Assignment 1 Writeup

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1 Discrete time resource dynamics: mine deepening

(a)
- State variables: x_t, r_t
- control variables: w_t, q_t

(b) Constraints:

- $r_{t+1} \leq r_t + f(w_t, x_t) - q_t$
- $x_{t+1} \geq x_t + w_t \rightarrow$ having more depth is bad because additions to reserves are decreasing in feet mined in prior periods

$$\begin{aligned}\mathcal{L} = &\sum_{t=1}^{T} \delta^t \left(pq_t - c\left(q_t, r_t\right) - k\left(w_t\right)\right) \\ &+ \sum_{t=1}^{T} \lambda_t \left(r_t - r_{t+1} + f\left(w_t, x_t\right) - g_t\right) \\ &+ \sum_{t=1}^{T} \mu_t \left(x_{t+1} - x_t - w_t\right)\end{aligned}$$

FOC:

(i) $\frac{\partial \mathcal{L}}{\partial q_t} = \delta^t(p - c_q\left(q_t, r_t\right)) - \lambda_t \leq 0; \quad q_t \geq 0; \quad c.s.$

(ii) $\frac{\partial \mathcal{L}}{\partial w_t} = -\delta^t k'\left(w_t\right) + \lambda_t f_w\left(w_t, x_t\right) - \mu_t \leq 0; \quad w_t \geq 0; \quad c.s.$

(iii) $\frac{\partial \mathcal{L}}{\partial r_t} = -\delta^t c_r\left(q_f, r_x\right) - \lambda_{t+1} + \lambda_t = 0$

(iv) $\frac{\partial \mathcal{L}}{\partial x_t} = \lambda_t f_x\left(w_{t+1} x_x\right) + \mu_{x-1} - \mu_t = 0$

(v) TVC for $r: \lambda_T \geq 0, \quad r_{T+1} \geq 0, \quad \lambda_\tau r_{T+1} = 0$

(vi) TVC for x: $\mu_T \leq 0, \quad x_{T+1} \geq 0, \quad \mu_T x_{T+1} = 0$

(c) Rearranging the second (ii), we have

$$\mu_t \geq \lambda_t f_w\left(w_t, x_t\right) - \delta^t - k'\left(w_t\right)$$

We see that μ_t, the marginal cost of more depth, must be greater than or equal to the marginal benefit of having more reserves, minus the marginal cost of digging.

(d)

$$\begin{aligned}-\delta^t c_r\left(q_t, r_x\right) - \lambda_{t-1} + \lambda_t &= 0 \\ \lambda_t - \lambda_{t-1} &= \delta^t c_r\left(q_t, r_t\right)\end{aligned}$$

Since $c_r < 0$, we have $\lambda_t > \lambda_{t-1}$, and thus λ_t is decreasing over time in present value.

The cost of extraction is cheaper when we have more reserves. Thus the stock provides a flow of value, and the miner likes getting flow utility sooner rather than later. The reserves here are essentially paying dividends, so having the stock is worth more now than later because they will pay out more dividends this way.

$$\begin{aligned} \lambda_t f_x\left(w_{t+1} x_1\right)+\mu_{x-1}-\mu_t=0 \\ \mu_t-\mu_{t+1}=\lambda_t f_x\left(w_t, x_t\right) \end{aligned}$$

Since $f_x < 0$, we again have $\mu_t > \mu_{t-1}$, and thus μ_t is decreasing over time in present value. Similarly, the depth (or lack thereof) pays dividends by making it easier to add new reserves. So you want shallower mines earlier on.

2 More discrete time resource dynamics: well drilling

(a) In each period after drilling, $(1-\lambda)^{t-1}x$ amount of oil is propelled to the surface. Thus, the amount of revenue the producer makes is

$$\begin{aligned} \sum_{t=1}^{\infty} \delta^t(1-\lambda)^{t-1} \times P &= \sum_{t=1}^{\infty} \delta^{1+}(1-\lambda)^{1-t} \delta P X \\ &= \frac{\delta P x}{1-\delta(1-\lambda)} \end{aligned}$$

(b) Since P and R_t are given, we have a bound on the marginal value of drilling a well. Thus there must reach a point where the costs outweigh the marginal value, and drilling must cease at that point, T. Prior to T, γ_t must be constant for extractors to be indifferent in different periods. Otherwise, all drilling would happen in the period when the present net value is the highest.

(c) As noted in (b), the marginal value of drilling a well must be equalized over time. Since price is fixed, the equilibrium rental rate must decline over time in order for the equality to hold. Since $R(a_t)$ is strictly upward sloping, the rate of drilling must therefore also decline over time.

(d) At the beginning, with the stock of drilled wells increasing, F_t increases since the flow rate from new wells outpaces the decay rate of old wells. Once all wells are drilled, $F_{t+1} = F_t(1-\lambda)$, where the underground pressure is declining in each period, so F_t declines asymptotically to zero.

If $R(a_t)$ were constant, then it would be most profitable to drill everything in the first period, so F_t would increase from period 1 to 2, but then start to fall every period after that since drilling ceases in the second period.

(e)

$$\mathcal{L}=\sum_{t=0}^{T}\left[\delta^t\left(P F_t-a_t R_t\right)+\theta_t\left(F_t(1-\lambda)-F_{t+1}+a_t X\right)+\gamma_t\left(w_t-w_{t+1}-a_t\right)\right]$$

(i) $FOC_{a_t}: -\delta^t R_t + \theta_t X - \gamma_t \leq 0 \quad a_t \geq 0, \quad c.s.$

(ii) $FOC_{F_t}: \quad \delta^t P + \theta_t(1-\lambda) - \theta_{+-1} = 0$

(iii) $FOC_{F_{T+1}}: \quad F_{T+1} \geq 0, \quad \theta_T \geq 0, \quad c.s.$

(iv) $FOC_{w_t}: \quad \gamma_t - \gamma_{t+1} = 0$

(v) $FOC_{w_T} \quad w_{T+1} \geq 0, \quad \gamma_T \geq 0, \quad c.s.$

(f) As long as $a_t > 0$, we have $\theta_t x - \delta^t R_t = \gamma_t$. This states that the marginal value of adding to the capacity constraint, less the cost of drilling, must be equal to the shadow value of the marginal undrilled well. Since P and R_t are bounded, there will come a point when $\theta_t x - \delta^t R_t < \gamma_t$, and drilling must stop.

(g) We have $\gamma_r = \gamma_{t+1}$ from our FOC, so γ_t is constant, call it γ. Since γ is the shadow value of the marginal undrilled well, γW gives the aggregate wealth attainable from all wells.

(h) From $FOC_{F_{T+1}}$ we have $F_{T+1} \geq 0, \theta_T \geq 0$, and $F_{T+1}\theta_T = 0$. It must be that $F_{T+1} > 0$ since it declines asymptotically to 0; thus it must be that $\theta_T = 0$. Using this, we can solve

see answer for intuition

$$\delta^t P + \theta_t(1-\lambda) - \theta_{t-1} = 0$$

$$\begin{aligned}
\theta_{t-1} &= \delta^t p + \theta_t(1-\lambda) \\
&= \delta^t P + (1-\lambda)\left(\delta^{t+1}P + \theta_{+\alpha}(1-\lambda)\right) \\
&= \delta^t P + (1-\lambda)\left(\delta^{t+1}P + (1-\lambda)\left(\delta^{t+2}P + \theta_{t+2}(1-\lambda)\right)\right) \\
&= \delta^t p + (1-\lambda)\delta^{t+1}P + (1-\lambda)^2\delta^{t+2}P + \theta_{t+2}(1-\lambda)^2 \\
&= \delta^t P\left(1 + (1-\lambda)\delta + (1-\lambda)^2\delta^2 + \ldots + (1-\lambda)^{T-t}\delta^{T-t}\right)
\end{aligned}$$

Update by 1 period, we have

$$\begin{aligned}
\theta_t &= \delta^{t=1}P\left(1 + (1-\lambda)f + (1-\lambda)^2\delta^2 + \ldots (1-\lambda)^{T+t-1}\delta^{T-t-1}\right) \\
&= \delta^{t+1}P\sum_{t=0}^{T-t-1}(1-\lambda)^t\delta^t \\
&= \delta^{t+1}P\frac{1-(1-\lambda)^{T-t}\delta^{T-t}}{1-(1-\lambda)\delta}
\end{aligned}$$

And we can see that

$$\lim_{T\to\infty}\theta_t = \frac{P\delta^{t+1}}{1-\delta(1-\lambda)}$$

which yields the same marginal value of adding to the flow constraint as (a) for $t = 0$.

3 Hotelling under monopoly, continuous time

(a) The monopolist problem is:

$$\begin{aligned}
\max_{y_t} \quad & p(y_t)y_t \\
\text{s.t.} \quad & x_t \geq 0 \\
& \dot{x}_t = -y_t
\end{aligned}$$

The current value Hamiltonian is

$$\mathcal{H} = p(y_t)y_t - \mu_t y_t$$

(i) $FOC_{y_t} : p'(y_t)y_t + p(y_t) - \mu_t \leq 0; \quad y_t \geq 0; \quad c.s.$
When $y_t \geq 0$, we see that the marginal benefit of extraction is equal to the marginal value of keeping the resource in the ground. = marginal revenue

(ii) FOC_{x_t}: $\dot{\mu}_t = \rho\mu_t$
The marginal value of the resource is growing at the discount rate, giving us the Hotelling rule.

(iii) TVC: $\lim_{t\to\infty} x_t \geq 0; \quad \lim_{t\to\infty} e^{-rt}\mu_t \geq 0; \quad \lim_{t\to\infty} e^{-rt}x_t\mu_t = 0$

(b) From constant elasticity, we get that

$$\frac{dy_t/y_t}{dp_t/p_t} = \varepsilon$$
$$\implies p'(y_t)y_t = \frac{1}{\varepsilon}p_t$$

Plugging this into our FOC, we get

$$\mu_t = \left(1 + \frac{1}{\varepsilon}\right)p_t$$

What does this tell us about the growth rate of p? We know that at $t = 0$, $\frac{p_0}{\mu_0} = 1 + \frac{1}{\epsilon}$. At time t, we have:

$$\frac{p_0 e^{\gamma t}}{\mu_0 e^{rt}} = 1 + \frac{1}{\varepsilon}$$
$$= \frac{p_0}{\mu_0}$$
$$\Rightarrow \gamma = r$$

and we see that p is growing at the interest rate, just as in the perfect competition model.

$\Rightarrow$ extraction rate is the same

(c) We can write our (inverse) demand function as

$$y_t = a - bp_t$$
$$\Rightarrow p_t = \frac{a - y_t}{b}$$
$$\Rightarrow p_t' = -\frac{1}{b}$$

Plugging this into our FOC, we get

$$-\frac{1}{b}(a - bp_t) + p_t = \mu_t$$
$$-\frac{a}{b} + p_t + p_t = \mu$$
$$p_t = \frac{1}{2}\left(\mu + \frac{a}{b}\right)$$

We know $a, b > 0$, and we can see that p_t is growing at a slower pace than μ_t. The price is higher initially than in perfect competition, and extraction is slower

(d) When facing a demand curve with constant elasticity of demand, the monopolist is facing the same trade-off at every point in time. With a linear demand curve, the monopolist faces a relatively inelastic part of demand during the earlier part of the extraction path. Therefore, the monopolist will want to set a higher price in earlier periods, and the extraction is slower.

4 Uranium production, continuous time

(a)

$$\max \int_0^\infty \left(u(y) - c\left(y_t, x_t\right)\right) e^{-\rho t} dt$$
$$\text{s.t.} \quad R_0 > 0$$
$$y_t \geq 0, \quad R_t \geq 0$$
$$\dot{R}_t = -y_t$$

Write the current value Hamiltonian:

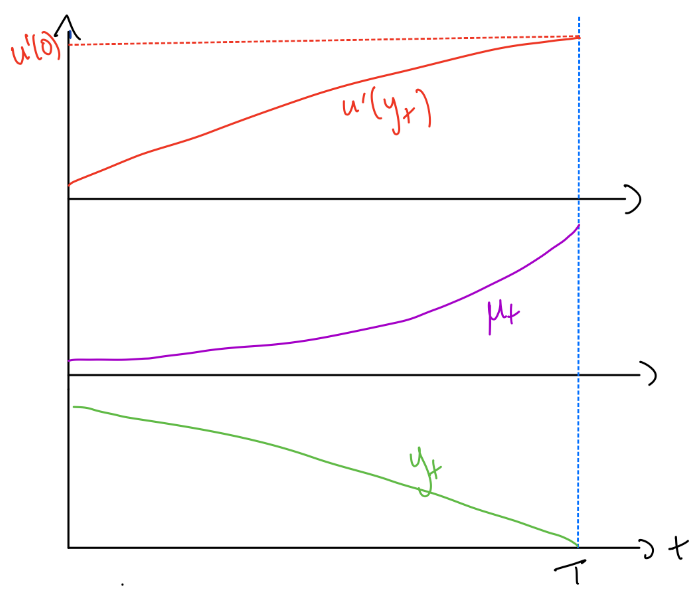
$$\mathcal{H} = u(y_t) - c(y_t) - \mu_t y_t$$

(b) (i) FOC_{y_t}: $u'(y) - c'\left(y_t\right) - \mu_t \leq 0, \quad y_t \geq 0, \quad c.s.$

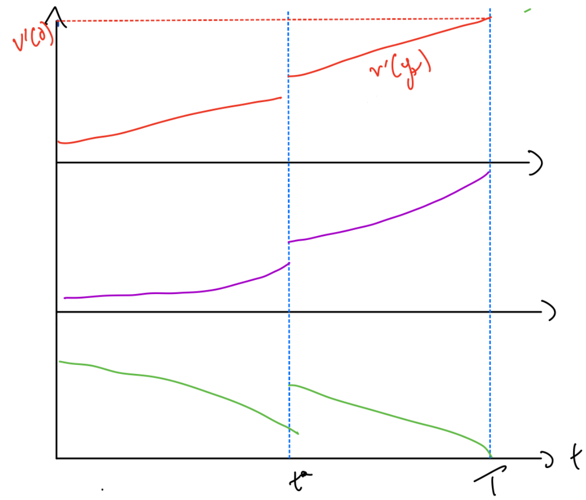
(ii) FOC_{R_t}: $\dot{\mu}_t = \rho\mu_t$

(iii) TVC: $\lim_{t\to\infty} R_t \geq 0; \quad \lim_{t\to\infty} e^{-\rho t}\mu_t \geq 0; \quad \lim_{t\to\infty} e^{-\rho t} R_t \mu_t = 0$

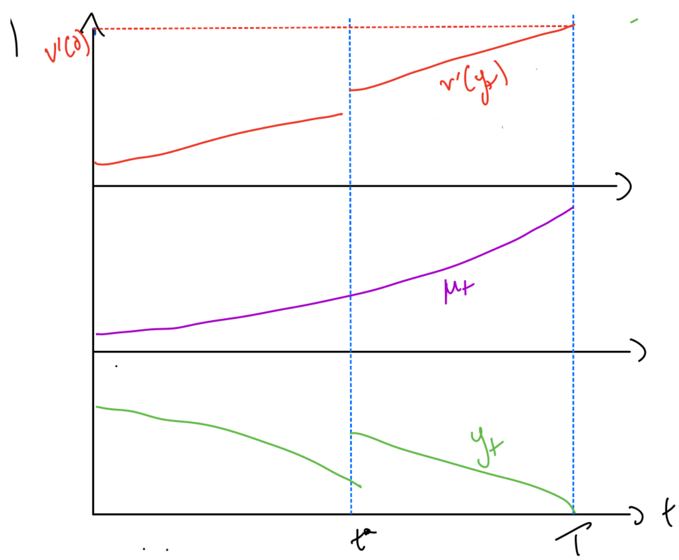
(c) We know μ_t is growing at rate ρ. Then $u'(y_t)$ is also increasing with time and y_t is decreasing. Since $u'(y_t)$ is bounded above by $u'(0)$, extraction must stop at time T.



(d) $u'(y_t)$ jumps by construction, which causes μ_t to jump at t^*, otherwise (i) would not hold. y_t must also jump, otherwise our transversality condition would be violated since we would extract everything before we hit $u'(0)$.



(e) Since we anticipate the jump, μ_t will be smoothed. y_t still jumps because the stock will become more valuable in the future, so we want to push extraction to that time.



(f)

$$\mathcal{H} = u(y_t) - c(y_t) - f(a_t) - \mu_t y_t + \theta_t a_t + \varphi_t (K_t - y_t)$$

(g) (i) $FOC_{y_t} : u'(y_t) - c'(y_t) - \mu_t - \varphi_t \leq 0; \quad y_t \geq 0; \quad c.s.$

(ii) $FOC_{a_t} : -f'(a_t) + \theta_t \leq 0, \quad y_t \geq 0, \quad c.s.$

(iii) $FOC_{R_t} : \dot{\mu}_t = \rho \mu_t$

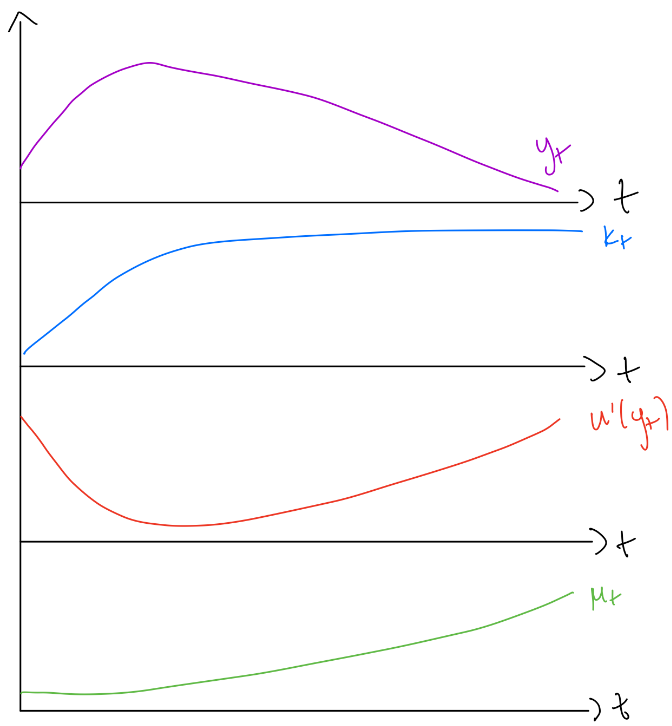
(iv) $FOC_{K_t}: \dot{\theta}_t = \rho\theta_t - \varphi_t$

(v) $TVC_{R_t}: \lim_{t\to\infty} R_t \geq 0; \quad \lim_{t\to\infty} e^{-\rho t}\mu_t \geq 0; \quad \lim_{t\to\infty} e^{-\rho t} R_t \mu_t = 0$

(vi) $TVC_{K_t}: \lim_{t\to\infty} K_t \geq 0; \quad \lim_{t\to\infty} e^{-\rho t}\theta_t \geq 0; \quad \lim_{t\to\infty} e^{-\rho t} K_t \theta_t = 0$

(h) Once capacity is built up, it never decreases, so K_t cannot be zero. Therefore, in order for the transversality condiiton to hold it must be that $lim_{t\to\infty}\theta_t = 0$. Intuitively, once we run out of uranium, the excess capacity is no longer useful, and thus the marginal value of capacity, θ_t, becomes zero. See answer

(i) Since y_t is now constrained by capacity, the two follow the same path until enough capacity has been built store all y_t. During this time, $u'(y_t)$ is decreasing, since u' is decreasing in y_t. Once enough capacity has been built, the paths all follow the shape shown in (c).



5 Storage and uncertainty

(a) Then the current value Hamiltonian is

$$\mathcal{H} = P_t Q_t e^{-\alpha t} - \mu_t Q_t$$

(i) $FOC_{Q_t}: P_t e^{-\alpha t} - \mu_t \leq 0; \quad Q_t \geq 0; \quad c.s.$

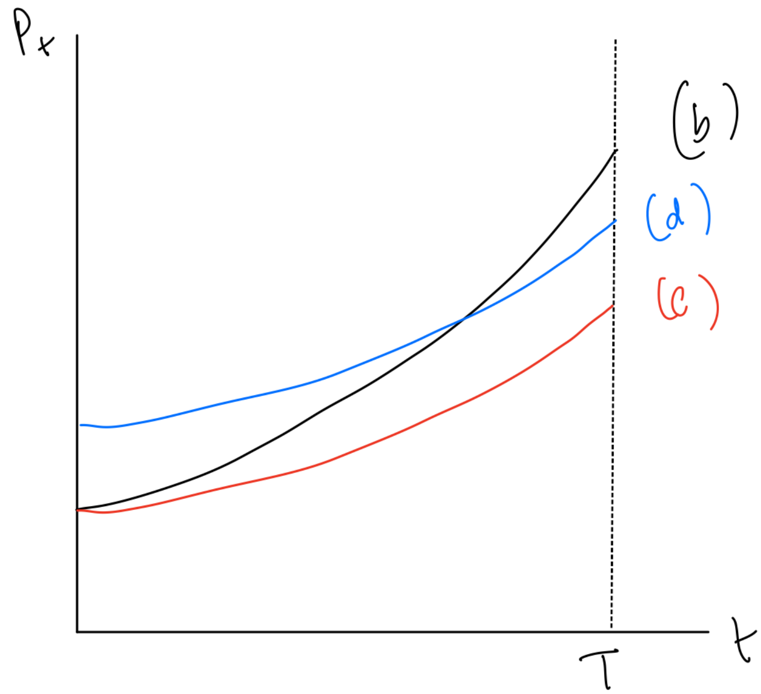
(ii) $FOC_{S_t}: \dot{\mu}_t = \rho\mu_t$

(iii) TVC: $S_T \geq 0; \quad \mu_T e^{\rho T} \geq 0; \quad S_T \mu_T e^{-\rho T} = 0$

(b) From FOC_{Q_t}, for $Q_t > 0$, we have

$$\begin{aligned} P_t &= \mu_t e^{\alpha t} \\ &= \mu_0 e^{(\alpha+\rho)t} \end{aligned}$$

so price is growing at $\alpha + \rho$.



(c) The probably in any period that no substitute has been discovered at that point (or anytime before) is $e^{-\alpha t}$, at which point the firm can earn profit $P_t = \mu_t e^{-\alpha t}$. There's a $1 - e^{-\alpha t}$ chance a substitute is discovered, at which point the firm cannot sell the unobtanium anymore. So the expected price is:

$$\begin{aligned} E[P_t] &= e^{-\alpha t} \times \mu e^{\alpha t} + (1 - e^{-\alpha t}) \times 0 \\ &= \mu_0 e^{\rho t} \end{aligned}$$

And we see a Hotelling result for the expected price.

(d) Without the threat of a discovery of a substitute, initial price is higher, and the initial extraction rate is lower. The growth rate is now back to ρ.