Comprehension Check 2

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1 Quantity and Revenue Per Worker

Given productivity ϕ, we have that price is

$$p(\phi) = \frac{w}{\rho\phi} = \frac{1}{\rho\phi}$$

Plugging this into the optimal consumption and revenue for individual varieties, we have

$$q(\phi) = Q(\rho\phi P)^{\sigma}$$
$$r(\phi) = R(\rho\phi P)^{\sigma-1}$$

We can plug in $q(\phi)$ into labor demand to get

$$\ell = f + \frac{Q(\rho\phi P)^{\sigma}}{\phi}$$

This gives us revenue and quantity per worker

$$\begin{aligned}
\frac{q(\phi)}{\ell(\phi)} &= \frac{Q(\rho\phi P)^{\sigma}}{f + \frac{1}{\phi}Q(\rho\phi P)^{\sigma}} \\
\frac{r(\phi)}{\ell(\phi)} &= \frac{R(\rho\phi P)^{\sigma-1}}{f + \frac{1}{\phi}Q(\rho\phi P)^{\sigma}} \\
&= \frac{R(\rho\phi P)^{\sigma-1}}{f + R\rho^{\sigma}(P\phi)^{\sigma-1}}
\end{aligned}$$

2 Isomorphic Quality Variant Model

<u>Price Index</u>: First set up the cost minimization problem

$$\min \int_{\omega\in\Omega} p(\omega)q(\omega)d\omega$$
$$\text{s.t. } \left[\int_{\omega\in\Omega} \varphi(\omega)^{\varepsilon}q(\omega)^{\rho}d\omega\right]^{\frac{1}{\rho}} = Q$$

Setting up the Lagrangian and taking FOC with respect to $q(\omega)$, we have:

$$p(\omega) = \lambda\varphi(\omega)^{\epsilon}q(\omega)^{\rho-1}Q^{1-\rho}$$

Raise both sides to $1-\sigma$:

$$\begin{aligned}
p(\omega)^{1-\sigma} &= \lambda^{1-\sigma}\varphi(\omega)^{\epsilon(1-\sigma)}q(\omega)^{\rho}Q^{-\rho} \\
\varphi(\omega)^{\epsilon\sigma}p(\omega)^{1-\sigma} &= \lambda^{1-\sigma}\varphi(\omega)^{\varepsilon}q(\omega)^{\rho}Q^{-\rho}
\end{aligned}$$

Integrate both sides to get

$$\int_{\omega\in\Omega} \phi(\omega)^{\varepsilon\sigma} p(\omega)^{1-\sigma} d\omega$$

$$P = \lambda = \left(\int_{\omega\in\Omega} \phi(\omega)^{\varepsilon\sigma} p(\omega)^{1-\sigma} d\omega\right)^{\frac{1}{1-\sigma}}$$

The last line is from the fact that λ is the shadow price of one additional unit of the aggregate good, thus giving us the price index.

Optimal Consumption and Expenditure: Rearrange the FOC from the minimization problem,

$$p(\omega) = P\varphi(\omega)^{\epsilon} q(\omega)^{\rho-1} Q^{1-\rho}$$
$$\Rightarrow q(\omega) = \left(\frac{p(\omega)}{PQ^{1-\rho}\varphi(\omega)^{\epsilon}}\right)^{\rho-1}$$

Recall that $\sigma = \frac{1}{1-\rho}$, so $\frac{1}{\sigma} = 1-\rho$:

$$q(\omega) = Q\left(\frac{p(\omega)}{P\varphi(\omega)^{\epsilon}}\right)^{-\sigma}$$

We know $R = PQ$ and $r(\omega) = p(\omega)q(\omega)$, so we can plug this in to get the optimal revenue:

$$r(\omega) = R\varphi(\omega)^{\epsilon\sigma}\left(\frac{p(\omega)}{P}\right)^{1-\sigma}$$

Equations (4) & (5) With the given production function, profits are

$$\pi(\varphi) = p(\varphi)q(\varphi) - f - q(\varphi)$$
$$FOC_p : p(\varphi) = 1 - \frac{q(\varphi)}{\partial q/\partial p}$$

Using the optimal consumption equation derived above, we have

$$\frac{\partial q}{\partial p} = -\sigma Q\varphi(\varphi)^{\varepsilon\sigma} P^{\sigma} p(\varphi)^{\sigma-1}$$
$$\Rightarrow \frac{q(\varphi)}{\partial q/\partial p} = \frac{p(\varphi)}{\sigma}$$

We can solve for price,

$$p(\varphi) = \frac{1}{\rho}$$

Plugging this into the equations for revenue and profits, we arrive at the analogous equations to (4) and (5):

$$r(\varphi) = R\varphi^{\varepsilon\sigma}(P\rho)^{\sigma-1}$$
$$\pi(\varphi) = \frac{R}{\sigma}\varphi^{\varepsilon\sigma}(P\rho)^{\sigma-1} - f$$

3 Quantity and Revenue Per Worker for Quality Variant

Following the same procedure as in section 1, we calculate revenue and quantity per worker,

$$
\begin{aligned}
\frac{q(\phi)}{\ell(\phi)} &= \frac{Q\left(P\rho\phi^{\varepsilon}\right)^{\sigma}}{f+Q\left(P\rho\phi^{\varepsilon}\right)^{\sigma}} \\
\frac{r(\phi)}{\ell(\phi)} &= \frac{R\phi^{\varepsilon\sigma}(P\rho)^{\sigma-1}}{f+Q\left(P\rho\phi^{\varepsilon}\right)^{\sigma}} \\
&= \frac{R\phi^{\varepsilon\sigma}(P\rho)^{\sigma-1}}{f+R\left(\rho\phi^{\varepsilon}\right)^{\sigma}P^{\sigma-1}}
\end{aligned}
$$

4 Isomorphism Implications

Comparing quantity and revenue per worker from section 3 to the results from section 1, we see that

$$
\begin{aligned}
\phi^{\epsilon\sigma} &= \phi^{\sigma-1} \\
\Rightarrow \epsilon = \rho &= \frac{\sigma-1}{\sigma}
\end{aligned}
$$

So the quantity and revenue per worker is the same in both models, regardless of whether we model productivity as quality variety (in which case the utility function captures relative demand for quality) or lower marginal cost (in which case the production function incorporates heterogeneity in costs).