Comprehension Check 1

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Question 1

In the paper, the authors define population of a city by

$$L(c) = S(\bar{\tau}(c)) = \pi\bar{\tau}^2$$
$$\Rightarrow \bar{\tau} = \sqrt{\frac{L}{\pi}}$$

The paper also shows, from Lemma 1, that

$$\begin{aligned}\underline{\gamma} &\equiv A(c)T(\bar{\tau}(c)) \\ &= 1 \times (d_1 - d_2\bar{\tau}) \\ &= d_1 - d_2\sqrt{\frac{L}{\pi}}\end{aligned}$$

Question 2

From Lemma 3, we are given the function for rental rates:

$$r_\Gamma(\gamma) = \int_{\underline{\gamma}}^{\gamma} G(K(x))dx$$

Combining this with Lemma 2 and the given functional form for $G(\omega)$, we can rewrite this as

$$r_\Gamma(\gamma) = \int_{\underline{\gamma}}^{\gamma} gF^{-1}\left(\frac{L - S_\Gamma(x)}{L}\right)dx$$

In this single country case, we can rewrite $S_\Gamma(x)$

$$\begin{aligned}S_\Gamma(\gamma) &= S\left(T^{-1}\left(\frac{\gamma}{A(c)}\right)\right) \\ &= S(T^{-1}(\gamma)) \\ &= S\left(\frac{d_1 - \gamma}{d_2}\right) \\ &= \pi\left(\frac{d_1 - \gamma}{d_2}\right)^2\end{aligned}$$

We want to evaluate this for the γ such that $\tau = 0$. Call this $\bar{\gamma}$. Plugging in given equations, we get

$$\begin{aligned}\bar{\gamma} &= d_1 - d_2\underline{\tau} \\ &= d_1\end{aligned}$$

Thus the rental rate can be written as

$$r(\bar{\gamma}) = \int_{\underline{\gamma}}^{d_1} gF^{-1}\left(\frac{L-\pi\left(\frac{d_1-\gamma}{d_2}\right)^2}{L}\right)dx$$

We can now use the fact that

$$\begin{aligned}\omega \sim Unif[\underline{\omega},\bar{\omega}] &\Rightarrow F(\omega) \equiv \frac{\omega-\underline{\omega}}{\bar{\omega}-\underline{\omega}}\\ &\Rightarrow F^{-1}(y) = y(\bar{\omega}-\underline{\omega})+\underline{\omega}\end{aligned}$$

We finally get the expression

$$\begin{aligned}r(\bar{\gamma}) &= \int_{\underline{\gamma}}^{d_1} g\left[\left(\frac{L-\pi\left(\frac{d_1-x}{d_2}\right)^2}{L}\right)(\bar{\omega}-\underline{\omega})+\underline{\omega}\right]dx\\ &= g(\bar{\omega}-\underline{\omega})\int_{\underline{\gamma}}^{d_1}dx + g\underline{\omega}\int_{\underline{\gamma}}^{d_1}dx - \frac{g\pi(\bar{\omega}-\underline{\omega})}{d_2^2L}\int_{\underline{\gamma}}^{d_1}(d_1-x)^2dx\\ &= (g(\bar{\omega}-\underline{\omega})x)\Big|_{\underline{\gamma}}^{d_1} + (g\underline{\omega}x)\Big|_{\underline{\gamma}}^{d_1} + \frac{g\pi(\bar{\omega}-\underline{\omega})}{Ld_2^2}\left(\frac{1}{3}(d_1-x)^3\right)\Big|_{\underline{\gamma}}^{d_1}\\ &= g\bar{\omega}d_1 - g\bar{\omega}\underline{\gamma} - \frac{g\pi(\bar{\omega}-\underline{\omega})}{Ld_2^2}\left(\frac{1}{3}(d_1-\underline{\gamma})^3\right)\\ &= g\bar{\omega}d_1 - g\bar{\omega}\left(d_1-d_2\sqrt{\frac{L}{\pi}}\right) + \frac{g\pi(\bar{\omega}-\underline{\omega})}{3Ld_2^2}d_2^3\left(\frac{L}{\pi}\right)^{3/2}\\ &= g\bar{\omega}d_2\sqrt{\frac{L}{\pi}} + \frac{gd_2(\bar{\omega}-\underline{\omega})}{3}\sqrt{\frac{L}{\pi}}\end{aligned}$$

Where the fifth line uses the results from part 1.

Question 3

Using the results from part 2, we know the rent schedule is

$$\begin{aligned}r(\gamma) &= \int_{\underline{\gamma}}^{\gamma} g\left[\left(\frac{L-\pi\left(\frac{\gamma-x}{d_2}\right)^2}{L}\right)(\bar{\omega}-\underline{\omega})+\underline{\omega}\right]dx\\ &= g(\bar{\omega}-\underline{\omega})\int_{\underline{\gamma}}^{\gamma}dx + g\underline{\omega}\int_{\underline{\gamma}}^{\gamma}dx - \frac{g\pi}{d_2^2L}\int_{\underline{\gamma}}^{\gamma}(\gamma-x)^2\,dx\\ &= g\bar{\omega}\gamma - g\bar{\omega}\underline{\gamma} + \frac{g\pi(\bar{\omega}+\underline{\omega})}{3d_2^2L}\left((d_1-\gamma)^3-(d_1-\underline{\gamma})^3\right)\\ &\Rightarrow \frac{\partial r(\gamma)}{\partial g} = \bar{\omega}\gamma - \bar{\omega}\underline{\gamma} + \frac{\pi(\bar{\omega}+\underline{\omega})}{3d_2^2L}\left((d_1-\gamma)^3-(d_1-\underline{\gamma})^3\right)\end{aligned}$$

The right hand side of the last line is positive, so an increase in g will lead to an increase in the rent schedule.

The equilibrium utility for skill level ω is:

$$U(c,\tau,\sigma;\omega) = A(c)T(\tau)H(\omega,\sigma)p(\sigma) - r(c,\tau)$$

We have defined

$$G(\omega) \equiv H(\omega, M(\omega))p(M(\omega))$$

So we can rewrite utility as

$$\begin{aligned} U(c,\tau,\sigma;\omega) &= T(\tau)G(\omega) - r(c,\tau) \\ \Rightarrow \quad U(\gamma;\omega) &= \gamma g\omega - r(\gamma) \\ \Rightarrow \frac{\partial U(\gamma;\omega)}{\partial g} &= \gamma\omega - \frac{\partial r(\gamma)}{\partial g} \end{aligned}$$

Similar to the rent schedule, an increase in g will lead to an increase in utility.

Question 4

First note that $K(\underline{\gamma}) = \underline{\omega}$ and $K(\bar{\gamma}) = \bar{\omega}$.

Then from part 3, we can find

$$\frac{\partial r(\gamma)}{\partial \underline{\omega}} = \frac{g\pi}{3d_2^2 L}\left((d_1-\gamma)^3 - \left(d_1 - \underline{\gamma}\right)^3\right)$$

For $\gamma > \underline{\gamma}$, an increase in $\underline{\omega}$ leads to an increase in the rent at that location. With no accompanying increase in income, this increase in rents leads to a decrease in utility for a worker with $\bar{\omega}$.

Question 5

Similarly,

$$\frac{\partial r(\gamma)}{\partial \bar{\omega}} = g(\gamma - \underline{\gamma}) + \frac{g\pi}{3d_2^2 L}\left((d_1-\gamma)^3 - \left(d_1 - \underline{\gamma}\right)^3\right)$$

Again, an increase in $\bar{\omega}$ increases rent for every $\gamma > \underline{\gamma}$, decreasing utility. At $\underline{\omega}$, there is no change in rent or utility because $\frac{\partial r(\underline{\gamma})}{\partial \bar{\omega}} = 0$.