MINE DEEPENING

a) State vars are r_t & x_t. Controls are q_t & w_t

b) Depth is bad here: having a deeper mine means that it's harder to add reserves. So constraint is $x_{t+1} \geq x_t + w_t$

$$\Rightarrow L = \sum_{t=1}^{T} \delta^t \left(pq_t - c(q_t, r_t) - k(w_t)\right)$$
$$+ \sum_{t=1}^{T} \lambda_t \left(r_t - r_{t+1} + F(w_t, x_t) - q_t\right)$$
$$+ \sum_{t=1}^{T} \mu_t \left(x_{t+1} - x_t - w_t\right)$$

FOC_{q_t}: $\delta^t(p - c_q(q_t, r_t)) - \lambda_t \leq 0$, $q_t \geq 0$, c.s.

FOC_{w_t}: $-\delta^t k'(w_t) + \lambda_t F_w(w_t, x_t) - \mu_t \leq 0$, $w_t \geq 0$, c.s.

FOC_{r_t}: $-\delta^t c_r(q_t, r_t) + \lambda_t - \lambda_{t-1} = 0$

FOC_{x_t}: $\lambda_t F_x(w_t, x_t) - \mu_t + \mu_{t-1} = 0$

TVC_r $(FOC_{r_{T+1}})$: $\lambda_T \geq 0$, $r_{T+1} \geq 0$, $\lambda_T r_{T+1} = 0$

TVC_x $(FOC_{x_{T+1}})$: $\mu_T \leq 0$, $x_{T+1} \geq 0$, $\mu_T x_{T+1} = 0$

$\Rightarrow \mu_T = 0$

c) FOC_{w_t}: Says that the value of additional reserves $\lambda_t F_w(w_t, x_t)$ less the marginal digging cost $\delta^t k'(w_t)$ equals the marginal cost of increasing depth μ_t

d) FOC_{r_t} has $\lambda_t - \lambda_{t-1} = \delta^t c_r(q_t, r_t) < 0$

λ_t decreases over time because reserves pay "dividends" by making production cheaper. These dividends mean that reserves today are worth more than reserves in the future, in present value.

FOC_{x_t} has $\mu_t - \mu_{t-1} = \lambda_t F_x(w_t, x_t) < 0$

μ_t is the marginal value of having a shallow mine. Shallowness pays "dividends" by improving the rate at which digging increases reserves. Thus, μ_t must decrease in present value.

DISCRETE TIME WELL DRILLING

a) Gross revenue $= P\sum_{t=1}^{\infty}\delta^t X(1-\lambda)^{t-1} = \frac{PX}{1-\lambda}\sum_{t=1}^{\infty}\delta^t(1-\lambda)^t$

$$= \frac{PX\delta}{1-\delta(1-\lambda)}$$

b) Taking P and $\{R_t\}$ as given, firms wish to drill all of their wells when PDV is greatest. The only way to avoid all drilling happen at the same time is to have $\{R_t\}$ such that firms are indifferent about when to drill at all times t such that total drilling >0.

Because P is constant and $\{R_t\}$ is bounded below, the current value of an undrilled well cannot rise forever. Thus, drilling must stop at some finite time.

c) Since P is constant, R_t must decrease so that the return to drilling increases at r. R_t decreasing implies a decreasing a_t

d) When a_t is initially large, the flowrate from new wells will exceed the decline from old wells, so $\Delta F_t > 0$.

If $R(a_t)$ were constant, all wells would be drilled at $t=0$, so flow would decline from $t=1$ to $t=2$.

The aggregate flow must decline to zero because, once drilling stops, $F_{t+1} = F_t(1-\lambda)$, which is exponential decline.

e) $L = \sum_{t=0}^{T} \left[\delta^t (pF_t - a_t R_t) + \theta_t (F_t(1-\lambda) - F_{t+1} + a_t X) + \gamma_t (w_t - w_{t+1} - a_t) \right]$

FOC_{F_t}: $\delta^t p + \theta_t (1-\lambda) - \theta_{t-1} = 0$

FOC_{a_t}: $-\delta^t R_t + \theta_t X - \gamma_t \leq 0$, $a_t \geq 0$, c.s.

FOC_{w_t}: $\gamma_t - \gamma_{t-1} = 0$

$FOC_{F_{T+1}}$: $\theta_T \geq 0$, $F_{T+1} \geq 0$, $\theta_T F_{T+1} = 0$

$FOC_{w_{T+1}}$: $\gamma_T \geq 0$, $w_{T+1} \geq 0$, $\gamma_T w_{T+1} = 0$

f) FOC_{a_t} says that the (marginal) value of the new flow capacity $\theta_t X$ equals the drilling cost plus the shadow value γ_t of an undrilled well. This equation explains why drilling must stop in finite time ~~($\theta_t X - \delta^t R_t$ is bounded but γ~~ ($\theta_t X(1+r)^t - R_t$ is bounded but $\gamma_t (1+r)^t$ rises without bound). To satisfy the indifference condition while $a_t > 0$, a_t must decrease so that R_t decreases

g) FOC_{w_t} implies that γ_t is constant. So every well has the same value. Also, $\frac{\partial L}{\partial w_0} = \gamma_0 = \gamma$. Thus, total wealth is γw.

h) F_{T+1} can't be zero due to exponential declines

So TVC $\Rightarrow \theta_T = 0$. Intuitively, any leftover capacity at T isn't worth anything, because the extraction period is over.

$$\theta_{T-1} = \theta_T(1-\lambda) + \delta^T p$$

$$\theta_{T-2} = \theta_{T-1}(1-\lambda) + \delta^{T-1} p$$

$$\Rightarrow \theta_{T-2} = \theta_T(1-\lambda)^2 + \delta^T p(1-\lambda) + \delta^{T-1} p$$

$$\theta_{T-3} = \theta_{T-2}(1-\lambda) + \delta^{T-2} p$$

$$\Rightarrow \theta_{T-3} = \theta_T(1-\lambda)^3 + \delta^T p(1-\lambda)^2 + \delta^{T-1} p(1-\lambda) + \delta^{T-2} p$$

Use $t = T-(T-t)$

$$\Rightarrow \theta_t = \underbrace{\theta_T(1-\lambda)^{T-t}}_{=0} + p\left(\delta^T(1-\lambda)^{T-t-1} + \delta^{T-1}(1-\lambda)^{T-t-2} + \ldots + \delta^{t+1}\right)$$

$$\Rightarrow \theta_t = p\delta^{t+1}\left(1 + \delta(1-\lambda) + \delta^2(1-\lambda)^2 + \ldots + \delta^{T-(t+1)}(1-\lambda)^{T-(t+1)}\right)$$

$$\Leftrightarrow \theta_t = \frac{p\delta^{t+1}\left(1 - \delta^{T-(t+1)}(1-\lambda)^{T-(t+1)}\right)}{1-\delta(1-\lambda)}$$

$$\Rightarrow \lim_{T\to\infty} \theta_t = \frac{p\delta^{t+1}}{1-\delta(1-\lambda)}$$

Same as (a) for $t=0$, expressed as \$ per unit of capacity

HOTELLING MONOPOLY

a) $\max_{\{y_t\}} \int_0^\infty p(y_t)y_t e^{-\rho t} dt$ s.t. $\dot{x}_t = -y_t$; $y_t \geq 0$, $\lim_{t\to\infty} x_t \geq 0$

$$H = p(y_t)y_t - \mu_t y_t$$

FOC_{y_t}: $p'(y_t)y_t + p(y_t) - \mu_t \leq 0$; $y_t \geq 0$; c.s.

FOC_{x_t}: $\dot{\mu}_t = \rho \mu_t$

TVC: $\lim_{t\to\infty} x_t \geq 0$, $\lim_{t\to\infty} \mu_t e^{-\rho t} \geq 0$, $\lim_{t\to\infty} x_t \mu_t e^{-\rho t} = 0$

FOC_{y_t} says that marginal revenue should increase at ρ to maximize profits. This is the monopolist's intertemporal arbitrage condition.

b) $\varepsilon = \frac{dy}{dp} \cdot \frac{p}{y}$ constant, < 0

$$FOC_{y_t} \Leftrightarrow \frac{dp_t}{dy_t} \cdot \frac{y_t}{p_t} + \frac{p_t}{p_t} - \frac{\mu_t}{p_t} \leq \frac{0}{p_t}$$

$$\Leftrightarrow p_t\left(\frac{1}{\varepsilon}\right) + p_t - \mu_t \leq 0$$

$$\Leftrightarrow p_t\left(1 + \frac{1}{\varepsilon}\right) = \mu_0 e^{\rho t}$$ whenever $y_t > 0$ (which is true $\forall t$ given constant elasticity demand)

$1 + \frac{1}{\varepsilon}$ is a constant

$\Rightarrow p_t(y_t)$ rises at ρ, just as in perfect competition

$\Rightarrow$ To satisfy the same exhaustion constraint (total stock is S), the path $\{y_t\}$ must be the <u>same</u> as with perfect competition.

c) Let demand be $P_t = a - by_t \iff y_t = \frac{a}{b} - \frac{P_t}{b}$

FOC $\Rightarrow a - 2by_t - \mu_t \leq 0$

$\iff a - 2a + 2P_t - \mu_t \leq 0$

$\iff P_t = \frac{a}{2} + \frac{\mu_t}{2}$ when $y_t > 0$

Since $a > 0$, P_t increases more slowly than ρ

$\Rightarrow$ The monopolist extracts more slowly than in perfect competition. The initial price is higher than in perfect competition.

d) The monopolist's ability to exercise market power is constrained by the stock constraint. Less production today implies more production tomorrow.

With constant elasticity demand, the static markup incentive is the same no matter how much stock is left. Any deviation from the competitive price path then implies that MR no longer increases at ρ.

With linear demand, demand is less elastic when p is low and q is high; i.e., early in the extraction path. Thus, the incentive to exercise market power is stronger early, so the initial production rate is low.

URANIUM PRODUCTION, CONTINUOUS TIME

a) $\max_{\{y_t\}} \int_0^\infty (u(y_t) - c(y_t)) e^{-\rho t} \, dt$ s.t. $\dot{R}_t = -y_t$

$y_t \geq 0$

$\lim_{t \to \infty} R_t \geq 0$

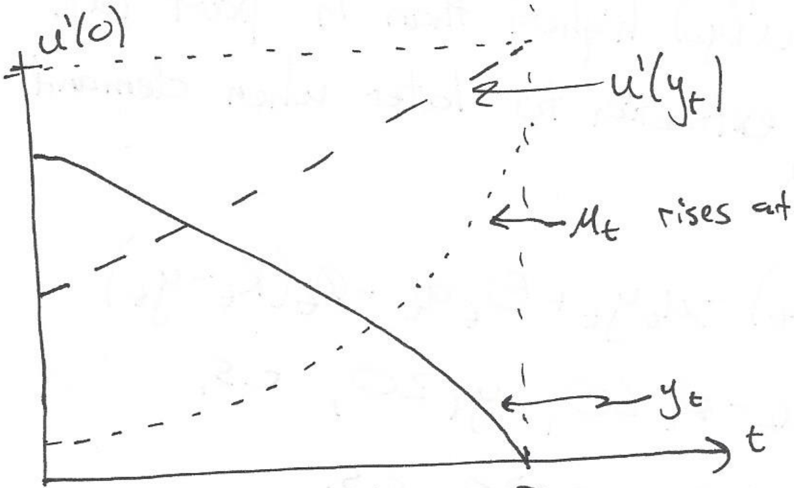
$H = u(y_t) - c(y_t) - \mu_t y_t$

b) FOC_{y_t}: $u'(y_t) - c'(y_t) - \mu_t \leq 0$, $y_t \geq 0$, c.s.

FOC_{R_t}: $\dot{\mu}_t = \rho \mu_t$

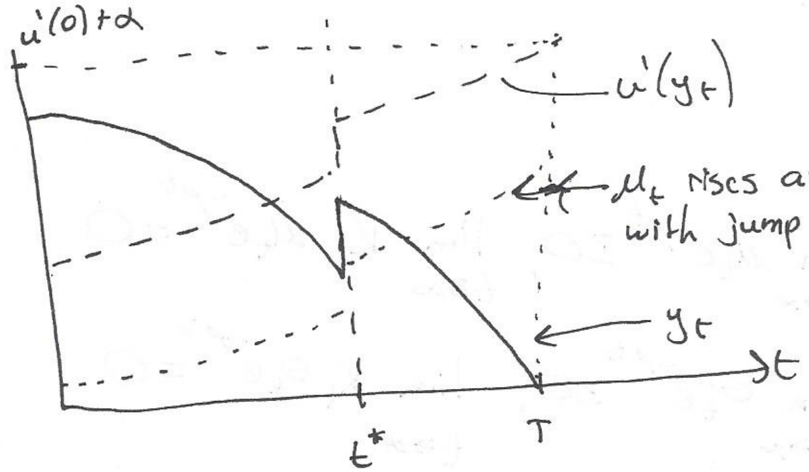
TVC: $\lim_{t \to \infty} R_t \geq 0$, $\lim_{t \to \infty} \mu_t e^{-\rho t} \geq 0$, $\lim_{t \to \infty} R_t \mu_t e^{-\rho t} = 0$

c)



Extraction must cease at some t because $u'(y_t)$ is bounded above

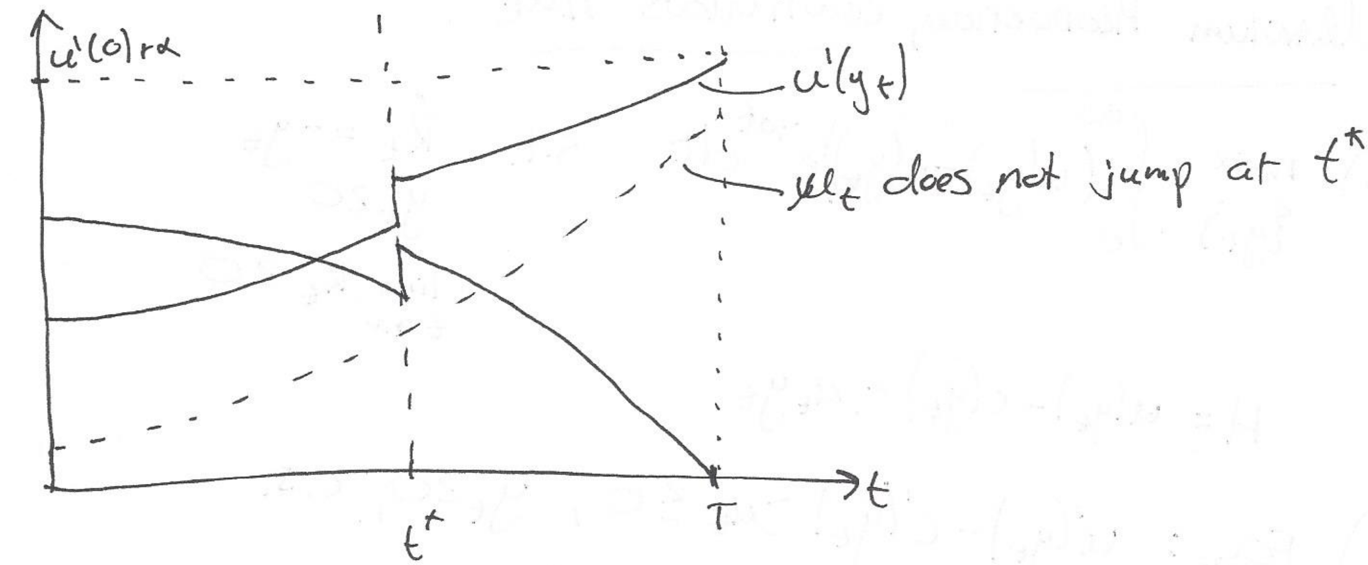
d)



μ_t must jump at t^* because stock is worth more. If it didn't jump, extraction would happen too quickly.

y_t must jump because, if not, not all uranium would be extracted before $u'(0)$ was reached.

e)



μ_t cannot jump at t^*, otherwise there would be an arbitrage opportunity. The value μ_0 is greater than in part (d)

y_0 is lower (and $u'(y_0)$ higher) than in part (d).
We want to delay extraction to later when demand will be larger.

f) $H = u(y_t) - c(y_t) - F(a_t) - \mu_t y_t + \theta_t a_t + \phi_t (K_t - y_t)$

g) FOC_{y_t}: $u'(y_t) - c'(y_t) - \mu_t - \phi_t \leq 0$, $y_t \geq 0$, c.s.

FOC_{a_t}: $-F'(a_t) + \theta_t \leq 0$, $a_t \geq 0$, c.s.

FOC_{R_t}: $\dot{\mu}_t = \rho \mu_t$

FOC_{K_t}: $\dot{\theta}_t = \rho \theta_t - \phi_t$

TVC_R: $\lim_{t \to \infty} R_t \geq 0$, $\lim_{t \to \infty} \mu_t e^{-\rho t} \geq 0$, $\lim_{t \to \infty} R_t \mu_t e^{-\rho t} = 0$

TVC_K: $\lim_{t \to \infty} K_t \geq 0$, $\lim_{t \to \infty} \theta_t e^{-\rho t} \geq 0$, $\lim_{t \to \infty} K_t \theta_t e^{-\rho t} = 0$

h) Since $f'(0) = \infty$, the initial rate of capacity investment will be strictly positive (else welfare is zero forever).

K_t can never decrease, so $\lim_{t \to \infty} K_t \geq 0$

Then $TVC_K \Rightarrow \lim_{t \to \infty} \theta_t e^{-\rho t} = 0$

Next, $u'(0) > c'(0)$ and TVC_R together imply that $\mu_t > 0 \;\; \forall t$ and $\lim_{t \to \infty} R_t = 0$

(o/w, FOC_{y_t} is eventually violated)

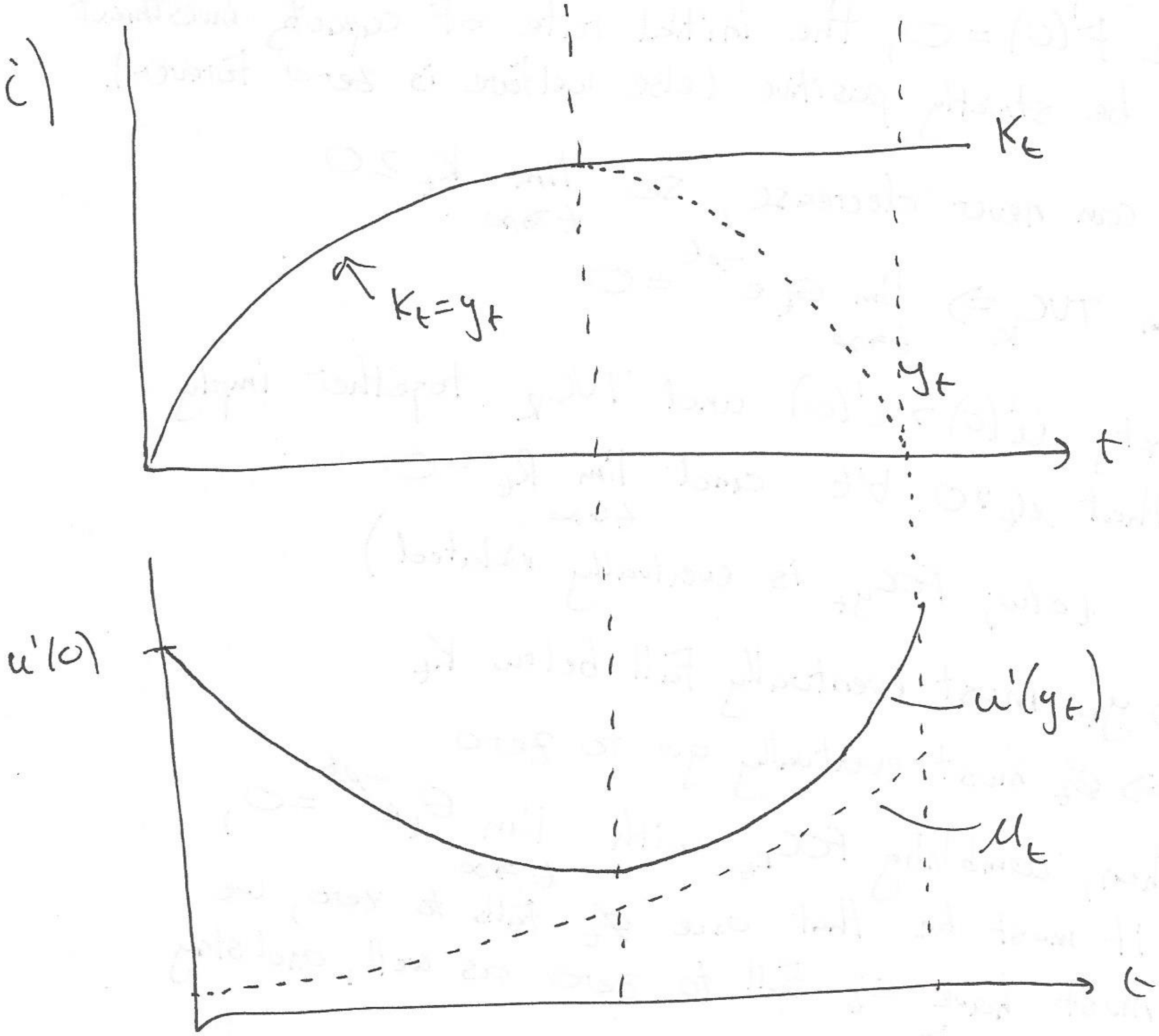
$\Rightarrow y_t$ must eventually fall below K_t

$\Rightarrow \phi_t$ must eventually go to zero

Then, combining FOC_{K_t} with $\lim_{t \to \infty} \theta_t e^{-\rho t} = 0$, it must be that once ϕ_t falls to zero, we must have θ_t fall to zero as well, and stay at zero forever.

Intuitively, there will eventually be excess capacity as reserves run out. The marginal value of ~~a~~ capacity, θ_t, must then be zero.

i)



STORAGE AND UNCERTAINTY

a) $$\max_{\{q_t\}} \int_0^T p_t q_t e^{-(\alpha+r)t}\,dt \quad \text{s.t.} \quad \dot{s}_t = -q_t,\; s_0 = S,\; q_t \geq 0,\; s_T \geq 0$$

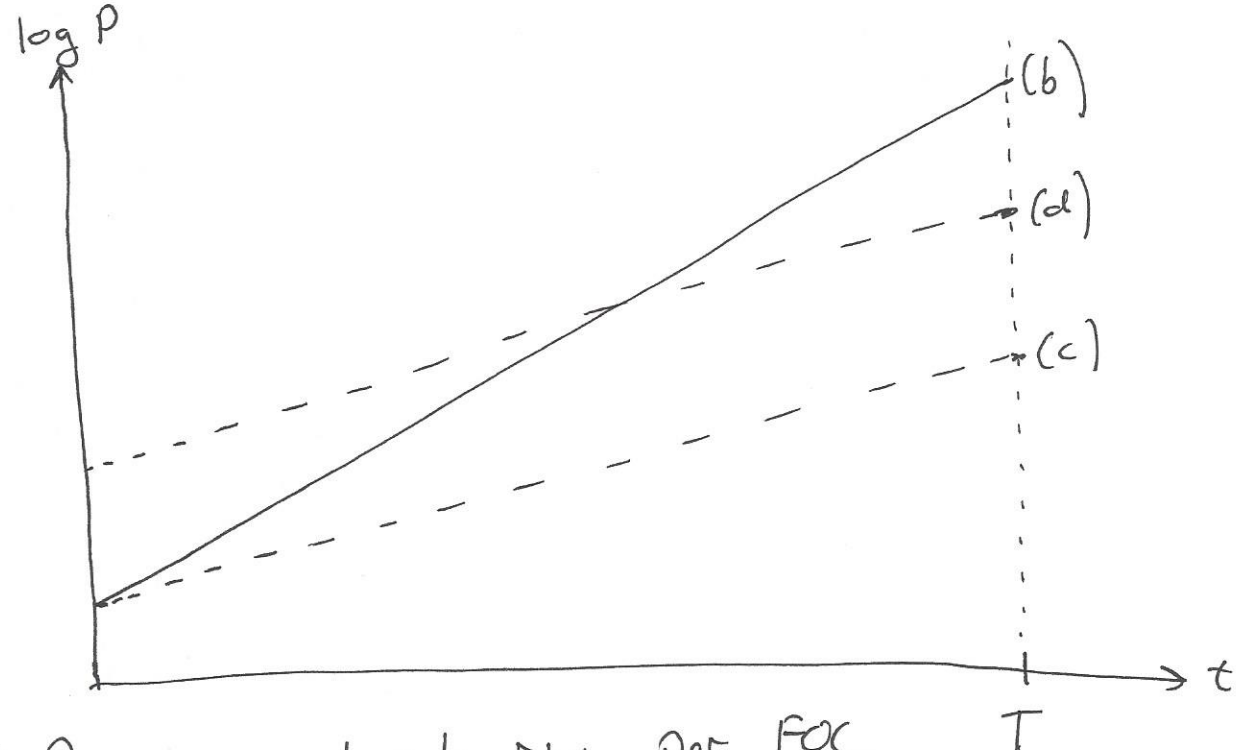
$$H = p_t q_t e^{-\alpha t} - \mu_t q_t$$

$$FOC_{q_t}: \; p_t e^{-\alpha t} - \mu_t \leq 0, \; q_t \geq 0, \; \text{c.s.}$$

$$FOC_{s_t}: \; \dot{\mu}_t = r\mu_t$$

$$TVC: \; s_T \geq 0, \; \mu_T e^{-rT} \geq 0, \; s_T \mu_T e^{-rT} = 0$$

b-d)



b) Price grows at rate $r+\alpha$, per FOC_{q_t}

c) $E[p_t] = e^{-\alpha t}\mu_0 e^{(\alpha+r)t} + (1-e^{-\alpha t})\cdot 0$
$= \mu_0 e^{rt}$. Grows at rate r, starting from same initial price

d) Price grows at r, per FOC_{q_t}.
Initial price must be higher, and initial extraction rate lower, to satisfy the exhaustion constraint