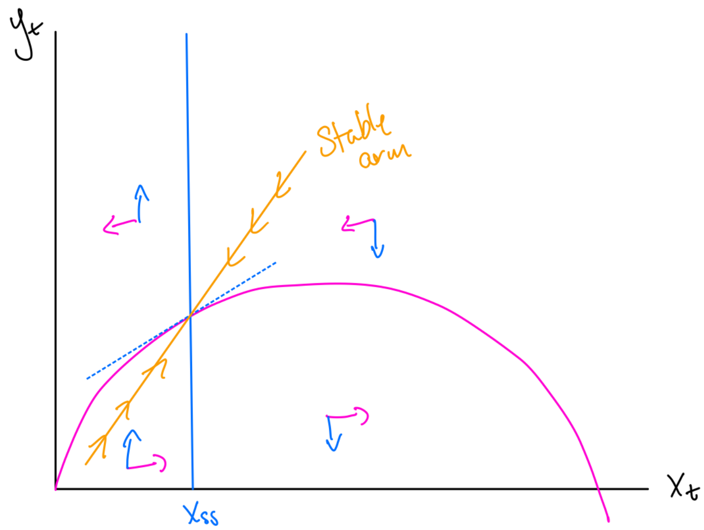
$\dot{y}_t = 0$:

- take total derivative with respect to t of $FOC_{y_t}: p - c'(y_t) = \mu_t$:

$$
\begin{aligned}
-c''(y_t)\dot{y}_t &= \dot{\mu}_t \\
&= \mu_t(r - f'(x_t)) \\
&= (p - c'(y_t))(r - f'(x_t))
\end{aligned}
$$

$$\Rightarrow \dot{y}_t = \frac{\overbrace{p - c'(y_t)}^{p>c'}}{\underbrace{-c''(y_t)}_{c''>0}}(r - f'(x_t))$$

- $r > f'(x_t) \Rightarrow \dot{y}_t < 0$
- $r < f'(x_t) \Rightarrow \dot{y}_t > 0$



(d) The new TVC is

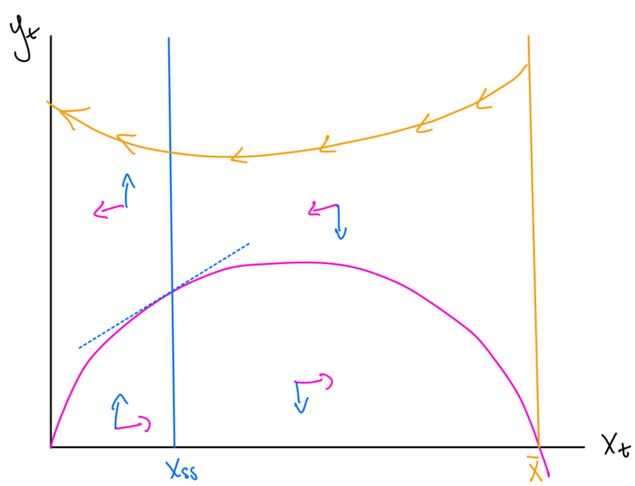
$$x_T \geq 0; \quad \mu_T e^{-rT} \geq 0; \quad x_T \mu_T e^{-rT} = 0$$

We know $p - c'(y_t) > 0$, so at time T,

$$\mu_T e^{-rT} \geq p - c'(y_t) > 0$$

Therefore, $x_T = 0$.

(e) Now, as we move closer to T, the fish next period is worth less and less



2 Common pool with exhaustible resource

(a)

$$H = py_{it} - c(y_{it}) - \mu_t(y_{it} + \sum_{j \neq i} y_j(x_t))$$

$$FOC_{y_t}: \quad p - c'(y_{it}) - \mu_t \leq 0; \quad y_t \geq 0,; \quad c.s.$$

$$FOC_{x_t}: \quad \dot{\mu}_t = r\mu_t + \mu_t \sum_{j \neq i} y_j'(x_t)$$

$$TVC: \quad \lim_{t \to \infty} x_t \geq 0, \quad \lim_{t \to \infty} \mu e^{-rt} \geq 0, \quad \lim_{t \to \infty} \mu_t x_t e^{-rt} \geq 0$$

Invoking symmetry, rewrite FOC_{x_t} :

$$\dot{\mu}_t = r\mu_t + \mu_t(N-1)\frac{dy_{it}}{dx_t}$$

(b) Imposing the plausible assumption that $\frac{dy_{it}}{dx_t} > 0$, this reduces the steady state stock, because if I leave some oil in the ground, this means other people will get the oil instead of me, causing an externality.

3 Highway congestion externalities

(a) The driver's problem is:

$$\min \{\alpha x, 1\}$$

If $\alpha x < 1$, I would take the bridge. So the number of people who take the bridge are

$$x = \frac{1}{\alpha}$$

(b) The social planner's problem is

$$\min_x \quad (\alpha x) \times x + (N - x) \times 1$$

$$FOC_x : x^\star = \frac{1}{2\alpha}$$

(c) The externality is the extra hours of driving imposed on people at $x^\star$ by having an extra driver on the road (and adding a price to it using w). Then we can write the externality as

$$\frac{dz}{dx} \times x^\star \times w = \frac{w}{2}$$

(d) Adding $\frac{w}{2}$ to the driver's problem, we have

$$\min \left\{ \alpha x + \frac{1}{2}, 1 \right\}$$

So we know that the number of people who will take the bridge is determined by

$$\alpha x + \frac{1}{2} = 1$$

$$x = \frac{1}{2\alpha}$$

Who is better off? Without the toll:

- People who take the bridge spend $wz = w\alpha\frac{1}{\alpha} = w$ hours
- People who take the alternative route spend w

With the toll:

- People who take the bridge spend $wz + \frac{1}{2} = w\left(\alpha\frac{1}{2\alpha} + \frac{1}{2}\right) = w$
- People who take the alternative route still spend w

So the drivers do not gain anything, and all of the efficiency gain goes to the government

4 Targeting Pigouvian taxes

(a) Social Planner's Problem:

$$\max_{k_i,e} af(k,e) - \frac{b}{2}f(k,e)^\alpha - \sum_i k_i r_i - \tau e$$

Firm's Problem:

$$\max_{k_i,e} af(k,e) - \frac{b}{2}f(k,e)^\alpha - \sum_i k_i r_i - te$$

The optimal tax is clearly $t^\star = \tau$.

(b) Now the firm's problem is:

$$\begin{aligned}\max_{k_i,e} \quad & af(k,e) - \frac{b}{2}f(k,e)^\alpha - \sum_i k_i r_i - tf(k,e) \\ & FOC_{k_i} : af_i(k,e) - bf_i(k,e)f(k,e) - r_i - tf_i(k,e) = 0 \\ & FOC_e : af_e(k,e) - bf_e(k,e)f(k,e) - tf_e(k,e) = 0\end{aligned}$$

The social planner's FOC are

$$\begin{aligned}\max_{k_i,e} \quad & af(k,e) - \frac{b}{2}f(k,e)^\alpha - \sum_i k_i r_i - \tau e \\ & FOC_{k_i} : af_i(k,e) - bf_i(k,e)f(k,e) - r_i = 0 \\ & FOC_e : af_e(k,e) - bf_e(k,e)f(k,e) - \tau = 0\end{aligned}$$

Matching FOC, we need

$$\begin{aligned}tf_i(k,e) &= 0 \\ tf_e(k,e) &= \tau\end{aligned}$$

Which can only hold if $t = 0$ and $\tau = 0$

(c) Now the firm's problem is:

$$\begin{aligned}\max_{k_i,e} \quad & af(k,e) - \frac{b}{2}f(k,e)^\alpha - \sum_i k_i r_i - \sum_i k_i t_i \\ & FOC_{k_i} : af_i(k,e) - bf_i(k,e)f(k,e) - r_i - t_i = 0 \\ & FOC_e : af_e(k,e) - bf_e(k,e)f(k,e) = 0\end{aligned}$$

Again, this can only match the Social Planner's FOC if $t_i = 0$ and $\tau = 0$

(d) Combining the two tax policies, the firm problem becomes

$$\max_{k_i,e} \quad af(k,e) - \frac{b}{2}f(k,e)^\alpha - \sum_i k_i r_i - t_0 f(k,e) - \sum_i k_i t_i$$
$$FOC_{k_i} : af_i(k,e) - bf_i(k,e)f(k,e) - r_i - t_i - t_0 f_i(k,e) = 0$$
$$FOC_e : af_e(k,e) - bf_e(k,e)f(k,e) - t_0 f_e(k,e) = 0$$

To match the social planner's FOC, we want to set

$$t_i = -t_0 f_i(k,e)$$
$$t_0 = \frac{\tau}{f_e(k,e)}$$

(e) The uncertainty around private benefit induces uncertainty around marginal cost for the social planner. Invoking the intuition from Weitzman's Prices Vs Quantities, in the world of uncertain marginal costs, if marginal benefit is relatively flat (which is the case here, since $MB = \tau$), then a tax policy is better than a quantities based policy.

5 Permit market dynamics

(a)

$$\int_0^{10} D(p(0)e^{rt})dt = \overline{S}$$

(b) Firms will want to use all permits by the time the price hits p^c (otherwise there is opportunity for arbitrage), which is at time $10 - \tau$. This gives us the equations:

$$\int_0^{10-\tau} D(p(0)e^{rt})dt = \overline{S}$$
$$\Delta = \tau D(p^c)$$

(c) If G is less than Δ, then firms will launch a speculative attack, which raises price up because they suddenly buy up the government's entire stock.

(d) In this scenario, price immediately starts rising, and is rising at r to still satisfy the condition that firms are indifferent from one moment to the next. This gives us the condition that

$$G < \int_{10-\tau}^{10} D(p^c e^{rt})dt$$

(e) Again to avoid arbitrage opportunities, the price is always rising at r, and the whole price curve shifts so that $p(0) = p^f$. At $t = 0$, governments purchase all the permits necessary so that $p(0) = p^f$. Then we have

$$\beta = \overline{S} - \int_0^{10} D(p^f e^{rt})dt$$

6 Clean backstop technology

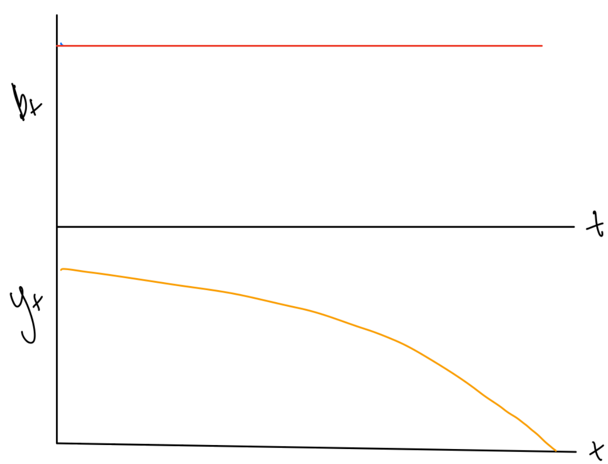
(a) Planner's problem:

$$
\begin{aligned}
max_{y_t,b_t} \int_0^\infty e^{-rt} \left(p[y_t + b_t] - c_y(y_t) - c_b(b_t)dt\right. \\
s.t. \quad \dot{S}_t = -y_t \\
S(0) = S_0 \\
y_t \geq 0 \\
b_t \geq 0
\end{aligned}
$$

The Hamiltonian and FOC are:

$$
\begin{aligned}
&H = p[y_t + b_t] - c_y(y_t) - c_b(b_t) - \mu_t y_t \\
&\quad FOC_{y_t}: \quad p - c_y'(y_t) - \mu_t \leq 0, \quad y_t \geq 0, \quad c.s. \\
&\quad FOC_{b_t}: \quad p - c_b'(b_t) \leq 0, \quad b_t \geq 0, \quad c.s. \\
&\quad FOC_{s_t}: \quad \dot{\mu}_t = r\mu_t \\
&\quad TVC: \quad \lim_{t\to\infty} \mu_t e^{-rt} \geq 0; \quad \lim_{t\to\infty} S_t \geq 0; \quad \lim_{t\to\infty} S_t \mu_t e^{-rt} = 0
\end{aligned}
$$

(b) y_t here is decreasing and concave. The concavity is due to the strict convexity of the cost function.



(c) Using the finite time TVC, we know $S_T > 0$ so it must be that $\mu_T = 0$. Combining this with FOC_{s_t}, we have that $\mu_t = 0$ for all t. Substituting this into FOC_{y_t}, we know

$$p = c_y'(y_t)$$

y_t found using this equation is a constant, call it $\bar{y}$, and we can use this to solve for the T such that there will be some stock left in the ground:

$$\bar{y} \times T \leq S_0$$

(d) The social planner's Hamiltonian and the firm's Hamiltonian are, respectively:

$$\begin{aligned} H^{SP} &= p[y_t + b_t] - c_y(y_t) - c_b(b_t) - \mu_t y_t - \tau(e_y y_t + e_b b_t) \\ H^F &= p[y_t + b_t] - c_y(y_t) - c_b(b_t) - \mu_t y_t - t_y e_y y_t - t_b e_b b_t \end{aligned}$$

Thus we can clearly see that by setting the Pigouvian tax to be $t_y = \tau e_y$ and $t_b = \tau e_b$, we can reach the social planner's optimal outcome.

(e) The firm's Hamiltonian is now:

$$\begin{aligned} H^F &= p[y_t + b_t] - c_y(y_t) - c_b(b_t) - \mu_t y_t - \lambda(\sigma[y_t + b_t] - e_y y_t - e_b b_t) \\ FOC_{y_t}: &\quad p - c_y'(y_t) - \mu_t + \lambda(\sigma - e_y) \leq 0, \quad y_t \geq 0, \quad c.s. \\ FOC_{b_t}: &\quad p - c_b'(y_t) + \lambda(\sigma - e_b) \leq 0, \quad b_t \geq 0, \quad c.s. \end{aligned}$$

The social planner's FOCs are:

$$\begin{aligned} FOC_{y_t}: &\quad p - c_y'(y_t) - \mu_t 0 \tau e_y \leq 0, \quad y_t \geq 0, \quad c.s. \\ FOC_{b_t}: &\quad p - c_b'(y_t) - \tau e_b \leq 0, \quad b_t \geq 0, \quad c.s. \end{aligned}$$

By matching terms in the FOCs, we see that to reach first best, we need

$$\tau e_b = \lambda(\sigma - e_b)$$

which only happens if $e_b = 0$ (otherwise there is always a subsidy). If $e_b = 0$, it must be that $\sigma = 0$, which means

$$\begin{aligned} \frac{e_y y_t}{y_t + b_t} &\leq 0 \\ \Rightarrow y_t &= 0 \end{aligned}$$

This can only be first best if $y_t = 0$ in the first best situation already. So when is optimal not to extract at all? This happens only when

$$p - c_y'(0) - \mu_t - \tau e_y < 0, \quad \forall t$$

Following a similar logic to part (c), from the TVC we know that if there's oil left in the ground in the limit, it must be that $\mu_t = 0 \quad \forall t$. Then we are left with

$$p + \tau e_y < c_y'(0)$$

7 Mustangs and Civics

(a) The planner's problem is:

$$\begin{aligned} \max_{m_f, m_s} \quad & U_F(m_F) + U_S(m_S) - (p + \tau\varphi_F)e_F m_f - (p + \tau\varphi_S)e_S m_S \\ & FOC_{m_F} : U_F'(m_F) - (p + \tau\varphi_F)e_F = 0 \\ & FOC_{m_S} : U_S'(m_S) - (p + \tau\varphi_S)e_S = 0 \end{aligned}$$

The student's problem is:

$$\max_{m_s} \quad U_S(m_s) - (p+t)e_S m_S$$
$$FOC_{m_S} : U_S'(m_S) - (p+t)e_S = 0$$

The faculty's problem is:

$$\max_{m_f} \quad U_F(m_F) - (p+t)e_F m_F$$
$$FOC_{m_F} : U_F'(m_F) - (p+t)e_F = 0$$

Matching FOCs, we see that we cannot simultaneously satisfy the following two equalities:

$$t = \tau\varphi_S$$
$$t = \tau\varphi_F$$

so a single gasoline tax alone cannot achieve the first-best welfare outcome.

We can find the second-best tax by trying to minimize the difference between the planner's FOC and the faculty/student's FOC. From above, we know that optimally, we want $t = \tau\varphi_S$ and $t = \tau\varphi_F$, so let's use this to set up a loss function and associated minimization problem:

$$\min_t \quad (t - \tau\varphi_S + t - \tau\varphi_F)^2$$
$$FOC_t : 4(2t - \tau\varphi_S - \tau\varphi_F) = 0$$
$$\Rightarrow t = \frac{\tau\varphi_S + \tau\varphi_F}{2}$$

(b) Under the mindset of Jacobsen, Knitte, Sallee, and Van Benthem, the DWL is

$$\text{DWL} = -\frac{1}{2}\overline{\frac{\partial x_j}{\partial t_j}}\sum_{j=1}^{J} e_j^2$$

Since the response to miles driven to a change in price of gasoline is identical, we can ignore that for the purposes of this DWL calculation. Let's think about the "error" under second-best and under no policy, as compared to first-best.

$$e_F^{2nd} = \tau\varphi_F - \frac{\tau\varphi_S + \tau\varphi_F}{2}$$
$$= \frac{\tau\varphi_S - \tau\varphi_F}{2}$$
$$e_S^{2nd} = \frac{\tau\varphi_F - \tau\varphi_S}{2}$$
$$e_F^{No} = \tau\varphi_F$$
$$e_S^{No} = \tau\varphi_S$$

Then we can compared DWL values between the second best and the no policy scenario by comparing squared error terms:

$$
\begin{aligned}
\frac{DWL^{2nd}}{DWL^{No}} &= \frac{\left(\frac{\tau\varphi_S - \tau\varphi_F}{2}\right)^2 + \left(\frac{\tau\varphi_F - \tau\varphi_S}{2}\right)^2}{(\tau\varphi_F)^2 + (\tau\varphi_S)^2} \\
&= \frac{\left(\frac{\varphi_S - \varphi_F}{2}\right)^2 + \left(\frac{\varphi_F - \varphi_S}{2}\right)^2}{\varphi_F^2 + \varphi_S^2} \\
&= \frac{\frac{1}{2}\varphi_F^2}{5\varphi_F^2} \\
&= 0.1
\end{aligned}
$$

Thus the percentage of the first-best welfare gain that is achievable with the second-best optimal policy is 1 - 0.1 = 0.9.

(c) Let's rewrite the planner's problem, noting that a student's miles driven are a function of the price and tax:

$$
\begin{aligned}
\max_t \quad & U_F(m_F) + U_S(m_S(p+t)) - (p + \tau\varphi_F)e_F m_f - (p + \tau\varphi_S)e_S m_S(p+t) \\
& FOC_t : \left(U_S'(m_S(p+t)) - (p + \tau\varphi_S)e_S\right)\frac{dm_S(p+t)}{dt} = 0
\end{aligned}
$$

We know from (a) that the FOC for students gives us:

$$U_S'(m_s) = (p+t)e_s$$

Substituting this in, we get

$$
\begin{aligned}
(p+t)e_s &= (p + \tau\varphi_S)e_s \\
\Rightarrow t &= \tau\varphi_S
\end{aligned}
$$

The faculty do not respond to the tax anyways, so the first-best mileage is the one they would pick under any tax policy. The social planner focuses the tax on gasoline to affect emissions from the students, which achieves the first best mileage driven by students.